AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 24137–24151.
DOI:10.3934/math.2026974
Received: 26 April 2026
Revised: 23 July 2026
Accepted: 28 July 2026
Published: 07 August 2026

Research article

Global existence and energy decay for a logarithmically damped wave equation arising in structural vibration models

Mohammad M. Al-Gharabli[1,2], **Adel M. Al-Mahdi**[1,2,*] **and Bashayer Aldossary**[1]

[1] Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

[2] The Interdisciplinary Research Center in Construction and Building Materials, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

* **Correspondence:** Email: almahdi@kfupm.edu.sa.

Abstract: Wave equations with nonlinear damping arise naturally in the mathematical modeling of vibrating structures, wave propagation in heterogeneous media, and structural vibration control. Among the recently proposed damping mechanisms, logarithmic damping provides a realistic description of weak dissipation near equilibrium together with enhanced energy absorption at larger velocities. Motivated by these applications, this paper investigated a wave equation with nonlinear logarithmic damping and established a rigorous proof of the global existence and uniqueness of strong solutions for this class of problems. The proof was based on the Faedo–Galerkin approximation method together with suitable a priori estimates and monotonicity arguments adapted to the logarithmic nonlinearity. This result filled an important gap in the existing literature by providing the analytical foundation required for the study of logarithmically damped wave equations. Furthermore, we established a polynomial decay rate for the associated energy by developing a multiplier approach specifically tailored to the logarithmic damping term. Unlike the techniques commonly used for classical linear or polynomial damping, our analysis overcame the nonhomogeneous nature of the logarithmic dissipation through refined estimates that captured its distinct growth behavior. These results complemented and extended the existing literature on logarithmic damping by providing both the well-posedness theory and the long-time stability analysis within a unified framework.

Keywords: wave equation; logarithmic nonlinearity; multiplier method; Faedo-Galerkin method; global existence
Mathematics Subject Classification: 5L05, 35L20, 45K05, 93D23

1. Introduction

The study of partial differential equations (PDEs) with damping mechanisms has attracted significant attention in the mathematical and physical literature due to their critical role in modeling energy dissipation in various dynamical systems, such as elastic structures, wave propagation, and fluid dynamics; see, for instance, [1, 2]. One of the central objectives in this field is to understand the existence, uniqueness, and long-time behavior of solutions under different types of damping effects. These investigations not only provide theoretical insights into the stability and asymptotic properties of the solutions but also contribute to the design of more effective control and stabilization strategies in applied problems; see, for example, [3, 4].

Traditionally, many studies have focused on nonlinear frictional damping mechanisms of the form $h(u_t)$, where h is a continuous monotone function that satisfies certain growth and coercivity conditions. Such damping functions are often used to model classical dissipation phenomena and have been instrumental in establishing global existence and decay properties of solutions to a variety of evolution equations. For instance, for the following nonlinear damped wave equation

$$u_{tt} - \Delta u + h(u_t) = 0, \quad (x, t) \in \Omega \times (0, \infty), \tag{1.1}$$

where h represents a nonlinear damping function, it is well known that the energy $E(t)$ corresponding to a solution u of Eq (1.1) decays at the rate $(1 + t)^{-2/m}$ as $t \to \infty$, and exhibits exponential decay when $m = 0$, assuming the damping term is given by $g(u_t) = u_t|u_t|^m$ for $0 \leq m < m^*$, where m^* denotes the critical Sobolev exponent; see [5–7] for details.

More generally, if the damping function h satisfies the growth condition

$$c_1 \min\{|s|, |s|^p\} \leq |h(s)| \leq c_2 \max\{|s|, |s|^{1/p}\}, \quad \forall s \in \mathbb{R},$$

for some constants $c_1, c_2 > 0$ and $p > 1$, it has been shown that the energy decays at the rate

$$E(t) \leq C(E(0))\, t^{-\frac{2}{p-1}}, \quad \forall t > 0.$$

In the limiting case $p = 1$, the decay becomes exponential.

Significant progress in understanding the decay phenomenon for arbitrary nonlinear damping functions was made by Lasiecka and Tataru [8], who considered the wave equation with boundary feedback. They demonstrated that the energy decays at the same rate as the solution of an associated differential equation, whose coefficients reflect the behavior of the damping function.

Further contributions were made by Martinez [9], who established general decay results under the assumption that $h \in C^1(\mathbb{R})$ is an odd, strictly increasing function satisfying:

$$|s| \leq |h(s)| \leq C|s| \quad \text{for } |s| \geq 1, \qquad H(|s|) \leq |h(s)| \leq CH^{-1}(|s|) \quad \text{for } |s| \leq 1,$$

where H^{-1} denotes the inverse of a given function H, and C is a positive constant. These results provide a unified framework for analyzing the decay of energy in wave equations under a broad class of nonlinear damping mechanisms. For additional results in this direction, we refer the reader to [11–13]. Besides the extensive literature on dissipative wave equations, recent years have also witnessed substantial advances in the qualitative analysis of nonlinear evolution equations, including the derivation of exact solutions, the study of nonlinear wave interactions, and the investigation of complex dynamical behaviors; see, for example, [14, 15]. These developments demonstrate the continued interest in nonlinear PDEs and further motivate the study of wave equations with novel damping mechanisms, such as the logarithmic damping considered in the present work.

Recently, attention has shifted toward more complex and nonstandard damping mechanisms, among which the logarithmic damping of the form $u_t \ln(1 + u_t^2)$ stands out as a physically and mathematically intriguing alternative. Unlike the classical nonlinear damping $h(u_t)$, the logarithmic damping captures a weaker dissipation effect for small velocities while maintaining a strong growth rate for large velocities. This dual behavior is particularly useful in applications where the system experiences mild dissipation near equilibrium but requires enhanced damping in high-energy regimes. Moreover, the logarithmic structure introduces additional analytical challenges due to its non-polynomial, non-Lipschitz nature, making the study of the corresponding PDEs both rich and nontrivial.

Understanding the impact of such damping on the existence and asymptotic stability of solutions is crucial, especially in bridging the gap between physically realistic modeling and rigorous mathematical analysis. This motivates the current investigation into PDEs equipped with logarithmic damping terms, where we aim to establish well-posedness and derive a polynomial decay rate of the associated energy functional.

Motivated by the subsequent work of Li et al. [16], Messaoudi and Al-Mahdi [17] and Messaoudi et al. [18] recently established the global existence of strong solutions and derived energy decay for the Timoshenko system with logarithmic damping. Also, weak logarithmic damping has been investigated in [19–21] for several one-dimensional systems.

From a physical and engineering viewpoint, logarithmic damping provides a realistic model for dissipation mechanisms whose intensity increases slowly for small velocities while becoming more effective as the velocity grows. Such behavior may arise in wave propagation through heterogeneous media, structural vibration control, smart materials, and mechanical systems exhibiting nonlinear frictional effects. Therefore, understanding the well-posedness and long-time behavior of wave equations with logarithmic damping is essential for predicting the stability of these systems and assessing the effectiveness of logarithmic feedback in suppressing unwanted oscillations.

From the stabilization viewpoint, logarithmic damping possesses a substantially different structure from the classical linear or polynomial damping mechanisms that have been extensively studied in the literature. Indeed, its growth is weaker than any power near the origin while increasing more slowly than polynomial damping for large velocities. Consequently, the techniques commonly employed for polynomial nonlinearities cannot be applied directly. In particular, the logarithmic term requires new estimates that combine its different behaviors in the low- and high-velocity regimes. The polynomial decay established in the present work therefore provides a rigorous stability result specifically adapted to the logarithmic damping law rather than being a straightforward consequence of existing decay theories.

Motivated by these considerations, we consider the following problem:

$$\begin{cases} u_{tt} - \Delta u + u_t \ln(1 + u_t^2) = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^N$ ($N \geq 1$) with a smooth boundary $\partial\Omega$.

The main contributions of this paper are summarized as follows:

1) We establish the global existence and uniqueness of solutions to problem (1.2). The proof is based on the Faedo–Galerkin approximation method together with the monotonicity properties of the logarithmic damping term. This result fills an important gap in the existing literature, since the existence and uniqueness of solutions were not addressed in [16].
2) We provide a complete and rigorous proof of the polynomial stability of solutions by developing an approach that differs substantially from that employed in [16].

The paper is organized as follows. In Section 2, we present some preliminary material essential for our analysis. In Section 3, we state and prove the existence results. Section 4 is devoted to the discussion of the stability of the solution energy.

2. Preliminaries

In this section, we introduce essential mathematical tools and preliminary results that will be used in the proofs of our main theorems. We consider the standard Lebesgue space $L^2(\Omega)$ and the Sobolev space $H_0^1(\Omega)$, equipped with their usual norms and inner products. Throughout the paper, we use c and C to denote generic positive constants.

Remark 1. *[16] The following inequalities are essential for establishing the existence and the decay results.*

(i) For $|s| \leq 1$, $m \geq 1$, and $0 < n \leq 2$, there exist positive constants C_p and C_q such that

$$C_p|s|^{1+m} \leq |s| \ln(1 + |s|) \leq C_q|s|^n. \tag{2.1}$$

(ii) For $|s| \geq 1$, $0 < k < 1$, and $\rho > 0$, there exist positive constants C_k and C_ρ such that

$$C_k|s|^k \leq |s| \ln(1 + |s|) \leq C_\rho|s|^{1+\rho}. \tag{2.2}$$

Lemma 1. *[22] (Aubin-Lions Theorem) Given three Banach spaces V, X, W such that*

$$V \subset X \subset W,$$

V and W are reflexive and $V \subset W$ is compact. If (u_m) is bounded in $L^p(0,T;V)$ and (u'_m) is bounded in $L^q(0,T;W)$, $1 < p, q < +\infty$, then there exists a subsequence (u_ℓ) such that

$$u_\ell \to u \quad \text{strongly in } L^p(0,T;X).$$

Lemma 2. *[22] (Lions Lemma) Let $\Omega \subset \mathbb{R}^n$ be a measurable set and let (g_n) be a sequence such that*

$$g_n \to g \quad a.e.\ in\ \Omega,$$

and

$$(g_n) \text{ is bounded in } L^q(\Omega), \quad 1 < q < \infty.$$

Then,

$$g_n \rightharpoonup g \quad \text{weakly in } L^q(\Omega).$$

The energy functional associated with problem (1.2) is

$$E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|\nabla u\|_2^2\right). \tag{2.3}$$

Direct differentiation of (2.3), using (1.2), leads to

$$E'(t) = -\int_\Omega u_t^2 \ln(1+u_t^2)dx \leq 0. \tag{2.4}$$

3. Existence

In this section, we state and prove an existence result for problem (1.2).

Theorem 1. *Let $(u_0, u_1) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$. Then, problem (1.2) has a unique "strong" solution satisfying*

$$u \in L^\infty(0,T; H^2(\Omega) \cap H_0^1(\Omega)),\ u_t \in L^\infty(0,T; H_0^1(\Omega)),\ u_{tt} \in L^\infty(0,T; L^2(\Omega)), \tag{3.1}$$

for any $T > 0$.

Proof. To establish the well-posedness of problem (1.2), we use the standard Faedo-Galerkin method. Let $\{w_j\}_{j=1}^\infty$ be an orthogonal basis of the "separable" space $H_0^1(\Omega)$ satisfying

$$-\Delta w_j = \lambda_j w_j$$

and let $V_m = span\{w_1, w_2, ..., w_m\}$. We search for an approximate solution of the form

$$u^m(x,t) = \sum_{j=1}^m h_j^m(t) w_j(x)$$

in V_m for the following approximate problem:

$$\begin{cases} \int_\Omega \left(u_{tt}^m w + \nabla u^m.\nabla w + w u_t^m \ln(1+(u_t^m)^2)\right) dx = 0,\ \forall w \in V_m \\ u^m(0) = u_0^m \\ u_t^m(0) = u_1^m, \end{cases} \tag{3.2}$$

such that

$$u_0^m \to u_0 \text{ in } H^2(\Omega) \cap H_0^1(\Omega) \text{ and } u_1^m \to u_1 \text{ in } H_0^1(\Omega), \text{ as } m \to \infty. \tag{3.3}$$

This leads to a system of ordinary differential equations for unknown functions $h_j^m(t)$. Based on standard existence theory for ODE, one can obtain C^2-functions

$$h_j : [0, t_m) \to \mathbb{R}, \ j = 1, 2, ..., m,$$

which satisfy (3.2) in a maximal interval $[0, t_m), t_m \in (0, T]$. Next, we show that $t_m = T$ and that the local solution is uniformly bounded independently of m and t. For this purpose, let us replace w by u_t^m in (3.2) and integrate by parts to get

$$\frac{d}{dt} E^m(t) = -\int_\Omega (u_t^m)^2 \ln(1 + (u_t^m)^2) dx \le 0, \tag{3.4}$$

where

$$E^m(t) = \frac{1}{2}\left(\|u_t^m\|_2^2 + \|\nabla u^m\|_2^2\right). \tag{3.5}$$

From (3.4), we have

$$E^m(t) \le E^m(0).$$

Integrate (3.4) over $(0, t)$ and use (3.3) to obtain

$$\begin{aligned} \frac{1}{2}\left(\|u_t^m\|_2^2 + \|\nabla u^m\|_2^2\right) + \int_0^t \int_\Omega (u_t^m)^2 \ln(1 + (u_t^m)^2) dx ds &= \frac{1}{2}\left(\|u_1^m\|_2^2 + \|\nabla u_0^m\|_2^2\right) \\ &\le \frac{1}{2}\left(\|u_1\|_2^2 + \|\nabla u_0\|_2^2\right) = C_0. \end{aligned} \tag{3.6}$$

This implies that, for some positive constant C_1,

$$\sup_{t\in(0,t_m)} \|u_t^m\|_2^2 + \sup_{t\in(0,t_m)} \|\nabla u^m\|_2^2 + 2\int_0^{t_m} \int_\Omega (u_t^m)^2 \ln(1 + (u_t^m)^2) dx ds \le C_1. \tag{3.7}$$

Substituting $w = \Delta u_t^m$ in (3.2), we have

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u_t^m\|_2^2 + \|\Delta u^m\|_2^2\right) - \int_\Omega \Delta u_t^m u_t^m \ln\left(1 + (u_t^m)^2\right) dx = 0, \tag{3.8}$$

which gives

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u_t^m\|_2^2 + \|\Delta u^m\|_2^2\right) + \int_\Omega |\nabla u_t^m|^2 \ln\left(1 + (u_t^m)^2\right) dx + \int_\Omega \frac{2|\nabla u_t^m|^2 (u_t^m)^2}{1 + (u_t^m)^2} dx = 0. \tag{3.9}$$

Integrate (3.9) over $(0, t)$ and use (3.3), to get

$$\begin{aligned} &\|\nabla u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + 2\int_0^t \int_\Omega |\nabla u_t^m|^2 \ln\left(1 + (u_t^m)^2\right) dx ds \\ &\qquad + 2\int_0^t \int_\Omega \frac{2|\nabla u_t^m|^2 (u_t^m)^2}{1 + (u_t^m)^2} dx ds \le \|\nabla u_1^m\|_2^2 + \|\Delta u_0^m\|_2^2 \\ &\qquad \le \|\nabla u_1\|_2^2 + \|\Delta u_0\|_2^2 = C_2. \end{aligned} \tag{3.10}$$

Next, substituting $w = u_{tt}^m$ in (3.2) and using Young's inequality, we arrive at

$$\begin{aligned}\|u_{tt}^m\|_2^2 &= \int_\Omega \Delta u^m u_{tt}^m - \int_\Omega u_{tt}^m u_t^m \ln(1+(u_t^m)^2)dx \\ &\le \frac{1}{4}\|u_{tt}^m\|_2^2 + \|\Delta u^m\|_2^2 + \frac{1}{4}\|u_{tt}^m\|_2^2 + \int_\Omega (u_t^m)^2 \ln^2(1+(u_t^m)^2)dx.\end{aligned} \tag{3.11}$$

Now, we want to show that the last term of (3.11) is bounded. For this purpose, we define as in [7], the partition

$$\Omega_1 = \{x \in \Omega \mid |u_t^m(x)| > 1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega \mid |u_t^m(x)| \le 1\}.$$

On Ω_2, we have

$$\int_{\Omega_2} (u_t^m)^2 \ln^2(1+(u_t^m)^2)dx \le |\Omega| \ln^2 2 < \infty. \tag{3.12}$$

By the algebraic inequality (2.1) with $s = u_t^2$, by choosing $\rho > 0$ such that

$$2(1+\rho) \le \frac{2N}{N-2}, \quad \text{if } N \ge 3, \quad \text{and } \rho > 0, \quad \text{if } N = 1, 2,$$

and using the Sobolev embedding theorem, we obtain, by (3.10),

$$\int_{\Omega_1} \left((u_t^m)\ln(1+(u_t^m)^2)\right)^2 dx \le c\int_{\Omega_1} |u_t^m|^{2(1+\rho)}dx \le c\|\nabla u_t^m\|_2^{2(1+\rho)} < \infty. \tag{3.13}$$

Combining (3.12) and (3.13), we deduce that

$$\int_\Omega (u_t^m)^2 \ln^2(1+(u_t^m)^2)dx < \infty. \tag{3.14}$$

Therefore, (3.11), (3.10), and (3.14) yield

$$\|u_{tt}^m\|_2^2 \le 2\|\Delta u^m\|_2^2 + 2C_3 \le C_4, \tag{3.15}$$

for some positive constant C_4.

Recalling (3.7), (3.10), (3.14), and (3.15), we conclude that the approximate solution is uniformly bounded independently of m and t. Therefore, we can extend t_m to T. Moreover, there exists a subsequence of (u^m) (still denoted by (u^m)) such that

$$\begin{cases} u^m \rightharpoonup u \quad \text{weakly * in } L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \\ u_t^m \rightharpoonup u_t \quad \text{weakly * in } L^\infty(0,T;H_0^1(\Omega)), \\ u^m \rightharpoonup u \quad \text{weakly in } L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \\ u_t^m \rightharpoonup u_t \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \\ u_{tt}^m \rightharpoonup u_{tt} \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega)), \\ u_{tt}^m \rightharpoonup u_{tt} \quad \text{weakly in } L^2(\Omega\times(0,T)), \\ (u_t^m)\ln(1+(u_t^m)^2) \rightharpoonup \chi \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega)), \\ (u_t^m)\ln(1+(u_t^m)^2) \rightharpoonup \chi \quad \text{weakly in } L^2(\Omega\times(0,T)). \end{cases} \tag{3.16}$$

An application of the Aubin–Lions lemma (Lemma 1) yields

$$u^m \to u, \quad \text{strongly in } L^2(0,T;H_0^1(\Omega)), \tag{3.17}$$

and

$$u_t^m \to u_t, \quad \text{strongly in } L^2(\Omega\times(0,T)). \tag{3.18}$$

Consequently,

$$u^m \to u, \quad \text{a.e. in } \Omega\times(0,T), \tag{3.19}$$

and

$$u_t^m \to u_t, \quad \text{a.e. in } \Omega\times(0,T). \tag{3.20}$$

Using (3.20) and the continuity of the map $s \mapsto s\ln(1+s^2)$, we obtain

$$u_t^m \ln(1+(u_t^m)^2) \to u_t \ln(1+(u_t)^2) \quad \text{a.e. in } \Omega\times(0,T). \tag{3.21}$$

Combining (3.21) and (3.14) $\left(u_t^m \ln(1+(u_t^m)^2) \text{ is uniformly bounded in } L^2(0,T;L^2(\Omega))\right)$ and applying Lions lemma (Lemma 2), we obtain

$$u_t^m \ln(1+(u_t^m)^2) \rightharpoonup u_t \ln(1+(u_t)^2) \;\; \text{weakly in } L^2(\Omega\times(0,T)). \tag{3.22}$$

Convergences (3.3), (3.16), and (3.22) are sufficient to pass to the limit in (3.2), as $m \to +\infty$, and get, for any $w \in V_m$,

$$\int_\Omega u_{tt}(x,t)w(x)dx + \int_\Omega \nabla u(x,t).\nabla w(x)dx + \int_\Omega u_t \ln(1+(u_t)^2)wdx = 0. \tag{3.23}$$

This remains true for any w in $H_0^1(\Omega)$, and, consequently, integration by parts leads to

$$u_{tt} - \Delta u + u_t \ln(1+(u_t)^2) = 0, \;\text{ in } L^2(\Omega). \tag{3.24}$$

To recover the initial data, we first note that, by the above convergences together with the uniform estimates (3.10)–(3.15),

$$(u^m) \quad \text{is bounded in} \quad L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega)),$$

while

$$(u_t^m) \quad \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)).$$

Since

$$H^2(\Omega)\cap H_0^1(\Omega) \hookrightarrow\hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega),$$

where the first embedding is compact and the second one is continuous, Simon's compactness theorem (or the Aubin–Lions lemma in its continuous version) yields, up to a subsequence,

$$u^m \longrightarrow u \quad \text{strongly in } C([0,T];H_0^1(\Omega)).$$

Consequently,

$$u^m(\cdot,0) \to u(\cdot,0) \quad \text{strongly in } H_0^1(\Omega). \tag{3.25}$$

Also, using (3.3), we have

$$u^m(x,0) = u_0^m(x) \to u_0(x) \quad \text{strongly in } H^2(\Omega) \cap H_0^1(\Omega). \tag{3.26}$$

Combining (3.25) and (3.26) and using the uniqueness of limit, we have $u(x,0) = u_0(x)$.
Similarly, by exploiting the convergence

$$\begin{cases} u_t^m \rightharpoonup u_t & \text{weakly * in } L^\infty(0,T;H_0^1(\Omega)),\\ u_{tt}^m \rightharpoonup u_{tt} & \text{weakly * in } L^\infty(0,T;L^2(\Omega)),\end{cases}$$

we get

$$u_t^m \to u_t \quad \text{strongly in } C([0,T];L^2(\Omega)),$$

thus,

$$u_t^m(x,0) \to u_t(x,0) \quad \text{strongly in } L^2(\Omega).$$

From (3.3), we have

$$u_t^m(x,0) = u_1^m(x) \to u_1(x) \quad \text{strongly in } H_0^1(\Omega),$$

which implies that $u_t(x,0) = u_1(x)$.
For uniqueness, we suppose that problem (1.2) has two solutions (u, v). So, $z = u - v$, satisfies the following problem

$$\begin{cases} z_{tt} - \Delta z + u_t \ln(1+u_t^2) - v_t \ln(1+v_t^2) = 0, & (x,t) \in \Omega \times (0,T),\\ z(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0 \\ z(x,0) = 0, \quad z_t(x,0) = 0, \quad x \in \Omega, \end{cases} \tag{3.27}$$

Multiplying (3.27) by z_t and integrating over $\Omega \times (0,t)$, we obtain

$$\frac{1}{2}\int_\Omega z_t^2\,dx + \frac{1}{2}\int_\Omega |\nabla z|^2\,dx + \int_0^t \int_\Omega (A(u_t) - A(v_t))\,(u_t - v_t)\,dx\,ds = 0, \tag{3.28}$$

where $A(s) = s\ln(1+s^2)$. Using the monotonicity of A, we infer

$$\|z_t\|_2^2 + \|\nabla z\|_2^2 = 0, \tag{3.29}$$

which gives $z = 0$ in Ω, using (3.27); hence $u = v$. This completes the proof of Theorem 1. □

4. Stability

In this section, we prove a decay estimate for the energy associated with problem (1.2). For this purpose, first we establish the following result:

Lemma 3. *The solution of* (1.2) *satisfies, for a positive constant* C,

$$\|u_t\|_2^2 \leq C(-E'(t))^{\frac{1}{2}} + C(-E'(t)). \tag{4.1}$$

Proof. We define the following partition of Ω

$$\Omega^+ = \{x \in \Omega \mid |u_t(x)| > 1\} \quad \text{and} \quad \Omega^- = \{x \in \Omega \mid |u_t(x)| \leq 1\}.$$

By the algebraic inequality (2.1), with $m = 1$ and $s = u_t^2$, we obtain

$$\begin{aligned} \int_{\Omega^-} u_t^2 dx \leq C\left(\int_{\Omega^-} u_t^4 dx\right)^{\frac{1}{2}} = C\left(\int_{\Omega^-} (u_t^2)^2 dx\right)^{\frac{1}{2}} &\leq C\left(\int_{\Omega^-} u_t^2 \ln(1+u_t^2) dx\right)^{\frac{1}{2}} \\ &\leq C\left(-E'(t)\right)^{\frac{1}{2}}. \end{aligned} \tag{4.2}$$

Also, we have

$$\begin{aligned} \int_{\Omega^+} u_t^2\, dx &= \int_{\Omega^+} u_t^2 \cdot \frac{\ln(1+u_t^2)}{\ln(1+u_t^2)}\, dx \\ &\leq \frac{1}{\ln 2} \int_{\Omega^+} u_t^2 \ln(1+u_t^2)\, dx \leq C(-E'(t)). \end{aligned} \tag{4.3}$$

Then, (4.1) is established by putting together (4.2) and (4.3).

□

Theorem 2. *Let* $(u_0, u_1) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega)$. *Then, there exists a positive constant c, such that the energy functional satisfies, for all* $t > 0$,

$$E(t) \leq \frac{c}{t}. \tag{4.4}$$

Proof. Define the functional

$$L(t) = NE(t) + \int_\Omega u u_t\, dx,$$

where $N > 0$ is a constant to be determined. A standard argument shows that, for N sufficiently large, $L(t)$ is equivalent to the energy $E(t)$, namely,

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t), \tag{4.5}$$

for some positive constants α_1 and α_2.

Indeed, by Young's inequality and Poincaré's inequality,

$$\left|\int_\Omega u u_t\, dx\right| \leq \frac{\varepsilon_1}{2}\|u_t\|_2^2 + \frac{c_p}{2\varepsilon_1}\|\nabla u\|_2^2,$$

where $c_p > 0$ denotes the Poincaré constant. Consequently,

$$L(t) \leq NE(t) + \frac{\varepsilon_1}{2}\|u_t\|_2^2 + \frac{c_p}{2\varepsilon_1}\|\nabla u\|_2^2 \leq \alpha_2 E(t),$$

and

$$L(t) \geq NE(t) - \frac{\varepsilon_1}{2}\|u_t\|_2^2 - \frac{c_p}{2\varepsilon_1}\|\nabla u\|_2^2 \geq \alpha_1 E(t),$$

provided that N is chosen sufficiently large. Now, we differentiate $L(t)$ and use (1.2), to obtain

$$\begin{aligned} L'(t) &= NE'(t) + \|u_t\|_2^2 - \|\nabla u\|_2^2 - \int_\Omega uu_t \ln(1+(u_t)^2)dx \\ &= NE'(t) - 2E(t) + 2\|u_t\|_2^2 - \int_\Omega uu_t \ln(1+(u_t)^2)dx. \end{aligned} \tag{4.6}$$

To estimate the last term of (4.6), we use Young's and Poincaré's inequalities, to arrive at

$$\begin{aligned} \int_{\Omega^-} uu_t \ln(1+u_t^2)dx &\le \epsilon c_p\|\nabla u\|^2 + C(\epsilon)\int_{\Omega^-} u_t^2 \ln^2(1+u_t^2)dx \\ &= \epsilon c_p\|\nabla u\|^2 + C(\epsilon)\int_{\Omega^-} \left[u_t^2 \ln(1+u_t^2)\right]\ln(1+u_t^2)dx \\ &\le \epsilon c_p\|\nabla u\|^2 + C(\epsilon)\int_{\Omega^-} u_t^2 \ln(1+u_t^2)dx \\ &= \epsilon c_p\|\nabla u\|^2 + C(\epsilon)(-E'(t)) \\ &\le 2\epsilon c_p E(t) + C(\epsilon)(-E'(t)). \end{aligned} \tag{4.7}$$

Next, let γ be such that $2 < \gamma \le \frac{2N}{N-2}$ if $N \ge 3$, and $\gamma > 2$ if $N = 1, 2$. So, by means of Hölder's inequality and (2.2), with $s = u_t^2$ and $\rho = \frac{\gamma}{2} - 1$, we have

$$\begin{aligned} \int_{\Omega^+} uu_t \ln(1+u_t^2)\,dx &\le \left(\int_{\Omega^+} u^\gamma\,dx\right)^{\frac{1}{\gamma}} \left(\int_{\Omega^+} (|u_t|\ln(1+u_t^2))^{\frac{\gamma}{\gamma-1}}\,dx\right)^{\frac{\gamma-1}{\gamma}} \\ &= \left(\int_{\Omega^+} u^\gamma\,dx\right)^{\frac{1}{\gamma}} \left(\int_{\Omega^+} (|u_t|)^{\frac{\gamma-2}{\gamma-1}}(|u_t|^2\ln(1+u_t^2))^{\frac{1}{\gamma-1}}\ln(1+u_t^2)\,dx\right)^{\frac{\gamma-1}{\gamma}} \\ &\le C\left(\int_{\Omega^+} u^\gamma\,dx\right)^{\frac{1}{\gamma}} \left(\int_{\Omega^+} (|u_t|)^{\frac{\gamma-2}{\gamma-1}}\left((u_t^2)^{1+\rho}\right)^{\frac{1}{\gamma-1}}\ln(1+u_t^2)\,dx\right)^{\frac{\gamma-1}{\gamma}}. \\ &= C\left(\int_{\Omega^+} u^\gamma\,dx\right)^{\frac{1}{\gamma}} \left(\int_{\Omega^+} u_t^2\ln(1+u_t^2)\,dx\right)^{\frac{\gamma-1}{\gamma}} \\ &\le C\|u\|_\gamma(-E'(t))^{\frac{\gamma-1}{\gamma}} \\ &\le C\|\nabla u\|_2(-E'(t))^{\frac{\gamma-1}{\gamma}} \\ &\le CE^{\frac{1}{2}}(0)(-E'(t))^{\frac{\gamma-1}{\gamma}}. \end{aligned} \tag{4.8}$$

Combining (4.1), (4.6), (4.7), and (4.8) leads to

$$L'(t) \le (N - C(\epsilon) - C)E'(t) - 2(1-c_p\epsilon)E(t) + C(-E'(t))^{\frac{1}{2}} + C(-E'(t))^{\frac{\gamma-1}{\gamma}}. \tag{4.9}$$

We choose $\epsilon > 0$ so small that $m := 1 - \epsilon c_p > 0$, then N large enough so that $N - C(\epsilon) \ge 0$ and (4.5) remains valid. Therefore, we get

$$L'(t) \le -2mE(t) + C(-E'(t))^{\frac{1}{2}} + C(-E'(t))^{\frac{\gamma-1}{\gamma}}. \tag{4.10}$$

Multiply (4.10) by $E(t)$ and apply Young's inequality to get

$$\begin{aligned} E(t)L'(t) &\le -2mE^2(t) + \delta E^2(t) + c_\delta(-E'(t)) + \delta E^\gamma(t) + c_\delta(-E'(t)) \\ &\le -\left(2m - \delta - \delta E^{\gamma-2}(0)\right)E^2(t) + c(-E'(t)). \end{aligned} \tag{4.11}$$

Choosing δ small enough and letting $\mathcal{L} = EL + cE \sim E$, we have, for some positive constant m_0,

$$\mathcal{L}'(t) \leq -m_0 \mathcal{L}^2(t). \tag{4.12}$$

Simple integration gives, for some positive constant m_1,

$$\mathcal{L}(t) \leq \frac{m_1}{t},$$

then, recalling the equivalence $\mathcal{L} \sim E$, (4.4) is established. □

Remark 2. *The polynomial decay established in Theorem 2 shows that the logarithmic damping mechanism is sufficiently strong to dissipate the total energy of the system over time. From an engineering perspective, this guarantees that oscillations are progressively attenuated and that the underlying structure approaches its equilibrium configuration. Although the decay is slower than the exponential decay obtained for stronger damping mechanisms, it accurately reflects the weaker nature of logarithmic dissipation and is consistent with systems in which moderate energy absorption is desired while preserving the dynamic response.*

Remark 3. *Theorem 2 establishes the polynomial decay estimate*

$$E(t) \leq \frac{c}{t}, \qquad t > 0.$$

Although the present analysis does not provide a rigorous proof of the optimality of this decay rate, we conjecture that it is optimal for the logarithmically damped wave equation considered in this work. Proving that no faster uniform decay can occur under the present assumptions, remains an interesting and challenging open problem that deserves further investigation.

Remark 4. *Although the present work is purely theoretical and does not include experimental validation, the obtained results are qualitatively consistent with the behavior observed in many dissipative mechanical and structural systems, where nonlinear damping gradually suppresses vibrations and drives the system toward equilibrium. The polynomial decay established in this work provides a mathematical characterization of this stabilization process.*

5. Conclusions

In this work, we investigated a nonlinear wave equation with logarithmic damping. We established, for the first time, the global existence and uniqueness of strong solutions using the Faedo–Galerkin approximation method together with monotonicity arguments adapted to the logarithmic damping term. We also derived a polynomial decay estimate for the associated energy by developing a multiplier technique specifically suited to the non-polynomial structure of the logarithmic dissipation. These results provide a rigorous analytical foundation for the study of wave equations with logarithmic damping and contribute to a better understanding of their long-time behavior.

From an engineering perspective, the obtained results provide theoretical support for the use of logarithmic damping as an effective stabilization mechanism in vibrating structures and wave propagation models. The global well-posedness guarantees the mathematical reliability of the model, while the polynomial decay confirms that the system dissipates energy and asymptotically approaches equilibrium.

Future research will focus on extending the present analysis to more general hyperbolic systems involving logarithmic damping, including models with memory effects, thermoelasticity, nonlinear source terms, boundary feedback, and time delays. It would also be of interest to investigate optimal decay rates and to develop numerical methods capable of validating the theoretical predictions obtained in this work.

Author contributions

Mohammad M. Al-Gharabli and Adel M. Al-Mahdi: Study design, conducted the research, wrote the first draft of the manuscript; Bashayer Aldossary: Development of the study, critically revised, edited the manuscript; Mohammad M. Al-Gharabli: Supervised the project. All authors read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to acknowledge the support provided by King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia. The support provided by the Interdisciplinary Research Center for Construction & Building Materials (IRC-CBM) at King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia is also greatly acknowledged.

Funding

This work is funded by KFUPM under Project (IN26021).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. I. Lasiecka, R. Triggiani, *Control theory for partial differential equations: continuous and approximation theories*, Cambridge: Cambridge University Press, 2013. https://doi.org/10.1017/CBO9780511574801
2. I. Chueshov, I. Lasiecka, *Long-time behavior of second-order evolution equations with nonlinear damping*, Providence: American Mathematical Society, 2008. https://doi.org/10.1090/memo/0912
3. I. Chueshov, I. Lasiecka, *Von Karman evolution equations: well-posedness and long-time dynamics*, New York: Springer, 2010. https://doi.org/10.1007/978-0-387-87712-9

4. S. A. Messaoudi, A. A. Talahmeh, On wave equation: review and recent results, *Arab. J. Math.*, **7** (2018), 113–145. https://doi.org/10.1007/s40065-017-0190-4

5. M. Nakao, Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative term, *J. Math. Anal. Appl*, **58** (1977), 336–343. https://doi.org/10.1016/0022-247x(77)90211-6

6. E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, *Commun. Part. Diff. Eq.*, **15** (1990), 205–235. https://doi.org/10.1080/03605309908820684

7. V. Komornik, *Exact controllability and stabilization: the multiplier method*, Paris: Wiley, 1994.

8. I. Lasiecka, D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping, *Differ. Integral Equ.*, **6** (1993), 507–533. https://doi.org/10.57262/die/1370378427

9. P. Martinez, A new method to obtain decay rate estimates for dissipative systems, *ESAIM-Control, Optimisation and Calculus of Variations*, **4** (1999), 419–444. https://doi.org/10.1051/cocv:1999116

10. W.-J. Liu, E. Zuazua, Decay rates for dissipative wave equations, *Ric. Mat.*, **48** (1999), 61–75.

11. F. Alabau-Boussouira, Convexity and weighted integral inequalities for energy decay rates of nonlinear dissipative hyperbolic systems, *Appl. Math. Optim.*, **51** (2005), 61–105. https://doi.org/10.1007/s00245

12. M. M. Cavalcanti, V. N. D. Cavalcanti, I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction, *J. Differ. Equations*, **236** (2007), 407–459. https://doi.org/10.1016/j.jde.2007.02.004

13. I. Lasiecka, D. Toundykov, Stability of higher-level energy norms of strong solutions to a wave equation with localized nonlinear damping and a nonlinear source term, *Control Cybern.*, **36** (2007), 681–710.

14. X. Gao, J. Liu, and G. Wang, "On a (2+1)-dimensional generalized variable-coefficient date–jimbo–kashiwara–miwa equation with the marine-engineering and ocean-dynamics applications," *China Ocean Engineering*, vol. 40, pp. 695–703, 2026. https://doi.org/10.1007/s13344-026-0069-2.

15. X. Xin, Y. Liu, Y. Xia, and H. Liu, "Integrability, darboux transformation and exact solutions for nonlocal couplings of akns equations," *Applied Mathematics Letters*, vol. 119, p. 107209, 2021. https://doi.org/10.1016/j.aml.2021.107209.

16. D. H. Li, C. X. Zhang, H. W. Zhang, Energy decay estimates for the wave equation with logarithmic feedback, *Math. Method. Appl. Sci.*, **48** (2025), 10110–10113. https://doi.org/10.1002/mma.10871

17. S. A. Messaoudi, A. M. Al-Mahdi, Well-posedness and asymptotic stability of a one-dimensional Timoshenko system with a logarithmic damping, *Discrete Cont. Dyn. S*, **21** (2025), 1–16. https://doi.org/10.3934/dcdss.2025155

18. S. A. Messaoudi, A. M. Al-Mahdi, M. Alotaibi, Well-posedness, exponential and polynomial decay for a Timoshenko system with logarithmic nonlinear damping: The cases of equal & nonequal-speed propagation, *Evol. Equ. Control The.*, **21** (2026), 223–243. https://doi.org/10.3934/eect.2026040

19. A. M. Al-Mahdi, M. M. Al-Gharabli, On the well-posedness and stability of piezoelectric beams with magnetic effects, logarithmic damping, and Fourier-type thermal conduction, *J. Therm. Stresses*, **49** (2026), 712–727. https://doi.org/10.1080/01495739.2025.2603426

20. A. M. Al-Mahdi, M. M. Al-Gharabli, T. A. Apalara, Existence, uniqueness, and polynomial decay results for a laminated beam with a logarithmic damping, *Ric. Mat.*, **2026** (2026), 1–21. https://doi.org/10.1007/s11587-026-01071-2

21. A. M. Al-Mahdi, M. M. Al-Gharabli, On the existence and long-time behavior of solutions to swelling models with weak logarithmic damping mechanism: Optimal polynomial decay rate, *Math. Method. Appl. Sci.*, **49** (2026), 8909–8921. https://doi.org/10.1002/mma.70507

22. J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Paris: Gauthier-Villars, 1969.