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Research article

On the composition of discrete distributions

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Abstract: In this paper, we introduced a parametric family of discrete probability distributions that had a simpler probabilistic construction than the classical Lagrangian family of distributions, obtained by composing some counting distributions with bounded support with the natural exponential family. This new family included, as a special case, the natural exponential family of distributions and consequently some of the well-known classical discrete distributions in the statistical literature, such as the Poisson, binomial, negative binomial, Hyper-Poisson, and others. A useful generalization of this family, encompassing quasi-binomial and generalized Poisson distributions, was studied. A few stochastic interpretations of these families were described. An extension of the proposed family that induced a dispersion parameter was also discussed. Finally, an extensive comparison with several standard count-data models illustrated the flexibility and competitiveness of the proposed family.

Keywords: decay transformation; Lagrangian family of distributions; Conway-Maxwell- Poisson distribution; probability generating function; dispersion index
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1. Introduction

The discrete Lagrangian probability distributions have been systematically studied in a number of papers by Consul and Shenton [1, 2]. Jain and Consul [3] used the Lagrange transformation $z = ug(z)$ to obtain Lagrangian negative binomial distribution by considering $g(z)$, where $g(z)$ is assumed to be a probability generating function (pgf); i.e., as $g(z) = \sum_{x=0}^{\infty} z^x p_x$ for some probability mass function p_x with support in $\mathbb{N} = \{0, 1, \dots\}$. The Lagrangian Poisson distribution was defined and studied by Consul and Jain [2]. Consul [4] showed that the Lagrangian Poisson distribution has been studied for its modes of genesis, properties, estimation, and applications under the title of a generalized Poisson distribution. The Lagrangian logarithmic distribution was defined by [5]. Consul [6] has shown that $g(z)$ need not be a pgf for obtaining the Lagrangian distributions. The class of Lagrangian probability distributions can be divided into three subclasses: (i) basic Lagrangian distributions, (ii) delta Lagrangian distributions, and (iii) general Lagrangian distributions, according to their probabilistic structure.

Recall that the general family of Lagrangian distributions of the second kind (see [7] and [8]) is given by

$$f(z) = \left(1 - \frac{zg^{Pime}(z)}{g(z)}\right) \sum_{x \in \mathcal{X}} \frac{u^x}{x!} \left[\frac{\partial^x}{\partial z} \{[g(z)]^x f(z)\}\right]_{z=0}, \tag{1.1}$$

where $f(z)$ and $g(z)$ are two probability generating functions defined on the nonnegative integers, with $g(0) \neq 0$, $\mathcal{X}$ being a subset of the nonnegative integers. For a complete reading about this family of distributions, see the text [8]. The corresponding probability mass function of the basic Lagrangian family of distributions, given by taking $f(z) = 1$, is given by

$$P(X = x) = \frac{1}{x!} \left[D^{x-1} (g(z))^x\right]_{z=0}, \quad x = 0, 1, \ldots,$$

where D represents the symbol of the derivative (see [8] for details). Most classical discrete statistical distributions belong to this family, and new ones can be derived from it. For example, recently [9] proposed the Lagrangian noncentral negative binomial distribution, [10] the Haight-Poisson distribution, and [11] the Lagrangian zero-truncated Poisson distribution.

In this paper, we introduce a parametric family of discrete probability distributions in the spirit of the well-known Lagrangian family [12–14], based on a novel construction arising from the natural exponential family. The proposed family admits a simple stochastic interpretation and a tractable probability generating function representation, in contrast to the predominantly algebraic construction underlying the classical Lagrangian family, while still encompassing several important discrete distributions as special cases. In particular, it contains the natural exponential family as a subclass and therefore includes many classical discrete distributions. We further develop extensions of the proposed family, leading to additional classes of discrete distributions.

The rest of the paper is structured as follows. The proposed parametric family of distributions is discussed in Section 2 along with its generating function, particular cases, moments, and a stochastic formulation. In Section 3 we provide a generalization of the proposed family, its particular cases, and an interesting relationship with the generalized Poisson distribution of [3]. In Section 4, we derive an extended version of the proposed family by introducing a dispersion parameter and present a few of its characteristics. Section 5 presents a general result that encompasses the cases studied previously and provides a numerical example. Finally, the article ends with concluding remarks in Section 7.
The main contributions of this paper are summarized as follows:

1. We introduce a new parametric family of discrete probability distributions obtained by compounding bounded-support distributions with members of the natural exponential family.
2. We derive a general extension based on quasi-binomial compounding mechanisms, thereby encompassing several important discrete models.

3. We develop a further extension based on exponential dispersion models and establish a general compounding theorem showing that the bounded-support kernel can be replaced by an arbitrary discrete distribution.
4. We investigate several probabilistic properties of the proposed family and illustrate its practical usefulness through an application to real count data.

2. The new family of distributions : $\mathcal{F}(\theta, p)$

Most of the classical continuous and discrete distributions used in the statistical literature belong to the natural exponential family (henceforth, NEF). In the discrete case, for random variables X taking values in some space such as $\mathbb{N}_0 = \{0, 1, \dots\}$, these distributions are the Poisson, binomial, negative binomial, and logarithmic series, among others. We say that a discrete random variable N relative to the canonical parameter θ is distributed according to the NEF of distributions if its probability mass function is given by

$$P(N = n) = p_n = q(n) \exp\left[-\theta\, n - \log(\eta(\theta))\right], \tag{2.1}$$

where $\eta(\theta)$ is the appropriate constant of normalization given by

$$\eta(\theta) = \sum_n q(n) \exp(-\theta\, n) < \infty, \tag{2.2}$$

while $q(n)$ is a function of $n \in \mathbb{N}_0$.

In advance, we will write $N \sim \text{NEF}(\theta)$ to denote that a random variable N follows the probability mass function (pmf) given in (2.1). The NEF is a broad class of distributions that has been studied by numerous authors (see, for example, [15], [16] and [17], among others). Excellent discussions of the NEF can be found in [18], [19], [20] and [21]. Their probability generating function is given by

$$G_X(z) = \frac{\eta(\theta - \log z)}{\eta(\theta)}, \quad 0 < z < 1. \tag{2.3}$$

We will use the NEF(θ) of distributions defined above to get a family of probability distributions that has a similar appearance to the family of Lagrangian distributions given in (1.1). This is provided in the following result.

Theorem 2.1. *Let X be a discrete random variable defined on the nonnegative integers, then*

$$P(X = x) = \frac{p^x}{x!\eta(\theta)} \left\{ \frac{d^x}{dz^x} \eta(\theta - \log z) \right\}_{z=1-p}, \quad x = 0, 1, \dots \tag{2.4}$$

represents a proper probability mass function for $0 < p < 1$*, being* $\eta(\theta)$ *as in* (2.2).

Proof. Let $N \sim \text{NEF}(\theta)$ with probability mass function given in Eq. (2.1), and suppose that, conditionally on $N = n$, $X \mid N = n \sim \text{Bin}(n, p)$. Then, by the law of total probability,

$$P(X = x) = \sum_{n=x}^{\infty} \binom{n}{x} p^x (1 - p)^{n-x} p_n.$$

Substituting the expression of p_n yields

$$P(X=x)=\frac{p^x}{\eta(\theta)}\sum_{n=x}^{\infty}\binom{n}{x}(1-p)^{n-x}q(n)e^{-\theta n}.$$

Now, using the identity

$$\frac{d^x}{dz^x}z^n=\frac{n!}{(n-x)!}z^{n-x}=x!\binom{n}{x}z^{n-x},$$

we obtain

$$P(X=x)=\frac{p^x}{x!\eta(\theta)}\,\frac{d^x}{dz^x}\left(\sum_{n=0}^{\infty}q(n)e^{-\theta n}z^n\right)\Bigg|_{z=1-p}.$$

Since

$$\sum_{n=0}^{\infty}q(n)e^{-\theta n}z^n=\eta(\theta-\log z),$$

it follows that

$$P(X=x)=\frac{p^x}{x!\eta(\theta)}\,\frac{d^x}{dz^x}\eta(\theta-\log z)\Big|_{z=1-p},$$

which is exactly the expression given in (2.4). Since it is obtained as a mixture of proper binomial distributions, it defines a proper probability mass function. □

When a discrete random variable X follows the pmf given in (2.4), we will write $X\sim\mathcal{F}(\theta,p)$.
The next result provides an alternative expression for (2.4), which may be useful for understanding certain features of the new family.

Theorem 2.2. *The pmf given in* (2.4) *can be rewritten as*

$$P(X=x)=\sum_{n=x}^{\infty}\binom{n}{x}p^x(1-p)^{n-x}p_n,\quad x=0,1,2,\dots \tag{2.5}$$

Proof. Let $G_N(z)$ be the pgf of the random variable N, defined on the nonnegative integers, with pmf p_n. Then, we have that

$$\begin{aligned}P(X=x)&=\frac{p^x}{x!}\left\{\frac{d^x}{dz^x}\,G_N(z)\right\}_{z=1-p}=\frac{p^x}{x!}\left\{\frac{d^x}{dz^x}\sum_{n=0}^{\infty}z^n p_n\right\}_{z=1-p}\\&=\sum_{n=x}^{\infty}\binom{n}{x}p^x(1-p)^{n-x}p_n,\end{aligned}$$

where we have used the expression given in (2.3). □

Expression (2.4) is reminiscent of the discrete Lagrangian family of distributions ([1], [7] and [22]; among others). Furthermore, this expression can also be considered a countable mixture distribution, with mixing distribution p_n. That is, if we denote $Bi(n,p)$ the binomial distribution with parameters

$n > 0$ and $p \in (0, 1)$, expression (2.4) results as $Bi(n, p) \bigwedge_n p_n$. This representation facilitates, for example, the computation of the distribution's moments. It also provides an intuitive probabilistic interpretation for the induced dispersion properties. Although the infinite mixture obtained by compounding a discrete or continuous distribution with a continuous distribution is the most useful application of mixture, it is also possible to obtain new distributions by compounding a discrete distribution with another discrete distribution. For details, see Chapter 8 in [7]. Furthermore, the mixture considered here differs from the countable mixtures treated in [7].

2.1. Probability generating function

Here we provide the pgf of the new family of distributions.

Proposition 2.1. *Let $X \sim \mathcal{F}(\theta, p)$ with pmf given in* (2.4)*. Then, the probability generating function of X is given by*

$$G_X(z) = \frac{\eta(\theta - \log(1 - p + pz))}{\eta(\theta)}, \quad 0 < z < 1. \tag{2.6}$$

Proof. Since the pgf of the binomial distribution is given by $G_X(z) = (1 - p + pz)^n$, using (2.5), the pgf of the random variable following the pmf given in (2.4) can be computed in the following way:

$$\begin{aligned} G_X(z) &= \sum_{x=0}^{\infty} z^x \sum_{n=x}^{\infty} \binom{n}{x} p^x (1-p)^{n-x} q(n) \exp\left[-\theta n - \log(\eta(\theta))\right] \\ &= \sum_{n=0}^{\infty} \left[\sum_{x=0}^{n} z^x \binom{n}{x} p^x (1-p)^{n-x}\right] q(n) \exp\left[-\theta n - \log(\eta(\theta))\right] \\ &= \sum_{n=0}^{\infty} (1 - p + pz)^n q(n) \exp\left[-\theta n - \log(\eta(\theta))\right], \end{aligned}$$

from which we obtain the result after simple algebra. □

Clearly, $\mathcal{F}(\theta, p)$ is a kind of generalization of the $Bi(n, p)$ distribution when $N = n$ follows NEF(θ) in (2.1) or random sum of $N = n$ independent $Bi(n, p)$ when $N \sim$ NEF(θ) in (2.1). From (2.6) we get (2.3) by taking $p = 1$. Thus, the family includes the NEF(θ) of distributions as a particular case.

2.2. Special cases of the family

For different choices of $\eta(\cdot)$, we can obtain different probability distributions, and weighted extensions of some well-known distributions as special cases. The following are illustrative examples.

Example 2.1. *Taking in* (2.6) $\eta(\theta) = \exp(-\theta r)$*,* $r > 0$*, it is easy to see that the binomial distribution with parameters* $r > 0$ *and* $0 < p < 1$ *is obtained.*

Example 2.2. *Also taking in* (2.6) $\eta(\theta) = (1 + \exp(-\theta))^r$*,* $r > 0$*, we get the binomial distribution with parameters r and* $p/(1 + \exp(\theta))$*.*

Example 2.3. *Taking in* (2.6) $\eta(\theta) = \exp(\exp(-\theta))$*, we get the Poisson distribution with parameters* $\lambda > 0$ *where* $\lambda = p\ \exp(-\theta)$*.*

Example 2.4. *Taking in* (2.6) $\eta(\theta) = (1 - \exp(-\theta))^{-r}$*,* $r > 0$*, we get the negative binomial distribution with parameters* r *and* $p/(p + \exp(\theta) - 1)$*. For* $r = 1$*, the geometric distribution also appears as a special case of the preceding one.*

Example 2.5. *Taking in* (2.6) $\eta(\theta) = S(\exp(-\theta), \alpha)$*,* $r > 0$*, we obtain the following pgf and pmf*

$$G_X(z) = \frac{S(\exp(-\theta)(1 - p + pz), \alpha)}{S(\exp(-\theta), \alpha)} = \frac{S(\lambda z + \xi), \alpha)}{S(\lambda + \xi, \alpha)}.$$

$$Pr(X = x) = \sum_{i \geq x}^{\infty} \binom{i}{x} \frac{\lambda^x \xi^{i-x}}{(i!)^{\alpha} S(\lambda, \alpha)},$$

where $S(a, b) = \sum_{i=0}^{\infty} \frac{a^i}{(i!)^b}$*,* $\lambda = p\ \exp(-\theta)$*,* $\xi = \exp(-\theta) - \lambda$*.*

This distribution reduces to the Conway-Maxwell Poisson distribution [23] when $\xi \to 0$*, that is, when* $p \to 1$*, and as such can be seen as a generalization of the same. It may be referred to as a quasi-Conway-Maxwell distribution. Unlike the Conway–Maxwell Poisson distribution, the proposed family admits a direct stochastic representation through compounding mechanisms.*

Example 2.6. *Take* $\eta(\theta) = {}_1F_1(\alpha, \beta, \exp(-\theta))$*,* $\alpha > 0$*,* $\beta > 0$*, where* ${}_1F_1(a, c, s)$ *is the Kummer confluent hypergeometric function [24] given by*

$${}_1F_1(a, b, s) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{s^k}{k!},$$

where $(a)_k = a(a + 1) \dots (a + k - 1)$ *represents the Pochhammer function. Now, it is simple to get the following pgf and pmf*

$$G_X(z) = \frac{{}_1F_1(\alpha, \beta, (1 - p + pz)\exp(-\theta))}{{}_1F_1(\alpha, \beta, \exp(-\theta))}. \tag{2.7}$$

$$P(X = x) = \frac{(\alpha)_x}{(\beta)_x} \frac{p^x \exp(-\theta x)}{x!} \frac{{}_1F_1(\alpha + x, \beta + x, (1 - p)\exp(-\theta))}{{}_1F_1(\alpha, \beta, \exp(-\theta))}. \tag{2.8}$$

From here, we can obtain the following:

(i) *Taking* $\alpha = \beta$ *and remembering that* ${}_1F_1(\alpha, \alpha, s) = \exp(s)$*, we get from* (2.8)*:*

$$P(X = x) = \frac{(p\exp(-\theta))^x\ \exp(-p\exp(-\theta))}{x!},$$

which is the Poisson distribution with parameter $p\ \exp(-\theta)$*.*

(ii) *Taking* $\exp(-\theta) = \lambda$ *and* $p \to 1$*, we get the confluent hypergeometric family from* (2.7) *as*

$$G_X(z) = \frac{{}_1F_1(\alpha, \beta, z\lambda)}{{}_1F_1(\alpha, \beta, \lambda)}.$$

This is a special case of the family of distributions provided by [25]. See also [7].

(*iii*) *Taking* $\exp(-\theta) = \lambda$ *and* $\alpha = 1$ *we get the hyper-Poisson distribution [26] with pgf*

$$G_X(z) = \frac{{}_1F_1(1, \beta, z\lambda)}{{}_1F_1(1, \beta, \lambda)}.$$

Therefore, the pmf in (2.8) can be considered as a further generalization of the different extensions of the Poisson distribution existing in the statistical literature, such as the confluent hypergeometric distributions of [27], the hyper-Poisson distribution of [26], and the Conway-Maxwell distribution [23].

An expression for the r^{th} factorial moment of this distribution in (2.8) can be easily derived as

$$\mathbb{E}(X_{(r)}) = \frac{(\alpha)_r}{(\beta)_r} \frac{p^r \exp(-\theta x)}{(p)_r} \frac{{}_1F_1(\alpha + r, \beta + r, \exp(-\theta))}{{}_1F_1(\alpha, \beta, \exp(-\theta))}.$$

2.3. Moments and dispersion index

Factorial moments of the random variable following the pmf given in (2.4) can be seen easily from (2.6) as

$$\mathbb{E}(X_{(r)}) = p^r\, \mathbb{E}(N_{(r)}). \tag{2.9}$$

Various other moments and related measures can be easily derived from the above formula. Since

$$\mathbb{E}(N) = -\frac{\eta'(\theta)}{\eta(\theta)},$$

the mean and variance are respectively given by

$$\begin{aligned} \mathbb{E}(X) &= -p\frac{\eta'(\theta)}{\eta(\theta)} = p\,\mathbb{E}(N), \\ \mathrm{var}(X) &= p^2\,\mathrm{var}(N) + p(1-p)\mathbb{E}(N), \end{aligned}$$

Observe that, if $\mathbb{E}(N) = var(N)$, then $\mathbb{E}(X) = var(X)$.

The index of dispersion (ID) of X can be expressed in terms of that of N as

$$ID(X) = p\, ID(N) + (1-p).$$

Thus, $ID(X) <>= 1$ according to $ID(N) <>= 1$. As $p \to 0(1), ID(X) \to 1(ID(N))$. Therefore, the dispersion structure of the proposed family is directly inherited from the parent distribution N. In particular, the model preserves overdispersion and underdispersion properties.

3. $\mathcal{F}(\theta, p, \phi)$: A generalization of $\mathcal{F}(\theta, p)$

The following result provides a more general result than the one provided in Theorem 2.1.

Theorem 3.1. *Let X be a discrete random variable taking values on the nonnegative integers, then*

$$P(X = x) = \frac{p\,(p + x\phi)^{x-1}}{\eta(\theta)x!} \left\{\frac{d^x}{dz^x}\eta(\theta - \log z)\right\}_{z=1-p-x\phi}, \quad x = 0, 1, \ldots \tag{3.1}$$

represents a proper probability mass function for $0 < p < 1$ *and* $-p/n < \phi < (1-p)/n$, $\eta(\theta)$ *being as in* (2.2).

Proof. Again, using (3.1) we get that

$$\begin{aligned}
\sum_{x=0}^{\infty} P(X=x) &= \sum_{x=0}^{\infty} \frac{p\,(p+x\phi)^{x-1}}{x!\eta(\theta)} \left\{\frac{d^x}{dz^x}\eta(\theta-\log z)\right\}_{z=1-p-x\,\phi} \\
&= \sum_{x=0}^{\infty} \frac{p\,(p+x\phi)^{x-1}}{x!\eta(\theta)} \frac{q(n)}{p_n} \left\{\frac{d^x}{dz^x}\exp\left[-(\theta-\log z)n\right]\right\}_{z=1-p-x\,\phi} \\
&= \sum_{x=0}^{\infty} \frac{p\,(p+x\phi)^{x-1}}{x!} \left\{\frac{d^x z^n}{dz^x}\right\}_{z=1-p-x\,\phi} \\
&= \sum_{x=0}^{n} \binom{n}{x} p\,(p+x\phi)^{x-1}(1-p-x\,\phi)^{n-x}.
\end{aligned}$$

Now, the last sum is equal to 1, since

$$p_x = \binom{n}{x} p(p+x\,\phi)^{x-1}(1-p-x\phi)^{n-x}, \quad x=0,1,\ldots \tag{3.2}$$

is the pmf of the quasi-binomial distribution (see [28]). □

Theorem 3.2. *The pmf given in* (3.1) *can be written as*

$$P(X=x) = \sum_{n=x}^{\infty} \binom{n}{x} p(p+x\,\phi)^{x-1}(1-p-x\phi)^{n-x} p_n, \quad x=0,1,2,\ldots \tag{3.3}$$

Proof. Similar to that in Theorem 2.2. □

We denote the family of distributions in (3.3) by $\mathcal{F}(\theta,p,\phi)$, and it reduces to $\mathcal{F}(\theta,p)$, for $\phi=0$.

Again, the result provided in Theorem 2.2 can also be extended by considering the dispersion exponential family of distributions, which is considered in Section 4.

$\mathcal{F}(\theta,p,\phi)$ is a generalization of the quasi-binomial(n,p,ϕ) distribution when $N=n$ follows NEF(θ) in (2.1) or random sum of $N=n$ independent quasi-binomial(n,p,ϕ) when $N \sim$ NEF(θ) in (2.1).

3.1. Moments

The expression for moments for the family $\mathcal{F}(\theta,p,\phi)$ is now given in the following result.

Theorem 3.3. *For $X \sim \mathcal{F}(\theta,p,\phi)$ the r^{th} factorial moment can be expressed as*

$$\mathbb{E}(X_{(r)}) = p(p+r\phi)^{r-1}\mathbb{E}(N_{(r)})\mathbb{E}\left\{1+\frac{\phi}{p+r\phi}Y\right\}^r, \tag{3.4}$$

where $Y \sim$ quasi-binomial $(n-r, p+r\phi, \phi)$.

Proof. The result follows in the following way,

$$\mathbb{E}(X_{(r)}) = \sum_{x=r}^{\infty} x_{(r)} \sum_{n=x}^{\infty} \binom{n}{x} p\,(p+x\phi)^{x-1}(1-p-x\,\phi)^{n-x} p_n$$

$$
\begin{aligned}
&= \sum_{x=r}^{\infty}\sum_{n=x}^{\infty} x_{(r)}\binom{n}{x} p\,(p+x\phi)^{x-1}(1-p-x\,\phi)^{n-x}p_n \\
&= \sum_{x=r}^{\infty} n_{(r)} \sum_{n=x}^{\infty}\binom{n-r}{x-r} p\,(p+x\phi)^{x-1}(1-p-x\,\phi)^{n-x}p_n \\
&= \frac{p}{p+r\phi}\sum_{n=r}^{\infty} n_{(r)} \sum_{x=r}^{\infty}(p+x\phi)^{r}\binom{n-r}{x-r}(p+r\phi) \\
&\quad (p+r\phi+(x-r)\phi)^{x-r-1} \\
&\quad (1-(p+r\phi)-(x-r)\,\phi)^{n-r-(x-r)}p_n \\
&= \frac{p}{p+r\phi}\sum_{n=r}^{\infty} n_{(r)}p_n \sum_{y=0}^{\infty}(p+r\phi+y\phi)^{r}\{\binom{n-r}{y}(p+r\phi) \\
&\quad (p+r\phi+y\phi)^{y-1}(1-(p+r\phi)-y\,\phi)^{n-r-y}\} \\
&= p(p+r\phi)^{r-1}\mathbb{E}(N_{(r)})\mathbb{E}\left\{1+\frac{\phi}{p+r\phi}Y\right\}^{r}.
\end{aligned}
$$

Hence, the result. □

Remark 3.1. *Note that for* $\phi = 0$*, quasi-binomial* $(n-r, p+r\phi, \phi) \to$ *binomial* $(n-r, p)$*, thus* (3.4) *reduces to* (2.9).

The mean and other moments can be derived from (3.4)*. In particular, the mean is seen as*

$$\mathbb{E}(X) = p\,\mathbb{E}(N)\left(1+\frac{\phi\,\mathbb{E}(Y)}{p+\phi}\right).$$

3.2. Special cases of $\mathcal{F}(\theta, p, \phi)$

In this section, we present some special cases of the family $F(\theta, p, \phi)$ obtained from different choices of the normalizing function $\eta(\theta)$.

Example 3.1. *Taking in* (3.1) $\eta(\theta) = \exp(-\theta\, r)$*,* $r > 0$*, we get the quasi-binomial distribution with parameters* $r > 0$*,* $0 < p < 1$*, and* $-p/n < \phi < (1-p)/n$*.*

Example 3.2. *Also, taking in* (3.1) $\eta(\theta) = (1+\exp(-\theta))^r$*,* $r > 0$*, we get the quasi-binomial distribution with parameters* r*,* $p/(1+\exp(\theta))$*, and* $\phi/(1+\exp(\theta))$*.*

Remark 3.2. *Therefore, the family in* (3.1) *can be viewed as a natural extension of the quasi-binomial distribution obtained by randomizing the number of trials through a natural exponential family.*

3.3. Stochastic interpretations of the family

Model (3.3) and model (2.5), the latter being a special case of the former, have a nice interpretation as described here.

In the binomial decay transformation (see [29]), N independent particles (for example, photons in laser pulses) pass through an attenuator. Let $0 \le p \le 1$ be the probability such that the particles penetrate the attenuator. If X is the number of particles that enter the attenuator, and N is the number of particles that come out of the attenuator, then the probability mass functions p_x and P_n of X and N, respectively, are connected by the binomial decay transformation given in (2.5).

Suppose now that the penetration probability is no longer constant but depends linearly on the number of particles that have already passed through the attenuator. This fact can be modeled using the quasi-binomial distribution [30], with pmf given by (3.2), yielding the quasi-binomial decay transformation in (3.3). Observe that for $\phi = 0$ the quasi-binomial distribution reduces to the binomial distribution, and in this case the probability that a particle penetrates the attenuator remains always the same, therefore giving to the model in [29].

On the other hand, for patients with cancer, the survival model describes the relationship between the absorbed dose and the number of surviving cells. Let us consider a cancer patient who has N cancer cells in the body. Suppose a dose is given to the patient to kill the same number of cancer cells. Here N is a random variable with a discrete probability mass function, and the destructive process is binomial with parameter $0 < p < 1$. Let X be the number of cells remaining. Then $P(X = x)$ is given by (2.5). Suppose now that the destructive process is not binomial, but that the probability of an infected cell increasing or decreasing is directly proportional to the number of infected cells. Then, this situation can be modeled by using (3.3). For examples in this line, see for instance [31], [32], [33], and [34]. However, the family $\mathcal{F}(\theta, p, \phi)$ is defined whenever the quasi-binomial kernel is well defined for every value in the support of the mixing distribution.

4. Extending the $\mathcal{F}(\theta, p)$ family by inducing a dispersion parameter

To add flexibility to the model described in the previous section, we may extend the result provided in Theorem 2.1 to the exponential dispersion family of distributions. This family, which has been studied by [35] and [36], among others, adds a parameter $\sigma > 0$ (the dispersion parameter) to the NEF of distributions, which is equal to 1 in the case of the NEF. The pmf of this extended family is given by

$$p_n = q(n, \sigma) \exp\left\{-\frac{1}{\sigma}\left[\theta n + \log(\eta(\theta))\right]\right\},$$

and the pgf results in

$$G_X(z) = \left[\frac{\eta(\theta - \log z)}{\eta(\theta)}\right]^{\sigma}, \quad 0 < z < 1.$$

Now, similar to that in Theorem 2.1, we have that

$$P(X = x) = \frac{p^x}{x![\eta(\theta)]^{\sigma}} \left\{\frac{d^x}{dz^x}[\eta(\theta - \log z)]^{\sigma}\right\}_{z=1-p}, \quad x = 0, 1, \dots \tag{4.1}$$

represents a proper probability mass function for $0 < p < 1$.

Remark 4.1. *Clearly,* (4.1) *reduces to the* $\mathcal{F}(\theta, p)$ *when* $\sigma = 1$. *For* σ*, a positive integer greater than 1, the distribution in Equation* (4.1) *can be seen as the distribution of the sum of* σ *identical and independent variables with pmf in Equation* (2.4).

Theorem 4.1. *For σ nonnegative integer, the pmf given in* (4.1) *can be written as*

$$P(X = x) = \sum_{m=x}^{\infty} \binom{m}{x} p^x (1-p)^{m-x} c_m, \quad x = 0, 1, 2, \ldots,$$

where

$$c_0 = p_0^{\sigma} \text{ and } c_m = \frac{1}{m p_0} \sum_{k=1}^{m} (k\sigma - m + k) p_k \, c_{m-k}.$$

Proof. Let $G_N(z)$ be the pgf of the random variable N, defined on the nonnegative integers, with pmf p_n. Then, we have that

$$\begin{aligned} P(X = x) &= \frac{p^x}{x!} \left\{ \frac{d^x}{dz^x} \, G_N(z)^{\sigma} \right\}_{z=1-p} = \frac{p^x}{x!} \left\{ \frac{d^x}{dz^x} (\sum_{n=0}^{\infty} z^n p_n)^{\sigma} \right\}_{z=1-p} \\ &= \sum_{m=x}^{\infty} \binom{m}{x} p^x (1-p)^{m-x} c_m, \end{aligned}$$

where we have used the result $(\sum_{n=0}^{\infty} z^n p_n)^{\sigma} = \sum_{m=x}^{\infty} c_m z^m$ wherein $c_0 = p_0^{\sigma}$ *and* $c_m = \frac{1}{m p_0} \sum_{k=1}^{m} (k\sigma - m + k) p_k \, c_{m-k}$ □

It is easily checked that $c_0 = p_0$, $c_1 = p_1$ for $\sigma = 1$. Then, the Theorem 4.1 reduces to the Theorem 2.2.

Theorem 4.2. *The r^{th} factorial moment of the distribution in* (4.1) *can be expressed as*

$$\mathbb{E}(X_{(r)}) = \sum_{m=r}^{\infty} p^r m_{(r)} c_m,$$

Proof. The proof follows immediately from Theorem 4.1. □

5. Beyond the binomial and quasi-binomial distributions

The composition mechanism developed in Sections 2 and 3 is not restricted to the binomial or quasi-binomial kernels. More generally, any family of bounded-support probability mass functions may be used as the conditional distribution, as established in the following theorem.

Theorem 5.1. *Let $\{p_n\}_{n \geq 0}$ be a probability mass function on $\mathbb{N}_0$ and, for each $n \geq 0$, let*

$$q_n(\cdot)$$

be a probability mass function supported on $\{0, \ldots, n\}$. *Then,*

$$p_x = \sum_{n=x}^{\infty} q_n(x) p_n, \qquad x = 0, 1, \ldots,$$

defines a probability mass function.

Proof. Since $q_n(x) = 0$ for $x > n$, we have

$$\sum_{x=0}^{\infty} p_x = \sum_{x=0}^{\infty}\sum_{n=x}^{\infty} q_n(x) p_n.$$

Interchanging the order of summation,

$$\sum_{x=0}^{\infty} p_x = \sum_{n=0}^{\infty}\sum_{x=0}^{n} q_n(x) p_n = \sum_{n=0}^{\infty} p_n \left(\sum_{x=0}^{n} q_n(x)\right).$$

Since $q_n(\cdot)$ is a probability mass function for every n,

$$\sum_{x=0}^{n} q_n(x) = 1,$$

and therefore

$$\sum_{x=0}^{\infty} p_x = \sum_{n=0}^{\infty} p_n = 1.$$

Moreover, $p_x \geq 0$, because both $q_n(x)$ and p_n are nonnegative. Hence, $\{p_x\}_{x\geq 0}$ is a probability mass function. $\square$

An immediate consequence of Theorem 5.1 is that the probability generating function of the compounded distribution is

$$G_X(s) = \sum_{n=0}^{\infty} G_{X|N=n}(s)\, p_n,$$

where $G_{X|N=n}(s)$ denotes the probability generating function of the conditional distribution $X \mid N = n$. Hence, the moments of the compounded distribution follow directly from the corresponding conditional moments by differentiation of $G_X(s)$.

The following examples illustrate the flexibility of Theorem 5.1. The case in which X follows a uniform discrete distribution with support in $\{0, 1, \ldots, N\}$ and N follows a Poisson distribution [37]. Consider now the case in which X follows a discrete distribution with support in $\{1, 2, \ldots, N\}$ and N follows a logarithmic series distribution with parameter $\theta \in (0, 1)$. Then, applying (5.1) we get

$$p_x = \sum_{n=x}^{\infty} \frac{1}{n}\frac{\theta^n}{n}\frac{-1}{\log(1-\theta)} = -\frac{\theta^x \Phi(\theta, 2, x)}{\log(1-\theta)}, \quad x = 1, 2, \ldots, \tag{5.1}$$

where $\Phi(z, s, a) = \sum_{k=0}^{\infty} z^k (k + a)^{-s}$ represents the Hurwitz-Lerch transcendent function. This special function is implemented in most statistical packages, such as Mathematica [38], R [39], and MATLAB [40].

6. Application

An application is shown that relates the number of injuries per accident to Zurich traffic accidents.

6.1. Numerical application

To illustrate the practical usefulness of the proposed distribution given in (5.1), we analyze the well-known dataset on the number of injuries per accident in Zurich traffic accidents (1961–62), reported by [41]. The proposed model was fitted by maximum likelihood using *Mathematica* (Version 13.2). For comparison purposes, the Poisson, logarithmic series, negative binomial, and Conway–Maxwell–Poisson (COM–Poisson) distributions were also fitted to the same data. Standard errors were computed from the inverse of the observed Fisher information matrix.

Table 1 presents the observed and fitted frequencies; the maximum likelihood (ML) estimates of the model parameters and their standard errors; the maximized log-likelihood; the Akaike information criterion (AIC) and Bayesian information criterion (BIC) values; and the Pearson χ^2 goodness-of-fit statistic with the corresponding p-value.

As expected, the Poisson distribution provides a poor fit due to the substantial overdispersion present in the data. The logarithmic series and COM–Poisson distributions improve the fit but still fail to adequately capture the observed frequencies, as reflected by their large Pearson statistics and negligible p-values. Among the classical competitors, the negative binomial distribution performs best. Nevertheless, the proposed distribution achieves the largest maximized log-likelihood (−2003.39), the smallest AIC (4008.77) and BIC (4015.22), and the smallest Pearson goodness-of-fit statistic ($\chi^2 = 7.26$), yielding the largest p-value (0.1229). These results indicate that the proposed model provides the best overall fit among the competing count distributions considered for this data.

Table 1. Observed and fitted frequencies for the Zurich traffic accident data.

number of injuries	number of accidents	Poisson	logarithmic	distribution neg. binomial	COM-Poisson	proposed
1	4121	4021.48	4084.31	4122.66	4060.1	4111.92
2	430	585.72	477.26	417.32	516.59	436.14
3	71	42.65	74.36	84.87	65.73	79.53
4	19	2.07	13.03	20.15	8.36	18.02
5	6	0.07	2.44	5.13	1.06	4.6
6+	5	0.01	0.6	1.87	0.16	1.79
$\widehat{\theta}$		0.1456	0.2337	0.6945	0.1272	0.3881
s.e.		0.0056	0.008	0.0277	0.0175	0.0116
$\widehat{\phi}$				0.3313	<0.0001	
s.e.				0.0405	0.2212	
log-likelihood		−2110.03	−2012.04	−2003.86	−2030.93	−2003.39
BIC		4228.50	4032.53	4024.61	4078.75	4015.22
AIC		4222.05	4026.09	4011.72	4065.86	4008.77
Pearson χ^2		3193.54	45.36	8.10	198.82	7.26
d.f.		4	4	3	3	4
p-value		<0.0001	<0.0001	0.0439	<0.0001	0.1229

Overall, these results show that the proposed distribution is competitive with the most widely used count-data models while maintaining a comparatively simple one-parameter structure.

7. Concluding remarks

A new parametric family of discrete probability distributions has been introduced, encompassing the natural exponential family and therefore many classical discrete models as special cases. A general extension based on exponential dispersion families has also been proposed, together with a general compounding theorem showing that the binomial kernel may be replaced by any bounded-support discrete distribution. Finally, a numerical application illustrates the flexibility of the proposed methodology. From an inferential viewpoint, the proposed family naturally admits likelihood-based estimation through its probability mass function. However, the complexity of the likelihood depends on the underlying natural exponential family, since evaluating the normalizing function and its derivatives may require numerical methods for certain choices of the parent distribution. Consequently, the development of efficient estimation algorithms, including maximum likelihood, Expectation–Maximization–type procedures, and Bayesian methods, constitutes an interesting direction for future research.

Author contributions

Héctor W. Gómez: Conceptualization of this study, Methodology, Writing—original draft preparation. Emilio Gómez–Déniz: Conceptualization of this study, Methodology, Investigation, Writing—original draft preparation. Diego I. Gallardo: Investigation, Software, Writing—review and editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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