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Research article

New results on Schur complements for Dashnic–Zusmanovich-type matrices and applications

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Abstract: The Schur complement has a wide range of applications. This paper establishes more general conditions under which the Schur complement of a Dashnic–Zusmanovich-type (DZT) matrix remains a DZT matrix. The proposed criteria cover many cases not previously addressed. We also characterize the cases in which the Schur complement is a strictly diagonally dominant matrix or a Dashnic–Zusmanovich (DZ) matrix. Numerical examples demonstrate that our results provide quite general criteria. As an application, we obtain an estimate for the determinant of DZT matrices and solve a large system of equations by a Schur-based method.

Keywords: Dashnic–Zusmanovich-type matrix; Schur complement; determinant
Mathematics Subject Classification: 15A45, 15A48

1. Introduction

The Schur complement has a wide range of applications in numerical analysis, control theory, and matrix theory [1, 2]. For instance, Schur complements can be used to reduce the order of large linear equations [3–5], compute the determinant of a matrix [6, 7], and estimate the infinity norm of the inverse [8–10]. It is well-known that the Schur complement of an H-matrix is also an H-matrix [1]. Moreover, this closure property holds for several subclasses of H-matrices, such as strictly diagonally dominant (SDD) [1, 11], γ-SDD [12], doubly strictly diagonally dominant (DSDD) [13, 14], and Dashnic–Zusmanovich (DZ) matrices [15, 16]. However, the Schur complement of a matrix from some subclasses does not necessarily belong to the same subclass. For example, Schur complements of Nekrasov matrices may not be Nekrasov matrices [16–18], and Schur complements of SDD_1 matrices may not be SDD_1 matrices [7, 19, 20].

The class of Dashnic–Zusmanovich-type (DZT) matrices was introduced by Zhao et al. [21], who proved that DZT matrices are H-matrices. Several results on DZT matrices have been established, including bounds on the infinity norm of the inverse and studies on linear complementary problems [22, 23], partitioned DZT matrices with applications [24], and diagonal-Schur complements [25]. For a DZT matrix A and an index set α, Li et al. [2, Theorem 3.4] showed that the Schur complement A/α is an SDD matrix if $N^+(A) \subseteq \alpha$, and a DZT matrix if $\alpha \subseteq N^+(A)$. Zeng et al. [23, Theorem 5] obtained that the Schur complement of a DZT matrix is an SDD matrix if $\bigcup\limits_{i\in N^-(A)} \Gamma_i(A) \subseteq \alpha$. Here,

$$r_i(A) = \sum_{j\neq i} |a_{ij}|,$$

$$N^+(A) = \{i \in [n] : |a_{ii}| > r_i(A)\},\ \ N^-(A) = [n] \setminus N^+(A)$$
$$\Gamma_i(A) = \{j \neq i : (|a_{ii}| - r_i^j(A))|a_{jj}| > |a_{ij}|r_j(A)\},$$

where $r_i^j(A) = \sum\limits_{k\neq i,j} |a_{ik}|$.

However, in the general case, the closure property of the Schur complement for DZT matrices remains open. Consider the following matrix:

$$A = \begin{bmatrix} 27 & 1 & 1 & 2 \\ 5 & 3 & 0 & 2 \\ 4 & 4 & 7 & 1 \\ 4 & 5 & 2 & 29 \end{bmatrix}. \tag{1.1}$$

By computation, we know that A is a DZT matrix with

$$N^+(A) = \{1, 4\}, \quad N^-(A) = \{2, 3\}, \quad \Gamma_2(A) = \Gamma_3(A) = \{1\}.$$

There are $2^4 - 2 = 14$ nonempty proper subsets of [4]. Among them, there are 6 index subsets ($\{2\}$, $\{3\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{2, 3, 4\}$) such that $\alpha \nsubseteq N^+(A)$, $N^+(A) \nsubseteq \alpha$, $\bigcup\limits_{i\in N^-(A)} \Gamma_i(A) \nsubseteq \alpha$. However, the Schur complements with respect to these six index subsets are also DZT matrices; as will be shown in Example 6.3, our results prove that all Schur complements of A (including these six) are DZT matrices. This finding suggests that the known conditions are not exhaustive, and a more comprehensive theory remains to be developed. This paper aims to establish more general sufficient conditions for the Schur complement of a DZT matrix to remain a DZT matrix, thereby extending the existing criteria in the literature. The conditions $N^+(A) \subseteq \alpha$, $\alpha \subseteq N^+(A)$, $\bigcup_{i\in N^-(A)} \Gamma_i(A) \subseteq \alpha$ are recovered as special cases of our criteria. We apply our results to estimating determinants of DZT matrices and solving large-scale systems of equations.

For a given matrix A and a set α, to investigate the conditions for A/α to be a DZT matrix, our consideration begins with the following two questions: (1) Under what conditions does u belong to $N^+(A/\alpha)$? (2) Under what conditions does $\Gamma_u(A/\alpha) \neq \emptyset$ hold for $u \in N^-(A/\alpha)$? By examining the relationship between $N^-(A)$ and α, we introduce two subsets of $N^-(A)$ (Θ and Υ) and use them to investigate these two questions. For the first question, we identify a subset S of $N^+(A/\alpha)$. For the second, we study the conditions under which $\Gamma_u(A/\alpha) \neq \emptyset$ holds for all $u \notin S$. Consequently, we establish a set of conditions that ensure that A/α is a DZT matrix. Numerical examples demonstrate that our results provide considerably more general criteria than those in [2] and [23].

The rest of this paper is organized as follows. Section 2 introduces notations and lemmas. In Section 3, we provide a partial characterization of $N^+(A/\alpha)$. Section 4 presents conditions under which the Schur complement of a DZT matrix remains a DZT matrix. Our criteria show that the DZT property is preserved under rather mild conditions; moreover, under additional assumptions, the Schur complement can even be an SDD or a DZ matrix. As applications of our main results, in Section 5, we derive bounds for the determinants of DZT matrices and solve a large system of equations using a Schur-based method. Section 6 provides extensive numerical examples to validate our theoretical results.

2. Preliminaries

Denote by $C^{n\times n}$ the set of all $n \times n$ complex matrices, and let $[n] := \{1, 2, \dots, n\}$. For given subsets α and β of $[n]$, $A(\alpha,\beta)$ denotes the submatrix of A lying in the rows indexed by α and the columns indexed by β. $A(\alpha, \alpha)$, abbreviated to $A(\alpha)$, is called the principal submatrix of A. Let $\bar{\alpha} = [n] \setminus \alpha$ be the complement of α. If $A(\alpha)$ is nonsingular, the Schur complement of A with respect to $A(\alpha)$ is denoted by A/α:

$$A/\alpha = A(\bar{\alpha}) - A(\bar{\alpha}, \alpha)[A(\alpha)]^{-1}A(\alpha, \bar{\alpha}).$$

For any $A = (a_{ij}) \in C^{n\times n}$, denote $|A| = (|a_{ij}|)$. The comparison matrix is defined by $\mu(A) = (u_{ij})_{n\times n}$, where $u_{ij} = |a_{ij}|$ for $i = j$, and $u_{ij} = -|a_{ij}|$ for $i \neq j$. Given a set $S \subseteq [n]$, we denote

$$r_i^S(A) = \sum_{j\in S,\neq i} |a_{ij}|. \tag{2.1}$$

If $S = [n]$, we denote

$$r_i(A) := r_i^{[n]}(A) = \sum_{j\neq i} |a_{ij}|. \tag{2.2}$$

If $S = [n] \setminus \{j\}$, we denote

$$r_i^j(A) := r_i^{[n]\setminus\{j\}}(A) = \sum_{k\neq i,j} |a_{ik}|. \tag{2.3}$$

Definition 2.1. [21] Let $A = (a_{ij}) \in C^{n\times n}$. We say that A is a DZT matrix if for each $i \in [n]$, there exists an index $j(\neq i)$ such that

$$(|a_{ii}| - r_i^j(A))|a_{jj}| > |a_{ij}|r_j(A). \tag{2.4}$$

For convenience, for an arbitrary index $i \in [n]$, we denote by $\Gamma_i(A)$ the collection of all indices j satisfying condition (2.4):

$$\Gamma_i(A) = \{j \neq i : (2.4) \text{ holds}\}. \tag{2.5}$$

Denote

$$N^+(A) = \{i \in [n] : |a_{ii}| > r_i(A)\}, \tag{2.6}$$

$$N^-(A) = \{i \in [n] : |a_{ii}| \leq r_i(A)\}. \tag{2.7}$$

In 2019, Kolotilina [26] gave an alternative equivalent expression for DZT matrices.

Theorem 2.1. *[26, Theorem 2.1] A matrix $A = (a_{ij}) \in C^{n\times n}$ is a DZT matrix if and only if either $N^-(A) = \emptyset$ or $\Gamma_i(A) \neq \emptyset$ for all $i \in N^-(A)$.*

In light of Theorem 2.1, for a given matrix A with $N^-(A) \neq \emptyset$, it suffices to show $\Gamma_i(A) \neq \emptyset$ for all $i \in N^-(A)$ to prove that A is a DZT matrix.

Definition 2.2. A matrix A is called an M-matrix if it can be written in the form $A = sI - B$, where I is the identity matrix, B is a nonnegative matrix, and $s > \rho(B)$, with $\rho(B)$ denoting the spectral radius of B.

Definition 2.3. A matrix A is called an H-matrix if $\mu(A)$ is an M-matrix.

Definition 2.4. Let $A = (a_{ij}) \in C^{n\times n}$. We say that A is an SDD matrix if for all $i \in [n]$, it holds that $|a_{ii}| > r_i(A)$.

Definition 2.5. Let $A = (a_{ij}) \in C^{n\times n}$. We say that A is a DZ matrix if there exists an index $j \in [n]$ such that (2.4) holds for all $i \neq j$.

Remark 2.1. Suppose A is a DZT matrix. If there exists $j \in N^+(A)$ such that (2.4) holds for all $i \in N^-(A)$, then A is also a DZ matrix.

It is known that

$$\{\text{SDD matrix}\} \subset \{\text{DZ matrix}\} \subset \{H\text{-matrix}\},$$

$$\{\text{SDD matrix}\} \subset \{\text{DZT matrix}\} \subset \{H\text{-matrix}\}.$$

Lemma 2.1. *[27, p. 131] If A is an H-matrix, then* $[\mu(A)]^{-1} \geq |A^{-1}|$.

Lemma 2.2. *[27, p. 117] If A is an M-matrix, then* $det(A) > 0$.

Lemma 2.3. *[28, p. 5] Let* $A \in C^{n\times n}$ *and* α *be a nonempty proper subset of* $[n]$. *If* $A(\alpha)$ *is nonsingular, then* $det(A) = det(A(\alpha))det(A/\alpha)$.

Remark 2.2. For any DZT matrix $A \in C^{n\times n}$, we always assume that A is not an SDD matrix because the Schur complements of SDD matrices are also SDD matrices [1, 11].

Remark 2.3. For any $A = (a_{ij}) \in C^{n\times n}$ and any nonempty $\alpha \subset [n]$, we assume throughout that $|\alpha| = k$, where $|\alpha|$ denotes the cardinality of α. Then $|\bar{\alpha}| = n - k$; for simplicity, set $l := n - k$. Without loss of generality, we denote

$$\alpha = \{i_1, i_2, \cdots, i_k\}, \quad i_1 < i_2 < \cdots < i_k; \tag{2.8}$$

$$\bar{\alpha} = \{j_1, j_2, \cdots, j_l\}, \quad j_1 < j_2 < \cdots < j_l, \ (l = n - k). \tag{2.9}$$

Lemma 2.4. *[26, Lemma 2.2] Given a DZT matrix* $A \in C^{n\times n}$, *then for any* $i \in [n]$, *we have*
(i) $a_{ii} \neq 0$;
(ii) $\Gamma_i(A) \subseteq N^+(A)$ for all $i \in N^-(A)$;
(iii) $|a_{ii}| - r_i^j(A) > \frac{|a_{ij}| r_j(A)}{|a_{jj}|}$ for all $j \in \Gamma_i(A)$.

Lemma 2.5. *Let* $A \in C^{n\times n}$ *be a DZT matrix. Then, for any* $\emptyset \neq \alpha \subset [n]$, $A(\alpha)$ *is also a DZT matrix.*

Proof. Denote $A = (a_{ij})$. Recall that α and $\bar{\alpha}$ are defined as in (2.8) and (2.9), respectively. Then, $A(\alpha) = (a_{i_t i_s})_{k\times k}$. There are three possible relationships between α and $N^+(A)$ (see Figure 1): (i) $\alpha \supseteq N^+(A)$, (ii) $\alpha \cap N^+(A) = \emptyset$, (iii) $\alpha \cap N^+(A) \neq \emptyset$, and $\alpha \cap N^-(A) \neq \emptyset$.

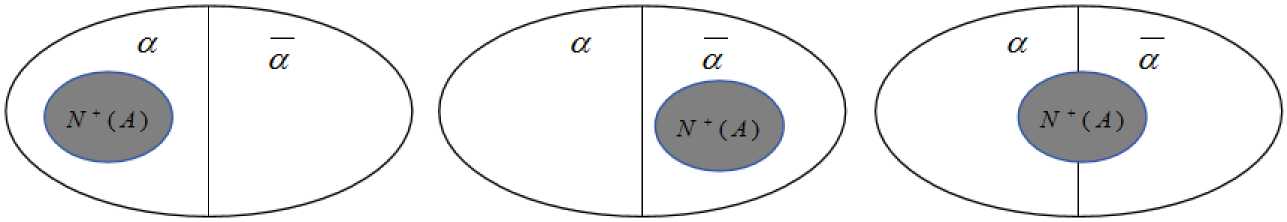

Figure 1. The three possible relationships between α and $N^+(A)$.

(i) $\alpha \supseteq N^+(A)$. For any $t \in [k]$, if $i_t \in N^+(A)$, it is clear that

$$|a_{i_t i_t}| > r_{i_t}(A) \geq r_{i_t}^{\alpha}(A) = r_t(A(\alpha)), \tag{2.10}$$

which implies $t \in N^+(A(\alpha))$. By Theorem 2.1, we only need to prove that $\Gamma_t(A(\alpha)) \neq \emptyset$ for all $i_t \in N^-(A)$. For any $i_t \in N^-(A)$, by Lemma 2.4(ii), there exists $i_u (u \neq t) \in N^+(A) \subseteq \alpha$ such that $i_u \in \Gamma_{i_t}(A)$. The following inequalities indicate $u \in \Gamma_t(A(\alpha))$:

$$(|a_{i_t i_t}| - r_t^u(A(\alpha)))|a_{i_u i_u}| \geq (|a_{i_t i_t}| - r_{i_t}^{i_u}(A))|a_{i_u i_u}| > |a_{i_t i_u}| r_{i_u}(A) \geq |a_{i_t i_u}| r_u(A(\alpha)). \tag{2.11}$$

Then, we have $A(\alpha)$ is a DZT matrix.

(ii) Since $\alpha \cap N^+(A) = \emptyset$, we have $\alpha \subseteq N^-(A)$. In this case, for any $i_t \in \alpha$, we have $i_t \in N^-(A)$. Then, by Lemma 2.4 (ii), we have $\Gamma_{i_t}(A) \subseteq N^+(A)$. By $\alpha \subseteq N^-(A)$, we obtain $N^+(A) \subseteq \bar{\alpha}$. Therefore,

$$\emptyset \neq \Gamma_{i_t}(A) \subseteq N^+(A) \subseteq \bar{\alpha}, \tag{2.12}$$

which implies that there exists $j_v \in \bar{\alpha} \subseteq N^+(A)$ such that $j_v \in \Gamma_{i_t}(A)$. Then, $[n] \setminus \{j_v\} \supseteq \alpha$. By Lemma 2.4(iii), we have

$$|a_{i_t i_t}| > r_{i_t}^{j_v}(A) = r_{i_t}^{[n]\setminus\{j_v\}}(A) \geq r_{i_t}^{\alpha}(A) = r_t(A(\alpha)), \tag{2.13}$$

which implies $t \in N^+(A(\alpha))$. We conclude that $A(\alpha)$ is an SDD matrix.

(iii) $\alpha \cap N^+(A) \neq \emptyset$, and $\alpha \cap N^-(A) \neq \emptyset$. For any $t \in [k]$, if $i_t \in N^+(A)$, (2.10) holds and $t \in N^+(A(\alpha))$. If $i_t \in N^-(A)$ with $\Gamma_{i_t}(A) \cap \bar{\alpha} \neq \emptyset$, there exists $j_v \in \bar{\alpha}$ such that $j_v \in \Gamma_{i_t}(A) \subseteq N^+(A)$. Then, (2.13) holds and then $t \in N^+(A(\alpha))$. If $i_t \in N^-(A)$ with $\Gamma_{i_t}(A) \cap \bar{\alpha} = \emptyset$, then $\Gamma_{i_t}(A) \cap \alpha \neq \emptyset$, which implies that there exists $i_u \in \alpha$ $(u \neq t)$ such that $i_u \in \Gamma_{i_t}(A) \subseteq N^+(A)$. Then, (2.11) holds and $u \in \Gamma_t(A(\alpha))$. Then, $A(\alpha)$ is a DZT matrix. □

Corollary 2.1. *Let $A \in C^{n\times n}$ be a DZT matrix. Then, $A(\alpha)$ is an SDD matrix if α satisfies one of the following conditions:*
(i) $\alpha \subseteq N^+(A)$;
(ii) $\Gamma_i(A) \cap \bar{\alpha} \neq \emptyset$ for all $i \in \alpha \cap N^-(A)$;
(iii) $\alpha \subseteq N^-(A)$.

Proof. (i) If $\alpha \subseteq N^+(A)$, then by (2.10), we have that $A(\alpha)$ is an SDD matrix.

(ii) For any $t \in [k]$, if $i_t \in N^+(A)$, then $t \in N^+(A(\alpha))$. If $i_t \in N^-(A)$, then $\Gamma_{i_t}(A) \cap \bar{\alpha} \neq \emptyset$. Then there exists $j_v \in \bar{\alpha}$ such that (2.13) holds. It follows that $A(\alpha)$ is an SDD matrix.

(iii) If $\alpha \subseteq N^-(A)$, then (2.12) holds for all $i_t \in \alpha$. Then, there exists $j_v \in \bar{\alpha}$ such that (2.13) holds, which implies that $A(\alpha)$ is an SDD matrix. □

Lemma 2.6. *For any DZT matrix $A \in C^{n\times n}$ with $\mu(A) = A$ and $\emptyset \neq \alpha \subset [n]$, we have $det(A/\alpha) > 0$.*

Proof. Since A is a DZT matrix, and hence an H-matrix, Definition 2.3 implies that $\mu(A)$ is an M-matrix. It follows from $\mu(A) = A$ that A is also an M-matrix. Then, by Lemma 2.2, we have $det(A) > 0$. By the same argument, we have $det(A(\alpha)) > 0$. Then, by Lemma 2.3, we have $det(A/\alpha) > 0$. □

Remark 2.4. The conclusion of Lemma 2.6 remains true if A is an H-matrix with $\mu(A) = A$.

3. A subset of $N^+(A/\alpha)$

This section is concerned with the set $N^+(A/\alpha)$ for a given DZT matrix $A \in C^{n\times n}$ and a set $\alpha \subset [n]$. We introduce two symbols:

$$x_u := (a_{j_u i_1}, \ldots, a_{j_u i_k})^T, \quad u \in [l], \tag{3.1}$$
$$y_u := (a_{i_1 j_u}, \ldots, a_{i_k j_u})^T, \quad u \in [l]. \tag{3.2}$$

Then, we have

$$\sum_{v=1}^{l} |y_v| = \sum_{v=1}^{l} \begin{bmatrix} |a_{i_1 j_v}| \\ \vdots \\ |a_{i_k j_v}| \end{bmatrix} = \begin{bmatrix} r_{i_1}^{\bar{\alpha}}(A) \\ \vdots \\ r_{i_k}^{\bar{\alpha}}(A) \end{bmatrix}. \tag{3.3}$$

Denote $A/\alpha = (a'_{uv})_{l\times l}$. For any $u \in [l]$, the following inequality plays a key role to examine whether $u \in N^+(A/\alpha)$:

$$\begin{aligned} & |a'_{uu}| - r_u(A/\alpha) \\ = \; & |a_{j_u j_u} - x_u^T[A(\alpha)]^{-1} y_u| - \sum_{v\neq u} |a_{j_u j_v} - x_u^T[A(\alpha)]^{-1} y_v| \\ \geq \; & |a_{j_u j_u}| - |x_u^T|[\mu(A(\alpha))]^{-1}|y_u| - \sum_{v\neq u} |a_{j_u j_v}| - \sum_{v\neq u} |x_u^T|[\mu(A(\alpha))]^{-1}|y_v| \\ = \; & |a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}}(A) - |x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v|. \end{aligned} \tag{3.4}$$

Similarly, we have

$$|a'_{uu}| + r_u(A/\alpha) \leq |a_{j_u j_u}| + r_{j_u}^{\bar{\alpha}}(A) + |x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v|. \tag{3.5}$$

In light of (3.4), the value of $|x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v|$ is key to determining whether u belongs to $N^+(A/\alpha)$. Then, we begin with the estimations of $|x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v|$.

Lemma 3.1. *Let $A = (a_{ij}) \in C^{n\times n}$ be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. Suppose $\alpha \subseteq N^+(A)$. Then, if $j_u \in N^+(A)$, we have*

$$|x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v| \leq r_{j_u}^{\alpha}(A). \tag{3.6}$$

If $j_u \in N^-(A)$, and $\alpha \cap \Gamma_{j_u}(A) \neq \emptyset$, we have

$$|x_u^T|[\mu(A(\alpha))]^{-1}\sum_{v=1}^{l}|y_v| \leq r_{j_u}^{\alpha}(A) - \min_{i_t\in\Gamma_{j_u}(A)}(1-\frac{r_{i_t}(A)}{|a_{i_ti_t}|})|a_{j_ui_t}|. \tag{3.7}$$

Proof. If $j_u \in N^+(A)$, for any $\varepsilon > 0$, define

$$D_u := \begin{bmatrix} \mu(A(\alpha)) & -\sum\limits_{v=1}^{l}|y_v| \\ -|x_u^T| & r_{j_u}^{\alpha}(A)+\varepsilon \end{bmatrix}. \tag{3.8}$$

It is easy to see that $D_u = \mu(D_u)$. Denote $D_u = (d_{ij})$. Because $\alpha \subseteq N^+(A)$, we have, for any $t \in [k]$,

$$|d_{tt}| = |a_{i_ti_t}| > r_{i_t}(A) \geq r_{i_t}^{\bar{\alpha}}(A) = r_t(D_u),$$

and

$$|d_{k+1,k+1}| = r_{j_u}^{\alpha}(A)+\varepsilon \geq r_{j_u}^{\alpha}(A) = r_{k+1}(D_u),$$

which together imply that D_u is an SDD matrix. By Lemma 2.6, we have

$$det(D_u/[k]) = r_{j_u}^{\alpha}(A)+\varepsilon - |x_u^T|[\mu(A(\alpha))]^{-1}\sum_{v=1}^{l}|y_v| > 0.$$

Because the determinant is continuous as a function of the entries of the matrix, $det(D_u)$ is continuous with respect to ε. Then, (3.6) holds by letting $\varepsilon \to 0^+$.

If $j_u \in N^-(A)$, and $\Gamma_{j_u}(A) \cap \alpha \neq \emptyset$, for any $0 < \varepsilon < \min\limits_{i_t\in\Gamma_{j_u}(A)}(1 - r_{i_t}(A)/|a_{i_ti_t}|)\,|a_{j_ui_t}|$, define

$$F_u := \begin{bmatrix} \mu(A(\alpha)) & -\sum\limits_{v=1}^{l}|y_v| \\ -|x_u^T| & r_{j_u}^{\alpha}(A)+\varepsilon - \min\limits_{i_t\in\Gamma_{j_u}(A)}\left(1-\frac{r_{i_t}(A)}{|a_{i_ti_t}|}\right)|a_{j_ui_t}| \end{bmatrix}. \tag{3.9}$$

Denote $F_u = (f_{ij})$. The fact that $0 < 1 - r_{i_t}(A)/|a_{i_ti_t}| < 1$ for all $i_t \in \Gamma_{j_u}(A) \subseteq N^+(A)$ leads to

$$r_{j_u}^{\alpha}(A) - \min_{i_t\in\Gamma_{j_u}(A)}\left(1-\frac{r_{i_t}(A)}{|a_{i_ti_t}|}\right)|a_{j_ui_t}| \geq r_{j_u}^{\alpha}(A) - \min_{i_t\in\Gamma_{j_u}(A)}|a_{j_ui_t}| \geq r_{j_u}^{\alpha}(A) - |a_{j_ui_t}| \geq 0.$$

Therefore, we have

$$0 < r_{j_u}^{\alpha}(A)+\varepsilon - \min_{i_t\in\Gamma_{j_u}(A)}\left(1-\frac{r_{i_t}(A)}{|a_{i_ti_t}|}\right)|a_{j_ui_t}| < r_{j_u}^{\alpha}(A),$$

which implies $k+1 \in N^-(F_u)$ and $F_u = \mu(F_u)$. Then, it is clear that

$$N^+(F_u) = [k],\ \ N^-(F_u) = \{k+1\}.$$

Recall that $\alpha \cap \Gamma_{j_u}(A) \neq \emptyset$. Take an arbitrary $s \in [k]$ with $i_s \in \Gamma_{j_u}(A)$. Then we have

$$(|f_{k+1,k+1}| - r_{k+1}^{s}(F_u))\,|f_{ss}|$$

$$
\begin{aligned}
&= \left(r^{\alpha}_{j_u}(A) + \varepsilon - \min_{i_t \in \Gamma_{j_u}(A)} \left(1 - \frac{r_{i_t}(A)}{|a_{i_t i_t}|}\right) |a_{j_u i_t}| - r^{s}_{k+1}(F_u)\right) |a_{i_s i_s}| \\
&= \left(r^{\alpha}_{j_u}(A) + \varepsilon - \min_{i_t \in \Gamma_{j_u}(A)} \left(1 - \frac{r_{i_t}(A)}{|a_{i_t i_t}|}\right) |a_{j_u i_t}| - r^{\alpha}_{j_u}(A) + |a_{j_u i_s}|\right) |a_{i_s i_s}| \\
&= \left(|a_{j_u i_s}| + \varepsilon - \min_{i_t \in \Gamma_{j_u}(A)} \left(1 - \frac{r_{i_t}(A)}{|a_{i_t i_t}|}\right) |a_{j_u i_t}|\right) |a_{i_s i_s}| \\
&> \left(|a_{j_u i_s}| - \min_{i_t \in \Gamma_{j_u}(A)} \left(1 - \frac{r_{i_t}(A)}{|a_{i_t i_t}|}\right) |a_{j_u i_t}|\right) |a_{i_s i_s}| \\
&\geq \left(|a_{j_u i_s}| - (1 - \frac{r_{i_s}(A)}{|a_{i_s i_s}|}) |a_{j_u i_s}|\right) |a_{i_s i_s}| = |a_{j_u i_s}| r_{i_s}(A) = |f_{k+1,s}| r_s(F_u),
\end{aligned}
$$

which implies that $s \in \Gamma_{k+1}(F_u)$. Then, F_u is a DZT matrix. By Lemma 2.6, and letting $\varepsilon \to 0^+$, we get (3.7). □

Lemma 3.2. *Let $A = (a_{ij}) \in C^{n\times n}$ be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. Suppose $\alpha \cap \Gamma_{i_t}(A) \neq \emptyset$ for all $i_t \in N^-(A)$. Then, if $j_u \in N^+(A)$, we have (3.6). If $j_u \in N^-(A)$, and $\alpha \cap \Gamma_{j_u}(A) \neq \emptyset$, we have (3.7).*

Proof. If $j_u \in N^+(A)$, let D_u be defined as in (3.8), and denote $D_u = (d_{ij})$. It is easy to see that

$$
\begin{aligned}
&N^+(D_u) \supseteq \{t \in [k] : i_t \in N^+(A)\} \cup \{k+1\}, \\
&N^-(D_u) \subseteq \{t \in [k] : i_t \in N^-(A)\}.
\end{aligned}
$$

For any $i_t \in N^-(A)$, because $\Gamma_{i_t}(A) \cap \alpha \neq \emptyset$, there exists an index $i_s \in \alpha$ $(s \neq t)$ such that $i_s \in \Gamma_{i_t}(A)$. Then, we have

$$
(|d_{tt}| - r^s_t(D_u))|d_{ss}| = (|a_{i_t i_t}| - r_{i_t} i_s(A))|a_{i_s i_s}| > |a_{i_t i_s}| r_{i_s}(A) = |d_{ts}| r_s(D_u),
$$

which implies that $s \in \Gamma_t(D_u)$. Then, D_u is a DZT matrix. Then, (3.6) holds by Lemma 2.6 and letting $\varepsilon \to 0^+$.

If $j_u \in N^-(A)$, and $\Gamma_{j_u}(A) \cap \alpha \neq \emptyset$, let F_u be defined as in (3.9), and denote $F_u = (f_{ij})$. By a similar deduction as in the proof of Lemma 3.1, we have

$$
\begin{aligned}
&N^+(F_u) \supseteq \{t \in [k] : i_t \in N^+(A)\}, \\
&N^-(F_u) \subseteq \{t \in [k] : i_t \in N^-(A)\} \cup \{k+1\}, \\
&F_u = \mu(F_u),\ k+1 \in N^-(F_u),\ \Gamma_{k+1}(F_u) \neq \emptyset.
\end{aligned}
$$

For any $t \in [k]$ with $i_t \in N^-(A)$, because $\Gamma_{i_t}(A) \cap \alpha \neq \emptyset$, there exists an index $i_s \in \alpha$ $(s \neq t)$ such that $i_s \in \Gamma_{i_t}(A)$. Then, we have

$$
(|f_{tt}| - r^s_t(F_u))|f_{ss}| = (|a_{i_t i_t}| - r^{i_s}_{i_t}(A))|a_{i_s i_s}| > |a_{i_t i_s}| r_{i_s}(A) = |f_{ts}| r_s(F_u),
$$

which implies that $s \in \Gamma_t(F_u)$. Hence, F_u is a DZT matrix. By Lemma 2.6, and letting $\varepsilon \to 0^+$, we have (3.7). □

Denote

$$\Upsilon = \{i \in \alpha \cap N^-(A) : \Gamma_i(A) \cap \alpha = \emptyset\}. \tag{3.10}$$

By (2.8), the set Υ can be equivalently written as

$$\Upsilon = \{i_t \in N^-(A) : t \in [k], \Gamma_{i_t}(A) \cap \alpha = \emptyset\}.$$

The set Υ consists of the non-row-dominant indices inside α that have no $j \in \alpha$ satisfying (2.4); in other words, their admissible DZT indices must lie in $\bar{\alpha}$. It is clear that $\Upsilon \subseteq \alpha \cap N^-(A)$. $\Upsilon = \emptyset$ implies that

$$\alpha \cap N^-(A) = \emptyset \ (i.e., \alpha \subseteq N^+(A)), \ \text{or} \ \alpha \cap \Gamma_{i_t}(A) \neq \emptyset \text{ for all } i_t \in N^-(A).$$

Lemmas 3.1 and 3.2 are concerned with the bound of $|x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v|$ under the case $\Upsilon = \emptyset$. Next, we study the case $\Upsilon \neq \emptyset$.

Lemma 3.3. *Let $A = (a_{ij}) \in C^{n\times n}$ be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. If $\Upsilon \neq \emptyset$, and $\bigcap_{i_t\in\Upsilon} \Gamma_{i_t}(A) \neq \emptyset$, then for any $j_u \in \bigcap_{i_t\in\Upsilon} \Gamma_{i_t}(A)$, we have*

$$|x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v| < \frac{|a_{j_u j_u}|}{r_{j_u}(A)} r_{j_u}^{\alpha}(A) \ \ \textit{if } r_{j_u}^{\alpha}(A) \neq 0, \tag{3.11}$$

and

$$|x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v| = 0 \text{ if } r_{j_u}^{\alpha}(A) = 0.$$

Proof. Because $\Upsilon \subseteq N^-(A) \cap \alpha$, for any $i_t \in \Upsilon$, we have $\Gamma_{i_t}(A) \subseteq N^+(A)$ and $\Gamma_{i_t}(A) \subseteq \bar{\alpha}$. Then, $\bigcap_{i_t\in\Upsilon} \Gamma_{i_t}(A) \subseteq N^+(A) \cap \bar{\alpha}$. For any $j_u \in \bigcap_{i_t\in\Upsilon} \Gamma_{i_t}(A)$, we have $j_u \in N^+(A)$. If $r_{j_u}^{\alpha}(A) = 0$, then $x_u = 0$, and thus, $|x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v=1}^{l} |y_v| = 0$. We now suppose $r_{j_u}^{\alpha}(A) \neq 0$. Then, under this case, we have $(|a_{j_u j_u}|/r_{j_u}(A)) r_{j_u}^{\alpha}(A) > r_{j_u}^{\alpha}(A)$. Let

$$E_u := \begin{bmatrix} \mu(A(\alpha)) & -\sum_{v=1}^{l} |y_v| \\ -|x_u^T| & \frac{|a_{j_u j_u}|}{r_{j_u}(A)} r_{j_u}^{\alpha}(A) \end{bmatrix} := (e_{ts})_{(k+1)\times(k+1)}. \tag{3.12}$$

It is easy to see that

$$\begin{aligned} N^+(E_u) &\supseteq \{t \in [k] : i_t \in N^+(A)\} \cup \{k+1\}, \\ N^-(E_u) &\subseteq \{t \in [k] : i_t \in N^-(A)\}. \end{aligned}$$

For any $i_t \in N^-(A)$, if $i_t \notin \Upsilon$, then $\Gamma_{i_t}(A) \cap \alpha \neq \emptyset$, which implies that there exists an index $i_s \in \alpha$ $(s \neq t)$ such that $i_s \in \Gamma_{i_t}(A)$. Then,

$$(|e_{tt}| - r_t^s(E_u))|e_{ss}| = (|a_{i_t i_t}| - r_{i_t}^{i_s}(A))|a_{i_s i_s}| > |a_{i_t i_s}| r_{i_s}(A) = |e_{ts}| r_s(E_u),$$

which implies that $s \in \Gamma_t(E_u)$. If $i_t \in \Upsilon$, by $j_u \in \Gamma_{i_t}(A)$, we have

$$\begin{aligned}
& (|e_{tt}| - r_t^{k+1}(E_u))|e_{k+1k+1}| \\
= \ & (|a_{i_ti_t}| - r_{i_t}^{\alpha}(A))\frac{|a_{j_uj_u}|}{r_{j_u}(A)}r_{j_u}^{\alpha}(A) \\
= \ & (|a_{i_ti_t}| - r_{i_t}(A) + r_{i_t}^{\bar{\alpha}}(A))\frac{|a_{j_uj_u}|}{r_{j_u}(A)}r_{j_u}^{\alpha}(A) \\
= \ & (|a_{i_ti_t}| - r_{i_t}^{j_u}(A) - |a_{i_tj_u} + r_{i_t}^{\bar{\alpha}}(A)|)\frac{|a_{j_uj_u}|}{r_{j_u}(A)}r_{j_u}^{\alpha}(A) \\
> \ & (\frac{r_{j_u}(A)}{|a_{j_uj_u}|}|a_{i_tj_u}| - |a_{i_tj_u} + r_{i_t}^{\bar{\alpha}}(A)|)\frac{|a_{j_uj_u}|}{r_{j_u}(A)}r_{j_u}^{\alpha}(A) \ \text{(by Lemma 2.4 (iii))} \\
= \ & (r_{i_t}^{\bar{\alpha}}(A) - (1 - \frac{r_{j_u}(A)}{|a_{j_uj_u}|})|a_{i_tj_u}|)\frac{|a_{j_uj_u}|}{r_{j_u}(A)}r_{j_u}^{\alpha}(A) \\
\geq \ & (r_{i_t}^{\bar{\alpha}}(A) - (1 - \frac{r_{j_u}(A)}{|a_{j_uj_u}|})r_{i_t}^{\bar{\alpha}}(A))\frac{|a_{j_uj_u}|}{r_{j_u}(A)}r_{j_u}^{\alpha}(A) \\
= \ & \frac{r_{j_u}(A)}{|a_{j_uj_u}|}r_{i_t}^{\bar{\alpha}}(A)\frac{|a_{j_uj_u}|}{r_{j_u}(A)}r_{j_u}^{\alpha}(A) = r_{i_t}^{\bar{\alpha}}(A)r_{j_u}^{\alpha}(A) \\
= \ & |e_{t,k+1}|r_{k+1}(E_u),
\end{aligned}$$

which implies that $k+1 \in \Gamma_t(E_u)$. Then, E_u is a DZT matrix. By Lemma 2.6 again, (3.11) holds. □

Theorem 3.1. *Let $A = (a_{ij}) \in C^{n\times n}$ be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. If $\Upsilon = \emptyset$, and $j_u \in N^+(A)$, we have $u \in N^+(A/\alpha)$ and*

$$|a'_{uu}| - r_u(A/\alpha) \geq |a_{j_uj_u}| - r_{j_u}(A) > 0,$$
$$|a'_{uu}| + r_u(A/\alpha) \leq |a_{j_uj_u}| + r_{j_u}(A).$$

Proof. Because $\Upsilon = \emptyset$, we have $\alpha \subseteq N^+(A)$ or $\alpha \cap \Gamma_{i_t}(A) \neq \emptyset$ for all $i_t \in N^-(A)$. By Lemma 3.1 and Lemma 3.2, we always have (3.6) for $j_u \in N^+(A)$. Then, by (3.4) and (3.5), the results hold trivially. □

Theorem 3.2. *Let $A = (a_{ij}) \in C^{n\times n}$ be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. If $\Upsilon = \emptyset$, and $j_u \in N^-(A)$ with $\alpha \cap \Gamma_{j_u}(A) \neq \emptyset$, we have $u \in N^+(A/\alpha)$ and*

$$|a'_{uu}| - r_u(A/\alpha) \ \geq \ |a_{j_uj_u}| - r_{j_u}(A) + \min_{i_t\in\Gamma_{j_u}(A)}\left(1 - \frac{r_{i_t}(A)}{|a_{i_ti_t}|}\right)|a_{j_ui_t}| > 0, \tag{3.13}$$

$$|a'_{uu}| + r_u(A/\alpha) \ \leq \ |a_{j_uj_u}| + r_{j_u}(A) - \min_{i_t\in\Gamma_{j_u}(A)}\left(1 - \frac{r_{i_t}(A)}{|a_{i_ti_t}|}\right)|a_{j_ui_t}|. \tag{3.14}$$

Proof. Whenever $\alpha \subseteq N^+(A)$ or $\alpha\cap\Gamma_{i_t}(A) \neq \emptyset$ for all $i_t \in N^-(A)$, for any $j_u \in N^-(A)$ with $\Gamma_{j_u}(A)\cap\alpha \neq \emptyset$, by Lemmas 3.1 and 3.2, we always have (3.7). Denote

$$(1 - \frac{r_{i_s}(A)}{|a_{i_si_s}|})|a_{j_ui_s}| := \min_{i_t\in\Gamma_{j_u}(A)}\left(1 - \frac{r_{i_t}(A)}{|a_{i_ti_t}|}\right)|a_{j_ui_t}|.$$

By (3.4), we have

$$|a'_{uu}| - r_u(A/\alpha)$$

$$
\begin{aligned}
&\geq |a_{j_u j_u}| - r_{j_u}(A) + \min_{i_t \in \Gamma_{j_u}(A)} \left(1 - \frac{r_{i_t}(A)}{|a_{i_t i_t}|}\right) |a_{j_u i_t}| \\
&= |a_{j_u j_u}| - r_{j_u}(A) + (1 - \frac{r_{i_s}(A)}{|a_{i_s i_s}|})|a_{j_u i_s}| \\
&= \frac{(|a_{j_u j_u}| - r^{i_s}_{j_u}(A))|a_{i_s i_s}| - |a_{j_u i_s}| r_{i_s}(A)}{|a_{i_s i_s}|} > 0.
\end{aligned}
$$

Similarly, we can get $|a'_{uu}| + r_u(A/\alpha) \leq |a_{j_u j_u}| + r_{j_u}(A) - \min_{i_t \in \Gamma_{j_u}(A)} (1 - r_{i_t}(A)/|a_{i_t i_t}|) |a_{j_u i_t}|$. □

Theorem 3.3. *Let $A = (a_{ij}) \in C^{n\times n}$ be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. If $\Upsilon \neq \emptyset$, and $j_u \in \bigcap_{i_t \in \Upsilon} \Gamma_{i_t}(A)$, then we have $u \in N^+(A/\alpha)$,*

$$
\begin{aligned}
|a'_{uu}| - r_u(A/\alpha) &> (\frac{|a_{j_u j_u}|}{r_{j_u}(A)} - 1) r^{\bar{\alpha}}_{j_u}(A) \geq 0, \quad r^{\alpha}_{j_u}(A) \neq 0, \\
|a'_{uu}| + r_u(A/\alpha) &< |a_{j_u j_u}| + r_{j_u}(A) + (\frac{|a_{j_u j_u}|}{r_{j_u}(A)} - 1) r^{\alpha}_{j_u}(A), \quad r^{\alpha}_{j_u}(A) \neq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
|a'_{uu}| - r_u(A/\alpha) &\geq |a_{j_u j_u}| - r^{\bar{\alpha}}_{j_u}(A) > 0, \quad r^{\alpha}_{j_u}(A) = 0, \\
|a'_{uu}| + r_u(A/\alpha) &\leq |a_{j_u j_u}| + r^{\bar{\alpha}}_{j_u}(A), \quad r^{\alpha}_{j_u}(A) = 0.
\end{aligned}
$$

Proof. Suppose $j_u \in \bigcap_{i_t \in \Upsilon} \Gamma_{i_t}(A)$, and $\Upsilon \neq \emptyset$. It follows that $j_u \in N^+(A)$ by Lemma 2.4 (ii). If $r^{\alpha}_{j_u}(A) = 0$, then $x_u = 0$. By (3.4), we have

$$
|a'_{uu}| - r_u(A/\alpha) \geq |a_{j_u j_u}| - r^{\bar{\alpha}}_{j_u}(A) = |a_{j_u j_u}| - r_{j_u}(A) > 0.
$$

By similar deduction, we can get $|a'_{uu}| + r_u(A/\alpha) \leq |a_{j_u j_u}| + r^{\bar{\alpha}}_{j_u}(A)$.

If $r^{\alpha}_{j_u}(A) \neq 0$, by (3.4) and (3.11), we have

$$
|a'_{uu}| - r_u(A/\alpha) \quad > \quad |a_{j_u j_u}| - r^{\bar{\alpha}}_{j_u}(A) - \frac{|a_{j_u j_u}|}{r_{j_u}(A)} r^{\alpha}_{j_u}(A) = \frac{|a_{j_u j_u}| r^{\bar{\alpha}}_{j_u}(A)}{r_{j_u}(A)} - r^{\bar{\alpha}}_{j_u}(A) \geq 0.
$$

Similarly, we can get $|a'_{uu}| + r_u(A/\alpha) < |a_{j_u j_u}| + r_{j_u}(A) + (|a_{j_u j_u}|/r_{j_u}(A) - 1) r^{\alpha}_{j_u}(A)$. □

In the proof of Lemmas 3.1 and 3.2, and in the results derived thereafter, the condition "$j_u \in N^-(A)$ with $\Gamma_{j_u}(A) \cap \alpha \neq \emptyset$" plays a crucial role. We collect all elements j_u satisfying this condition into a set, denoted by Θ:

$$
\Theta = \{i \in \bar{\alpha} \cap N^-(A) : \Gamma_i \cap \alpha = \emptyset\}. \tag{3.15}
$$

By (2.9), Θ can be equivalently written as

$$
\Theta = \{j_u \in N^-(A) : u \in [l], \Gamma_{j_u} \cap \alpha = \emptyset\}.
$$

The set Θ consists of the non-row-dominant indices in $\bar{\alpha}$ that have no $j \in \alpha$ satisfying (2.4); hence their admissible DZT indices are located in $\bar{\alpha}$. It is clear that $\Theta \subseteq \bar{\alpha} \cap N^{-}(A)$. $\Theta = \emptyset$ implies that

$$\bar{\alpha} \cap N^{-}(A) = \emptyset \ (i.e.,\ \bar{\alpha} \subseteq N^{+}(A)),\ \text{or}\ \Gamma_{j_u} \cap \alpha \neq \emptyset \text{ for all } j_u \in N^{-}(A).$$

Therefore, we have

$$\Upsilon = \Theta = \emptyset \quad \Leftrightarrow \quad \Gamma_i \cap \alpha \neq \emptyset \text{ for all } i \in N^{-}(A).$$

The relationship among α, $\bar{\alpha}$, Θ, Υ, and $N^{-}(A)$ is shown in Figure 2.

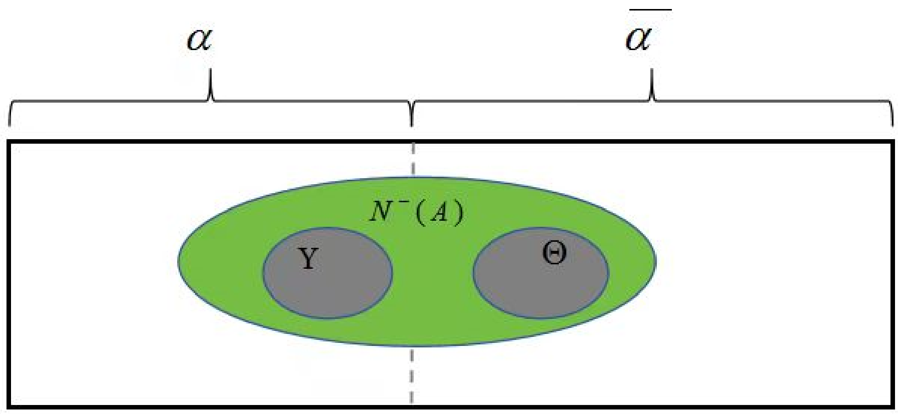

Figure 2. The relationship among α, $\bar{\alpha}$, Θ, Υ, and $N^{-}(A)$.

Corollary 3.1. *Let A be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. Denote*

$$\begin{aligned} S_1 &= \{u \in [l] : \Upsilon = \emptyset, j_u \in N^{+}(A)\}, \\ S_2 &= \{u \in [l] : \Upsilon = \emptyset, j_u \in N^{-}(A), j_u \notin \Theta\}, \\ S_3 &= \{u \in [l] : \Upsilon \neq \emptyset, j_u \in \bigcap_{i_t \in \Upsilon} \Gamma_{i_t}(A)\}. \end{aligned}$$

Then, $(S_1 \cup S_2 \cup S_3) \subseteq N^{+}(A/\alpha)$.

4. Schur complements of DZT matrices

Given a DZT matrix $A = (a_{ij}) \in C^{n \times n}$ and $\emptyset \neq \alpha \subset [n]$, the symbols α, $\bar{\alpha}$, Υ, and Θ are defined as in (2.8), (2.9), (3.10), and (3.15), respectively. Recall that S_1, S_2, and S_3 are defined in Corollary 3.1. We investigate the conditions under which A/α is a DZT matrix by examining the conditions under which $\Gamma_u(A/\alpha) \neq \emptyset$ for all $u \notin S_1 \cup S_2 \cup S_3$ whenever $S_1 \cup S_2 \cup S_3 \neq [l]$. Denote $A/\alpha = (a'_{uv})_{l \times l}$. We have

$$a'_{uv} = a_{j_u j_v} - x_u^T [A(\alpha)]^{-1} y_v, \ \ u, v \in [l].$$

For ease of presentation, we introduce the following notation:

$$G_{\Upsilon} := \bigcap_{i \in \Upsilon} \Gamma_i(A), \ \ G_{\Theta} := \bigcap_{j \in \Theta} \Gamma_j(A).$$

Lemma 4.1. *Let A be a DZT matrix. Let $\emptyset \neq \alpha \subset [n]$, $j_s \in \bar{\alpha}$, and $j_u \in \bar{\alpha}$. Then,*

$$G_u := \begin{bmatrix} \mu(A(\alpha)) & O & -\sum\limits_{v\neq s} |y_v| & -|y_s| \\ O & \mu(A(\alpha)) & -\sum\limits_{v\neq s} |y_v| & -|y_s| \\ -|x_u^T| & O & |a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}\backslash\{j_s\}}(A) & -|a_{j_u j_s}| \\ O & -|x_s^T| & -r_{j_s}^{\bar{\alpha}}(A) & |a_{j_s j_s}| \end{bmatrix} = (g_{ts})_{(2k+2)\times(2k+2)}. \tag{4.1}$$

is a DZT matrix if one of the following four conditions holds:
(i) $\Upsilon = \emptyset$, $\Theta \neq \emptyset$, $j_u \in \Theta$, and $j_s \in \Gamma_{j_u}(A)$;
(ii) $\Upsilon \neq \emptyset$, $G_\Upsilon \neq \emptyset$, $j_u \in N^+(A)\backslash G_\Upsilon$, and $j_s \in G_\Upsilon$;
(iii) $\Upsilon \neq \emptyset$, $G_\Upsilon \neq \emptyset$, $j_u \in N^-(A)\backslash \Theta$, and $j_s \in G_\Upsilon$;
(iv) $\Upsilon \neq \emptyset$, $G_\Upsilon \neq \emptyset$, $j_u \in \Theta$, and $j_s \in \Gamma_{j_u}(A) \cap G_\Upsilon$.
Moreover, whenever G_u satisfies any one of the above four conditions, it necessarily follows that $s \in \Gamma_u(A/\alpha)$.

Proof. For any $t \in [k]$, it holds that $|g_{tt}| = |a_{i_t i_t}| > r_{i_t}(A) = r_t(G_u)$, $|g_{t+k,t+k}| = |a_{i_t i_t}| > r_{i_t}(A) = r_{t+k}(G_u)$, and $|g_{2k+2,2k+2}| = |a_{j_s j_s}| > r_{j_s}(A) = r_{2k+2}(G_u)$. Then,

$$N^+(G_u) \supseteq \{t \in [k] : i_t \in N^+(A)\} \cup \{t + k : t \in [k], i_t \in N^+(A)\} \cup \{2k + 2\}, \tag{4.2}$$

$$N^-(G_u) \subseteq \{t \in [k] : i_t \in N^-(A)\} \cup \{t + k : t \in [k], i_t \in N^-(A)\} \cup \{2k + 1\}. \tag{4.3}$$

(i) If $i_t \in N^-(A)$, by $\Upsilon = \emptyset$, we have $\Gamma_{i_t}(A) \cap \alpha \neq \emptyset$, which implies that there exists $h \in [k]$ $(h \neq k)$ such that $i_h \in \Gamma_{i_t}(A)$. Then,

$$\left(|a_{i_t i_t}| - r_{i_t}^{i_h}(A)\right)|a_{i_h i_h}| > r_{i_h}(A)|a_{i_t i_h}|.$$

Noticing that

$$r_{i_h}(A)|a_{i_t i_h}| = g_h(G_u)|g_{th}| = g_{h+k}(G_u)|g_{t+k,h+k}|,$$

it holds that

$$\left(|g_{tt}| - r_t^h(G_u)\right)|g_{hh}| = \left(|a_{i_t i_t}| - r_{i_t}^{i_h}(A)\right)|a_{i_h i_h}| > g_h(G_u)|g_{th}|, \tag{4.4}$$

and

$$\left(|g_{t+k,t+k}| - r_{t+k}^{h+k}(G_u)\right)|g_{h+k,h+k}| = \left(|a_{i_t i_t}| - r_{i_t}^{i_h}(A)\right)|a_{i_h i_h}| > g_{h+k}(G_u)|g_{t+k,h+k}|, \tag{4.5}$$

which imply $h \in \Gamma_t(G_u)$ and $h + k \in \Gamma_{t+k}(G_u)$. For the $(2k + 1)$-th row, because $j_u \in \Theta$, that is, $\Gamma_{j_u}(A) \cap \alpha = \emptyset$. Then, we have $\Gamma_{j_u}(A) \subseteq \bar{\alpha} \cap N^+(A)$. By $j_s \in \Gamma_{j_u}(A)$, we have

$$\begin{aligned} & (|g_{2k+1,2k+1}| - r_{2k+1}^{2k+2}(G_u))|g_{2k+2,2k+2}| \\ = \ & (|a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}\backslash\{j_s\}}(A) - r_{j_u}^{\alpha}(A))|a_{j_s j_s}| \\ = \ & (|a_{j_u j_u}| - r_{j_u}^{j_s}(A))|a_{j_s j_s}| \\ > \ & |a_{j_u j_s}|r_{j_s}(A) \\ = \ & |g_{2k+1,2k+2}|r_{2k+2}(G_u). \end{aligned} \tag{4.6}$$

That is, $2k + 2 \in \Gamma_{2k+1}(G_u)$. Then, G_u is a DZT matrix.

(ii) Since $j_u \in N^+(A)$, from (4.2)–(4.3) we have

$$N^+(G_u) \supseteq \{t \in [k] : i_t \in N^+(A)\} \cup \{t + k : t \in [k], i_t \in N^+(A)\} \cup \{2k + 1, 2k + 2\},$$

$$N^-(G_u) \subseteq \{t \in [k] : i_t \in N^-(A)\} \cup \{t+k : t \in [k], i_t \in N^-(A)\}.$$

For any $i_t \in N^-(A)$ with $i_t \in \Upsilon$, by $j_s \in G_\Upsilon$, we have

$$\left(|a_{i_t i_t}| - r_{i_t}^{j_s}(A)\right)|a_{j_s j_s}| > r_{j_s}(A)|a_{i_t j_s}| = r_{2k+2}(G_u)|g_{t,2k+2}| = r_{2k+2}(G_u)|g_{t+k,2k+2}|.$$

Then,

$$\left(|g_{tt}| - r_t^{2k+2}(G_u)\right)|g_{2k+2,2k+2}| = \left(|a_{i_t i_t}| - r_{i_t}^{j_s}(A)\right)|a_{j_s j_s}| > r_{2k+2}(G_u)|g_{t,2k+2}| \tag{4.7}$$

and

$$\left(|g_{t+k,t+kt}| - r_{t+k}^{2k+2}(G_u)\right)|g_{2k+2,2k+2}| = \left(|a_{i_t i_t}| - r_{i_t}^{i_h}(A)\right)|a_{i_h i_h}| > r_{2k+2}(G_u)|g_{t+k,2k+2}|, \tag{4.8}$$

which imply $2k+2 \in \Gamma_t(G_u)$ and $2k+2 \in \Gamma_{t+k}(G_u)$. For any $i_t \in N^-(A)$ with $i_t \notin \Upsilon$, there exists $h \in [k]$ ($h \neq k$) such that $i_h \in \Gamma_{i_t}(A)$, we have (4.4) and (4.5). Then $h \in \Gamma_t(G_u)$, and $h+k \in \Gamma_{t+k}(G_u)$. We conclude that G_u is a DZT matrix.

(iii) For any $t \in [k]$ with $i_t \in N^-(A)$, by $j_s \in G_\Upsilon$, we have (4.7) and (4.8). Then, we have $2k+2 \in \Gamma_t(G_u)$ and $2k+2 \in \Gamma_{k+t}(G_u)$. For the $(2k+1)$-th row, because $j_u \notin \Theta$, we have $\Gamma_{j_u}(A) \cap \alpha \neq \emptyset$, which implies that there exists an index $i_t \in \alpha$ such that $i_t \in \Gamma_{j_u}(A)$. Then,

$$\begin{aligned} & (|g_{2k+1,2k+1}| - r_{2k+1}^t(G_u))|g_{tt}| \\ = \; & (|a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}\setminus\{j_s\}}(A) - r_{j_u}^{\{j_s\}\cup(\alpha\setminus\{i_t\})}(A))|a_{i_t i_t}| \\ = \; & (|a_{j_u j_u}| - r_{j_u}^{i_t}(A))|a_{i_t i_t}| \\ > \; & |a_{j_u i_t}| r_{i_t}(A) \\ = \; & |g_{2k+1,t}| r_t(G_u). \end{aligned} \tag{4.9}$$

which implies that $t \in \Gamma_{2k+1}(G_u)$. We obtain that G_u is a DZT matrix.

(iv) For any $t \in [k]$ with $i_t \in N^-(A)$, by $j_s \in G_\Upsilon$, (4.7), and (4.8) again, we have $2k+2 \in \Gamma_t(G_u)$ and $2k+2 \in \Gamma_{k+t}(G_u)$. For the $(2k+1)$-th row, because $j_u \in \Theta$, we have $\Gamma_{j_u}(A) \cap \alpha = \emptyset$. Then, we have $\Gamma_{j_u}(A) \subseteq \bar{\alpha} \cap N^+(A)$. By $j_s \in \Gamma_{j_u}(A)$, we have (4.6). That is, $2k+2 \in \Gamma_{2k+1}(G_u)$. We obtain that G_u is a DZT matrix.

Now, we prove $s \in \Gamma_u(A/\alpha)$ whenever G_u satisfies any one of the above four conditions (i.e., G_u is a DZT matrix). It is clear that $G_u = \mu(G_u)$. By Lemma 2.6, we have $det(G_u/[2k]) > 0$, where

$$\begin{aligned} & G_u/[2k] \\ = \; & \begin{bmatrix} |a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}\setminus\{j_s\}}(A) & -|a_{j_u j_s}| \\ -r_{j_s}^{\bar{\alpha}}(A) & |a_{j_s j_s}| \end{bmatrix} \\ & - \begin{bmatrix} -|x_u^T| & O \\ O & -|x_s^T| \end{bmatrix} \begin{bmatrix} [\mu(A(\alpha))]^{-1} & O \\ O & [\mu(A(\alpha))]^{-1} \end{bmatrix} \begin{bmatrix} -\sum\limits_{v\neq s} |y_v| & -|y_s| \\ -\sum\limits_{v\neq s} |y_v| & -|y_s| \end{bmatrix} \\ = \; & \begin{bmatrix} |a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}\setminus\{j_s\}}(A) - |x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v\neq s}|y_v| & -|a_{j_u j_s}| - |x_u^T|[\mu(A(\alpha))]^{-1}|y_s| \\ -r_{j_s}^{\bar{\alpha}}(A) - |x_s^T|[\mu(A(\alpha))]^{-1} \sum_{v\neq s}|y_v| & |a_{j_s j_s}| - |x_s^T|[\mu(A(\alpha))]^{-1}|y_s| \end{bmatrix}. \end{aligned}$$

It then holds that

$$(|a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}\setminus\{j_s\}}(A) - |x_u^T|[\mu(A(\alpha))]^{-1} \sum_{v\neq s} |y_v|)(|a_{j_s j_s}| - |x_s^T|[\mu(A(\alpha))]^{-1}|y_s|)$$

$$-(|a_{j_u j_s}| + |x_u^T|[\mu(A(\alpha))]^{-1}|y_s|)(r_{j_s}^{\bar{\alpha}} + |x_s^T|[\mu(A(\alpha))]^{-1}\sum_{v\neq s}|y_v|) > 0, \tag{4.10}$$

which leads to

$$\begin{aligned}
& (|a'_{uu}| - r_u^s(A/\alpha))|a'_{ss}| - |a'_{us}|r_s(A/\alpha) \\
= \;& (|a_{j_u j_u} - x_u^T[A(\alpha)]^{-1}y_u| - \sum_{v\neq u,s}|a_{j_u j_v} - x_u^T[A(\alpha)]^{-1}y_v|)|a'_{ss}| - |a'_{us}|r_s(A/\alpha) \\
\geq \;& (|a_{j_u j_u}| - |x_u^T|[\mu(A(\alpha))]^{-1}|y_u| - \sum_{v\neq u,s}|a_{j_u j_v}| - \sum_{v\neq u,s}|x_u^T|[\mu(A(\alpha))]^{-1}|y_v|)|a'_{ss}| - |a'_{us}|r_s(A/\alpha) \\
= \;& (|a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}\setminus\{j_s\}}(A) - |x_u^T|[\mu(A(\alpha))]^{-1}\sum_{v\neq s}|y_v|)|a'_{ss}| - |a'_{us}|r_s(A/\alpha) \\
\geq \;& (|a_{j_u j_u}| - r_{j_u}^{\bar{\alpha}\setminus\{j_s\}}(A) - |x_u^T|[\mu(A(\alpha))]^{-1}\sum_{v\neq s}|y_v|)(|a_{j_s j_s}| - |x_s^T|[\mu(A(\alpha))]^{-1}|y_s|) \\
& -(|a_{j_u j_s}| + |x_u^T|[\mu(A(\alpha))]^{-1}|y_s|)(r_{j_s}^{\bar{\alpha}} + |x_s^T|[\mu(A(\alpha))]^{-1}\sum_{v\neq s}|y_v|).
\end{aligned}$$

Together with (4.10), we obtain

$$(|a'_{uu}| - r_u^s(A/\alpha))|a'_{ss}| - |a'_{us}|r_s(A/\alpha) > 0,$$

which implies $s \in \Gamma_u(A/\alpha)$. □

Theorem 4.1. *Let A be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. If $\Upsilon = \Theta = \emptyset$ (i.e., $\Gamma_i \cap \alpha \neq \emptyset$ for all $i \in N^-(A)$), then A/α is an SDD matrix.*

Proof. Because $\Theta = \emptyset$, for every $u \in [l]$, either $j_u \in N^+(A)$, or $j_u \in N^-(A)$ with $\Gamma_{j_u}(A) \cap \alpha \neq \emptyset$. Combined with $\Upsilon = \emptyset$, this implies that either $u \in S_1$ or $u \in S_2$ for each $u \in [l]$. It follows from Corollary 3.1 that A/α is an SDD matrix. □

Corollary 4.1. *Let A be a DZT matrix, and let $\alpha \supseteq N^+(A)$. Then, A/α is an SDD matrix.*

Proof. If $\alpha \supseteq N^+(A)$, for any $i \in N^-(A)$, we have $\Gamma_i(A) \subseteq N^+(A) \subseteq \alpha$. Then, $\Upsilon = \Theta = \emptyset$. Then, by Theorem 4.1, A/α is an SDD matrix. □

Corollary 4.2. *Let A be a DZT matrix, and let $\bigcup_{i\in N^-(A)} \Gamma_i(A) \subseteq \alpha$. Then, A/α is an SDD matrix.*

Proof. If $\bigcup_{i\in N^-(A)} \Gamma_i(A) \subseteq \alpha$, then we have $\Gamma_i(A) \subseteq N^+(A) \subseteq \alpha$ for any $i \in N^-(A)$, and we have $\Upsilon = \Theta = \emptyset$. By Theorem 4.1, A/α is an SDD matrix. □

Remark 4.1. It is clear that Corollaries 4.1 and 4.2 coincide with [2, Theorem 3.4] and [23, Theorem 5], respectively.

Theorem 4.2. *Let A be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. If $\bar{\alpha} \subseteq G_\Upsilon$. Then, A/α is an SDD matrix.*

Proof. By $\bar{\alpha} \subseteq G_\Upsilon$, we get

$$\bar{\alpha} \subseteq N^+(A),\ j_u \in G_\Upsilon \text{ for all } u \in [l],\ \Theta = \emptyset.$$

If $\Upsilon = \emptyset$, by Theorem 4.1, A/α is an SDD matrix. If $\Upsilon \neq \emptyset$, then $S_3 = [l]$. It follows that A/α is an SDD matrix by Corollary 3.1. □

Theorem 4.3. *Let A be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. If $\Upsilon = \emptyset$, and $\Theta \neq \emptyset$, then A/α is a DZT matrix. Moreover, if $G_\Theta \neq \emptyset$, then A/α is also a DZ matrix.*

Proof. For any $u \in [l]$, we have $u \in S_1 \subseteq N^+(A/\alpha)$ if $j_u \in N^+(A)$ and $u \in S_2 \subseteq N^+(A/\alpha)$ if $j_u \in N^-(A) \setminus \Theta$ by Corollary 3.1. Hence, we have

$$N^-(A/\alpha) \subseteq \{u \in [l] : j_u \in \Theta\}.$$

For any $j_u \in \Theta$, we have $\Gamma_u(A/\alpha) \neq \emptyset$ by Lemma 4.1 (i). Therefore, A/α is a DZT matrix.

Particularly, if $G_\Theta \neq \emptyset$, then in the proof of Lemma 4.1, we can choose $j_s \in G_\Theta$. Then, $s \in \bigcap\limits_{j_u \in \Theta} \Gamma_u(A/\alpha)$. Hence, A/α is also a DZ matrix. □

Corollary 4.3. *Let A be a DZT matrix, and let $\emptyset \neq \alpha \subset N^+(A)$. Then, A/α is a DZT matrix.*

Proof. Because $\alpha \subset N^+(A)$, we have $\Upsilon = \emptyset$. Then, A/α is an SDD matrix if $\Theta = \emptyset$ by Theorem 4.1, and A/α is a DZT matrix if $\Theta \neq \emptyset$ by Theorem 4.3. □

Remark 4.2. Corollary 4.3 coincides with [2, Theorem 3.6].

Theorem 4.4. *Let A be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. Suppose $\Upsilon \neq \emptyset$, $G_\Upsilon \neq \emptyset$, and $\Theta = \emptyset$. Then, A/α is a DZT matrix.*

Proof. For any $u \in [l]$, if $j_u \in G_\Upsilon$, we have $u \in S_3 \subseteq N^+(A/\alpha)$ by Corollary 3.1. Hence, we have by $\Theta = \emptyset$,

$$N^-(A/\alpha) \subseteq \{u \in [l] : j_u \in N^+(A) \setminus G_\Upsilon\} \cup \{u \in [l] : j_u \in N^-(A), \Gamma_{j_u} \cap \alpha \neq \emptyset\}.$$

If $j_u \in N^+(A) \setminus G_\Upsilon$, we have $\Gamma_u(A/\alpha) \neq \emptyset$ by Lemma 4.1 (ii). If $j_u \in N^-(A)$ with $\Gamma_{j_u} \cap \alpha \neq \emptyset$, we have $\Gamma_u(A/\alpha) \neq \emptyset$ by Lemma 4.1 (iii). Hence, A/α is a DZT matrix. □

Theorem 4.5. *Let A be a DZT matrix, and let $\emptyset \neq \alpha \subset [n]$. Suppose $\Upsilon \neq \emptyset$, $G_\Upsilon \neq \emptyset$, and $\Theta \neq \emptyset$. If $\Gamma_j(A) \cap G_\Upsilon \neq \emptyset$ for all $j \in \Theta$, then A/α is a DZT matrix. Moreover, if $G_\Theta \cap G_\Upsilon \neq \emptyset$, then A/α is also a DZ matrix.*

Proof. Because $\Upsilon \neq \emptyset$, for any $u \in [l]$, if $j_u \in G_\Upsilon$, we have $u \in S_3 \subseteq N^+(A/\alpha)$ by Corollary 3.1. By $\Theta \neq \emptyset$, we have

$$N^-(A/\alpha) \subseteq \{u \in [l] : j_u \in N^+(A) \setminus G_\Upsilon\} \cup \{u \in [l] : j_u \in N^-(A) \setminus \Theta\} \cup \{u \in [l] : j_u \in \Theta\}.$$

If $j_u \in N^+(A) \setminus G_\Upsilon$, we have $\Gamma_u(A/\alpha) \neq \emptyset$ by Lemma 4.1 (ii). If $j_u \in \Theta$, by $\Gamma_{j_u}(A) \cap G_\Upsilon \neq \emptyset$, we have $\Gamma_u(A/\alpha) \neq \emptyset$ by Lemma 4.1 (iv). If $j_u \in N^+(A) \setminus \Theta$, we have $\Gamma_u(A/\alpha) \neq \emptyset$ by Lemma 4.1 (iii). Then, A/α is a DZT matrix.

Particularly, if there exists $j_s \in G_\Theta \cap G_\Upsilon$, then we have $s \in \Gamma_u(A/\alpha)$ for all $u \notin S_3$ by Lemma 4.1(i)–(iii). Then, A/α is also a DZ matrix. □

Remark 4.3. For a given matrix A and a set α, Li et al. established that the Schur complement of a DZT matrix is an SDD matrix if $N^+(A) \subseteq \alpha$ [2, Theorem 3.4]. Zeng et al. established that the Schur complement of a DZT matrix is an SDD matrix if $\bigcup\limits_{i \in N^-(A)} \Gamma_i(A) \subseteq \alpha$ [23, Theorem 5]. It is easy to see that the conditions $N^+(A) \subseteq \alpha$ and $\bigcup\limits_{i \in N^-(A)} \Gamma_i(A) \subseteq \alpha$ are two special cases of $\Upsilon = \Theta = \emptyset$.

Moreover, Li et al. established that the Schur complement of a DZT matrix remains a DZT matrix if $\alpha \subseteq N^{+}(A)$ [2, Theorem 3.6]. It is easy to see that $\alpha \subseteq N^{+}(A)$ is a special case of $\Upsilon = \emptyset$. We consider an example. Let

$$A = \begin{bmatrix} 32 & 4 & 1 & 7 & 0 & 0 & 0 & 0 \\ 1 & 30 & 3 & 3 & 0 & 0 & 0 & 0 \\ 9 & 1 & 8 & 3 & 0 & 0 & 0 & 0 \\ 1 & 8 & 9 & 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 32 & 4 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 & 30 & 3 & 3 \\ 0 & 0 & 0 & 0 & 9 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 & 1 & 8 & 9 & 12 \end{bmatrix}. \tag{4.11}$$

It is easy to testify that A is a DZT matrix with $N^{+}(A) = \{1, 2, 5, 6\}$, $N^{-}(A) = \{3, 4, 7, 8\}$, $\Gamma_3(A) = \{1\}, \Gamma_4(A) = \{2\}, \Gamma_7(A) = \{5\}$, and $\Gamma_8(A) = \{6\}$. Take $\alpha = \{1, 2, 3, 4\}$. By computation, we have that

$$\Upsilon = \Theta = \emptyset,\ N^{+}(A) \nsubseteq \alpha,\ \bigcup_{i \in N^{-}(A)} \Gamma_i(A) \nsubseteq \alpha,$$

and A/α is an SDD matrix. Take $\beta = \{1, 2, 3, 4, 5, 7, 8\}$. By computation, we have that

$$\bar{\beta} \subseteq G_{\Upsilon},\ \Upsilon \neq \emptyset,\ N^{+}(A) \nsubseteq \alpha,\ \bigcup_{i \in N^{-}(A)} \Gamma_i(A) \nsubseteq \alpha,$$

and A/β is an SDD matrix. Here, α satisfies the condition in Theorem 4.1, β satisfies the condition in Theorem 4.2, and neither of them satisfies any condition in [2] and [23]. In fact, Theorems 4.1–4.5 indicate 115 Schur complements to be DZT matrices and 25 Schur complements to be SDD matrices. In contrast, the results given by [2] and [23] are applicable to only 29 Schur complements, of which 15 are SDD matrices. For further detail, one can see Example 6.4.

Remark 4.4. For a DZT matrix A and a proper index subset α, the class that A/α belongs to can be summarized as in Table 1, where (C1) and (C2) denote the conditions under which A/α does not necessarily yield a DZT matrix.

Table 1. The Schur complements for DZT matrices.

Υ	Θ	Additional condition	A/α
$\emptyset$	$\emptyset$	–	SDD
$\emptyset$	$\neq \emptyset$	$G_{\Theta} = \emptyset$	DZT
$\emptyset$	$\neq \emptyset$	$G_{\Theta} \neq \emptyset$	DZT, DZ
$\neq \emptyset$	$\emptyset$	$G_{\Upsilon} \neq \emptyset$	DZT
$\neq \emptyset$	$\emptyset$	$\bar{\alpha} \subseteq G_{\Upsilon}$	SDD
$\neq \emptyset$	$\neq \emptyset$	$G_{\Upsilon} \neq \emptyset, \Gamma_j(A) \cap G_{\Upsilon} \neq \emptyset, \forall j \in \Theta$	DZT
$\neq \emptyset$	$\neq \emptyset$	$G_{\Upsilon} \neq \emptyset, \Gamma_j(A) \cap G_{\Upsilon} \neq \emptyset, \forall j \in \Theta, \Gamma_{\Theta}(A) \cap G_{\Upsilon} \neq \emptyset$	DZT, DZ
$\neq \emptyset$	$\neq \emptyset$	$G_{\Upsilon} \neq \emptyset, \Gamma_j(A) \cap G_{\Upsilon} \neq \emptyset, \exists j \in \Theta$	(C1)
$\neq \emptyset$	–	$G_{\Upsilon} \neq \emptyset$	(C2)

We provide a method for determining the class of the Schur complement for any given DZT matrix A and nonempty proper subset α of $[n]$:

1) Compute the sets $N^+(A)$, $N^-(A)$, $\Gamma_i(A)$ for $i \in N^-(A)$;
2) Compute Υ, Θ, G_Υ, G_Θ;
3) Match the computed quantities with one of the rows in Table 1; the corresponding entry in the last column gives the class of A/α (SDD, DZT, DZ, or one of the combined cases (C1), (C2)).

5. Applications

In this section, we apply our obtained results to determinant estimation and solving large linear systems.

5.1. The determinants of DZT matrices

The dominant degrees of the Schur complements can be used to estimate matrices' determinants. For instance, the bounds for determinants of SDD matrices [4,29], γ-SDD matrices [12], and Nekrasov matrices [17], were investigated by using diagonally dominant degrees of the Schur complements. Motivated by these pioneering works, we estimate the determinants of DZT matrices.

Lemma 5.1. *[30] Let $A = (a_{ij}) \in C^{n\times n}$ be an SDD matrix. Then,*

$$\prod_{i\in[n]} (|a_{ii}| - r_i(A)) \le |det(A)| \le \prod_{i\in[n]} (|a_{ii}| + r_i(A)).$$

Lemma 5.2. *[31] Let $A = (a_{ij}) \in C^{n\times n}$ be an SDD matrix. Then,*

$$\prod_{i\in[n]} (|a_{ii}| - \sum_{j>i} |a_{ij}|) \le |det(A)| \le \prod_{i\in[n]} (|a_{ii}| + \sum_{j>i} |a_{ij}|).$$

Theorem 5.1. *Let $A = (a_{ij}) \in C^{n\times n}$ be a DZT matrix. Then,*

$$\begin{aligned} |det(A)| &\ge \prod_{j\in N^-(A)} (|a_{jj}| - r_j(A) + \min_{i\in\Gamma_j(A)} (1 - \frac{r_i(A)}{|a_{ii}|})|a_{ji}|) \prod_{i\in N^+(A)} (|a_{ii}| - \sum_{v>i, v\in N^+(A)} |a_{iv}|), \\ |det(A)| &\le \prod_{j\in N^-(A)} (|a_{jj}| + r_j(A) - \min_{i\in\Gamma_j(A)} (1 - \frac{r_i(A)}{|a_{ii}|})|a_{ji}|) \prod_{i\in N^+(A)} (|a_{ii}| + \sum_{v>i, v\in N^+(A)} |a_{iv}|). \end{aligned}$$

Proof. Let $\alpha = N^+(A)$. Then, $A(N^+(A))$ is an SDD matrix by Corollary 2.1, and $A/N^+(A)$ is also an SDD matrix by Corollary 4.1. Moreover, we have (3.13) and (3.14) for any $u \in [l]$. By Lemma 2.3, we have

$$|det(A)| = det(A(N^+(A)))det(A/N^+(A)).$$

Then, by Lemma 5.2, we have

$$\prod_{i\in N^+(A)} (|a_{ii}| - \sum_{v>i, v\in N^+(A)} |a_{iv}|) \le det(A(N^+(A))) \le \prod_{i\in N^+(A)} (|a_{ii}| + \sum_{v>i, v\in N^+(A)} |a_{iv}|),$$

By (3.13), (3.14), and Lemma 5.1, we have

$$det(A/N^+(A))$$

$$
\begin{aligned}
&\leq \prod_{j_u \in N^-(A)} (|a'_{uu}| + r_u(A/\alpha)) \\
&\leq \prod_{j_u \in N^-(A)} (|a_{j_u j_u}| + r_{j_u}(A) - \min_{i_t \in \Gamma_{j_u}(A)} (1 - \frac{r_{i_t}(A)}{|a_{i_t i_t}|})|a_{j_u i_t}|), \\
&= \prod_{j \in N^-(A)} (|a_{jj}| + r_j(A) - \min_{i \in \Gamma_j(A)} (1 - \frac{r_i(A)}{|a_{ii}|})|a_{ji}|),
\end{aligned}
$$

and

$$
\begin{aligned}
&det(A/N^+(A)) \\
&\geq \prod_{j_u \in N^-(A)} (|a'_{uu}| - r_u(A/\alpha)) \\
&\geq \prod_{j_u \in N^-(A)} (|a_{j_u j_u}| - r_{j_u}(A) + \min_{i_t \in \Gamma_{j_u}(A)} (1 - \frac{r_{i_t}(A)}{|a_{i_t i_t}|})|a_{j_u i_t}|), \\
&= \prod_{j \in N^-(A)} (|a_{jj}| - r_j(A) + \min_{i \in \Gamma_j(A)} (1 - \frac{r_i(A)}{|a_{ii}|})|a_{ji}|).
\end{aligned}
$$

The result then holds. □

Remark 5.1. For any SDD matrix, it is easy to see that the bound in Theorem 5.1 is the same as that in Lemma 5.2 and sharper than that in Lemma 5.1.

Remark 5.2. In Theorem 5.1 and Lemma 5.2, the ordering $v > i$ is taken as the natural order of the entries of A (i.e., the original increasing order of the indices). In fact, for any permutation matrix P, we have $det(p^T AP) = det(A)$; consequently, the determinant bound can be applied to any permuted version of the matrix. Theoretically, if we exhaust all possible orderings (permutations), we would obtain an optimal bound that is closest to the true determinant. However, this exhaustive search is generally computationally prohibitive, as it requires $n!$ evaluations. Therefore, in practice, we may implement several special orderings, including the natural orders that are likely to yield sharper bounds. For instance, we may consider the reverse natural order (i.e., $n, n-1, \cdots, 1$).

Example 5.1. *Consider the following DZT matrices, in which A_5 is an SDD matrix:*

$$
A_1 = \begin{bmatrix} 0.9 & -0.1 \\ -6 & 1.4 \end{bmatrix}, \; A_2 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 3 & 4 \end{bmatrix},
$$

$$
A_3 = \begin{bmatrix} 27 & 1 & 1 & 2 \\ 5 & 3 & 0 & 2 \\ 4 & 4 & 7 & 1 \\ 4 & 5 & 2 & 29 \end{bmatrix}, \; A_4 = \begin{bmatrix} -2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 7 & 0 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 & 5 & 0 \\ 1 & 1 & 1 & 1 & 1 & 10 \end{bmatrix}, \; A_5 = \begin{bmatrix} 10 & 2 & 1 & 2 \\ 1 & 6 & 0 & 2 \\ 1 & 4 & 12 & 1 \\ 4 & 0 & 0 & 6 \end{bmatrix}.
$$

The bounds of determinants for A_1–A_5 are estimated by Theorem 5.1. See Table 2.

Table 2. The determinants of A_1–A_5.

	A_1	A_2	A_3	A_4	A_5
$\lvert det(A_i)\rvert$	0.66	19	14235	10050	3772
Theorem 5.1	[0.66, 4.86]	[2, 60]	[71.10,59807.41]	[450,12150]	[180,24300]
Lemma 5.1	\	\	\	\	[1320,9360]
Lemma 5.2	\	\	\	\	[1320,9360]

5.2. *Solving linear equations*

Solving large-scale linear systems is an important topic in scientific computing. We consider the following linear system:

$$Ax = b, \tag{5.1}$$

where $A \in R^{n\times n}$, and $b \in R^n$. If $A(\alpha)$ is nonsingular, (5.1) can be transformed into two systems with smaller size:

$$(A/\alpha)x(\bar{\alpha}) = b(\bar{\alpha}) - A(\bar{\alpha},\alpha)A(\alpha)^{-1}b(\alpha), \tag{5.2}$$

$$A(\alpha)x(\alpha) = -A(\alpha,\bar{\alpha})x(\bar{\alpha}) + b(\alpha). \tag{5.3}$$

The method of transforming (5.1) into (5.2) and (5.3) was called the Schur-based method [2–6]. This paper uses the Jacobi method and the Gauss–Seidel (GS) method to solve (5.2) and (5.3) (which was called the Schur-based Jacobi (S-Jacobi) method and the Schur-based Gauss–Seidel (S-GS) iteration method). It is shown in Table 3 that Schur-based methods may reduce iterations and shorten computation time.

Here, we list several notes on the numerical experiments.

(i) All experiments are carried out via MATLAB 2023b on a Windows 11 (64 bit) PC with the following configuration: Intel(R) Core(TM) i5-8250U 1.60GHz CPU and 8 GB RAM.

(ii) When computing A/α, it is necessary to compute $A(\alpha)^{-1}$. Hence, we try to choose some α such that $A(\alpha)$ has special structures, such as diagonal matrix, upper (lower) triangular matrix, etc.

(iii) In solving (5.2), we first solve $A(\alpha)y = b(\alpha)$. Then, (5.2) is converted to $(A/\alpha)x(\bar{\alpha}) = b(\bar{\alpha}) - A(\bar{\alpha},\alpha)y$. Then, the Jacobi or GS method is implemented to solve it. In solving (5.3), if $A(\alpha)$ is a diagonal matrix, we multiply both sides of (5.3) by $A(\alpha)^{-1}$; otherwise, we apply the same iteration method to that in solving (5.2).

(iv) The cputime is the sum of time to compute A/α and solve (5.2) and (5.3).

(v) The GS and S-GS methods terminate if the residual error satisfies $\|x^{k+1} - x^k\|_\infty < 10^{-6}$. The Jacobi and S-Jacobi method terminate if the residual error satisfies $\|x^{k+1} - x^k\|_\infty < 10^{-6}$.

Example 5.2. *Now, we consider the linear system $Ax = b$, where $b = (1, 1, \dots, 1)^T \in R^{2n\times 1}$,*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in R^{2n\times 2n},$$

where $A_{ij} \in R^{n\times n}$ for $i, j \in \langle 2\rangle$ (n is an even number), and

$$A_{11} = \begin{bmatrix} 2.7 & 0.6 & & & & \\ 0.5 & 2 & 0.6 & & & \\ & 0.5 & 2.7 & 0.6 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0.5 & 2.7 & 0.6 \\ & & & & 0.5 & 2 \end{bmatrix},$$

$$A_{12} = A_{21} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{bmatrix},$$

It is easy to see that

$$N^{-}(A) = \{2, 4 \ldots, n-2\}, \ N^{+}(A) = \{1, 3, \ldots, n-3, n-1, n, n+1, n+2, \ldots, 2n\},$$

$$\Gamma_i(A) = \{i-1, i+1, n+1, n+2, \ldots, 2n\} \text{ for all } i \in N^{-}(A).$$

For $\alpha = \{n+1, n+2, \ldots, 2n\}$, we have $\alpha \cap \Gamma_i(A) \neq \emptyset$ for all $i \in N^{-}(A)$; then, A/α is an SDD matrix. Noticing that $A(\alpha)$ is a diagonal matrix, it saves a lot of time to compute the Schur complement A/α and solve (5.3). The results are shown in Table 3. We can see that the effectiveness of Schur-based methods is significant.

Table 3. Results of Example 5.2 ($\alpha = \{n+1, n+2, \ldots, 2n\}$).

		Jacobi method	S-Jacobi method	GS method	S-GS method
n=200	iterations	59	31	25	16
	cputime	0.0059597	0.0075971	0.14367	0.032477
	$\|\|Ax-b\|\|_2$	2.1192e-05	1.1393e-05	9.4439e-06	3.0851e-06
n=500	iterations	59	31	25	16
	cputime	0.082402	0.038591	0.38547	0.10738
	$\|\|Ax-b\|\|_2$	3.406e-05	1.8348e-05	1.5149e-05	4.977e-06
n=1000	iterations	59	31	25	16
	cputime	0.37722	0.13904	1.4039	0.31503
	$\|\|Ax-b\|\|_2$	4.8426e-05	2.6102e-05	2.1526e-05	7.0846e-06
n=2000	iterations	59	31	25	16
	cputime	2.3866	0.79711	5.4443	1.1983
	$\|\|Ax-b\|\|_2$	6.8666e-05	3.7024e-05	3.0513e-05	1.0052e-05
n=5000	iterations	59	31	25	16
	cputime	26.213	11.25	43.712	12.496
	$\|\|Ax-b\|\|_2$	0.00010874	5.8643e-05	4.8314e-05	1.5924e-05

6. Examples

Examples 6.1 and 6.2 show that the Schur complements of DZT matrices may not be DZT matrices under the conditions (C1) and (C2). Examples 6.3–6.5 illustrate the correctness of Theorems 4.1–4.5.

Example 6.1. *We consider the DZT matrix [2]:*

$$A = \begin{bmatrix} 32 & 4 & 1 & 7 \\ 1 & 30 & 3 & 3 \\ 9 & 1 & 8 & 3 \\ 1 & 8 & 9 & 12 \end{bmatrix}.$$

We have $N^+(A) = \{1,2\}$, $N^-(A) = \{3,4\}$, $\Gamma_3(A) = \{1\}$, and $\Gamma_4(A) = \{2\}$. For $\alpha = \{3\} := \{i_1\}$, we have $\Upsilon = \{3\}$, $\bar{\alpha} = \{1,2,4\} := \{j_1, j_2, j_3\}$, $\Theta = \{4\}$. Then, we have $\Gamma_{j_3}(A) \bigcap (\bigcap_{i_t \in \Upsilon} \Gamma_{i_t}(A)) = \{2\} \cap \{1\} = \emptyset$. It follows from $\Gamma_3(A/\alpha) = \emptyset$ that the Schur complement of a DZT matrix may be not a DZT matrix when the condition (C1) occurs.

Example 6.2. *Let A be defined as in (4.11) and $\alpha = \{3,8\} := \{i_1, i_2\}$. Then we have $\Upsilon = \{3,8\}$ and $\bigcap_{i_t \in \Upsilon} \Gamma_{i_t}(A) = \emptyset$. By computation, we get $\Gamma_3(A/\alpha) = \Gamma_6(A/\alpha) = \emptyset$, which implies that the Schur complement of a DZT matrix may be not a DZT matrix when the condition (C2) occurs.*

Example 6.3. *Let A be defined as in (1.1). There are $2^4 - 2 = 14$ nonempty proper index subsets of* [4]. *The results are shown in Table 4. The boldface in the table indicates that the index subset does not satisfy the conditions stated in [2] and [23]. We can see that the* 14 *Schur complements are all DZT matrices, in which only* 8 *index subsets satisfy the conditions stated in [2] and [23], that is,* $N^+(A) \nsubseteq \alpha$, $\alpha \nsubseteq N^+(A)$, $\bigcup_{i \in N^-(A)} \Gamma_i(A) \nsubseteq \alpha$.

Table 4. The results of Example 6.3.

α	Theorem	A/α
$\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}$, $\{1,2,4\}, \{1,3,4\}$	Theorem 4.1	SDD
$\{4\}$	Theorem 4.3	DZT,DZ
$\{2,3,4\}$	Corollary 4.1	SDD
$\{2,3\}$	Theorem 4.4	DZT
$\{2\}$, **{3}**, $\{2,4\}$, **{3,4}**	Theorem 4.5	DZT,DZ

Example 6.4. *Let A be defined as in (4.11). We study $2^8 - 2 = 254$ nonempty proper index subsets one by one. The results are shown in Table 5. There are* 115 *index subsets satisfying the conditions in Theorems 4.1–4.5, in which only* 29 *index subsets satisfy the conditions stated in [2] and [23]. The example also shows that: A/α is a DZT matrix but not necessarily a DZ matrix when α satisfies $\Upsilon = \emptyset$, $\Theta \neq \emptyset$, and $G_\Theta = \emptyset$ (see Theorem 4.3); and that A/α is a DZT matrix but not necessarily an SDD matrix when α satisfies $\Upsilon \neq \emptyset$, $\Theta = \emptyset$, $G_\Upsilon \neq \emptyset$ (see Theorem 4.4).*

Table 5. The results of Example 6.4.

α	Theorem	A/α
$\{1,2,5,6\},\{1,2,3,5,6\},\{1,2,4,5,6\},$ $\{1,2,5,6,8\},\{1,2,3,4,5,6\},\{1,2,3,5,6,7\},$ $\{1,2,4,5,6,7\},\{1,2,4,5,6,8\},\{1,2,4,6,7,8\},$ $\{1,2,5,6,7,8\},\{1,2,3,4,5,6,7\},\{1,2,3,4\},$ $\{1,2,3,5,6,7,8\},\{1,2,4,5,6,7,8\},$ $\{1,2,5,6,7\},\{1,2,3,4,5,6,8\},\{1,2,3,5,6,8\}$	Theorem 4.1	SDD
$\{1\},\{2\},\{5\},\{6\},\{1,2\},\{1,3\},\{1,5\},\{1,6\},$ $\{2,4\},\{2,5\},\{2,6\},\{5,6\},\{5,7\},\{1,2,3\},$ $\{1,3,6\},\{1,5,7\},\{1,6,8\},\{2,4,5\},$ $\{2,6,8\},\{5,6,7\},\{5,6,8\},\{2,4,5,7\},$ $\{1,3,6,8\},\{2,4,6,8\},\{5,6,7,8\},\{1,3,5\},$ $\{1,2,4\},\{2,5,7\},\{1,3,5,7\},\{2,4,6\}$	Theorem 4.3 $G_\Theta = \emptyset$	DZT (not necessarily DZ, such as $A/\{1\}$)
$\{1,2,5\},\{1,2,6\},\{1,5,6\},\{2,5,6\},\{1,2,3,5\},$ $\{2,4,5,6\},\{1,2,4,5\},\{1,2,4,6\},\{1,2,5,7\},$ $\{2,5,6,8\},\{1,2,3,4,5\},\{1,2,3,4,6\},$ $\{1,2,4,5,7\},\{1,5,6,7\},\{1,5,6,8\},$ $\{2,4,6,7,8\},\{2,5,6,7,8\},\{1,2,3,4,5,7\},$ $\{1,3,5,6,8\},\{1,5,6,7,8\},\{2,4,5,6,7\},$ $\{1,3,5,6\},\{1,3,5,6,7\},\{1,3,5,6,7,8\},$ $\{1,2,3,6\},\{1,2,6,8\},\{1,2,3,6,8\},$ $\{1,2,3,4,6,8\},\{2,4,5,6,8\},\{2,4,5,6,7,8\},$ $\{1,2,3,5,7\},\{1,2,4,6,8\},\{2,5,6,7\}$	Theorem 4.3 $G_\Theta \neq \emptyset$	DZT, DZ (not necessarily SDD, such as $A/\{1,2,5\}$)
$\{1,2,3,4,5,7,8\},\{1,2,3,4,6,7,8\},$ $\{1,3,4,5,6,7,8\},\{2,3,4,5,6,7,8\},$	Theorem 4.2	SDD
$\{1,2,5,8\},\{1,4,5,6\},\{2,3,5,6\},$ $\{1,2,3,6,7\},\{1,2,4,5,8\},\{1,2,4,6,7\},$ $\{1,2,6,7,8\},\{1,3,4,5,6\},\{1,4,5,6,7\},$ $\{2,3,5,6,7\},\{2,3,5,6,8\},\{1,2,3,4,5,8\},$ $\{1,3,4,5,6,7\},\{1,3,4,5,6,8\},$ $\{1,2,3,6,7,8\},\{2,3,4,5,6,8\},$ $\{2,3,4,5,6\},\{1,2,5,7,8\},\{1,3,4,5,7,8\}$ $\{1,2,3,5,8\},\{1,2,3,4,6,7\},\{2,3,4,5,6,7\},$ $\{1,2,4,5,7,8\},\{1,4,5,6,8\},\{2,4,6,7\}.$ $\{1,4,5,6,7,8\},\{2,3,5,6,7,8\}$	Theorem 4.4 $\bar{\alpha} \nsubseteq G_\Upsilon$	DZT (not necessarily SDD, such as $A/\{2,4,6,7\}$)
Others	\	\

Example 6.5. *Assume a DZT matrix*

$$A = \begin{bmatrix} 52 & 0 & 3 & 3 & 3 & 6 \\ -5 & -18 & 1 & -4 & -5 & -5 \\ -6 & -4 & -14 & 0 & 4 & 4 \\ -5 & -5 & -1 & 15 & 1 & -4 \\ 1 & 5 & -5 & 0 & -14 & 6 \\ -3 & 2 & -2 & -2 & -5 & 34 \end{bmatrix}.$$

with

$$N^{+}(A) = \{1,6\},\ N^{-}(A) = \{2,3,4,5\},\ \Gamma_2(A) = \Gamma_4(A) = \{1,6\},\ \Gamma_3(A) = \{1\},\ \Gamma_5(A) = \{6\}.$$

The results are shown in Table 6. We can see there are 50 index subsets satisfying the conditions in Theorems 4.1–4.5, in which only 17 index subsets satisfy the conditions stated in [2] and [23]. This example shows that A/α is a DZT matrix but not necessarily a DZ matrix when α satisfies (see Theorem 4.5):

$$\Upsilon \neq \emptyset, \Theta \neq \emptyset, \Gamma_{j_u}(A) \cap G_{\Upsilon} \neq \emptyset \textit{ for all } j_u \in \Theta, G_{\Theta} \cap G_{\Upsilon} = \emptyset.$$

Table 6. The results of Example 6.5.

α	Theorem	A/α
$\{1,6\},\{1,2,6\},\{1,3,6\}$, $\{1,2,4,6\},\{1,2,5,6\}$, $\{1,2,3,4,6\},\{1,3,4,5,6\}$, $\{1,2,3,6\},\{1,4,5,6\}$, $\{1,5,6\},\{1,3,5,6\}$, $\{1,4,6\},\{1,3,4,6\}$, $\{1,2,4,5,6\},\{1,2,3,5,6\}$	Theorem 4.1	SDD
$\{1\},\{6\},\{1,2\},\{1,3\},\{1,4\}$, $\{1,2,3\},\{1,2,4\},\{1,3,4\}$, $\{1,2,3,4\},\{2,4,5,6\},\{5,6\}$, $\{4,6\},\{2,5,6\},\{4,5,6\}$ $\{2,6\},\{2,4,6\}$	Theorem 4.3 $G_{\Theta} \neq \emptyset$	DZT, DZ
$\{1,2,3,4,5\},\{2,3,4,5,6\}$	Theorem 4.2	SDD
$\{3,6\},\{1,2,5\},\{1,3,5\}$, $\{3,4,6\},\{3,5,6\},\{1,2,3,5\}$, $\{2,3,4,6\},\{2,3,5,6\}$, $\{2,3,6\},\{1,3,4,5\},\{1,4,5\}$, $\{1,5\},\{1,2,4,5\},\{3,4,5,6\}$	Theorem 4.4	DZT
$\{2\},\{4\},\{2,4\}$	Theorem 4.5 $\Gamma_{j_u}(A) \cap G_{\Upsilon} \neq \emptyset$ $G_{\Theta} \cap G_{\Upsilon} = \emptyset$	DZT (not necessarily DZ, such as $A/\{4\}$)
Others	\	\

The numbers of total nonempty proper subsets (N_{total}), the subsets covered by previous results (N_{pre}), and the subsets covered by our results (N_{our}) in Examples 6.3–6.5 are shown in Table 7. From these examples, we can see that when α does not satisfy the current criteria, the Schur complement may or may not be DZT. For instance, if we take $\alpha = \{5\}$ in Example 6.5, the Schur complement is a DZT matrix; if we take $\alpha = \{3\}$ in Example 6.1, it is not a DZT matrix.

Table 7. Quantitative summaries.

Example	N_{total}	N_{pre}	N_{our}
6.3	14	6	14
6.4	254	29	111
6.5	62	17	50

7. Conclusions

In this paper, we investigate the closure property of the Schur complement for DZT matrices. We classify elements in $N^{-}(A)$ into those belonging to Υ, those belonging to Θ, and others. Through a discussion of Υ and Θ, we first provide a partial characterization of $N^{+}(A/\alpha)$, and then consider the conditions under which $\Gamma_u(A/\alpha)$ is not empty for $u \in N^{-}(A/\alpha)$. Following these conditions, we establish the criteria for A/α to be DZT matrix. The results showed that: the proposed criteria cover many previously untreated cases, and that the examples show a larger range of applicability than earlier sufficient conditions. As an application, we obtain the upper and lower bounds for determinants of DZT matrices, which are very elegant. Several possible extensions of the present work are worth exploring. Our conditions are more general than existing ones, so it would be interesting to investigate whether they can be further relaxed or adapted to other structured matrix classes beyond DZT matrices.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author has no competing interests to declare that are relevant to the content of this article.

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