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Research article

New inequalities for differentiable functions with values in a Banach space

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Abstract: This article estimates the deviation of a function taking values in a real Banach space from its Bochner integral mean by establishing new Ostrowski-type inequalities for three times differentiable functions. A new integral identity involving a parameter and the third derivative is presented, proved by means of the Bochner integration by parts formula. Based on this identity, three Ostrowski-type inequalities are derived under convexity assumptions imposed on the norm of the third derivative, by employing Hölder's inequality and the power mean inequality. Classical results from the literature are recovered as particular cases of the inequalities obtained.

Keywords: Ostrowski inequality; Banach space; Bochner integral; convex function
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1. Introduction

In [1], Ostrowski proved the following inequality for functions with bounded derivative:

Theorem 1.1. *[1] Let $\mathfrak{F} : [\mathfrak{c}, \mathfrak{d}] \to \mathbb{R}$ be differentiable on $(\mathfrak{c}, \mathfrak{d})$ with $\|\mathfrak{F}'\|_\infty := \sup_{\varsigma\in(\mathfrak{c},\mathfrak{d})} |\mathfrak{F}'(\varsigma)| < \infty$. Then*

$$\left|\mathfrak{F}(\chi) - \frac{1}{\mathfrak{d}-\mathfrak{c}}\int_{\mathfrak{c}}^{\mathfrak{d}} \mathfrak{F}(\varsigma)\,d\varsigma\right| \le \left[\frac{1}{4} + \frac{(\chi - \frac{\mathfrak{c}+\mathfrak{d}}{2})^2}{(\mathfrak{d}-\mathfrak{c})^2}\right](\mathfrak{d}-\mathfrak{c})\,\|\mathfrak{F}'\|_\infty$$

for all $\chi \in [\mathfrak{c}, \mathfrak{d}]$. The constant $\frac{1}{4}$ is sharp.

The Ostrowski inequality has had a profound effect on several topics in mathematical analysis, particularly in the estimation of error bounds in numerical integration and approximation theory, attracting much attention in recent decades [10–13]. Banach spaces have proven to be a fruitful framework for extending such inequalities. Working within a Banach space $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ allows the resulting estimates to apply to a much wider class of functions, since the scalar absolute value is replaced by the norm $\|\cdot\|_{\mathfrak{X}}$, and the classical Riemann integral is replaced by the Bochner integral.

In [6], an Ostrowski-type inequality for strongly differentiable functions with values in a Banach space was obtained as follows:

Theorem 1.2 ([6])**.** *Suppose* $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ *is a real Banach space and* $\mathfrak{F} : [\mathfrak{c}, \mathfrak{d}] \to \mathfrak{X}$ *is strongly differentiable on* $(\mathfrak{c}, \mathfrak{d})$*. Then for each* $\omega \in [\mathfrak{c}, \mathfrak{d}]$*,*

$$\left\| \int_{\mathfrak{c}}^{\mathfrak{d}} \mathfrak{F}(\varsigma)\, d\varsigma - (\mathfrak{d} - \mathfrak{c})\, \mathfrak{F}(\omega) \right\|_{\mathfrak{X}} \leq \left[\frac{(\mathfrak{d} - \mathfrak{c})^2}{4} + \left(\omega - \frac{\mathfrak{c} + \mathfrak{d}}{2} \right)^2 \right] \sup_{\varsigma \in [\mathfrak{c}, \mathfrak{d}]} \|\mathfrak{F}'(\varsigma)\|_{\mathfrak{X}}.$$

In [7], an Ostrowski-type inequality for functions taking values in a Banach space was established for the first derivative as follows:

Theorem 1.3. *Suppose* $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ *is a real Banach space and* $\mathfrak{F} \in C^1([\mathfrak{c}, \mathfrak{d}], \mathfrak{X})$*,* $\chi \in [\mathfrak{c}, \mathfrak{d}]$*. Then*

$$\left\| \frac{1}{\mathfrak{d} - \mathfrak{c}} \int_{\mathfrak{c}}^{\mathfrak{d}} \mathfrak{F}(\varsigma)\, d\varsigma - \mathfrak{F}(\chi) \right\|_{\mathfrak{X}} \leq \frac{(\chi - \mathfrak{c})^2 + (\mathfrak{d} - \chi)^2}{2(\mathfrak{d} - \mathfrak{c})} \sup_{\varsigma \in [\mathfrak{c}, \mathfrak{d}]} \|\mathfrak{F}'(\varsigma)\|_{\mathfrak{X}} \leq \frac{\mathfrak{d} - \mathfrak{c}}{2} \sup_{\varsigma \in [\mathfrak{c}, \mathfrak{d}]} \|\mathfrak{F}'(\varsigma)\|_{\mathfrak{X}}.$$

The Ostrowski inequality has also been studied within the framework of Bochner integration and convexity conditions on the norm of the derivative. Several contributions in this direction have appeared in the literature [8, 13, 14]. We recall that a function $\mathfrak{F} : [\mathfrak{c}, \mathfrak{d}] \to \mathfrak{X}$ is called strongly measurable if it is the almost everywhere limit of a sequence of simple functions, and $\mathfrak{F}$ is said to be Bochner integrable if it is strongly measurable and $\int_{\mathfrak{c}}^{\mathfrak{d}} \|\mathfrak{F}(\varsigma)\|_{\mathfrak{X}}\, d\varsigma < \infty$; in this case, its Bochner integral $\int_{\mathfrak{c}}^{\mathfrak{d}} \mathfrak{F}(\varsigma)\, d\varsigma \in \mathfrak{X}$ is defined as the limit of the integrals of the approximating simple functions; see [19] for details. A function $\mathfrak{G} : I \subseteq \mathbb{R} \to \mathbb{R}$ is called *convex* if

$$\mathfrak{G}(\varsigma\chi + (1 - \varsigma)\omega) \leq \varsigma\, \mathfrak{G}(\chi) + (1 - \varsigma)\, \mathfrak{G}(\omega)$$

holds for each $\chi, \omega \in I$ and $\varsigma \in [0, 1]$. Convexity has a long history in mathematical analysis, with connections to optimization, mechanics, economics, and probability theory. In the context of Banach space-valued functions, a natural convexity condition is to require that the real-valued function $\varsigma \mapsto \|\mathfrak{F}^{(k)}(\varsigma)\|_{\mathfrak{X}}$ be convex on $[\mathfrak{c}, \mathfrak{d}]$, and many results in the literature rely on this assumption. Ostrowski's inequality for vector-valued functions was studied in [3]. Ostrowski-type inequalities for functions of two variables taking values in Banach spaces were established in [8]. A sharper form of the Ostrowski inequality for absolutely continuous functions was derived in [4]. The study of Ostrowski-type inequalities remains an active area of research. In recent years, such inequalities have been developed in several directions: for 3-convex functions [18], within multiplicative calculus [16], on time scales [17], and via generalized fractional integral operators [15]. New Ostrowski type integral inequalities via a new Montgomery identity were obtained in [5]. The results of [5] were extended in [2] by introducing a parameter $\varkappa \in [0, 1] \setminus \{\frac{1}{2}\}$, as follows:

Theorem 1.4. *[2] Suppose* $\mathfrak{F} : I \subset \mathbb{R} \to \mathbb{R}$ *is twice differentiable on* I^o *with* $\mathfrak{F}'' \in L_1[\mathfrak{c}, \mathfrak{d}]$ *for some* $\mathfrak{c}, \mathfrak{d} \in I$*,* $\mathfrak{c} < \mathfrak{d}$*. Provided* $|\mathfrak{F}''|$ *is convex on* $[\mathfrak{c}, \mathfrak{d}]$*, then*

$$\left| \frac{1}{(\mathfrak{d} - \mathfrak{c})(1 - 2\varkappa)^3} \int_{\varkappa\mathfrak{d} + (1 - \varkappa)\mathfrak{c}}^{\varkappa\mathfrak{c} + (1 - \varkappa)\mathfrak{d}} \mathfrak{F}(\omega)\, d\omega \right.$$
$$- \frac{1}{2(1 - 2\varkappa)^2} \Big[\mathfrak{F}(\varkappa\chi + (1 - \varkappa)(\mathfrak{c} + \mathfrak{d} - \chi)) - \mathfrak{F}(\varkappa(\mathfrak{c} + \mathfrak{d} - \chi) + (1 - \varkappa)\chi) \Big]$$

$$+\frac{1}{2(1-2\varkappa)}\left(\frac{\mathfrak{c}+3\mathfrak{d}}{4}-\chi\right)\left[\mathfrak{F}'(\varkappa\chi+(1-\varkappa)(\mathfrak{c}+\mathfrak{d}-\chi))-\mathfrak{F}'(\varkappa(\mathfrak{c}+\mathfrak{d}-\chi)+(1-\varkappa)\chi)\right]\Bigg|$$
$$\leq\frac{1}{\mathfrak{d}-\mathfrak{c}}\left[(\mathfrak{d}-\chi)^3+\left(\chi-\frac{\mathfrak{c}+\mathfrak{d}}{2}\right)^3\right]\frac{|\mathfrak{F}''(\varkappa\mathfrak{c}+(1-\varkappa)\mathfrak{d})|+|\mathfrak{F}''(\varkappa\mathfrak{d}+(1-\varkappa)\mathfrak{c})|}{6}$$

for each $\chi\in\left[\frac{\mathfrak{c}+\mathfrak{d}}{2},\mathfrak{d}\right]$ *and* $\varkappa\in[0,1]\setminus\{\frac{1}{2}\}$.

Even though the Ostrowski inequality has been extended in many directions [2, 3, 5–7], Ostrowski-type inequalities for three times differentiable Banach space-valued functions are still lacking. Therefore, this paper derives new Ostrowski-type inequalities for functions $\mathfrak{F}:[\mathfrak{c},\mathfrak{d}]\to\mathfrak{X}$ taking values in a real Banach space $\mathfrak{X}$, that are three times differentiable, involving the parameter $\varkappa\in[0,1]\setminus\{\frac{1}{2}\}$ and a convexity condition on $\|\mathfrak{F}'''(\cdot)\|_{\mathfrak{X}}$. The significance of extending these results to the Banach space setting lies in the fact that the obtained inequalities provide computable error bounds for the approximation of Bochner integrals of vector-valued functions, for which the classical scalar inequalities cannot be used. Moreover, the parameter $\varkappa$ is significant because it unifies several known configurations, and at the endpoints $\varkappa=0$ and $\varkappa=1$ the results reduce to the standard statements on the whole interval $[\mathfrak{c},\mathfrak{d}]$. Our results extend the results of Set and Ekinci [2] to the Banach space setting and to the third derivative, and in turn generalize many of the classical results mentioned above, for instance [1, 5–7] and the references cited therein. The main difficulty in this extension is that a Banach space carries no order structure, so the arguments based on the absolute value and pointwise comparison of real numbers are replaced by norm estimates for the Bochner integral, and the integral identity has to be established through a triple application of the Bochner integration by parts formula with a suitably constructed three-piece cubic kernel. When $\mathfrak{X}=\mathbb{R}$, all results reduce to their scalar counterparts. The results of this study have applications in numerical integration and approximation theory, particularly in the estimation of error bounds for quadrature rules involving Banach space-valued integrands.

2. Main results

Throughout this section, $(\mathfrak{X},\|\cdot\|_{\mathfrak{X}})$ denotes a real Banach space, and all integrals of $\mathfrak{X}$-valued functions are taken in the Bochner sense [9]. Moreover, $\mathfrak{F}:[\mathfrak{c},\mathfrak{d}]\to\mathfrak{X}$ denotes a three times differentiable function and $\varkappa$ a parameter; the points $\mathfrak{P}_\varkappa$ and $\mathfrak{Q}_\varkappa$ defined below are convex combinations of the endpoints $\mathfrak{c}$ and $\mathfrak{d}$, and $\vartheta=\mathfrak{P}_\varkappa-\mathfrak{Q}_\varkappa$ is the signed length of the integration interval. We set

$$\mathfrak{P}_\varkappa:=\varkappa\mathfrak{c}+(1-\varkappa)\mathfrak{d},\quad \mathfrak{Q}_\varkappa:=\varkappa\mathfrak{d}+(1-\varkappa)\mathfrak{c},$$
$$\Lambda_\varkappa(\chi):=\varkappa\chi+(1-\varkappa)(\mathfrak{c}+\mathfrak{d}-\chi),\quad \Xi_\varkappa(\chi):=\varkappa(\mathfrak{c}+\mathfrak{d}-\chi)+(1-\varkappa)\chi,$$
$$\vartheta:=(\mathfrak{d}-\mathfrak{c})(1-2\varkappa),\quad \imath:=\frac{\mathfrak{d}-\chi}{\mathfrak{d}-\mathfrak{c}},\quad \jmath:=\frac{\chi-\frac{\mathfrak{c}+\mathfrak{d}}{2}}{\mathfrak{d}-\mathfrak{c}}$$

for $\varkappa\in[0,1]\setminus\{\frac{1}{2}\}$ and $\chi\in[\frac{\mathfrak{c}+\mathfrak{d}}{2},\mathfrak{d}]$.

Note that with these definitions,

$$\imath+\jmath=\frac{\mathfrak{d}-\chi}{\mathfrak{d}-\mathfrak{c}}+\frac{\chi-\frac{\mathfrak{c}+\mathfrak{d}}{2}}{\mathfrak{d}-\mathfrak{c}}=\frac{\mathfrak{d}-\frac{\mathfrak{c}+\mathfrak{d}}{2}}{\mathfrak{d}-\mathfrak{c}}=\frac{1}{2},\tag{2.1}$$

a relation that will be used repeatedly in the proofs below. Before stating the main results, we present the following lemma, which provides an integral identity for three times differentiable functions with values in $\mathfrak{X}$.

Lemma 2.1. *Suppose* $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ *is a real Banach space and* $\mathfrak{F} : [\mathfrak{c}, \mathfrak{d}] \to \mathfrak{X}$ *is three times differentiable on* $(\mathfrak{c}, \mathfrak{d})$ *with* $\mathfrak{F}''' \in L_1([\mathfrak{c}, \mathfrak{d}], \mathfrak{X})$*,* $\mathfrak{c} < \mathfrak{d}$*. Then for* $\varkappa \in [0,1] \setminus \{\frac{1}{2}\}$ *and* $\chi \in [\frac{\mathfrak{c}+\mathfrak{d}}{2}, \mathfrak{d}]$*, the following identity holds in* $\mathfrak{X}$*:*

$$\begin{aligned}
&-\frac{1}{\vartheta}\int_{\mathfrak{Q}_\varkappa}^{\mathfrak{P}_\varkappa} \mathfrak{F}(\omega)\, d\omega + \frac{1}{2}[\mathfrak{F}(\Lambda_\varkappa(\chi)) + \mathfrak{F}(\Xi_\varkappa(\chi))] \\
&+ \frac{\vartheta}{2}(\jmath^2 - \imath^2)[\mathfrak{F}'(\Lambda_\varkappa(\chi)) - \mathfrak{F}'(\Xi_\varkappa(\chi))] \\
&+ \frac{\vartheta^2}{6}(\imath^3 + \jmath^3)[\mathfrak{F}''(\Lambda_\varkappa(\chi)) + \mathfrak{F}''(\Xi_\varkappa(\chi))] \\
&= \frac{\vartheta^3}{6}\int_0^1 \mathscr{K}_3(\varsigma)\, \mathfrak{F}'''[\varsigma\, \mathfrak{P}_\varkappa + (1-\varsigma)\, \mathfrak{Q}_\varkappa]\, d\varsigma,
\end{aligned} \tag{2.2}$$

where

$$\mathscr{K}_3(\varsigma) := \begin{cases} \varsigma^3, & 0 \le \varsigma \le \dfrac{\mathfrak{d}-\chi}{\mathfrak{d}-\mathfrak{c}}, \\ \left(\varsigma - \dfrac{1}{2}\right)^3, & \dfrac{\mathfrak{d}-\chi}{\mathfrak{d}-\mathfrak{c}} \le \varsigma \le \dfrac{\chi-\mathfrak{c}}{\mathfrak{d}-\mathfrak{c}}, \\ (\varsigma-1)^3, & \dfrac{\chi-\mathfrak{c}}{\mathfrak{d}-\mathfrak{c}} \le \varsigma \le 1, \end{cases}$$

for each $\chi \in [\frac{\mathfrak{c}+\mathfrak{d}}{2}, \mathfrak{d}]$ *and* $\varkappa \in [0,1] \setminus \{\frac{1}{2}\}$*.*

Proof. Let $\phi(\varsigma) = \varsigma\, \mathfrak{P}_\varkappa + (1-\varsigma)\, \mathfrak{Q}_\varkappa$ for $\varsigma \in [0,1]$. Then $\phi'(\varsigma) = \vartheta$ and

$$\frac{d}{d\varsigma}\mathfrak{F}^{(k)}(\phi(\varsigma)) = \vartheta\, \mathfrak{F}^{(k+1)}(\phi(\varsigma)), \quad k = 0, 1, 2,$$

with $\phi(\imath) = \Lambda_\varkappa(\chi)$ and $\phi(1-\imath) = \Xi_\varkappa(\chi)$. We write

$$\mathcal{I} = \int_0^1 \mathscr{K}_3(\varsigma)\, \mathfrak{F}'''(\phi(\varsigma))\, d\varsigma = \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3,$$

where

$$\mathfrak{I}_1 = \int_0^\imath \varsigma^3\, \mathfrak{F}'''(\phi(\varsigma))\, d\varsigma, \quad \mathfrak{I}_2 = \int_\imath^{1-\imath} \left(\varsigma - \tfrac{1}{2}\right)^3 \mathfrak{F}'''(\phi(\varsigma))\, d\varsigma, \quad \mathfrak{I}_3 = \int_{1-\imath}^1 (\varsigma-1)^3\, \mathfrak{F}'''(\phi(\varsigma))\, d\varsigma.$$

Note that since $\imath + \jmath = \frac{1}{2}$ by (2.1), we have $1 - \imath = \frac{1}{2} + \jmath$, so the upper limit of $\mathfrak{I}_2$ equals $\frac{1}{2} + \jmath$, and the boundary values used below satisfy:

$$\phi(\imath) = \Lambda_\varkappa(\chi), \quad \phi(1-\imath) = \Xi_\varkappa(\chi), \quad (\imath - \tfrac{1}{2})^3 = -\jmath^3, \quad (1 - \imath - \tfrac{1}{2})^3 = \jmath^3.$$

Computation of $\mathfrak{I}_1$. By the Bochner integration by parts formula,

$$\mathfrak{I}_1 = \frac{\varsigma^3}{\vartheta}\, \mathfrak{F}''(\phi(\varsigma))\Big|_0^\imath - \frac{3}{\vartheta}\int_0^\imath \varsigma^2\, \mathfrak{F}''(\phi(\varsigma))\, d\varsigma = \frac{\imath^3}{\vartheta}\, \mathfrak{F}''(\Lambda_\varkappa(\chi)) - \frac{3}{\vartheta}\int_0^\imath \varsigma^2\, \mathfrak{F}''(\phi(\varsigma))\, d\varsigma,$$

since $\varsigma^3\big|_{\varsigma=0} = 0$. Integrating by parts a second time,

$$\int_0^\iota \varsigma^2\, \mathfrak{F}''(\phi(\varsigma))\, d\varsigma = \frac{\iota^2}{\vartheta}\, \mathfrak{F}'(\Lambda_\varkappa(\chi)) - \frac{2}{\vartheta}\int_0^\iota \varsigma\, \mathfrak{F}'(\phi(\varsigma))\, d\varsigma.$$

Integrating by parts a third time,

$$\int_0^\iota \varsigma\, \mathfrak{F}'(\phi(\varsigma))\, d\varsigma = \frac{\iota}{\vartheta}\, \mathfrak{F}(\Lambda_\varkappa(\chi)) - \frac{1}{\vartheta}\int_0^\iota \mathfrak{F}(\phi(\varsigma))\, d\varsigma.$$

Substituting back,

$$\mathfrak{I}_1 = \frac{\iota^3}{\vartheta}\, \mathfrak{F}''(\Lambda_\varkappa(\chi)) - \frac{3\iota^2}{\vartheta^2}\, \mathfrak{F}'(\Lambda_\varkappa(\chi)) + \frac{6\iota}{\vartheta^3}\, \mathfrak{F}(\Lambda_\varkappa(\chi)) - \frac{6}{\vartheta^3}\int_0^\iota \mathfrak{F}(\phi(\varsigma))\, d\varsigma. \tag{2.3}$$

Computation of $\mathfrak{I}_2$. By the Bochner integration by parts formula, using $(1-\iota-\frac{1}{2})^3 = \jmath^3$ and $(\iota-\frac{1}{2})^3 = -\jmath^3$,

$$\mathfrak{I}_2 = \frac{\jmath^3}{\vartheta}[\mathfrak{F}''(\Lambda_\varkappa(\chi)) + \mathfrak{F}''(\Xi_\varkappa(\chi))] - \frac{3}{\vartheta}\int_\iota^{1-\iota} \left(\varsigma - \tfrac{1}{2}\right)^2 \mathfrak{F}''(\phi(\varsigma))\, d\varsigma.$$

Integrating by parts a second time, using $(1-\iota-\frac{1}{2})^2 = \jmath^2$ and $(\iota-\frac{1}{2})^2 = \jmath^2$,

$$\int_\iota^{1-\iota} \left(\varsigma - \tfrac{1}{2}\right)^2 \mathfrak{F}''(\phi(\varsigma))\, d\varsigma = \frac{\jmath^2}{\vartheta}[\mathfrak{F}'(\Xi_\varkappa(\chi)) - \mathfrak{F}'(\Lambda_\varkappa(\chi))] - \frac{2}{\vartheta}\int_\iota^{1-\iota} \left(\varsigma - \tfrac{1}{2}\right) \mathfrak{F}'(\phi(\varsigma))\, d\varsigma.$$

Integrating by parts a third time, using $1-\iota-\frac{1}{2} = \jmath$ and $\iota - \frac{1}{2} = -\jmath$,

$$\int_\iota^{1-\iota} \left(\varsigma - \tfrac{1}{2}\right) \mathfrak{F}'(\phi(\varsigma))\, d\varsigma = \frac{\jmath}{\vartheta}[\mathfrak{F}(\Xi_\varkappa(\chi)) + \mathfrak{F}(\Lambda_\varkappa(\chi))] - \frac{1}{\vartheta}\int_\iota^{1-\iota} \mathfrak{F}(\phi(\varsigma))\, d\varsigma.$$

Substituting back,

$$\begin{aligned}\mathfrak{I}_2 = {}& \frac{\jmath^3}{\vartheta}[\mathfrak{F}''(\Lambda_\varkappa(\chi)) + \mathfrak{F}''(\Xi_\varkappa(\chi))] + \frac{3\jmath^2}{\vartheta^2}[\mathfrak{F}'(\Lambda_\varkappa(\chi)) - \mathfrak{F}'(\Xi_\varkappa(\chi))] \\ & + \frac{6\jmath}{\vartheta^3}[\mathfrak{F}(\Lambda_\varkappa(\chi)) + \mathfrak{F}(\Xi_\varkappa(\chi))] - \frac{6}{\vartheta^3}\int_\iota^{1-\iota} \mathfrak{F}(\phi(\varsigma))\, d\varsigma.\end{aligned} \tag{2.4}$$

Computation of $\mathfrak{I}_3$. By the Bochner integration by parts formula, using $(\varsigma-1)^3\big|_{\varsigma=1} = 0$ and $(1-\iota-1)^3 = -\iota^3$,

$$\mathfrak{I}_3 = \frac{\iota^3}{\vartheta}\, \mathfrak{F}''(\Xi_\varkappa(\chi)) - \frac{3}{\vartheta}\int_{1-\iota}^1 (\varsigma-1)^2\, \mathfrak{F}''(\phi(\varsigma))\, d\varsigma.$$

Integrating by parts a second time, using $(\varsigma-1)^2\big|_{\varsigma=1} = 0$ and $(1-\iota-1)^2 = \iota^2$,

$$\int_{1-\iota}^1 (\varsigma-1)^2\, \mathfrak{F}''(\phi(\varsigma))\, d\varsigma = -\frac{\iota^2}{\vartheta}\, \mathfrak{F}'(\Xi_\varkappa(\chi)) - \frac{2}{\vartheta}\int_{1-\iota}^1 (\varsigma-1)\, \mathfrak{F}'(\phi(\varsigma))\, d\varsigma.$$

Integrating by parts a third time, using $(\varsigma-1)\big|_{\varsigma=1}=0$ and $(1-\imath-1)=-\imath$,

$$\int_{1-\imath}^{1}(\varsigma-1)\,\mathfrak{F}'(\phi(\varsigma))\,d\varsigma=\frac{\imath}{\vartheta}\,\mathfrak{F}(\Xi_\varkappa(\chi))-\frac{1}{\vartheta}\int_{1-\imath}^{1}\mathfrak{F}(\phi(\varsigma))\,d\varsigma.$$

Substituting back,

$$\mathfrak{I}_3=\frac{\imath^3}{\vartheta}\,\mathfrak{F}''(\Xi_\varkappa(\chi))+\frac{3\imath^2}{\vartheta^2}\,\mathfrak{F}'(\Xi_\varkappa(\chi))+\frac{6\imath}{\vartheta^3}\,\mathfrak{F}(\Xi_\varkappa(\chi))-\frac{6}{\vartheta^3}\int_{1-\imath}^{1}\mathfrak{F}(\phi(\varsigma))\,d\varsigma. \tag{2.5}$$

Combining $\mathcal{I}=\mathfrak{I}_1+\mathfrak{I}_2+\mathfrak{I}_3$. Adding Eqs (2.3)–(2.5), and using $\imath+\jmath=\frac{1}{2}$ from (2.1), we observe that the interior boundary terms arising at $\varsigma=\imath$ and $\varsigma=1-\imath$ combine in pairs, so that

$$\begin{aligned}\mathcal{I}=&\frac{\imath^3+\jmath^3}{\vartheta}[\mathfrak{F}''(\Lambda_\varkappa(\chi))+\mathfrak{F}''(\Xi_\varkappa(\chi))]+\frac{3(\jmath^2-\imath^2)}{\vartheta^2}[\mathfrak{F}'(\Lambda_\varkappa(\chi))-\mathfrak{F}'(\Xi_\varkappa(\chi))]\\&+\frac{3}{\vartheta^3}[\mathfrak{F}(\Lambda_\varkappa(\chi))+\mathfrak{F}(\Xi_\varkappa(\chi))]-\frac{6}{\vartheta^3}\int_0^1\mathfrak{F}(\phi(\varsigma))\,d\varsigma.\end{aligned}$$

Applying the substitution $\omega=\varsigma\,\mathfrak{P}_\varkappa+(1-\varsigma)\,\mathfrak{Q}_\varkappa$, so that $d\omega=\vartheta\,d\varsigma$,

$$\int_0^1\mathfrak{F}(\phi(\varsigma))\,d\varsigma=\frac{1}{\vartheta}\int_{\mathfrak{Q}_\varkappa}^{\mathfrak{P}_\varkappa}\mathfrak{F}(\omega)\,d\omega.$$

Multiplying both sides by $\frac{\vartheta^3}{6}$ yields (2.2). $\square$

Remark 2.2. *We record the following special cases of Lemma 2.1.*

(i) *When $\mathfrak{X}=\mathbb{R}$, Lemma 2.1 reduces to a scalar identity for the third derivative with parameter $\varkappa$.*

(ii) *Choosing $\varkappa=0$ or $\varkappa=1$ and setting $\mathfrak{X}=\mathbb{R}$, we have $\vartheta=\mathfrak{d}-\mathfrak{c}$ and the identity reduces to a scalar integral identity for the third derivative; see also [2] for related scalar identities involving the second derivative.*

(iii) *Choosing $\chi=\frac{\mathfrak{c}+\mathfrak{d}}{2}$ gives $\imath=\frac{1}{2}$ and $\jmath=0$, so that $\jmath^2-\imath^2=-\frac{1}{4}$, $\imath^3+\jmath^3=\frac{1}{8}$, and the identity reduces to*

$$\begin{aligned}&-\frac{1}{\vartheta}\int_{\mathfrak{Q}_\varkappa}^{\mathfrak{P}_\varkappa}\mathfrak{F}(\omega)\,d\omega+\frac{1}{2}[\mathfrak{F}(\mathfrak{P}_\varkappa)+\mathfrak{F}(\mathfrak{Q}_\varkappa)]-\frac{\vartheta}{8}[\mathfrak{F}'(\mathfrak{P}_\varkappa)-\mathfrak{F}'(\mathfrak{Q}_\varkappa)]+\frac{\vartheta^2}{48}[\mathfrak{F}''(\mathfrak{P}_\varkappa)+\mathfrak{F}''(\mathfrak{Q}_\varkappa)]\\&=\frac{\vartheta^3}{6}\int_0^1\mathscr{K}_3(\varsigma)\,\mathfrak{F}'''[\varsigma\,\mathfrak{P}_\varkappa+(1-\varsigma)\,\mathfrak{Q}_\varkappa]\,d\varsigma.\end{aligned}$$

Theorem 2.3. *Suppose $(\mathfrak{X},\|\cdot\|_{\mathfrak{X}})$ is a real Banach space and $\mathfrak{F}:[\mathfrak{c},\mathfrak{d}]\to\mathfrak{X}$ is three times differentiable on $(\mathfrak{c},\mathfrak{d})$ with $\mathfrak{F}'''\in L_1([\mathfrak{c},\mathfrak{d}],\mathfrak{X})$, $\mathfrak{c}<\mathfrak{d}$. When $\|\mathfrak{F}'''(\cdot)\|_{\mathfrak{X}}$ is convex on $[\mathfrak{c},\mathfrak{d}]$,*

$$\begin{aligned}&\Bigg\|-\frac{1}{\vartheta}\int_{\mathfrak{Q}_\varkappa}^{\mathfrak{P}_\varkappa}\mathfrak{F}(\omega)\,d\omega+\frac{1}{2}[\mathfrak{F}(\Lambda_\varkappa(\chi))+\mathfrak{F}(\Xi_\varkappa(\chi))]\\&+\frac{\vartheta}{2}(\jmath^2-\imath^2)[\mathfrak{F}'(\Lambda_\varkappa(\chi))-\mathfrak{F}'(\Xi_\varkappa(\chi))]+\frac{\vartheta^2}{6}(\imath^3+\jmath^3)[\mathfrak{F}''(\Lambda_\varkappa(\chi))+\mathfrak{F}''(\Xi_\varkappa(\chi))]\Bigg\|_{\mathfrak{X}}\\&\le\frac{\vartheta^3(\imath^4+\jmath^4)}{24}\Big[\|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}}+\|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}\Big]\end{aligned} \tag{2.6}$$

for each $\chi\in[\frac{\mathfrak{c}+\mathfrak{d}}{2},\mathfrak{d}]$ and $\varkappa\in[0,1]\setminus\{\frac{1}{2}\}$.

Proof. An application of Lemma 2.1 together with the triangle inequality for the Bochner integral yields,

$$\Bigg\| -\frac{1}{\vartheta}\int_{\mathfrak{Q}_\varkappa}^{\mathfrak{P}_\varkappa} \mathfrak{F}(\omega)\,d\omega + \frac{1}{2}[\mathfrak{F}(\Lambda_\varkappa(\chi)) + \mathfrak{F}(\Xi_\varkappa(\chi))]$$
$$+\frac{\vartheta}{2}(\jmath^2-\imath^2)[\mathfrak{F}'(\Lambda_\varkappa(\chi)) - \mathfrak{F}'(\Xi_\varkappa(\chi))] + \frac{\vartheta^2}{6}(\imath^3+\jmath^3)[\mathfrak{F}''(\Lambda_\varkappa(\chi)) + \mathfrak{F}''(\Xi_\varkappa(\chi))]\Bigg\|_{\mathfrak{X}}$$
$$\le \frac{\vartheta^3}{6}\int_0^1 |\mathscr{K}_3(\varsigma)|\,\big\|\mathfrak{F}'''(\phi(\varsigma))\big\|_{\mathfrak{X}}\,d\varsigma = \frac{\vartheta^3}{6}(\mathfrak{I}_1+\mathfrak{I}_2+\mathfrak{I}_3),$$

where

$$\mathfrak{I}_1 = \int_0^\imath \varsigma^3\,\big\|\mathfrak{F}'''(\phi(\varsigma))\big\|_{\mathfrak{X}}\,d\varsigma, \quad \mathfrak{I}_2 = \int_\imath^{1-\imath}\big|\varsigma-\tfrac{1}{2}\big|^3\,\big\|\mathfrak{F}'''(\phi(\varsigma))\big\|_{\mathfrak{X}}\,d\varsigma,$$
$$\mathfrak{I}_3 = \int_{1-\imath}^1 (1-\varsigma)^3\,\big\|\mathfrak{F}'''(\phi(\varsigma))\big\|_{\mathfrak{X}}\,d\varsigma.$$

To estimate the three integrals, we observe that $\phi(\varsigma)$ traverses the segment joining $\mathfrak{Q}_\varkappa$ and $\mathfrak{P}_\varkappa$, so by the convexity of $\|\mathfrak{F}'''(\cdot)\|_{\mathfrak{X}}$ on $[\mathfrak{c},\mathfrak{d}]$,

$$\big\|\mathfrak{F}'''(\phi(\varsigma))\big\|_{\mathfrak{X}} \le \varsigma\,\|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}} + (1-\varsigma)\,\|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}$$

for all $\varsigma\in[0,1]$. Using this inequality and computing each integral,

$$\mathfrak{I}_1 \le \|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}}\int_0^\imath \varsigma^4\,d\varsigma + \|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}\int_0^\imath \varsigma^3(1-\varsigma)\,d\varsigma$$
$$= \frac{\imath^5}{5}\,\|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}} + \left(\frac{\imath^4}{4}-\frac{\imath^5}{5}\right)\|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}.$$

Since $\imath\le\frac{1}{2}$ for $\chi\in[\frac{\mathfrak{c}+\mathfrak{d}}{2},\mathfrak{d}]$, we have $|\varsigma-\frac{1}{2}| = \frac{1}{2}-\varsigma$ on $[\imath,\frac{1}{2}]$ and $=\varsigma-\frac{1}{2}$ on $[\frac{1}{2},1-\imath]$. Setting $s=\varsigma-\frac{1}{2}$ and $\jmath=\frac{1}{2}-\imath$,

$$\int_\imath^{1-\imath}\big|\varsigma-\tfrac{1}{2}\big|^3\,\varsigma\,d\varsigma = \int_{-\jmath}^{\jmath}|s|^3\big(s+\tfrac{1}{2}\big)\,ds = \frac{\jmath^4}{4},$$

and similarly $\int_\imath^{1-\imath}|\varsigma-\frac{1}{2}|^3(1-\varsigma)\,d\varsigma = \frac{\jmath^4}{4}$, so that

$$\mathfrak{I}_2 = \frac{\jmath^4}{4}\Big[\|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}} + \|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}\Big].$$

In the same way, for the third integral, we obtain

$$\mathfrak{I}_3 \le \|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}}\int_{1-\imath}^1 (1-\varsigma)^3\varsigma\,d\varsigma + \|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}\int_{1-\imath}^1(1-\varsigma)^4\,d\varsigma$$
$$= \left(\frac{\imath^4}{4}-\frac{\imath^5}{5}\right)\|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}} + \frac{\imath^5}{5}\,\|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}.$$

Collecting terms,

$$\mathfrak{I}_1+\mathfrak{I}_2+\mathfrak{I}_3 \le \frac{\imath^4+\jmath^4}{4}\Big[\|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}} + \|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}\Big].$$

Multiplying by $\frac{\vartheta^3}{6}$ yields (2.6). □

Corollary 2.4. *Under the assumptions of Theorem 2.3, choosing $\varkappa = 0$ gives $\vartheta = \mathfrak{d}-\mathfrak{c}$, $\Lambda_0(\chi) = \mathfrak{c}+\mathfrak{d}-\chi$, $\Xi_0(\chi) = \chi$, and*

$$\begin{aligned}&\Bigg\|-\frac{1}{\mathfrak{d}-\mathfrak{c}}\int_{\mathfrak{c}}^{\mathfrak{d}} \mathfrak{F}(\omega)\,d\omega + \frac{1}{2}[\mathfrak{F}(\mathfrak{c}+\mathfrak{d}-\chi) + \mathfrak{F}(\chi)] + \frac{\mathfrak{d}-\mathfrak{c}}{2}(\jmath^2-\imath^2)[\mathfrak{F}'(\mathfrak{c}+\mathfrak{d}-\chi) - \mathfrak{F}'(\chi)]\\ &+\frac{(\mathfrak{d}-\mathfrak{c})^2}{6}(\imath^3+\jmath^3)[\mathfrak{F}''(\mathfrak{c}+\mathfrak{d}-\chi) + \mathfrak{F}''(\chi)]\Bigg\|_{\mathfrak{X}}\\ &\leq \frac{(\mathfrak{d}-\mathfrak{c})^3(\imath^4+\jmath^4)}{24}\Big[\|\mathfrak{F}'''(\mathfrak{d})\|_{\mathfrak{X}} + \|\mathfrak{F}'''(\mathfrak{c})\|_{\mathfrak{X}}\Big]\end{aligned}$$

for any $\chi \in [\frac{\mathfrak{c}+\mathfrak{d}}{2}, \mathfrak{d}]$.

Corollary 2.5. *In the setting of Theorem 2.3, setting $\chi = \frac{\mathfrak{c}+\mathfrak{d}}{2}$ yields $\imath = \frac{1}{2}$, $\jmath = 0$, $\jmath^2-\imath^2 = -\frac{1}{4}$, $\imath^3+\jmath^3 = \frac{1}{8}$, $\imath^4+\jmath^4 = \frac{1}{16}$, $\Lambda_\varkappa\Big(\frac{\mathfrak{c}+\mathfrak{d}}{2}\Big) = \mathfrak{P}_\varkappa$, $\Xi_\varkappa\Big(\frac{\mathfrak{c}+\mathfrak{d}}{2}\Big) = \mathfrak{Q}_\varkappa$, and*

$$\begin{aligned}&\Bigg\|-\frac{1}{\vartheta}\int_{\mathfrak{Q}_\varkappa}^{\mathfrak{P}_\varkappa} \mathfrak{F}(\omega)\,d\omega + \frac{1}{2}[\mathfrak{F}(\mathfrak{P}_\varkappa) + \mathfrak{F}(\mathfrak{Q}_\varkappa)] - \frac{\vartheta}{8}[\mathfrak{F}'(\mathfrak{P}_\varkappa) - \mathfrak{F}'(\mathfrak{Q}_\varkappa)] + \frac{\vartheta^2}{48}[\mathfrak{F}''(\mathfrak{P}_\varkappa) + \mathfrak{F}''(\mathfrak{Q}_\varkappa)]\Bigg\|_{\mathfrak{X}}\\ &\leq \frac{\vartheta^3}{384}\Big[\|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}} + \|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}\Big]\end{aligned}$$

for each $\varkappa \in [0,1] \setminus \{\frac{1}{2}\}$.

Theorem 2.6. *Suppose $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ is a real Banach space and $\mathfrak{F} : [\mathfrak{c},\mathfrak{d}] \to \mathfrak{X}$ is three times differentiable on $(\mathfrak{c},\mathfrak{d})$ with $\mathfrak{F}''' \in L_1([\mathfrak{c},\mathfrak{d}],\mathfrak{X})$, $\mathfrak{c} < \mathfrak{d}$. Let $\mathfrak{s} > 1$ and $\frac{1}{\mathfrak{r}} + \frac{1}{\mathfrak{s}} = 1$. When $\|\mathfrak{F}'''(\cdot)\|_{\mathfrak{X}}^{\mathfrak{s}}$ is convex on $[\mathfrak{c},\mathfrak{d}]$, then for each $\chi \in [\frac{\mathfrak{c}+\mathfrak{d}}{2}, \mathfrak{d}]$ and $\varkappa \in [0,1] \setminus \{\frac{1}{2}\}$,*

$$\begin{aligned}&\Bigg\|-\frac{1}{\vartheta}\int_{\mathfrak{Q}_\varkappa}^{\mathfrak{P}_\varkappa} \mathfrak{F}(\omega)\,d\omega + \frac{1}{2}[\mathfrak{F}(\Lambda_\varkappa(\chi)) + \mathfrak{F}(\Xi_\varkappa(\chi))]\\ &+\frac{\vartheta}{2}(\jmath^2-\imath^2)[\mathfrak{F}'(\Lambda_\varkappa(\chi)) - \mathfrak{F}'(\Xi_\varkappa(\chi))] + \frac{\vartheta^2}{6}(\imath^3+\jmath^3)[\mathfrak{F}''(\Lambda_\varkappa(\chi)) + \mathfrak{F}''(\Xi_\varkappa(\chi))]\Bigg\|_{\mathfrak{X}}\\ &\leq \frac{\vartheta^3}{6}\left(\frac{2(\imath^{3\mathfrak{r}+1}+\jmath^{3\mathfrak{r}+1})}{3\mathfrak{r}+1}\right)^{\frac{1}{\mathfrak{r}}}\left(\frac{\|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}}^{\mathfrak{s}} + \|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}^{\mathfrak{s}}}{2}\right)^{\frac{1}{\mathfrak{s}}}.\end{aligned} \tag{2.7}$$

Proof. From Lemma 2.1 and the Bochner integral triangle inequality,

$$\begin{aligned}&\Bigg\|-\frac{1}{\vartheta}\int_{\mathfrak{Q}_\varkappa}^{\mathfrak{P}_\varkappa} \mathfrak{F}(\omega)\,d\omega + \frac{1}{2}[\mathfrak{F}(\Lambda_\varkappa(\chi)) + \mathfrak{F}(\Xi_\varkappa(\chi))]\\ &+\frac{\vartheta}{2}(\jmath^2-\imath^2)[\mathfrak{F}'(\Lambda_\varkappa(\chi)) - \mathfrak{F}'(\Xi_\varkappa(\chi))] + \frac{\vartheta^2}{6}(\imath^3+\jmath^3)[\mathfrak{F}''(\Lambda_\varkappa(\chi)) + \mathfrak{F}''(\Xi_\varkappa(\chi))]\Bigg\|_{\mathfrak{X}}\\ &\leq \frac{\vartheta^3}{6}\int_0^1 |\mathscr{K}_3(\varsigma)|\left\|\mathfrak{F}'''(\phi(\varsigma))\right\|_{\mathfrak{X}} d\varsigma.\end{aligned}$$

Applying Hölder's inequality,

$$\int_0^1 |\mathscr{K}_3(\varsigma)|\left\|\mathfrak{F}'''(\phi(\varsigma))\right\|_{\mathfrak{X}} d\varsigma \leq \left(\int_0^1 |\mathscr{K}_3(\varsigma)|^{\mathfrak{r}}\,d\varsigma\right)^{\frac{1}{\mathfrak{r}}}\left(\int_0^1 \left\|\mathfrak{F}'''(\phi(\varsigma))\right\|_{\mathfrak{X}}^{\mathfrak{s}} d\varsigma\right)^{\frac{1}{\mathfrak{s}}}.$$

Since $|\mathscr{K}_3(\varsigma)|^{\mathfrak{r}} = \varsigma^{3\mathfrak{r}}$ on $[0, \imath]$, $|\varsigma - \frac{1}{2}|^{3\mathfrak{r}}$ on $[\imath, 1-\imath]$, and $(1-\varsigma)^{3\mathfrak{r}}$ on $[1-\imath, 1]$,

$$\int_0^1 |\mathscr{K}_3(\varsigma)|^{\mathfrak{r}}\, d\varsigma = \int_0^{\imath} \varsigma^{3\mathfrak{r}}\, d\varsigma + \int_{\imath}^{1-\imath} \left|\varsigma - \tfrac{1}{2}\right|^{3\mathfrak{r}} d\varsigma + \int_{1-\imath}^{1} (1-\varsigma)^{3\mathfrak{r}}\, d\varsigma = \frac{2(\imath^{3\mathfrak{r}+1} + \jmath^{3\mathfrak{r}+1})}{3\mathfrak{r}+1}.$$

Since $\|\mathfrak{F}'''(\cdot)\|_{\mathfrak{X}}^{\mathfrak{s}}$ is convex on $[\mathfrak{c}, \mathfrak{d}]$,

$$\left\|\mathfrak{F}'''(\phi(\varsigma))\right\|_{\mathfrak{X}}^{\mathfrak{s}} \leq \varsigma\, \|\mathfrak{F}'''(\mathfrak{P}_{\varkappa})\|_{\mathfrak{X}}^{\mathfrak{s}} + (1-\varsigma)\, \|\mathfrak{F}'''(\mathfrak{Q}_{\varkappa})\|_{\mathfrak{X}}^{\mathfrak{s}}$$

for all $\varsigma \in [0, 1]$, hence

$$\int_0^1 \left\|\mathfrak{F}'''(\phi(\varsigma))\right\|_{\mathfrak{X}}^{\mathfrak{s}} d\varsigma \leq \frac{\|\mathfrak{F}'''(\mathfrak{P}_{\varkappa})\|_{\mathfrak{X}}^{\mathfrak{s}} + \|\mathfrak{F}'''(\mathfrak{Q}_{\varkappa})\|_{\mathfrak{X}}^{\mathfrak{s}}}{2}.$$

Combining these estimates yields (2.7). □

Corollary 2.7. *In the setting of Theorem 2.6, taking* $\varkappa = 0$ *gives* $\vartheta = \mathfrak{d} - \mathfrak{c}$, $\Lambda_0(\chi) = \mathfrak{c} + \mathfrak{d} - \chi$, $\Xi_0(\chi) = \chi$, *and*

$$\begin{aligned}
&\Bigg\| -\frac{1}{\mathfrak{d}-\mathfrak{c}} \int_{\mathfrak{c}}^{\mathfrak{d}} \mathfrak{F}(\omega)\, d\omega + \frac{1}{2}[\mathfrak{F}(\mathfrak{c}+\mathfrak{d}-\chi) + \mathfrak{F}(\chi)] + \frac{\mathfrak{d}-\mathfrak{c}}{2}(\jmath^2 - \imath^2)[\mathfrak{F}'(\mathfrak{c}+\mathfrak{d}-\chi) - \mathfrak{F}'(\chi)] \\
&+ \frac{(\mathfrak{d}-\mathfrak{c})^2}{6}(\imath^3 + \jmath^3)[\mathfrak{F}''(\mathfrak{c}+\mathfrak{d}-\chi) + \mathfrak{F}''(\chi)] \Bigg\|_{\mathfrak{X}} \\
&\leq \frac{(\mathfrak{d}-\mathfrak{c})^3}{6} \left(\frac{2(\imath^{3\mathfrak{r}+1} + \jmath^{3\mathfrak{r}+1})}{3\mathfrak{r}+1}\right)^{\frac{1}{\mathfrak{r}}} \left(\frac{\|\mathfrak{F}'''(\mathfrak{d})\|_{\mathfrak{X}}^{\mathfrak{s}} + \|\mathfrak{F}'''(\mathfrak{c})\|_{\mathfrak{X}}^{\mathfrak{s}}}{2}\right)^{\frac{1}{\mathfrak{s}}}
\end{aligned}$$

for each $\chi \in [\frac{\mathfrak{c}+\mathfrak{d}}{2}, \mathfrak{d}]$.

Corollary 2.8. *In the setting of Theorem 2.6, taking* $\chi = \frac{\mathfrak{c}+\mathfrak{d}}{2}$ *yields* $\imath = \frac{1}{2}$, $\jmath = 0$, *so that* $\imath^{3\mathfrak{r}+1} + \jmath^{3\mathfrak{r}+1} = \frac{1}{2^{3\mathfrak{r}+1}}$, *and*

$$\begin{aligned}
&\left\| -\frac{1}{\vartheta} \int_{\mathfrak{Q}_{\varkappa}}^{\mathfrak{P}_{\varkappa}} \mathfrak{F}(\omega)\, d\omega + \frac{1}{2}[\mathfrak{F}(\mathfrak{P}_{\varkappa}) + \mathfrak{F}(\mathfrak{Q}_{\varkappa})] - \frac{\vartheta}{8}[\mathfrak{F}'(\mathfrak{P}_{\varkappa}) - \mathfrak{F}'(\mathfrak{Q}_{\varkappa})] + \frac{\vartheta^2}{48}[\mathfrak{F}''(\mathfrak{P}_{\varkappa}) + \mathfrak{F}''(\mathfrak{Q}_{\varkappa})] \right\|_{\mathfrak{X}} \\
&\leq \frac{\vartheta^3}{6} \left(\frac{1}{(3\mathfrak{r}+1)\cdot 2^{3\mathfrak{r}}}\right)^{\frac{1}{\mathfrak{r}}} \left(\frac{\|\mathfrak{F}'''(\mathfrak{P}_{\varkappa})\|_{\mathfrak{X}}^{\mathfrak{s}} + \|\mathfrak{F}'''(\mathfrak{Q}_{\varkappa})\|_{\mathfrak{X}}^{\mathfrak{s}}}{2}\right)^{\frac{1}{\mathfrak{s}}}
\end{aligned}$$

for each $\varkappa \in [0, 1] \setminus \{\frac{1}{2}\}$.

Theorem 2.9. *Suppose* $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ *is a real Banach space and* $\mathfrak{F} : [\mathfrak{c}, \mathfrak{d}] \to \mathfrak{X}$ *is three times differentiable on* $(\mathfrak{c}, \mathfrak{d})$ *with* $\mathfrak{F}''' \in L_1([\mathfrak{c}, \mathfrak{d}], \mathfrak{X})$, $\mathfrak{c} < \mathfrak{d}$. *Let* $\mathfrak{s} \geq 1$. *If* $\|\mathfrak{F}'''(\cdot)\|_{\mathfrak{X}}^{\mathfrak{s}}$ *is convex on* $[\mathfrak{c}, \mathfrak{d}]$, *then*

$$\begin{aligned}
&\Bigg\| -\frac{1}{\vartheta} \int_{\mathfrak{Q}_{\varkappa}}^{\mathfrak{P}_{\varkappa}} \mathfrak{F}(\omega)\, d\omega + \frac{1}{2}[\mathfrak{F}(\Lambda_{\varkappa}(\chi)) + \mathfrak{F}(\Xi_{\varkappa}(\chi))] \\
&+ \frac{\vartheta}{2}(\jmath^2 - \imath^2)[\mathfrak{F}'(\Lambda_{\varkappa}(\chi)) - \mathfrak{F}'(\Xi_{\varkappa}(\chi))] + \frac{\vartheta^2}{6}(\imath^3 + \jmath^3)[\mathfrak{F}''(\Lambda_{\varkappa}(\chi)) + \mathfrak{F}''(\Xi_{\varkappa}(\chi))] \Bigg\|_{\mathfrak{X}} \\
&\leq \frac{\vartheta^3(\imath^4 + \jmath^4)}{6\cdot 2^{1+\frac{1}{\mathfrak{s}}}} \left(\|\mathfrak{F}'''(\mathfrak{P}_{\varkappa})\|_{\mathfrak{X}}^{\mathfrak{s}} + \|\mathfrak{F}'''(\mathfrak{Q}_{\varkappa})\|_{\mathfrak{X}}^{\mathfrak{s}}\right)^{\frac{1}{\mathfrak{s}}}
\end{aligned} \tag{2.8}$$

for any $\chi \in [\frac{\mathfrak{c}+\mathfrak{d}}{2}, \mathfrak{d}]$ *and* $\varkappa \in [0, 1] \setminus \{\frac{1}{2}\}$.

Proof. From Lemma 2.1 and the triangle inequality for the Bochner integral,

$$\Bigg\| -\frac{1}{\vartheta}\int_{\mathfrak{Q}_\varkappa}^{\mathfrak{P}_\varkappa} \mathfrak{F}(\omega)\,d\omega + \frac{1}{2}[\mathfrak{F}(\Lambda_\varkappa(\chi)) + \mathfrak{F}(\Xi_\varkappa(\chi))]$$
$$+\frac{\vartheta}{2}(\jmath^2 - \imath^2)[\mathfrak{F}'(\Lambda_\varkappa(\chi)) - \mathfrak{F}'(\Xi_\varkappa(\chi))] + \frac{\vartheta^2}{6}(\imath^3 + \jmath^3)[\mathfrak{F}''(\Lambda_\varkappa(\chi)) + \mathfrak{F}''(\Xi_\varkappa(\chi))]\Bigg\|_{\mathfrak{X}}$$
$$\leq \frac{\vartheta^3}{6}\int_0^1 |\mathscr{K}_3(\varsigma)|\,\big\|\mathfrak{F}'''(\phi(\varsigma))\big\|_{\mathfrak{X}}\,d\varsigma.$$

By the power mean inequality,

$$\int_0^1 |\mathscr{K}_3(\varsigma)|\,\big\|\mathfrak{F}'''(\phi(\varsigma))\big\|_{\mathfrak{X}}\,d\varsigma \leq \left(\int_0^1 |\mathscr{K}_3(\varsigma)|\,d\varsigma\right)^{1-\frac{1}{\mathfrak{s}}}\left(\int_0^1 |\mathscr{K}_3(\varsigma)|\,\big\|\mathfrak{F}'''(\phi(\varsigma))\big\|_{\mathfrak{X}}^{\mathfrak{s}}\,d\varsigma\right)^{\frac{1}{\mathfrak{s}}}.$$

For the first integral, we have

$$\int_0^1 |\mathscr{K}_3(\varsigma)|\,d\varsigma = \int_0^{\imath} \varsigma^3\,d\varsigma + \int_{\imath}^{\frac{1}{2}}\left(\tfrac{1}{2}-\varsigma\right)^3 d\varsigma + \int_{\frac{1}{2}}^{1-\imath}\left(\varsigma-\tfrac{1}{2}\right)^3 d\varsigma + \int_{1-\imath}^1 (1-\varsigma)^3\,d\varsigma = \frac{\imath^4+\jmath^4}{2}.$$

Since $\|\mathfrak{F}'''(\cdot)\|_{\mathfrak{X}}^{\mathfrak{s}}$ is convex on $[\mathfrak{c}, \mathfrak{d}]$,

$$\big\|\mathfrak{F}'''(\phi(\varsigma))\big\|_{\mathfrak{X}}^{\mathfrak{s}} \leq \varsigma\,\|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}}^{\mathfrak{s}} + (1-\varsigma)\,\|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}^{\mathfrak{s}}, \quad \varsigma \in [0,1],$$

hence

$$\int_0^1 |\mathscr{K}_3(\varsigma)|\,\big\|\mathfrak{F}'''(\phi(\varsigma))\big\|_{\mathfrak{X}}^{\mathfrak{s}}\,d\varsigma \leq \|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}}^{\mathfrak{s}}\int_0^1 |\mathscr{K}_3(\varsigma)|\,\varsigma\,d\varsigma + \|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}^{\mathfrak{s}}\int_0^1 |\mathscr{K}_3(\varsigma)|(1-\varsigma)\,d\varsigma$$
$$= \frac{\imath^4+\jmath^4}{4}\left(\|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}}^{\mathfrak{s}} + \|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}^{\mathfrak{s}}\right),$$

where we used $\int_0^1 |\mathscr{K}_3(\varsigma)|\,\varsigma\,d\varsigma = \int_0^1 |\mathscr{K}_3(\varsigma)|(1-\varsigma)\,d\varsigma = \frac{\imath^4+\jmath^4}{4}$. Substituting and using

$$\left(\frac{\imath^4+\jmath^4}{2}\right)^{1-\frac{1}{\mathfrak{s}}}\left(\frac{\imath^4+\jmath^4}{4}\right)^{\frac{1}{\mathfrak{s}}} = \frac{\imath^4+\jmath^4}{2^{1+\frac{1}{\mathfrak{s}}}}$$

yields (2.8). □

Remark 2.10. *When $\mathfrak{s} = 1$ in* (2.8)*, one obtains $2^{1+\frac{1}{\mathfrak{s}}} = 4$, so inequality* (2.8) *reduces to inequality* (2.6)*. Hence inequality* (2.6) *is a particular case of inequality* (2.8)*, corresponding to the endpoint $\mathfrak{s} = 1$ of the power mean inequality.*

Corollary 2.11. *In the setting of Theorem 2.9, taking $\varkappa = 0$ gives $\vartheta = \mathfrak{d} - \mathfrak{c}$, $\Lambda_0(\chi) = \mathfrak{c} + \mathfrak{d} - \chi$, $\Xi_0(\chi) = \chi$, and*

$$\Bigg\| -\frac{1}{\mathfrak{d}-\mathfrak{c}}\int_{\mathfrak{c}}^{\mathfrak{d}} \mathfrak{F}(\omega)\,d\omega + \frac{1}{2}[\mathfrak{F}(\mathfrak{c}+\mathfrak{d}-\chi) + \mathfrak{F}(\chi)] + \frac{\mathfrak{d}-\mathfrak{c}}{2}(\jmath^2 - \imath^2)[\mathfrak{F}'(\mathfrak{c}+\mathfrak{d}-\chi) - \mathfrak{F}'(\chi)]$$

$$+\frac{(\mathfrak{d}-\mathfrak{c})^2}{6}(\imath^3+\jmath^3)[\mathfrak{F}''(\mathfrak{c}+\mathfrak{d}-\chi)+\mathfrak{F}''(\chi)]\Bigg\|_{\mathfrak{X}}$$
$$\leq \frac{(\mathfrak{d}-\mathfrak{c})^3(\imath^4+\jmath^4)}{6\cdot 2^{1+\frac{1}{\mathfrak{s}}}}\left(\|\mathfrak{F}'''(\mathfrak{d})\|_{\mathfrak{X}}^{\mathfrak{s}}+\|\mathfrak{F}'''(\mathfrak{c})\|_{\mathfrak{X}}^{\mathfrak{s}}\right)^{\frac{1}{\mathfrak{s}}}$$

for any $\chi \in [\frac{\mathfrak{c}+\mathfrak{d}}{2}, \mathfrak{d}]$.

Corollary 2.12. *Under the assumptions of Theorem 2.9, choosing* $\chi = \frac{\mathfrak{c}+\mathfrak{d}}{2}$ *gives* $\imath = \frac{1}{2}$, $\jmath = 0$, *so that* $\imath^4 + \jmath^4 = \frac{1}{16}$, *and*

$$\left\|-\frac{1}{\vartheta}\int_{\mathfrak{Q}_\varkappa}^{\mathfrak{P}_\varkappa}\mathfrak{F}(\omega)\,d\omega+\frac{1}{2}[\mathfrak{F}(\mathfrak{P}_\varkappa)+\mathfrak{F}(\mathfrak{Q}_\varkappa)]-\frac{\vartheta}{8}[\mathfrak{F}'(\mathfrak{P}_\varkappa)-\mathfrak{F}'(\mathfrak{Q}_\varkappa)]+\frac{\vartheta^2}{48}[\mathfrak{F}''(\mathfrak{P}_\varkappa)+\mathfrak{F}''(\mathfrak{Q}_\varkappa)]\right\|_{\mathfrak{X}}$$
$$\leq \frac{\vartheta^3}{6\cdot 2^{4+\frac{1}{\mathfrak{s}}}}\left(\|\mathfrak{F}'''(\mathfrak{P}_\varkappa)\|_{\mathfrak{X}}^{\mathfrak{s}}+\|\mathfrak{F}'''(\mathfrak{Q}_\varkappa)\|_{\mathfrak{X}}^{\mathfrak{s}}\right)^{\frac{1}{\mathfrak{s}}}$$

for any $\varkappa \in [0,1] \setminus \{\frac{1}{2}\}$.

3. Applications

Beyond numerical quadrature, the inequalities proved above admit further practical uses. Solutions of abstract differential equations in a Banach space constitute a natural class of vector-valued functions, and for such solutions, our bounds estimate the difference between the time average of a solution over an interval and its value at a given point; in approximation theory, they give error estimates for the approximation of vector-valued functions by their values at finitely many points. Three concrete applications are developed below: an error estimate for a composite quadrature rule, inequalities for special means, and a numerical example illustrating the obtained bounds.

3.1. Error bounds for a composite quadrature rule

Let $\mathfrak{F} : [\mathfrak{c}, \mathfrak{d}] \to \mathfrak{X}$ be three times differentiable with $\mathfrak{F}''' \in L_1([\mathfrak{c}, \mathfrak{d}], \mathfrak{X})$. Divide $[\mathfrak{c}, \mathfrak{d}]$ into ν equal subintervals by setting $\chi_k = \mathfrak{c} + k\,\frac{\mathfrak{d}-\mathfrak{c}}{\nu}$ for $k = 0, 1, \ldots, \nu$, let $\rho_\nu = \frac{\mathfrak{d}-\mathfrak{c}}{\nu}$, and set $\mu_k = \frac{\chi_k+\chi_{k+1}}{2}$ for $k = 0, 1, \ldots, \nu-1$. Consider the composite quadrature approximation

$$Q_\nu(\mathfrak{F}) := \rho_\nu \sum_{k=0}^{\nu-1}\left[\mathfrak{F}(\mu_k)+\frac{\rho_\nu^2}{24}\,\mathfrak{F}''(\mu_k)\right].$$

Proposition 3.1. *Let* $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ *be a real Banach space and let* $\mathfrak{F} : [\mathfrak{c}, \mathfrak{d}] \to \mathfrak{X}$ *be three times differentiable on* $(\mathfrak{c}, \mathfrak{d})$ *with* $\mathfrak{F}''' \in L_1([\mathfrak{c}, \mathfrak{d}], \mathfrak{X})$, $\mathfrak{c} < \mathfrak{d}$. *If* $\|\mathfrak{F}'''(\cdot)\|_{\mathfrak{X}}$ *is convex on* $[\mathfrak{c}, \mathfrak{d}]$, *then*

$$\left\|\int_{\mathfrak{c}}^{\mathfrak{d}}\mathfrak{F}(\varsigma)\,d\varsigma - Q_\nu(\mathfrak{F})\right\|_{\mathfrak{X}} \leq \frac{(\mathfrak{d}-\mathfrak{c})^4}{384\,\nu^3}\left[\|\mathfrak{F}'''(\mathfrak{c})\|_{\mathfrak{X}}+\|\mathfrak{F}'''(\mathfrak{d})\|_{\mathfrak{X}}\right].$$

Proof. Applying Theorem 2.3 on each subinterval $[\chi_k, \chi_{k+1}]$ with $\varkappa = 0$ and $\chi = \mu_k$, so that $\vartheta = \rho_\nu$, $\imath = \frac{1}{2}$, $\jmath = 0$, $\Lambda_0(\mu_k) = \Xi_0(\mu_k) = \mu_k$, $\jmath^2 - \imath^2 = -\frac{1}{4}$, $\imath^3 + \jmath^3 = \frac{1}{8}$, and $\imath^4 + \jmath^4 = \frac{1}{16}$, the identity (2.2) reduces to

$$-\frac{1}{\rho_\nu}\int_{\chi_k}^{\chi_{k+1}}\mathfrak{F}(\varsigma)\,d\varsigma+\mathfrak{F}(\mu_k)+\frac{\rho_\nu^2}{24}\,\mathfrak{F}''(\mu_k)=\frac{\rho_\nu^3}{6}\int_0^1\mathscr{K}_3(\varsigma)\,\mathfrak{F}'''(\phi_k(\varsigma))\,d\varsigma,$$

where $\phi_k(\varsigma) = \varsigma\chi_{k+1} + (1-\varsigma)\chi_k$. Taking norms and applying Theorem 2.3 with $\imath^4 + \jmath^4 = \frac{1}{16}$ gives

$$\left\| -\frac{1}{\rho_\nu}\int_{\chi_k}^{\chi_{k+1}} \mathfrak{F}(\varsigma)\,d\varsigma + \mathfrak{F}(\mu_k) + \frac{\rho_\nu^2}{24}\,\mathfrak{F}''(\mu_k) \right\|_{\mathfrak{X}} \leq \frac{\rho_\nu^3}{384}\Big[\|\mathfrak{F}'''(\chi_k)\|_{\mathfrak{X}} + \|\mathfrak{F}'''(\chi_{k+1})\|_{\mathfrak{X}}\Big].$$

Multiplying both sides by ρ_ν and summing over $k = 0, 1, \ldots, \nu-1$,

$$\left\| \int_{\mathfrak{c}}^{\mathfrak{d}} \mathfrak{F}(\varsigma)\,d\varsigma - Q_\nu(\mathfrak{F}) \right\|_{\mathfrak{X}} \leq \frac{\rho_\nu^4}{384}\sum_{k=0}^{\nu-1}\Big[\|\mathfrak{F}'''(\chi_k)\|_{\mathfrak{X}} + \|\mathfrak{F}'''(\chi_{k+1})\|_{\mathfrak{X}}\Big].$$

Since $\|\mathfrak{F}'''(\cdot)\|_{\mathfrak{X}}$ is convex on $[\mathfrak{c}, \mathfrak{d}]$,

$$\|\mathfrak{F}'''(\chi_k)\|_{\mathfrak{X}} \leq \left(1 - \frac{k}{\nu}\right)\|\mathfrak{F}'''(\mathfrak{c})\|_{\mathfrak{X}} + \frac{k}{\nu}\,\|\mathfrak{F}'''(\mathfrak{d})\|_{\mathfrak{X}}$$

for each $k = 0, 1, \ldots, \nu$, and thus

$$\begin{aligned}\sum_{k=0}^{\nu-1}\Big[\|\mathfrak{F}'''(\chi_k)\|_{\mathfrak{X}} + \|\mathfrak{F}'''(\chi_{k+1})\|_{\mathfrak{X}}\Big] &\leq \sum_{k=0}^{\nu-1}\left[\left(2 - \frac{2k+1}{\nu}\right)\|\mathfrak{F}'''(\mathfrak{c})\|_{\mathfrak{X}} + \frac{2k+1}{\nu}\,\|\mathfrak{F}'''(\mathfrak{d})\|_{\mathfrak{X}}\right] \\ &= \nu\left[\|\mathfrak{F}'''(\mathfrak{c})\|_{\mathfrak{X}} + \|\mathfrak{F}'''(\mathfrak{d})\|_{\mathfrak{X}}\right],\end{aligned}$$

where we used $\sum_{k=0}^{\nu-1}(2k+1) = \nu^2$. Substituting $\rho_\nu = \frac{\mathfrak{d}-\mathfrak{c}}{\nu}$ gives the result. □

Remark 3.2. *When $\mathfrak{X} = \mathbb{R}$, Proposition 3.1 gives a corrected midpoint quadrature rule for three times differentiable functions. The correction term $\frac{\rho_\nu^2}{24}\,\mathfrak{F}''(\mu_k)$ yields an error of order $O(\nu^{-3})$ as $\nu \to \infty$.*

3.2. Illustrative examples involving special means

For two positive real numbers $\mathfrak{c} < \mathfrak{d}$, we recall the following means:

1) $A(\mathfrak{c}, \mathfrak{d}) := \dfrac{\mathfrak{c}+\mathfrak{d}}{2}$,
2) $L_\nu(\mathfrak{c}, \mathfrak{d}) := \left(\dfrac{\mathfrak{d}^{\nu+1} - \mathfrak{c}^{\nu+1}}{(\nu+1)(\mathfrak{d}-\mathfrak{c})}\right)^{1/\nu}, \quad \nu \in \mathbb{Z},\ \nu \neq -1, 0,$
3) $L(\mathfrak{c}, \mathfrak{d}) := \dfrac{\mathfrak{d}-\mathfrak{c}}{\ln\mathfrak{d} - \ln\mathfrak{c}}$.

Setting $\varkappa = 0$ and $\chi = \frac{\mathfrak{c}+\mathfrak{d}}{2}$ in Corollary 2.4 with $\mathfrak{X} = \mathbb{R}$, so that $\imath = \frac{1}{2}$, $\jmath = 0$, $\Lambda_0\left(\frac{\mathfrak{c}+\mathfrak{d}}{2}\right) = \Xi_0\left(\frac{\mathfrak{c}+\mathfrak{d}}{2}\right) = A(\mathfrak{c}, \mathfrak{d})$, we obtain

$$\left| -\frac{1}{\mathfrak{d}-\mathfrak{c}}\int_{\mathfrak{c}}^{\mathfrak{d}} \mathfrak{F}(\varsigma)\,d\varsigma + \mathfrak{F}(A(\mathfrak{c},\mathfrak{d})) + \frac{(\mathfrak{d}-\mathfrak{c})^2}{24}\,\mathfrak{F}''(A(\mathfrak{c},\mathfrak{d})) \right| \leq \frac{(\mathfrak{d}-\mathfrak{c})^3}{384}\Big[|\mathfrak{F}'''(\mathfrak{c})| + |\mathfrak{F}'''(\mathfrak{d})|\Big] \tag{3.1}$$

for any $\mathfrak{F} : [\mathfrak{c}, \mathfrak{d}] \to \mathbb{R}$ with $|\mathfrak{F}'''|$ convex on $[\mathfrak{c}, \mathfrak{d}]$.

Proposition 3.3. *Let $\nu \in \mathbb{N}$, $\nu \geq 1$, and $0 < \mathfrak{c} < \mathfrak{d}$. Then*

$$\begin{aligned}&\left| A^{\nu+3}(\mathfrak{c},\mathfrak{d}) - L_{\nu+3}^{\nu+3}(\mathfrak{c},\mathfrak{d}) + \frac{(\nu+3)(\nu+2)(\mathfrak{d}-\mathfrak{c})^2}{24}\,A^{\nu+1}(\mathfrak{c},\mathfrak{d}) \right| \\ &\leq \frac{(\nu+3)(\nu+2)(\nu+1)(\mathfrak{d}-\mathfrak{c})^3}{384}\left[\mathfrak{c}^\nu + \mathfrak{d}^\nu\right].\end{aligned}$$

Proof. Apply (3.1) with $\mathfrak{F}(\varsigma) = \varsigma^{\nu+3}$. Then $|\mathfrak{F}'''(\varsigma)| = (\nu+3)(\nu+2)(\nu+1)\varsigma^{\nu}$ is convex on $[\mathfrak{c}, \mathfrak{d}]$ since $\nu \geq 1$ and $\mathfrak{c} > 0$, and

$$\frac{1}{\mathfrak{d}-\mathfrak{c}} \int_{\mathfrak{c}}^{\mathfrak{d}} \varsigma^{\nu+3}\, d\varsigma = L_{\nu+3}^{\nu+3}(\mathfrak{c}, \mathfrak{d}), \quad \mathfrak{F}(A(\mathfrak{c}, \mathfrak{d})) = A^{\nu+3}(\mathfrak{c}, \mathfrak{d}), \quad \mathfrak{F}''(A(\mathfrak{c}, \mathfrak{d})) = (\nu+3)(\nu+2)\, A^{\nu+1}(\mathfrak{c}, \mathfrak{d}).$$

Substituting into (3.1) gives the result. □

Proposition 3.4. *For* $0 < \mathfrak{c} < \mathfrak{d}$,

$$\left| \left(1 + \frac{(\mathfrak{d}-\mathfrak{c})^2}{24}\right) e^{A(\mathfrak{c},\mathfrak{d})} - L(e^{\mathfrak{c}}, e^{\mathfrak{d}}) \right| \leq \frac{(\mathfrak{d}-\mathfrak{c})^3}{384} [e^{\mathfrak{c}} + e^{\mathfrak{d}}]. \tag{3.2}$$

Proof. Apply (3.1) with $\mathfrak{F}(\varsigma) = e^{\varsigma}$. Then $|\mathfrak{F}'''(\varsigma)| = e^{\varsigma}$ is convex on $[\mathfrak{c}, \mathfrak{d}]$, and

$$\frac{1}{\mathfrak{d}-\mathfrak{c}} \int_{\mathfrak{c}}^{\mathfrak{d}} e^{\varsigma}\, d\varsigma = L(e^{\mathfrak{c}}, e^{\mathfrak{d}}), \quad \mathfrak{F}(A(\mathfrak{c}, \mathfrak{d})) = \mathfrak{F}''(A(\mathfrak{c}, \mathfrak{d})) = e^{A(\mathfrak{c},\mathfrak{d})}.$$

Substituting into (3.1) and rearranging gives (3.2). □

3.3. A numerical example

We illustrate the obtained inequalities by the following numerical example.

Example 3.5. *Let* $\mathfrak{X} = \mathbb{R}^2$ *endowed with the maximum norm* $\|(u, v)\|_\infty = \max\{|u|, |v|\}$, *and consider* $\mathfrak{F} : [0, 1] \to \mathbb{R}^2$ *given by* $\mathfrak{F}(\varsigma) = (e^{\varsigma}, \varsigma^4)$, *so that* $\mathfrak{F}'''(\varsigma) = (e^{\varsigma}, 24\varsigma)$ *and* $\|\mathfrak{F}'''(\varsigma)\|_\infty = \max\{e^{\varsigma}, 24\varsigma\}$ *is convex on* $[0, 1]$ *as the maximum of two convex functions. Applying Proposition 3.1 with* $\nu = 1$, *so that* $\rho_1 = 1$ *and* $\mu_0 = \frac{1}{2}$, *the left-hand side equals*

$$\begin{aligned} &\left\| \int_0^1 \mathfrak{F}(\varsigma)\, d\varsigma - \mathfrak{F}\left(\tfrac{1}{2}\right) - \tfrac{1}{24}\, \mathfrak{F}''\left(\tfrac{1}{2}\right) \right\|_\infty \\ &= \left\| \left(e - 1 - e^{1/2} - \tfrac{1}{24} e^{1/2},\ \tfrac{1}{5} - \tfrac{1}{16} - \tfrac{1}{8}\right) \right\|_\infty = \max\{0.000864\ldots, 0.0125\} = 0.0125, \end{aligned}$$

while the right-hand side equals $\frac{1}{384}[\|\mathfrak{F}'''(0)\|_\infty + \|\mathfrak{F}'''(1)\|_\infty] = \frac{1+24}{384} = 0.065104\ldots$, *so the inequality holds. Similarly, in the scalar case with* $\mathfrak{F}(\varsigma) = e^{\varsigma}$ *on* $[0, 1]$, *inequality* (3.1) *reads* $0.000864\ldots \leq 0.009683\ldots$, *which is again valid.*

4. Conclusions

In this paper, we established a new parameterized integral identity for three times differentiable Banach space-valued functions. The identity was obtained by applying the Bochner integration by parts formula three successive times. Using this identity, we derived three Ostrowski-type inequalities under the assumption that either $\|\mathfrak{F}'''(\cdot)\|_{\mathfrak{X}}$ or its $\mathfrak{s}$-th power is convex. The proofs rely on the triangle inequality for the Bochner integral, Hölder's inequality, and the power mean inequality.

The obtained results extend the scalar second-derivative inequalities of Set and Ekinci [2] to Banach space-valued functions involving third-order derivatives. Moreover, several known inequalities are recovered as special cases of the main results. As applications, we derived an error estimate for a composite quadrature rule for Banach space-valued integrands and obtained inequalities for certain special means. A numerical example was also provided to illustrate the validity of the derived estimates. Future research may consider extensions to derivatives of order higher than three, broader classes of generalized convexity for the norm of the derivative, and multivariable functions taking values in Banach spaces.

Use of Generative-AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were used in the creation of this article.

Conflict of interest

The author confirms that there is no conflict of interest.

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