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Research article

On subnormal transcendental meromorphic solutions of delay differential equations with higher order derivative

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Abstract: The existence of subnormal transcendental meromorphic solutions for the following two classes of delay differential equations

$$\omega(z+1)-\omega(z-1)+a(z)\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^n = R(z,\omega(z)) = \frac{P(z,\omega(z))}{Q(z,\omega(z))}$$

and

$$(\omega(z)\omega(z+1)-1)(\omega(z)\omega(z-1)-1)+a(z)\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^n = R(z,\omega(z)) = \frac{P(z,\omega(z))}{Q(z,\omega(z))}$$

is studied, where $k \geq 1$ and $n \geq 1$ are integers, $a(z)$ is non-zero rational, and $P(z,\omega)$, $Q(z,\omega)$ are polynomials in ω with coefficients rational functions in z, and the roots of $Q(z,\omega)$ are non-zero functions of z and are distinct from the roots of $P(z,\omega)$. The necessary conditions for the existence of solutions of these equations mentioned above are characterized by using the degrees of $P(z,\omega)$ and $Q(z,\omega)$. Moreover, some examples are provided to illustrate these results.

Keywords: Nevanlinna theory; delay differential equations; Painlevé equations; subnormal solutions; existence
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1. Introduction

Malmquist proved the classical result for the differential equation $\omega'(z) = R(z,\omega)$ in [1], which was called Malmquist theorem. Later, Yosida [2] and Laine [3] gave elegant alternative proofs for the classical Malmquist theorem by refining methods within the Nevanlinna theory. Subsequently, as the Nevanlinna theory was systematically extended to the discrete setting, Ablowitz et al. [4] regarded the existence of finite-order meromorphic solutions as a manifestation of the Painlevé property. For the difference equation

$$\omega(z+1)+\omega(z-1)=R(z,\omega), \tag{1.1}$$

where $R(z,\omega)$ is a rational function of ω, Halburd and Korhonen [5] classified its finite-order meromorphic solutions. Moreover, reductions of integrable differential difference equations are known to yield delay differential equations with formal continuum limits to Painlevé equations, as exemplified by Quispel et al. [6]. The properties of solutions of delay differential equations become a very interesting topic, attracting the attention of many researchers; for instance, [7–9] have conducted extensive studies from various aspects, while [10, 11] have recently focused on subnormal solutions. Therefore, subnormal solutions of two classes of delay differential equations are studied in this paper.

2. Subnormal solutions of higher order delay differential equations

Halburd and Korhonen [12] further generalized the delay differential equation proposed by Quispel [6] and established the following theorem concerning its non-rational meromorphic solutions.

Theorem A. *[12] Let ω be a non-rational meromorphic solution of the following:*

$$\omega(z+1)-\omega(z-1)+a(z)\frac{\omega'(z)}{\omega(z)}=R(z,\omega(z))=\frac{P(z,\omega(z))}{Q(z,\omega(z))}, \tag{2.1}$$

where $a(z)$ is a rational function, $P(z,\omega(z))$ is a polynomial in ω with rational coefficients in z, and $Q(z,\omega(z))$ is a monic polynomial in ω satisfying $Q(z,0)\not\equiv 0$, with all roots being rational in z and none of them being roots of $P(z,\omega(z))$. If the hyper-order $\rho_2(\omega)<1$, then either $\deg_\omega(P)=\deg_\omega(Q)+1\le 3$ *or* $\deg_\omega(R)=\max\{\deg_\omega(P),\deg_\omega(Q)\}\in\{0,1\}$.

The hyper-order of a transcendental meromorphic function ω is defined as follows:

$$\rho_2(\omega)=\limsup_{r\to\infty}\frac{\log\log T(r,\omega)}{\log r}.$$

This is the growth condition used in Theorem A, whereas subsequent results rely on subnormality, as shown in [13–15]. A transcendental meromorphic function ω is said to be subnormal if it satisfies the condition $\limsup\limits_{r\to\infty}\dfrac{\log T(r,\omega)}{r}=0$, which is also called minimal hyper-type [15, 16]. In 2022, Chen and Cao [14] established the degree relations of polynomial coefficients for the following equation involving the *n-th* power of the logarithmic derivative terms

$$\omega(z+1)-\omega(z-1)+a(z)\left(\frac{\omega'(z)}{\omega(z)}\right)^n=R(z,\omega(z))=\frac{P(z,\omega(z))}{Q(z,\omega(z))}.$$

More recently, Cao and Chen [13] extended the framework to the case of higher-order logarithmic derivatives $\frac{\omega^{(k)}(z)}{\omega(z)}$, thereby proving the following theorem.

Theorem B. *[13] Let ω be a subnormal transcendental meromorphic solution of the following:*

$$\omega(z+1)-\omega(z-1)+a(z)\frac{\omega^{(k)}(z)}{\omega(z)}=R(z,\omega(z))=\frac{P(z,\omega(z))}{Q(z,\omega(z))}, \tag{2.2}$$

where k is a positive integer, $a(z)$ is rational in z, $P(z,\omega(z))$ is a polynomial in ω having rational coefficients in z, and $Q(z,\omega(z))$ is of degree zero or a polynomial in ω with roots that are nonzero rational functions of z and not roots of $P(z,\omega(z))$. If $\deg_\omega(Q) \le 1$*, then* $\deg_\omega(P) \le \deg_\omega(Q) + k$*. If* $\deg_\omega(Q) \ge 2$*, then* $\deg_\omega(Q) < \deg_\omega(P) \le \deg_\omega(Q) + k$ *and* $\deg_\omega(P) \le k + 3$*. In addition, if* $\deg_\omega(P) = \deg_\omega(Q) + k$*, then* $\deg_\omega(Q) \le 2$.

Inspired by these developments, we consider the existence of subnormal transcendental meromorphic solutions to delay differential equations involving higher order derivatives and n-power, and prove the following result.

Theorem 2.1. *Let ω be a subnormal transcendental meromorphic solution of the following:*

$$\omega(z+1) - \omega(z-1) + a(z)\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^n = R(z,\omega(z)) = \frac{P(z,\omega(z))}{Q(z,\omega(z))}, \tag{2.3}$$

where $k \ge 1$ and $n \ge 1$ are integers, $a(z)$ is non-zero rational function in z, $P(z,\omega(z))$ is a polynomial in ω having rational coefficients in z, and $Q(z,\omega(z))$ is of a polynomial in ω with roots that are nonzero rational functions of z and not roots of $P(z,\omega(z))$, then $\deg_\omega(R) \le 2 + nk$*. In addition, the following statements hold:*

(i) *If* $\deg_\omega(Q) \le 1$*, then* $\deg_\omega(P) \le nk$*;*
(ii) *If* $\deg_\omega(Q) \ge 2$*, then* $\deg_\omega(Q) < \deg_\omega(P) \le nk + \deg_\omega(Q)$*.*

Now, we clarify the concrete mathematical advances over Theorems A and B, as well as the relation to Theorem 4 of [14].

(1) If $n = k = 1$ and $\deg_\omega(Q) \ge 2$, then Theorem 2.1 gives $\deg_\omega(P) = \deg_\omega(Q) + 1$, which reduces to the Theorem A; if $k = 1$, then $\deg_\omega(P) \le n$ or $\deg_\omega(Q) < \deg_\omega(P) \le \deg_\omega(Q) + n$, which reduces to the Theorem 4 in [14].
(2) If $n = 1$ and $\deg_\omega(Q) \ge 2$, then $\deg_\omega(P) \le k + 2$, which gives a sharper estimate than Theorem B. Moreover, Example 2.3 below shows that the equality can be attained.

The following two examples show that case (i) of Theorem 2.1 can occur, and Example 2.1 proves that the equality $\deg_\omega(P) = nk$ is sharp for transcendental solutions.

Example 2.1. $\omega(z) = \tan(\frac{\pi}{2}z)$ *satisfies the equation*

$$\omega(z+1) - \omega(z-1) + 2\cdot\frac{\omega''(z)}{\omega(z)} = \pi^2(\omega^2+1),$$

where $k = 2$, $n = 1$. Then, $\deg_\omega(P) = 2 = nk$*.*

Example 2.2. $\omega(z) = \sec^2(\pi z)$ *satisfies the equation*

$$\omega(z+1) - \omega(z-1) + \left(\frac{\omega'(z)}{\omega(z)}\right)^4 = 16\pi^4(\omega-1)^2,$$

where $k = 1$, $n = 4$. Then, $\deg_\omega(P) = 2 < nk$*.*

Next, we give Examples 2.3 and 2.4 to illustrate the existence of solutions satisfying conclusion (ii) in Theorem 2.1, and we illustrate it by taking the solution $\omega(z) = \frac{1}{e^z+1}$ with two distinct parameter pairs (k, n).

Example 2.3. *$\omega(z) = \frac{1}{e^z+1}$ satisfies the equation*

$$\omega(z+1) - \omega(z-1) + 3 \cdot \frac{\omega^{(3)}(z)}{\omega(z)} = \frac{P(z, \omega(z))}{Q(z, \omega(z))},$$

where $P(z,\omega) = 18(e-1)^2\omega^5 - 54(e-1)^2\omega^4 + 3(19(e-1)^2 - 6e)\omega^3 + (36e - 23(e-1)^2)\omega^2 + (2(e-1)^2 - 21e)\omega + 2e$, $Q(z,\omega) = (e-1)^2\omega^2 - (e-1)^2\omega - e$, $k = 3$, $n = 1$. Then, we have $\deg_\omega(Q) < \deg_\omega(P) = nk + \deg_\omega(Q)$.

Example 2.4. *$\omega(z) = \frac{1}{e^z+1}$ satisfies the equation*

$$\omega(z+1) - \omega(z-1) + \left(\frac{\omega'(z)}{\omega(z)}\right)^2 = \frac{P(z, \omega(z))}{Q(z, \omega(z))},$$

where $P(z,\omega) = (2 - e - \frac{1}{e})\omega^4 + (3e + 3\frac{1}{e} - 6)\omega^3 + ((3 - e - \frac{1}{e}) + 2 - e - \frac{1}{e})\omega^2 + (e + \frac{1}{e})\omega + 1$, $Q(z,\omega) = (2-e-\frac{1}{e})\omega^2 + (e+\frac{1}{e}-2)\omega + 1$, $k = 1$, $n = 2$. Then, we have $\deg_\omega(Q) < \deg_\omega(P) = nk + \deg_\omega(Q)$.

3. Proof of Theorem 2.1

In this section, we will give the proof of Theorem 2.1. First, we present some lemmas for the proof. The Valiron-Mohon'ko identity [17] establishes the relationship between the characteristic function of the rational function $R(z,\omega)$ in ω and the characteristic function of ω itself.

Lemma 3.1. *[17] Let ω be a meromorphic function. Then, for all irreducible rational functions in ω,*

$$R(z,\omega) = \frac{P(z,\omega)}{Q(z,\omega)} = \frac{\sum_{i=0}^{p} a_i(z)\omega^i}{\sum_{j=0}^{q} b_j(z)\omega^j},$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$ such that $a_i(z)$ and $b_j(z)$ are small with respect to ω. Then, the Nevanlinna characteristic function of $R(z,\omega)$ satisfies the following:

$$T(r, R(z,\omega)) = \deg_\omega(R)T(r,\omega) + S(r,\omega),$$

where $\deg_\omega(R) = \max\{p, q\}$ is the degree of $R(z,\omega)$.

The following lemma is useful in the proof of our results.

Lemma 3.2. *[13] A differential difference polynomial in ω is defined by the following:*

$$P(z,\omega) = \sum_{l\in L} b_l(z)\omega(z)^{l_{0,0}}\omega(z+c_1)^{l_{1,0}}\cdots\omega(z+c_\nu)^{l_{\nu,0}}\omega'(z)^{l_{0,1}}\cdots\omega^{(\mu)}(z+c_\nu)^{l_{\nu,\mu}},$$

where $c_1, \cdots, c_\nu$ are distinct complex constants, L is a finite index set consisting of elements of the form $l = (l_{0,0}, \cdots, l_{\nu,\mu})$ and the coefficients b_l are rational functions for all $l \in L$. Let ω be a transcendental meromorphic solution of the differential difference equation

$$P(z,\omega) = 0, \tag{3.1}$$

$a_1, \ldots, a_k$ be rational functions satisfying $P(z, a_j) \neq 0$ for all $j \in \{1, \ldots, k\}$. If there exists $s > 0$ and $\tau \in (0, 1)$ such that

$$\sum_{j=1}^{k} n\left(r, \frac{1}{\omega - a_j}\right) \leq k\tau \cdot n(r + s, \omega) + O(1), \tag{3.2}$$

then

$$\limsup_{r\to\infty} \frac{\log T(r, \omega)}{r} > 0.$$

The following two lemmas are from [16]. Lemma 3.3 is the difference analogue of the logarithmic derivative.

Lemma 3.3. *[16] Let ω be a meromorphic function. If ω is subnormal, then*

$$m\left(r, \frac{\omega(z + c)}{\omega(z)}\right) = S(r, \omega)$$

holds as $r \notin E$ and $r \to \infty$, where E is a subset of $[1, +\infty)$ with zero upper density, that is,

$$\overline{dens}E = \limsup_{r\to\infty} \frac{1}{r} \int_{E\cap[1,r]} dt = 0.$$

Lemma 3.4. *[16] Let $T(r)$ be a non-decreasing positive function in $[1, +\infty)$ and logarithmic convex with $T(r) \to +\infty$ as $r \to \infty$. Assume that*

$$\limsup_{r\to\infty} \frac{\log T(r)}{r} = 0.$$

Set $\phi(r) = \max_{1\leq t\leq r}\left\{\frac{t}{\log T(t)}\right\}$. Then, given a constant $\delta \in (0, \frac{1}{2})$, we have the following:

$$T(r) \leq T(r + \phi^{\delta}(r)) \leq (1 + 4\phi^{\delta-\frac{1}{2}}(r))T(r), r \notin E_{\delta},$$

where E_{δ} is a subset of $[1, +\infty)$ with the zero upper density.

Now, we present delay differential analogues of the Clunie lemma [18] and the Mohon'ko theorem [19], both of which are essential to analyze meromorphic solutions.

Lemma 3.5. *[18] Let ω be a subnormal transcendental meromorphic solution of the following:*

$$P(z, \omega)U(z, \omega) = M(z, \omega),$$

where $P(z, \omega)$ is a difference polynomial that contains precisely one term of maximal total degree in ω and its shifts, $U(z, \omega)$, $M(z, \omega)$ are delay differential polynomials. All coefficients α_{λ} of the three polynomials satisfy $m(r, \alpha_{\lambda}) = S(r, w)$ and $\deg(M(z, \omega)) \leq \deg(P(z, \omega))$; then,

$$m(r, U(z, \omega)) = S(r, \omega).$$

Lemma 3.6. *[19] Let ω be a subnormal meromorphic solution of the following:*

$$\varphi(r,\omega)=0,$$

where $\varphi(z,\omega)$ is a delay differential polynomial in ω with rational coefficients. If $\varphi(z,a)\not\equiv 0$ for some meromorphic functions $a(z)$ small with respect to ω, then

$$m\left(r,\frac{1}{\omega-a}\right)=S(r,\omega).$$

The following inequality, which relates $N(r,\omega)$, $n(r,\omega)$ and $T(r,\omega)$, is available in [20].

Lemma 3.7. *[20] Let ω be a meromorphic function. Then,*

$$n(s,\omega)\leq\frac{r}{r-s}N(r,\omega)+O(1)\leq\frac{r}{r-s}T(r,\omega)+O(1),$$

for all sufficiently large positive real numbers r, s with $r>s$.

Before proving Theorem 2.1, we consider the case where Eq (2.3) satisfies $\deg_{\omega}(Q)=0$. Without loss of generality, we may assume that $R(z,\omega)=P(z,\omega)$, and thus obtain the following lemma.

Lemma 3.8. *Let ω be a transcendental meromorphic solution of the following:*

$$\omega(z+1)-\omega(z-1)+a(z)\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^{n}=P(z,\omega(z)),\tag{3.3}$$

where $k\geq 1$ and $n\geq 1$ are integers, $a(z)$ is a non-zero rational, and $P(z,\omega)$ is a polynomial in ω with rational coefficients in z. If ω is subnormal, then $\deg_{\omega}(P)\leq nk$.

Proof. We assume $\deg_{\omega}(P)=p\geq nk+1$, and aim for a contradiction. Equation (3.3) can be rewritten as follows:

$$\omega(z)^{n}(\omega(z+1)-\omega(z-1))+a(z)\left(\omega^{(k)}(z)\right)^{n}=P(z,\omega(z))\omega(z)^{n}.$$

By Lemma 3.5, we obtain $m(r,\omega)=S(r,\omega)$, which implies that $\omega(z)$ has infinitely many generic poles. Let $\tilde{z}$ be a pole of $\omega(z)$ of order t, which is not a zero or pole of the coefficients in (3.3) or any of their shifts. Then, $\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^{n}$ has a pole at $\tilde{z}$ of order nk. By Eq (3.3), at least one of $\tilde{z}+1$ and $\tilde{z}-1$ is a pole of $\omega(z)$. Without loss of generality, assume that $\omega(z)$ has a pole at $z=\tilde{z}+1$ of multiplicity at least pt. By iterating Eq (3.3), we obtain that $z=\widetilde{z}+2$ and $z=\widetilde{z}+3$ are poles of $\omega(z)$ with orders at least $p^{2}t$ and $p^{3}t$, respectively. By continuing the iteration, we conclude that $z=\widetilde{z}+d$ is a pole of $\omega(z)$ of order at least $p^{d}t$ for every $d\in\mathbb{N}^{+}$.

By Lemma 3.7, let $r=2d+2|\tilde{z}|$ and $s=d+|\tilde{z}|$; then, we can get

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log^{+}T(r,\omega)}{r}&\geq\limsup_{d\to\infty}\frac{\log^{+}[\frac{1}{2}n(d+|\tilde{z}|,\omega)]}{2d+2|\tilde{z}|}\\&\geq\limsup_{d\to\infty}\frac{\log(\frac{1}{2}p^{d}t)-\log 2}{2d+2|\tilde{z}|}\\&=\frac{1}{2}\log p>0.\end{aligned}\tag{3.4}$$

This is a contradiction. Therefore, $\deg_{\omega}(P)\leq nk$. □

Using a method analogous to that employed in the proof of Lemma 3.3 in [12], we establish the following lemma under the assumption that $Q(z,\omega)$ in (2.3) admits multiple roots.

Lemma 3.9. *Let ω be a subnormal transcendental meromorphic solution of the following:*

$$\omega(z+1)-\omega(z-1)+a(z)\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^n=\frac{P(z,\omega(z))}{(\omega(z)-b(z))^l Q_1(z,\omega(z))}, \tag{3.5}$$

where $a(z)$ and $b(z)$ are non-zero rational functions, $k \geq 1$, $n \geq 1$, and $l > 1$ are positive integers. Suppose that $P(z,\omega)$ and $Q_1(z,\omega)$ are polynomials in ω with coefficients that are rational functions of z, and $\omega-b$, $P(z,\omega)$, and $Q_1(z,\omega)$ are pairwise coprime. Then, $l+\deg_\omega(Q_1) < \deg_\omega(P) \leq nk+l+\deg_\omega(Q_1)$.

Proof. Clearly, $\omega(z)=b(z)$ is not a root of Eq (3.5). From Lemma 3.6, we obtain the following:

$$m(r,\frac{1}{\omega(z)-b(z)})=S(r,\omega(z)).$$

Combining it with Nevanlinna's First Fundamental Theorem yields the following:

$$\begin{aligned} T(r,\omega(z)) &= T\left(r,\frac{1}{\omega(z)-b(z)}\right)+S(r,\omega(z)) \\ &= m\left(r,\frac{1}{\omega(z)-b(z)}\right)+N\left(r,\frac{1}{\omega(z)-b(z)}\right)+S(r,\omega) \\ &= N\left(r,\frac{1}{\omega(z)-b(z)}\right)+S(r,\omega). \end{aligned}$$

Let z_1 be a zero of multiplicity t of $\omega(z)-b(z)$, where z_1 is neither a zero nor a pole of the coefficients and their shift of the equation. From (3.5), we obtain that $\omega(z)$ has a pole of multiplicity at least $t_1 = lt$ at either z_1+1 or z_1-1. Without loss of generality, we assume that $\omega(z)$ has a pole of multiplicity t_1 at z_1+1. By iterating (3.5), we obtain the following:

$$\omega(z+2)-\omega(z)+a(z+1)\left(\frac{\omega^{(k)}(z+1)}{\omega(z+1)}\right)^n=\frac{P(z+1,\omega(z+1))}{(\omega(z+1)-b(z+1))^l Q_1(z+1,\omega(z+1))}. \tag{3.6}$$

Case 1. $\deg_\omega(P) \leq l+\deg_\omega(Q_1)$. We will discuss these cases.

From Eq (3.6), $\omega(z)$ has an $nk-$order pole at z_1+2. By iterating Eq (3.6), we obtain the following:

$$\omega(z+3)-\omega(z+1)+a(z+2)\left(\frac{\omega^{(k)}(z+2)}{\omega(z+2)}\right)^n=\frac{P(z+2,\omega(z+2))}{(\omega(z+2)-b(z+2))^l Q_1((z+2),\omega(z+2))}. \tag{3.7}$$

Subcase 1.1. $nk=t_1$.

If we continue the iteration, then it's possible that $\omega(z_1+3)=b(z_1+3)$, which, in principle, implies that $\omega(z_1+4)$ could be a finite value that terminates the iteration. In this case, the zero multiplicity of $\omega(z)-b(z)$ at $z=z_1+3$ is at most $\frac{nk}{l}$. By considering all poles of $\omega(z)$ and zeros of $\omega(z)-b(z)$, we obtain the following:

$$n\left(r,\frac{1}{\omega(z)-b(z)}\right) \leq \frac{t+\frac{nk}{l}}{lt+nk}n(r+3,\omega(z))+O(1). \tag{3.8}$$

Satisfying conditions of Lemma 3.2 yields $\limsup_{r\to\infty}\frac{\log^{+}T(r,\omega)}{r}>0$, which contradicts the condition that ω is subnormal.

Subcase 1.2. $nk > t_1$.

From Eq (3.7), $\omega(z)$ has an $nk-$order pole at z_1+3. By iterating Eq (3.7), it's possible that the zero multiplicity of $\omega(z)-b(z)$ at z_1+4 is at most $\frac{nk}{l}$, which, in principle, implies that $\omega(z_1+5)$ could be a finite value that terminates the iteration. By considering all poles of ω and zeros of $\omega(z)-b(z)$, we obtain

$$n\left(r,\frac{1}{\omega(z)-b(z)}\right)\leq\frac{t+\frac{nk}{l}}{lt+2nk}n(r+4,\omega(z))+O(1); \tag{3.9}$$

thus, from Lemma 3.2, this leads to a contradiction.

Subcase 1.3. $nk < t_1$.

By Eq (3.7), we have that $\omega(z_1+3)$ has a pole at z_1 with an order of $t_1 = lt$. By iterating Eq (3.7), it's possible that the zero multiplicity of $\omega(z)-b(z)$ at z_1+4 is $\frac{t_1}{l}$, which, in principle, implies that $\omega(z_1+5)$ could be a finite value that terminates the iteration. Thus, by considering all poles of $\omega(z)$ and zeros of $\omega(z)-b(z)$, we derive the following:

$$n\left(r,\frac{1}{\omega(z)-b(z)}\right)\leq\frac{t+\frac{t_1}{l}}{2lt+nk}n(r+4,\omega(z))+O(1),$$

where $\frac{t+\frac{t_1}{l}}{2lt+nk}<\frac{1}{l}<1$. This yields a contradiction.

Case 2. $\deg_\omega(P) > nk+l+\deg_\omega(Q_1)$. Let $p_1=\deg_\omega(P)-\deg_\omega(Q_1)-l$.

From (3.6), we obtain that $\omega(z)$ has a pole at z_1+2 with an order of at least p_1t_1. By Eq (3.7), $\omega(z)$ has a pole at z_1+3 with an order of at least $p_1^2t_1$. By continuing the iteration and discussing it in this way, we consider all poles of $\omega(z)$ and zeros of $\omega(z)-b(z)$. Thus, we derive the following:

$$n\left(r,\frac{1}{\omega(z)-b(z)}\right)\leq\frac{1}{l+p_1l+p_1^2l}n(r+3,\omega(z))+O(1).$$

This yields a contradiction. By combining the above results, we obtain the following:

$$l+\deg_\omega(Q_1)<\deg_\omega(P)\leq nk+l+\deg_\omega(Q_1).$$

This completes the proof. □

By using the above lemmas and preliminary discussions, we now proceed to prove Theorem 2.1.

Proof of Theorem 2.1. Let ω be a subnormal transcendental solution of (2.3). Applying Lemma 3.1, we obtain the following:

$$\deg_\omega(R)\cdot T(r,\omega)=T(r,R(z,\omega))+S(r,\omega)\leq T\left(r,\overline{\omega}-\underline{\omega}\right)+T\left(r,a(z)\left(\frac{\omega^{(k)}}{\omega}\right)^n\right)+S(r,\omega), \tag{3.10}$$

where $\overline{\omega}=\omega(z+1)$ and $\underline{\omega}=\omega(z-1)$. By (2.3), all zeros of ω are not poles of $R(z,\omega)$. Hence, when estimating the poles of $R(z,\omega)$, it suffices to consider the poles of ω. If ω has a pole of order p at $\widetilde{z}$, then $\left(\frac{\omega^{(k)}}{\omega}\right)^n$ has a pole of order nk at $\widetilde{z}$. Therefore,

$$N(r,R(z,\omega))\leq N(r,\overline{\omega})+N(r,\underline{\omega})+nk\overline{N}(r,\omega)+S(r,\omega). \tag{3.11}$$

By applying Lemma 3.4 and combining (3.10) and (3.11), we deduce that

$$\begin{aligned} \deg_{\omega}(R)T(r,\omega) &\leq m(r,\omega) + N(r,\overline{\omega} - \underline{\omega}) + nk\overline{N}(r,w) + S(r,w) \\ &\leq m(r,w) + 2T(r,\omega) + nk\overline{N}(r,\omega) + S(r,\omega) \\ &\leq (2+nk)T(r,\omega) + S(r,\omega), \end{aligned} \tag{3.12}$$

that is, $\deg_{\omega}(R) \leq 2 + nk$.

If $\deg_{\omega}(Q) = 0$, then by Lemma 3.8, we directly obtain $\deg_{\omega}(P) \leq nk$. Now, we consider the solution of $\deg_{\omega}(Q) \geq 1$. Suppose that $Q(z,\omega)$ has at least one non-zero root. If $Q(z,\omega)$ has a multiple root, then from Lemma 3.9, we obtain $\deg_{\omega}(Q) < \deg_{\omega}(P) \leq nk + \deg_{\omega}(Q)$. Therefore, we only need to consider the case that $Q(z,w)$ has simple roots in the subsequent discussion. First, we assume that $\deg_{\omega}(Q) = 1$ (and thus without loss of generality assume $Q(z,\omega) = \omega - b$) and $\deg_{\omega}(P) \geq nk+1$. Then, it follows that b is not a solution of (2.3). Let $\tilde{z}$ be a zero of $\omega - b$ of order t. Then, ω has a pole of order of at least t at $\tilde{z}+1$ or $\tilde{z}-1$. Without loss of generality, we assume that $\tilde{z}+1$ is such a pole. Then, $\tilde{z}+2$ is a pole of order of at least $(\deg_{\omega}(P)-1)t(\geq nkt)$ and $\tilde{z}+3$ is a pole of order of at least $(\deg_{\omega}(P)-1)^2t(\geq (nk)^2t)$, and so on. In this case, we obtain

$$n(r,\frac{1}{\omega - b}) \leq \frac{t}{t + nkt + (nk)^2t}n(r+3,\omega) + O(1),$$

and Lemma 3.2 contradicts the condition that ω is subnormal.

Next, we prove that if $\deg_{\omega}(Q) > 1$, then $\deg_{\omega}(Q) < \deg_{\omega}(P) \leq nk + \deg_{\omega}(Q)$. Suppose $Q(z,\omega)$ has h distinct roots b_i; then, (2.3) can be expressed as follows:

$$\omega(z+1) - \omega(z-1) + a(z)\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^n = \frac{P(z,\omega(z))}{\prod_{i=1}^{h}[\omega(z) - b_i(z)]\tilde{Q}(z,\omega(z))}, \tag{3.13}$$

where $h \geq 2$, $h + \deg_{\omega}(\tilde{Q}) = \deg_{\omega}(Q)$. We obtain that $\omega = b_i$ is not a root of (3.13). Applying Lemma 3.6, we have $m(r, \frac{1}{\omega - b_i}) = S(r,\omega)$, $i \in \{1,\ldots,h\}$. Hence, we can take one $z_3 \in \mathbb{C}$ is a zero of $\omega(z) - b_i(z)$ $(i \in \{1,...,h\})$, say $\omega(z) - b_{i_1}(z)$ with multiplicity p, which is not a zero or pole of the coefficient of (3.13) and its shift, and $P(z_3,\omega(z_3)) \neq 0$. From (3.13), we obtain that ω has a pole of multiplicity $p_1(\geq p)$ at either z_3+1 or z_3-1. Without loss of generality, we assume that ω has a pole of multiplicity p_1 at z_3+1. By iterating (3.13), we obtain the following:

$$\omega(z+2) - \omega(z) + a(z+1)\left(\frac{\omega^{(k)}(z+1)}{\omega(z+1)}\right)^n = \frac{P(z+1,\omega(z+1))}{\prod_{i=1}^{h}[\omega(z+1) - b_i(z+1)]\tilde{Q}(z+1,\omega(z+1))}. \tag{3.14}$$

From Eq (3.14), $\omega(z)$ has an $nk-$order pole at z_3+2.

Case 1. $\deg_{\omega}(P) \leq \deg_{\omega}(Q)$. We will discuss these cases to derive a contradiction.

By iterating Eq (3.14), we obtain the following:

$$\omega(z+3) - \omega(z+1) + a(z+2)\left(\frac{\omega^{(k)}(z+2)}{\omega(z+2)}\right)^n = \frac{P(z+2,\omega(z+2))}{\prod_{i=1}^{h}[\omega(z+2) - b_i(z+2)]\tilde{Q}(z+2,\omega(z+2))}. \tag{3.15}$$

Subcase 1.1. $nk = p_1$.

From Eq (3.15), if we continue the iteration, then we obtain the following:

$$\omega(z+4)-\omega(z+2)+a(z+3)\left(\frac{\omega^{(k)}(z+3)}{\omega(z+3)}\right)^n=\frac{P(z+3,\omega(z+3))}{\prod_{i=1}^h[\omega(z+3)-b_i(z+3)]\tilde{Q}(z+3,\omega(z+3))}. \quad (3.16)$$

It's possible that $\omega(z_3+3)=b_i(z_3+3)$, $i=1,\ldots,h$, which implies that $\omega(z_3+4)$ could be a finite value that terminates the iteration. In this case, one of the $\omega-b_i$ has a zero multiplicity of nk at z_3+3. Considering all poles of ω and zeros of $\omega-b_i$, we have

$$\sum_{i=1}^h n\left(r,\frac{1}{\omega-b_i}\right)\leq h\cdot\frac{1}{h}\cdot\frac{p+p_1}{2p_1}n(r+3,\omega)+O(1), \quad (3.17)$$

$h\geq 2$, so from Lemma 3.2, it leads a contradiction.

Subcase 1.2. $nk>p_1$.

From Eq (3.15), $\omega(z)$ has an $nk-$order pole at z_3+3. By iterating Eq (3.15), it's possible that one of the $\omega-b_i$ has a zero multiplicity of nk at z_3+4. By considering all poles of ω and zeros of $\omega-b_i$, we obtain

$$\sum_{i=1}^h n\left(r,\frac{1}{\omega-b_i}\right)\leq\frac{nk+p}{2nk+p}n(r+4,\omega)+O(1), \quad (3.18)$$

so from Lemma 3.2, this leads to a contradiction.

Subcase 1.3. $1\leq nk<p_1$.

By Eq (3.15), we see that $\omega(z_3+3)$ has a pole at z_3 of order p_1. By iterating Eq (3.15), it's possible that one of the $\omega-b_i$ has a zero multiplicity of p_1 at z_3+4. Thus, by considering all poles of ω and zeros of $\omega-b_i$, we derive the following:

$$\sum_{i=1}^h n\left(r,\frac{1}{\omega-b_i}\right)\leq\frac{p_1}{3p_1+nk}n(r+3,\omega)+O(1).$$

This yields a contradiction.

Case 2. $\deg_\omega(P)>nk+\deg_\omega(Q)$. Let $q_1=\deg_\omega(P)-\deg_\omega(Q)>nk$.

By (3.14), we obtain that $\omega(z)$ has a pole at z_3+2 with an order of at least p_1q_1. By (3.15), ω has a pole at z_3+3 with an order of at least $p_1q_1^2$. Thus, by considering all poles of ω and zeros of $\omega-b_i$, we derive the following:

$$\sum_{i=1}^h n\left(r,\frac{1}{\omega-b_i}\right)\leq\frac{p_1}{p_1+p_1q_1+p_1(q_1)^2}n(r+3,\omega)+O(1).$$

This yields a contradiction. Combining the conclusions of Case 1 and Case 2 yields $\deg_\omega(Q)<\deg_\omega(P)\leq nk+\deg_\omega(Q)$. □

4. Subnormal solutions of the delay differential Painlevé V equation

In 2010, Ronkainen [9] studied the discrete counterpart of the fifth Painlevé equation. Later, Du and Zhang [21] researched the delay differential painlevé type equation

$$(\omega(z)\omega(z+1)-1)(\omega(z)\omega(z-1)-1)+a(z)\frac{\omega'(z)}{\omega(z)}=R(z,\omega(z))=\frac{P(z,\omega(z))}{Q(z,\omega(z))}. \quad (4.1)$$

Du and Zhang [21] worked out the necessary conditions for Eq (4.1) to admit subnormal solutions. Subsequently, Xia et al. [11] analyzed a higher order generalization and proved the following theorem.

Theorem 4.1. *[11] Let ω be a subnormal transcendental meromorphic solution of the following:*

$$(\omega(z)\omega(z+1)-1)(\omega(z)\omega(z-1)-1)+a(z)\frac{\omega^{(k)}(z)}{\omega(z)}=R(z,\omega(z))=\frac{P(z,\omega(z))}{Q(z,\omega(z))}, \tag{4.2}$$

where $k \geq 1$ is an integer, $a(z)$ is a non-zero rational, $P(z,\omega(z))$ and $Q(z,\omega(z))$ are polynomials in ω having meromorphic coefficients, and that the roots of $Q(z,\omega(z))$ are nonzero meromorphic functions of z and are not the roots of $P(z,\omega(z))$. If $\deg_\omega(Q)=0$, then $\deg_\omega(P)\leq k+3$; if $\deg_\omega(Q)\geq 1$, then $\deg_\omega(P)\leq \deg_\omega(Q)+2$ or $\deg_\omega(Q)+2<\deg_\omega(P)\leq \deg_\omega(Q)+k+2$.

In addition, if $k=1$, then $\deg_\omega(P)=\deg_\omega(Q)+2$; if $k>2$ is an odd number, then $\deg_\omega(Q)+2<\deg_\omega(P)\leq\deg_\omega(Q)+k+2$.

Building on the work of Xia et al. [11] and Chen et al. [14], we consider the necessary conditions for the existence of subnormal transcendental meromorphic solutions of the equation

$$(\omega(z)\omega(z+1)-1)(\omega(z)\omega(z-1)-1)+a(z)\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^n=R(z,\omega(z))=\frac{P(z,\omega(z))}{Q(z,\omega(z))}, \tag{4.3}$$

and prove the following result.

Theorem 4.2. *Let ω be a subnormal transcendental meromorphic solution of Eq (4.3), where $k\geq 1$ and $n\geq 1$ are integers, and $a(z)$ is a non-zero rational. Suppose that $P(z,\omega(z))$ and $Q(z,\omega(z))$ are polynomials in ω having rational coefficients, and that the roots of $Q(z,\omega(z))$ are non-zero rational functions of z and are not the roots of $P(z,\omega(z))$. Then, the following statements hold:*

(i) *If $\deg_\omega(Q)=0$, then $\deg_\omega(P)\leq 3+nk$;*
(ii) *If $\deg_\omega(Q)\geq 1$ and $\deg_\omega(P)>\deg_\omega(Q)+2$, then $\deg_\omega(P)\leq\deg_\omega(Q)+nk+2$.*

The following examples illustrate the existence of solutions satisfying the case (i) of Theorem 4.2.

Example 4.1. *$\omega(z)=e^{-2\pi iz}$ satisfies*

$$(\omega(z+1)\omega(z)-1)(\omega(z)\omega(z-1)-1)+\left(\frac{\omega'(z)}{\omega(z)}\right)^2=\omega^4-2\omega^2+1-4\pi^2,$$

where $P(z,\omega)=\omega^4-2\omega^2+1-4\pi^2$ and $Q(z,\omega)=1$, $k=1$, $n=2$. Then $\deg_\omega(P)=4<3+nk$.

Example 4.2. *$\omega(z)=e^{2\pi z}$ satisfies*

$$(\omega(z+1)\omega(z)-1)(\omega(z)\omega(z-1)-1)+z\cdot\frac{\omega''(z)}{\omega(z)}=\omega^4-e^{2\pi}\omega^2-\frac{1}{e^{2\pi}}\omega^2+4\pi^2z+1,$$

where $P(z,\omega)=\omega^4-e^{2\pi}\omega^2-\frac{1}{e^{2\pi}}\omega^2+4\pi^2z+1$ and $Q(z,\omega)=1$, $k=2$, $n=1$. Then, $\deg_\omega(P)=4<3+nk$, which is also consistent with the conclusion (i) in Theorem 4.2.

The existence of case (ii) in Theorem 4.2 is confirmed by an example below.

Example 4.3. *$\omega(z)=\frac{1}{e^z+1}$ satisfies*

$$(\omega(z+1)\omega(z)-1)(\omega(z)\omega(z-1)-1)+\left(\frac{\omega'(z)}{\omega(z)}\right)^3=\frac{P(z,\omega(z))}{Q(z,\omega(z))},$$

where $P(z,\omega)=[-(e-1)^2\omega^3+(2e^2-3e+2)\omega^2+3e\omega](\omega-1)^2$ and $Q(z,\omega)=\omega^2+\omega-2e\omega+e$, $k=1$, $n=3$. Then, $\deg_\omega(Q)+2<\deg_\omega(P)$ and $\deg_\omega(P)<\deg_\omega(Q)+nk+2$.

5. Proof of Theorem 4.2

Before proving Theorem 4.2, we establish the following preliminary lemma derived from the conditions of the theorem. For the case of $\deg_\omega(Q) = 0$ in Eq (4.3), we have the following result.

Lemma 5.1. *Let ω be a non-rational meromorphic solution of the following:*

$$(\omega(z+1)\omega(z)-1)(\omega(z)\omega(z-1)-1)+a(z)\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^n = P(z,\omega(z)), \tag{5.1}$$

where $k \geq 1$ and $n \geq 1$ are integers, $a(z)$ is a non-zero rational, and $P(z,\omega)$ is a polynomial in ω with rational coefficients in z. If ω is subnormal, then $\deg_\omega(P) \leq 3+nk$.

Proof. Suppose that $\deg_\omega(P) = p \geq 4+nk$, and aim for a contradiction. Equation (5.1) can be rewritten as follows:

$$\omega(z)^n((\omega(z+1)\omega(z)-1)(\omega(z)\omega(z-1)-1)) + a(z)(\omega^{(k)}(z))^n = P(z,\omega(z))\omega(z)^n.$$

Applying Lemma 3.5, we obtain $m(r,\omega) = S(r,\omega)$, which implies $N(r,\omega) = T(r,\omega)+S(r,\omega)$. Let $\omega(z)$ have a pole of order ν at z_2. From (5.1), $\omega(z)$ has a pole of multiplicity $\nu_2 = \frac{(p-2)\nu}{2}(> \nu \geq 1)$ at z_2-1 or z_2+1. Without loss of generality, we assume that $\omega(z)$ has a pole of multiplicity ν_2 at z_2+1. By iterating Eq (5.1), we have the following:

$$(\omega(z+2)\omega(z+1)-1)(\omega(z+1)\omega(z)-1)+a(z+1)\left(\frac{\omega^{(k)}(z+1)}{\omega(z+1)}\right)^n = P(z+1,\omega(z+1)). \tag{5.2}$$

$\omega(z+2)$ has a pole of order $\nu_3 = (p-2)\nu_2 - \nu (> (p-3)\nu)$ at z_2, $\omega(z+3)$ has a pole of order $\nu_4 = (p-2)\nu_3 - \nu_2 (> (p-3)^2\nu)$ at z_2. Then, for all $d \in \mathbb{N}^+$,

$$n(|z_2|+d,\omega) \geq (p-3)^{d-1}\nu + O(1). \tag{5.3}$$

By Lemma 3.7, let $r = 2d+2|z_2|$, $s = \frac{r}{2}$; then, we can obtain the following:

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log^+ T(r,\omega)}{r} &\geq \limsup_{d\to\infty}\frac{\log((p-3)^{d-1}\nu)-\log 2}{2|z_2|+2d}\\ &= \frac{\log(p-3)}{2} > 0.\end{aligned} \tag{5.4}$$

This is a contradiction. Thus, $\deg_\omega(P) \leq 3+nk$. □

With the above lemma established, we now proceed to the proof of Theorem 4.2.

Proof of Theorem 4.2. Let ω be a subnormal transcendental solution of (4.3). Applying Lemma 3.1, we obtain the following:

$$\begin{aligned}\deg_\omega(R)\cdot T(r,\omega) &= T(r,R(z,\omega)) + S(r,\omega)\\ &\leq T(r,(\omega(z+1)\omega(z)-1)(\omega(z)\omega(z-1)-1))\\ &\quad + T\left(r, a(z)\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^n\right) + S(r,\omega(z)).\end{aligned} \tag{5.5}$$

By (4.3), we can find that all zeros of ω are not poles of $R(z, \omega(z))$, and thus are not poles of $\omega(z+1)\omega(z) - \omega(z)\omega(z-1) + a(z)\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^n$. Additionally, we have the following:

$$N(r, (\omega(z+1)\omega(z) - 1)(\omega(z)\omega(z-1) - 1) \leq 4T(r, \omega(z)) + S(r, \omega(z)). \tag{5.6}$$

Applying Lemma 3.4, we deduce that

$$\deg_\omega(R)T(r, \omega(z)) \leq (4 + nk)T(r, \omega(z)) + S(r, \omega(z)), \tag{5.7}$$

that is, $\deg_\omega(R) \leq 4 + nk$.

If $\deg_\omega(Q) = 0$, then we obtain $\deg_\omega(R) \leq 3 + nk$ from Lemma 5.1, which is the conclusion (i) of Theorem 4.2. Now, let $\deg_\omega(Q) = q \geq 1, \deg_\omega(P) = p$, and $p > q + 2$. Assume that $Q(z, \omega(z))$ has h distinct non-zero meromorphic roots $b_1(z), b_2(z), \dots b_h(z)$, where $1 \leq h \leq q$. Equation (4.3) can be rewritten as follows:

$$(\omega(z+1)\omega(z) - 1)(\omega(z)\omega(z-1) - 1) + a(z)\left[\frac{\omega^{(k)}(z)}{\omega(z)}\right]^n = \frac{P(z, \omega(z))}{\prod_{i=1}^h [\omega(z) - b_i(z)]^{m_i} \tilde{Q}(z, \omega(z))}, \tag{5.8}$$

where $q = \sum_{i=1}^h m_i + \deg_\omega(\tilde{Q})$, $(m_1, ..., m_h) \in \mathbb{N}$, $b_1, ..., b_h$ are not roots of $P(z, \omega(z))$. Notice that b_i ($i \in \{1, ..., h\}$) are not solutions of (5.8); applying Lemma 3.6, we obtain $m(r, \frac{1}{\omega - b_i}) = S(r, \omega)$ for $i \in \{1, ..., h\}$. Thus, $N(r, \frac{1}{\omega - b_i}) = T(r, \omega) + S(r, \omega)$. Hence, we can take one $z_4 \in \mathbb{C}$ is a zero of $\omega(z) - b_i(z)$ ($i \in \{1, ..., h\}$), say $\omega(z) - b_{i_1}(z)$ with multiplicity u, which is not a zero or pole of the coefficient of (5.8) and its shift, and $P(z_4, \omega(z_4)) \neq 0$. From (5.8), it is easy to get that z_4 is a pole of $\omega(z+1)$ or $\omega(z-1)$. Without loss of generality, $\omega(z)$ has a pole at $z_4 + 1$ with order $m_{i_1}u$. After iterating Eq (5.8), we obtain the following:

$$\begin{aligned} &(\omega(z+2)\omega(z+1) - 1)(\omega(z+1)\omega(z) - 1) + a(z+1)\left[\frac{\omega^{(k)}(z+1)}{\omega(z+1)}\right]^n \\ =&\frac{P(z+1, \omega(z+1))}{\prod_{i=1}^h [\omega(z+1) - b_i(z+1)]^{m_i} \tilde{Q}(z+1, \omega(z+1))}. \end{aligned} \tag{5.9}$$

Suppose that $p \geq q + nk + 3$. We distinguish two cases as follows.

Case 1. $1 \leq nk \leq 2m_{i_1}u$.

Then, by (5.9), we get that $\omega(z+2)$ has a pole at z_4 with multiplicity at least $2m_{i_1}u$. By iterating (5.9), $\omega(z+3)$ has a pole at z_4 with multiplicity at least $3m_{i_1}u$, and so on. Hence, in this case, it follows that

$$\sum_{i=1}^h n\left(r, \frac{1}{\omega - b_i}\right) \leq \max_{i=1}^h \left\{\frac{1}{6m_i}\right\} n(r+3, \omega) + O(1).$$

Thus, it's a contradiction.

Case 2. $nk > 2m_{i_1}u$.

From Eq (5.9), $\omega(z+2)$ has a pole at z_4 with multiplicity at least $(nk+1)m_{i_1}u$. By iterating Eq (5.9), we obtain the following:

$$\begin{aligned} &(\omega(z+3)\omega(z+2) - 1)(\omega(z+2)\omega(z+1) - 1) + a(z+2)\left[\frac{\omega^{(k)}(z+2)}{\omega(z+2)}\right]^n \\ =&\frac{P(z+2, \omega(z+2))}{\prod_{i=1}^h [\omega(z+2) - b_i(z+2)]^{m_i} \tilde{Q}(z+2, \omega(z+2))}. \end{aligned} \tag{5.10}$$

From (5.10), $\omega(z+3)$ has a pole at z_4 with multiplicity at least $nk(nk+2)m_{i_1}u$, and so on. Hence, in this case, it follows that

$$\sum_{i=1}^{h} n\left(r, \frac{1}{\omega - b_i}\right) \leq \max_{i=1}^{h}\left\{\frac{1}{(nk+1)(nk+2)m_{i_1}}\right\} n(r+3,\omega) + O(1).$$

From Lemma 3.2, we get a contradiction. In summary, we have that if $\deg_\omega(Q) + 2 < \deg_\omega(P)$, then

$$\deg_\omega(P) \leq \deg_\omega(Q) + nk + 2.$$

This is the conclusion of Theorem 4.2. Therefore, the proof is completed. □

6. Conclusions and further discussion

In this paper, we established necessary conditions for the existence of subnormal transcendental meromorphic solutions of the single-shift delay differential equations (2.3) and (4.3), where all coefficients are rational functions.

Hu and Liu [7] proved that if the delay differential equation

$$\omega^m(z)\sum_{\mu=1}^{s} e_\mu(z)\omega(z+c_\mu) + a(z)\frac{\omega^{(n)}(z)}{\omega(z)} = R(z,\omega(z)) \tag{6.1}$$

admits a transcendental entire solution with hyper-order less than one, then it necessarily reduces to

$$\omega^m(z)\sum_{\mu=1}^{s} e_\mu(z)\omega(z+c_\mu) + a(z)\frac{\omega^{(n)}(z)}{\omega(z)} = \frac{1}{\omega(z)}\sum_{i=0}^{m+2} A_i(z)\omega^i(z),$$

where the coefficients $A_i(z)$ are rational functions. Consequently, one obtains the estimate $\deg_\omega(P) \leq m+2$. This naturally raises the following question.

Problem 1. If Eqs (2.3) and (4.3) admit transcendental entire solutions, can the degree bounds obtained in Theorems 2.1 and 4.2 be further improved?

This motivates the extension of our methods to multiple-shift delay differential equations.

Problem 2. Let $m \geq 0$, $n \geq 1$, $s \geq 1$, and $k \geq 1$ be integers, and let $c_1, \ldots, c_s$ be nonzero complex constants. Suppose that $a, e_1, \ldots, e_s$ are rational functions satisfying $|e_1| + \cdots + |e_s| \not\equiv 0$, and let $R(z,\omega) \not\equiv 0$ be an irreducible rational function in ω with rational coefficients in z. If the delay differential equation

$$\omega^m(z)\sum_{\mu=1}^{s} e_\mu(z)\omega(z+c_\mu) + a(z)\left(\frac{\omega^{(k)}(z)}{\omega(z)}\right)^n = R(z,\omega(z)) = \frac{P(z,\omega(z))}{Q(z,\omega(z))}$$

admits a subnormal transcendental meromorphic solution, then can one obtain analogous degree estimates for $P(z,\omega(z))$ and $Q(z,\omega(z))$?

For the case $n = 1$, related degree estimates were obtained by Xia et al. [11]. Therefore, it would be interesting to investigate whether analogous results remain valid for arbitrary positive integers n. These questions will be investigated in future work.

Author contributions

Yi Li: Formal analysis, methodology, writing – original draft; Jianren Long: Conceptualization, supervision, writing – review & editing; Xuxu Xiang: Validation, writing – review & editing. All authors have read and approved the final version of the manuscript for publication.

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The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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