Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 24106–24119.
DOI: 10.3934/math.2026972
Received: 29 April 2026
Revised: 10 July 2026
Accepted: 17 July 2026
Published: 06 August 2026

Research article

Oriented discrepancy of Hamilton cycles in oriented graphs satisfying Ore-type condition

Jiangdong Ai[1], **Qiwen Guo**[2], **Gregory Gutin**[1,2,*], **Yongxin Lan**[3], **Qi Shao**[4], **Anders Yeo**[5,6] **and Yacong Zhou**[7]

[1] School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, China
[2] Department of Computer Science, Royal Holloway, University of London, Surrey TW20 0EX, United Kingdom
[3] School of Science, Hebei University of Technology, Tianjin 300401, China
[4] Center for Combinatorics and LPMC, Nankai University, Tianjin 300071, China
[5] Department of Mathematics and Computer Science, University of Southern Denmark, Odense 5230, Denmark
[6] Department of Mathematics, University of Johannesburg, Johannesburg 2006, South Africa
[7] Shenzhen Institutes of Advanced Technology, Chinese Academy of Sciences, Shenzhen 518055, China

* **Correspondence:** Email: g.gutin@rhul.ac.uk.

Abstract: Erdős (1963) initiated extensive graph discrepancy research on 2-edge-colored graphs. Gishboliner, Krivelevich, and Michaeli (2023) launched similar research on oriented graphs. They conjectured the following extension of Dirac's theorem: If D is an oriented graph on $n \geq 3$ vertices with minimum degree $\delta(D) \geq n/2$, then D contains a Hamilton oriented cycle with at least $\delta(D)$ arcs in the same direction. This conjecture was proved by Freschi and Lo (2024) who posed an open problem to extend their result to an Ore-type condition. We proposed two conjectures for such extensions and proved results which provided support to the conjectures.

Keywords: oriented graphs; oriented discrepancy; Hamilton oriented cycles; Ore-type condition
Mathematics Subject Classification: 05C20, 05C45

1. Introduction

For a hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ and coloring $c : V(\mathcal{H}) \to \{-1, 1\}$, the discrepancy of an edge $e \in E(\mathcal{H})$ is $D_c(e) = |\sum_{v \in e} c(v)|$. Combinatorial discrepancy studies the so-called discrepancy of $\mathcal{H}$

defined as

$$\min_{c} \max_{e \in E(\mathcal{H})} D_c(e).$$

For an excellent exposition of the topic, see, e.g., [1]. Erdős [2] launched an extensive study of graph discrepancy, which is a special case of combinatorial discrepancy: $V(\mathcal{H})$ is the edge set of some graph G and $E(\mathcal{H})$ consists of the edge sets of specific subgraphs of G. For recent papers on the topic, see, e.g., [3–5].

Gishboliner, Krivelevich, and Michaeli [6] introduced an oriented version of graph discrepancy and conjectured the statement of Theorem 1.1, stated below, which was proved by Freschi and Lo [7].

Let us now introduce some terminology and notation; terminology and notation not introduced in this paper can be found in [8–10]. The underlying graph of a digraph $D = (V(D), A(D))$ is an undirected multigraph $U(D)$ obtained from D by removing the orientations of all arcs in D. The degree of a vertex x in a digraph D, denoted by $d_D(x)$, is the degree $d_{U(D)}(x)$ of x in $U(D)$. In what follows, we will often omit directed or undirected graph subscripts if such graphs are clear from the context. The minimum degree of a directed or undirected graph H is denoted by $\delta(H)$. For any $k \in \mathbb{N}$, let $[k] := \{1, \ldots, k\}$.

The neighborhood $N(x)$ of a vertex x in a directed or undirected graph D is the set of vertices adjacent to x. The closed neighborhood of v is $N[x] = N(x) \cup \{x\}$. We use K_n to denote the complete undirected graph of order n. For an undirected graph G, let $\overline{G}$ denote its complement.

A digraph D is an oriented graph if there is no more than one arc between any pair of vertices in D. An oriented cycle (oriented path, respectively) in D is a subdigraph Q of D such that $U(Q)$ is a cycle (path, respectively) in $U(D)$. An oriented cycle (oriented path, respectively) in D is Hamilton if it is a spanning subdigraph of D. Note that a digraph D has a Hamilton oriented cycle (path, respectively) if, and only if, $U(D)$ has a Hamilton cycle (path, respectively).

For an oriented cycle $C = x_1x_2 \ldots x_tx_1$, the arc a between x_i and x_{i+1} (all subscripts are taken modulo t) on C is forward if $a = x_ix_{i+1}$ and backward if $a = x_{i+1}x_i$. In some parts of the literature (e.g., [6]), an arc from x to y is written as (x, y) and an edge between x and y as $\{x, y\}$. Following [7], we instead use the shorthand xy for both (x, y) and $\{x, y\}$ whenever the meaning is clear from context. This convention is also consistent with the standard notation for oriented, directed, and undirected paths and cycles. The number of forward (backward, respectively) arcs of C is denoted by $\sigma^+(C)$ ($\sigma^-(C)$, respectively). Let $\sigma_{\min}(C) = \min\{\sigma^+(C), \sigma^-(C)\}$ and $\sigma_{\max}(C) = \max\{\sigma^+(C), \sigma^-(C)\}$. Note that $\sigma_{\min}(C)$ and $\sigma_{\max}(C)$ do not depend on the order of the vertices of C. If all arcs of C are forward, then C is a *directed cycle*. Similar definitions and notation can be introduced for oriented paths.

Theorem 1.1. *[7] Let D be an oriented graph on $n \geq 3$ vertices. If $\delta(D) \geq \frac{n}{2}$, then there exists a Hamilton oriented cycle C in D such that $\sigma_{\max}(C) \geq \delta(D)$.*

Note that Theorem 1.1 is a strengthening of Dirac's theorem [11]: A graph G on $n \geq 3$ vertices has a Hamilton cycle if $\delta(G) \geq n/2$. Ore [12] extended Dirac's theorem as follows: A graph G on $n \geq 3$ vertices has a Hamilton cycle, if $d(u) + d(v) \geq n$ for every pair of non-adjacent vertices u and v of G. In a digraph D, a pair of vertices are non-adjacent if there is no arc between them.

Freschi and Lo [7] stated an open problem to extend their result to an Ore-type condition, i.e., a lower bound on the sum of degrees of any pair of non-adjacent vertices. Now we pose two conjectures on the open problem. While we were unable to prove either conjecture, we show some results providing support to the conjectures.

Taking the minimum degree of D into account, we propose the following Ore-type analogue of Theorem 1.1.

Conjecture 1.2. *Let D be an oriented graph on $n \geq 3$ vertices with minimum degree δ. If $d(u)+d(v) \geq n$ for each pair of non-adjacent vertices u and v, then there exists a Hamilton oriented cycle C in D such that $\sigma_{\max}(C) \geq \max\{\delta, n-\delta\}$.*

When $\delta \geq n/2$, Conjecture 1.2 holds and is sharp, as it directly follows from Theorem 1.1. The following construction shows that Conjecture 1.2 is sharp when $\delta < n/2$, as $\sigma_{\max}(C) \leq n-\delta$ for all Hamilton oriented cycles C in the construction below.

Construction 1.3. *Given integers n and δ with $\delta \geq 2$ and $n > 2\delta$, let G be a graph on n vertices with vertex set $V(G) = A \cup B \cup \{v_0\}$, where $G[A]$ is isomorphic to $\overline{K}_\delta$, $G[B]$ is isomorphic to $K_{n-\delta-1}$, $N(v_0) = A$, and every vertex in A is adjacent to all vertices in B. Let D be an orientation of G, such that every edge between A and B is oriented from B to A, every edge between v_0 and A is oriented from v_0 to A (see Figure 1) and the orientation of every edge in $G[B]$ is arbitrary.*

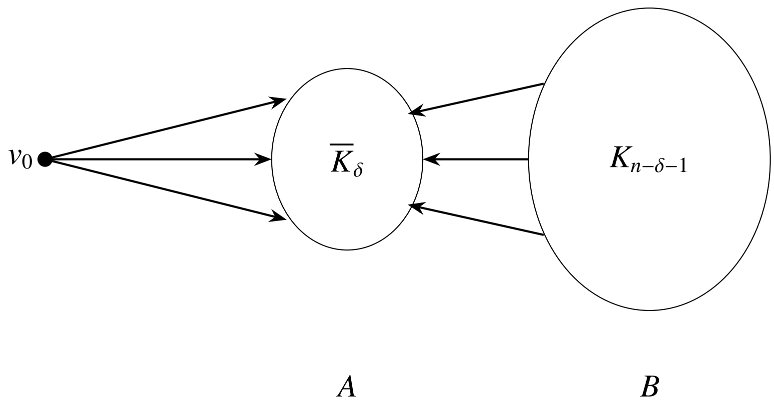


Figure 1. Tight examples for Conjecture 1.2.

It is not hard to check that $d(u) + d(v) \geq n$ for each pair of non-adjacent vertices u and v in D and $\delta(D) = \delta$. Let C be a Hamilton oriented cycle in D. Observe that every vertex in A is incident to two edges of C which are oriented in opposite directions. Since A is an independent set, $\sigma_{\min}(C) \geq |A| = \delta$. It follows that $\sigma_{\max}(C) \leq n-\delta$.

In the following result, we show that the conclusion of Conjecture 1.2 holds under a stronger Ore-type condition.

Theorem 1.4. *Let D be an oriented graph on n vertices with minimum degree $1 < \delta < \frac{n}{2}$. If $d(u)+d(v) \geq n+\delta-2$ for each pair of non-adjacent vertices u and v, then there exists a Hamilton oriented cycle C in D such that $\sigma_{\max}(C) \geq n-\delta$.*

Observe that Theorem 1.4 implies that Conjecture 1.2 is true for $\delta = 2$, since in this case the lower bound of degree sum of the condition in Theorem 1.4 is precisely n.

When the subgraph induced by the neighborhood of a vertex with the minimum degree in the graph is isomorphic to a tournament, we can derive a slightly weaker bound than in Conjecture 1.2.

Theorem 1.5. *Let D be an oriented graph on n vertices with minimum degree $3 \le \delta \le \frac{n}{3}$, which satisfies that $d(u) + d(v) \ge n$ for each pair of non-adjacent vertices u and v in D. If there is a vertex $v_0 \in V(D)$ such that $d(v_0) = \delta$ and $D[N(v_0)]$ is isomorphic to a tournament, then there is a Hamilton oriented cycle C in D such that $\sigma_{\max}(C) \ge n - \delta - 2$.*

For every n-vertex oriented graph D which is not a tournament, we define

$$s^*(D) = \min\{d(u) + d(v) - n : u \neq v \in V(D), uv, vu \notin A(D)\}.$$

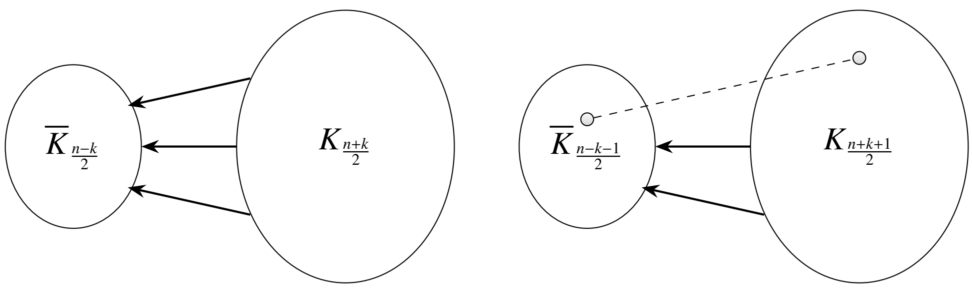
If D is a tournament, let $s^*(D) = n - 2$. The following conjecture is another possible generalization of Theorem 1.1.

Conjecture 1.6. *Let D be an oriented graph on $n \ge 3$ vertices. If $s^*(D) \ge 0$, then there is a Hamilton oriented cycle C in D such that $\sigma_{\max}(C) \ge \lceil \frac{n+s^*(D)}{2} \rceil$.*

If Conjecture 1.6 is true, it also implies Theorem 1.1. Indeed, if $\delta(D) \ge n/2$, then $s^*(D) \ge 2\delta(D) - n \ge 0$, and so Conjecture 1.6 implies that $\sigma_{\max}(C) \ge n/2 + (2\delta(D) - n)/2 = \delta(D)$.

The following construction shows the lower bound on $\sigma_{\max}(C)$ in Conjecture 1.6 is tight for every positive integer n and integer $0 \le s^*(D) \le n - 4$ (note that if $s^*(D) \ge n - 3$, since it is impossible that $s^*(D) = n - 3$, we are considering tournaments, in which case the lower bound is achieved by the transitive tournament).

Construction 1.7. *Given a positive integer n and a nonnegative integer k with $n \ge k + 4$, let $c \in \{0, 1\}$ with $c \equiv n + k \pmod 2$. Let G be a graph on n vertices with vertex set $V(G) = A \cup B$, where $G[A]$ is isomorphic to $\overline{K}_{\frac{n-k-c}{2}}$, $G[B]$ is isomorphic to $K_{\frac{n+k+c}{2}}$, and all possible edges with one end-vertex in A and the other in B exist except one edge if $c = 1$. Let D be an orientation of G, such that every edge between A and B is oriented from B to A, and the orientation of every edge in $G[B]$ is arbitrary (see Figure 2).*



(a) $n + k$ is even (b) $n + k$ is odd

Figure 2. Tight examples for Conjecture 1.6.

It is not hard to check that $s^*(D) = k$, and for every Hamilton oriented cycle C in D, every vertex in A is incident to two edges of C which are oriented in opposite directions. Since A is an independent set, we have $\sigma_{\min}(C) \ge |A| = \left\lfloor \frac{n-k}{2} \right\rfloor$ and then $\sigma_{\max}(C) \le \lceil \frac{n+k}{2} \rceil$.

By applying an approach from [6], we can obtain the following approximate version of Conjecture 1.6.

Theorem 1.8. *Let $k \in \mathbb{N}$ and let D be an oriented graph with order $n \geq 30 + 4(k-1)$. If $s^*(D) \geq 8k$, then there is a Hamilton oriented cycle C in D such that $\sigma_{\max}(C) \geq \lceil \frac{n+k}{2} \rceil$.*

The rest of this paper is organized as follows: In Section 2, we prove Theorem 1.4; in Section 3, we prove Theorem 1.5; and in Section 4, we prove Theorem 1.8. In Section 5, we briefly discuss further research of digraph discrepancy, and Section 6 concludes the paper.

2. Proof of Theorem 1.4

The core technique in the proof of Theorem 1.4 is to extend an oriented path or cycle by adding new vertices to it. This extension preserves the order of its original vertices without decreasing its discrepancy. Specifically, we first identify a large oriented cycle C with high discrepancy in the digraph D. Then, we add all the vertices in $V(D) \setminus V(C)$ to C to extend it to a Hamilton oriented cycle C' while preserving high discrepancy.

To accomplish this, Lemma 2.1 is needed. It allows us to add a sufficient number of vertices to an oriented path or cycle while not decreasing its discrepancy.

Lemma 2.1. *Let D be an oriented graph which contains an oriented path P and a set of vertices K disjoint from $V(P)$, such that $V(P) \subseteq N(v)$ for each $v \in K$. If $|V(P)| \geq |K| + 2$, then D contains an oriented path P' with $V(P') = V(P) \cup K$ and same initial and terminal vertices as P, satisfying $\sigma^+(P') \geq \sigma^+(P)$. Similarly, an oriented path P'' satisfying $\sigma^-(P'') \geq \sigma^-(P)$ also exists.*

Proof. Let $K = \{v_1, v_2, \ldots, v_k\}$ and $P = u_1 u_2 \ldots u_\ell$, where $\ell \geq k + 2$. Let $A = \{a_1, a_2, \ldots, a_k\}$ be k distinct arcs chosen randomly and uniformly from $A(P)$. For each $i \in [k]$, let a_i be the arc between u_{p_i} and u_{p_i+1} and let P' be obtained from the path P, by substituting a_i with the oriented path $u_{p_i} v_i u_{p_i+1}$ for each $i \in [k]$.

Let q and r be any integers such that $2 \leq q \leq \ell - 1$ and $r \in [k]$. If $u_q v_r \in A(D)$, then $u_q v_r$ will be a forward arc in P' with probability $\frac{1}{\ell-1}$, as this is the case if, and only if, a_r is the arc between u_q and u_{q+1}. And, if $v_r u_q \in A(D)$, then $v_r u_q$ will be a forward arc in P' with probability $\frac{1}{\ell-1}$, as this is the case if a_r is the arc between u_{q-1} and u_q. So every arc between K and $\{u_2, u_3, u_4, \ldots, u_{\ell-1}\}$ will be a forward arc with probability $\frac{1}{\ell-1}$ and there are $k \cdot (\ell - 2)$ such arcs. We remove at most k forward arcs from P when we remove the arcs in A. Therefore, by linearity of expectation, P' satisfies the following (as $\ell \geq k + 2$),

$$\mathbb{E}(\sigma^+(P')) \geq \sigma^+(P) + \frac{k \cdot (\ell - 2)}{\ell - 1} - k \geq \sigma^+(P) + \frac{k \cdot k}{k+1} - k > \sigma^+(P) - 1.$$

As $\sigma^+(P')$ is an integer, this implies that $\mathbb{P}(\sigma^+(P') \geq \sigma^+(P))$ is positive, and there exists an oriented path P' with $\sigma^+(P') \geq \sigma^+(P)$. By considering the reverse of P, instead of P, we conclude that $\sigma^-(P'') \geq \sigma^-(P)$. □

Remark 2.2. *Clearly, Lemma 2.1 is of interest in itself in study of discrepancy of oriented paths and cycles. In algorithmic applications, naturally we may ask whether the proof of Lemma 2.1 can be derandomized leading to a polynomial-time algorithm. Consider the complete bipartite graph B with partite sets $A(P)$ and K. For an arc $a \in A(P)$ with end-vertices u_t, u_{t+1} and a vertex $v \in K$, let $P(a, v)$ be the oriented path obtained from P by replacing a with the oriented path $u_t v u_{t+1}$ and let the weight $w(av)$ be equal to $\sigma^+(P(a, v)) - \sigma^+(P)$. Note that the set $A = \{a_1, a_2, \ldots, a_k\}$ of chosen arcs* *of P can be viewed as a matching $\{a_i v_i \mid i \in [k]\}$ in B. The proof of Lemma 2.1 essentially shows that the average weight of a matching in B that saturates K is strictly greater than -1. By applying the Hungarian algorithm for maximum weight assignment problem, in polynomial time, we can find a maximum weight matching that saturates K whose weight is at least the average weight of such a matching, so we can, in polynomial time, create a path P' with $\sigma^+(P') \geq \sigma^+(P)$ (by substituting each a_i by $u_{p_i} v_i u_{p_i+1}$ for each edge $a_i v_i$ in the maximum weight matching).*

With Lemma 2.1 established, we are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Observe that the assumptions $1 < \delta < \frac{n}{2}$ imply $n \geq 5$, and if $n = 5$, every Hamilton oriented cycle C in D satisfies $\sigma_{\max}(C) \geq 3 \geq n - \delta$. Therefore, we may assume that $n \geq 6$.

Let $v_0 \in V(D)$ and $d(v_0) = \delta$. Let $N(v_0) = \{v_1, v_2, \ldots, v_\delta\}$ and $U = V(D) \backslash N[v_0] = \{u_1, u_2, \ldots, u_{n-\delta-1}\}$. Since $d(u) + d(v) \geq n + \delta - 2$ for any non-adjacent $u, v \in V(D)$, $d(u) \geq n - 2$ for all $u \in U$, which means that u is adjacent to all vertices except v_0 in D and $D[U]$ is a tournament. Since $|U| = n - \delta - 1 \geq 3$, Theorem 1.1 implies that there is a Hamilton oriented cycle C_0 in $D[U]$ such that $\sigma_{\max}(C_0) \geq n - \delta - 2$. Without loss of generality, we assume that $C_0 = u_1 u_2 \ldots u_{n-\delta-1} u_1$, $\sigma_{\max}(C_0) = \sigma^+(C_0)$ and $u_2 u_1$ is the only backward arc in C_0 (if $\sigma_{\max}(C_0) = n - \delta - 2$). Since $\delta \geq 2$, v_1 and v_2 are well-defined and distinct. For each $i \in [n - \delta - 1]$, we get Hamilton oriented cycles $C^1_{i,i+1} = u_i v_1 v_0 v_2 u_{i+1} u_{i+2} \ldots u_{i-1} u_i$ and $C^2_{i,i+1} = u_i v_2 v_0 v_1 u_{i+1} u_{i+2} \ldots u_{i-1} u_i$ (all subscripts are taken modulo $n - \delta - 1$) in $D[U \cup \{v_0, v_1, v_2\}]$.

Claim 2.1. *There exist $i \in [n - \delta - 1]$ and $t \in [2]$ such that $\sigma^+(C^t_{i,i+1}) \geq n - \delta$.*

Suppose that $\sigma^+(C^t_{i,i+1}) \leq n - \delta - 1$ for each $i \in [n - \delta - 1]$ and $t \in [2]$. We may assume $\{v_1 v_0, v_2 v_0\} \subset A(D)$ or $\{v_0 v_1, v_0 v_2\} \subset A(D)$, since otherwise $\sigma^+(C^1_{1,2}) \geq n - \delta$ if $v_1 v_0 v_2$ is a directed path or $\sigma^+(C^2_{1,2}) \geq n - \delta$ if $v_2 v_0 v_1$ is a directed path. Without loss of generality, assume $\{v_1 v_0, v_2 v_0\} \subset A(D)$. Since $v_1 v_0 \in A(D)$, $\sigma^+(C^1_{1,2}) \geq n - \delta - 1$ and $\sigma^+(C^1_{1,2}) = n - \delta - 1$. Thus, $\{v_1 u_1, u_2 v_2\} \subset A(D)$. Similarly, since $v_2 v_0 \in A(D)$, $\sigma^+(C^2_{1,2}) = n - \delta - 1$ and $\{v_2 u_1, u_2 v_1\} \subset A(D)$. Note that, for any $i \in [n - \delta - 1] \setminus \{1\}$, if $\{u_i v_1, v_1 v_0\} \subset A(D)$, we have $\sigma^+(C^1_{i,i+1}) = n - \delta - 1$ and then $u_{i+1} v_2 \in A(D)$, and if $\{u_i v_2, v_2 v_0\} \subset A(D)$, we have $\sigma^+(C^2_{i,i+1}) = n - \delta - 1$ and then $u_{i+1} v_1 \in A(D)$. Now, since $\{v_1 v_0, v_2 v_0\} \subset A(D)$ and $\{u_2 v_1, u_2 v_2\} \subset A(D)$, we have $\{u_{i+1} v_1, u_{i+1} v_2\} \subset A(D)$ for every $i \in [n - \delta - 1]$. Thus, $\{u_1 v_1, u_1 v_2, v_1 u_1, v_2 u_1\} \subset A(D)$, which contradicts the condition that D is an oriented graph. Thus, Claim 2.1 holds. ■

Let $C^{t_0}_{i_0,i_0+1}$ be the oriented cycle such that $\sigma^+(C^{t_0}_{i_0,i_0+1}) \geq n - \delta$ as in Claim 2.1. Consider the oriented path $P = u_{i_0+1} u_{i_0+2} \ldots u_{i_0}$ (all subscripts are taken modulo $n - \delta - 1$). Since every vertex in P is adjacent to every vertex in $\{v_3, \ldots, v_\delta\}$ and $\delta - 2 \leq (n - \delta - 1) - 2$, Lemma 2.1 implies that there exists an oriented path P' with $V(P') = V(P) \cup \{v_3, \ldots, v_\delta\}$ and the same initial and terminal vertices as P, satisfying $\sigma^+(P') \geq \sigma^+(P)$. Let C be the Hamilton oriented cycle in D, whose arc set consists of $A(C^{t_0}_{i_0,i_0+1}) \setminus A(P)$ and $A(P')$. Thus, $\sigma_{\max}(C) \geq \sigma^+(C^{t_0}_{i_0,i_0+1}) \geq n - \delta$. □

3. Proof of Theorem 1.5

First, we need the following lemma.

Lemma 3.1. *Let D be a tournament on $n \geq 3$ vertices. Given a Hamilton oriented cycle $C = v_1 v_2 \ldots v_{i-1} v_i v_{i+1} \ldots v_{j-1} v_j v_{j+1} \ldots v_n v_1$ in D, let C' be a Hamilton oriented cycle which is obtained from C by swapping v_i and v_j, that is, $C' = v_1 v_2 \ldots v_{i-1} v_j v_{i+1} \ldots v_{j-1} v_i v_{j+1} \ldots v_n v_1$. Then, $\sigma^+(C') \geq \sigma^+(C) - 4$*

and $\sigma^-(C') \geq \sigma^-(C) - 4$. *In particular, if* v_i *and* v_j *are adjacent in* C*, then* $\sigma^+(C') \geq \sigma^+(C) - 3$ *and* $\sigma^-(C') \geq \sigma^-(C) - 3$.

Proof. Since the relative order of every pair of vertices other than v_i and v_j is unchanged, only the four cycle arcs incident with v_i or v_j may change their contributions and then we have that $\sigma^+(C') \geq \sigma^+(C) - 4$ and $\sigma^-(C') \geq \sigma^-(C) - 4$. In particular, if v_i and v_j are adjacent in C, there are three arcs in C incident with v_i or v_j, and then $\sigma^+(C') \geq \sigma^+(C) - 3$ and $\sigma^-(C') \geq \sigma^-(C) - 3$. □

Proof of Theorem 1.5. Suppose that $v_0 \in V(D)$ with $d(v_0) = \delta$ and $D[N(v_0)]$ is isomorphic to a tournament. We denote $V(D)\backslash N[v_0]$ by W. Then, $|W| = n - \delta - 1$. Since for every vertex $w \in W$, we have $d(w) \geq n - \delta$ and $n - 2\delta \leq d(w) - |N(v_0)| \leq d_{D[W]}(w) \leq n - \delta - 2$, then w must be adjacent to at least two vertices in $N(v_0)$.

Since $\delta(D[W]) \geq n - 2\delta \geq \frac{n-\delta-1}{2} = \frac{|W|}{2}$ when $\delta \leq \frac{n}{3}$, Theorem 1.1 implies that there is a Hamilton oriented cycle C_1 in $D[W]$ such that $\sigma_{\max}(C_1) \geq \delta(D[W])$. Let $C_1 = w_1w_2 \dots w_{n-\delta-1}w_1$. Similarly, there exists a Hamilton oriented cycle $C_2 = v_0v_1v_2 \dots v_\delta v_0$ with $\sigma_{\max}(C_2) \geq \delta$ in $D[N[v_0]]$. Since $V(C_1) \cap V(C_2) = \emptyset$, without loss of generality, we may assume that $\sigma^+(C_i) = \sigma_{\max}(C_i)$, for each $i \in [2]$. Let $w_k \in W$ such that $d_{D[W]}(w_k) = \delta(D[W])$. In what follows, we adopt the convention that $w_{k+1} = w_1$ if $k = n - \delta - 1$.

We first establish the following five claims. Each claim rules out one possible configuration involving the neighbors of w_k and w_{k+1} on C_2. After excluding these configurations one by one, we obtain increasingly stronger lower bounds on $\delta(D[W])$. These bounds are used in the proofs of the subsequent claims and in the remainder of the proof.

Claim 3.1. *If there exist* v_j *and* v_{j+1} *for* $j \in [\delta - 1]$ *in* C_2 *that are adjacent to* w_{k+1} *and* w_k*, respectively, then there is a Hamilton oriented cycle* C *in* D *such that* $\sigma_{\max}(C) \geq n - \delta - 2$.

Consider $C = w_1 \dots w_kv_{j+1}v_{j+2} \dots v_\delta v_0v_1 \dots v_jw_{k+1} \dots w_{n-\delta-1}w_1$, which is obtained from C_1 by deleting the arc between w_k and w_{k+1}, and adding the arc between w_k and v_{j+1}, the arc between w_{k+1} and v_j, and the longer oriented path on C_2 from v_{j+1} to v_j. We have $\sigma_{\max}(C) \geq \sigma^+(C) \geq n - 2\delta - 1 + \delta - 1 = n - \delta - 2$. Thus, Claim 3.1 holds. ■

By Claim 3.1, we may assume that for each $j \in [\delta - 1]$, if w_{k+1} is adjacent to v_j, then w_k is not adjacent to v_{j+1}. Then, w_k is not adjacent to at least one vertex in $V(C_2) \setminus \{v_0\}$ since w_{k+1} has at least two neighbors in $V(C_2) \setminus \{v_0\}$. In this case, $\delta(D[W]) \geq n - 2\delta + 1$.

Claim 3.2. *If* v_1 *and* v_δ *are adjacent to* w_k *and* w_{k+1}*, respectively, then there is a Hamilton oriented cycle* C *in* D *such that* $\sigma_{\max}(C) \geq n - \delta - 2$.

Consider $C' = w_1 \dots w_kv_1v_2 \dots v_\delta w_{k+1} \dots w_{n-\delta-1}w_1$, which is obtained from C_1 by deleting the arc between w_k and w_{k+1}, and adding the arc between w_k and v_1, the arc between w_{k+1} and v_δ, and the oriented path $C_2 - \{v_0\}$. We have $\sigma^+(C') \geq n - 2\delta + 1 - 1 + \delta - 2 = n - \delta - 2$. Applying Lemma 2.1 with $P = v_1v_2 \dots v_\delta$ and $K = \{v_0\}$, we can add v_0 to C' when $\delta \geq 3$, to obtain the desired Hamilton oriented cycle C in D such that $\sigma_{\max}(C) \geq \sigma_{\max}(C') \geq n - \delta - 2$. Thus, Claim 3.2 holds. ■

If $\delta(D[W]) = n - 2\delta + 1$, then w_k is not adjacent to at most one vertex in $V(C_2) \setminus \{v_0\}$. By the assumption before, w_k is not adjacent to exactly one vertex v_{i_1} in $V(C_2) \setminus \{v_0\}$. Since w_{k+1} is adjacent to at least two vertices in $V(C_2) \setminus \{v_0\}$, we may assume that v_{j_1} and v_{j_2} are adjacent to w_{k+1} and $0 < j_1 < j_2$. By Claim 3.1, we may assume that $i_1 = j_1 + 1$ and $j_2 = \delta$. Then, v_1 and v_δ are adjacent to w_k and w_{k+1}, respectively. By Claim 3.2, we are done.

Hence, we may assume that $\delta(D[W]) \geq n-2\delta+2$. We may also assume that either w_k is not adjacent to v_1 or w_{k+1} is not adjacent to v_δ.

Claim 3.3. *If there exist v_j and v_{j+1} for $j \in [\delta-1]$ in C_2 that are adjacent to w_k and w_{k+1}, respectively, then there is a Hamilton oriented cycle C in D such that $\sigma_{\max}(C) \geq n-\delta-2$.*

Let C_2' be an oriented cycle which is obtained from C_2 in Lemma 3.1, by exchanging v_j and v_{j+1} in C_2. By Lemma 3.1, $\sigma^+(C_2') \geq \sigma^+(C_2) - 3 \geq \delta - 3$. Note that the longer oriented path on C_2' from v_j to v_{j+1} has at least $\delta-3$ forward arcs by the construction of C_2'. Consider $C = w_1 \dots w_k v_j v_{j+2} \dots v_\delta v_0 v_1 \dots v_{j-1} v_{j+1} w_{k+1} \dots w_{n-\delta-1} w_1$, which is obtained from C_1 by deleting the arc between w_k and w_{k+1}, and adding the arc between w_k and v_j, the arc between w_{k+1} and v_{j+1}, and the longer oriented path on C_2' from v_j to v_{j+1}. We have $\sigma_{\max}(C) \geq \sigma^+(C) \geq n-2\delta+2-1+\delta-3 = n-\delta-2$. Thus, Claim 3.3 holds. ■

By Claim 3.3, we may assume that for each $j \in [\delta-1]$, if w_{k+1} is adjacent to v_{j+1}, then w_k is not adjacent to v_j.

Claim 3.4. *If both of w_k and w_{k+1} are adjacent to v_1 and v_3, then there is a Hamilton oriented cycle C in D such that $\sigma_{\max}(C) \geq n-\delta-2$.*

Let C_2' be an oriented cycle which is obtained from C_2 in Lemma 3.1, by exchanging v_2 and v_3 in C_2. Note that v_1 and v_3 are consecutive on C_2' and the longer oriented path on C_2' from v_3 to v_1 has at least $\delta-3$ forward arcs by the construction of C_2'. Consider $C = w_1 \dots w_k v_3 v_2 v_4 \dots v_\delta v_0 v_1 w_{k+1} \dots w_{n-\delta-1} w_1$, which is obtained from C_1 by deleting the arc between w_k and w_{k+1}, and adding the arc between w_k and v_3, the arc between w_{k+1} and v_1, and the longer oriented path of C_2' from v_3 to v_1. We have $\sigma_{\max}(C) \geq \sigma^+(C) \geq n-2\delta+2-1+\delta-3 = n-\delta-2$. ■

Claim 3.5. *If both of w_k and w_{k+1} are adjacent to $v_{\delta-2}$ and v_δ, then there is a Hamilton oriented cycle C in D such that $\sigma_{\max}(C) \geq n-\delta-2$.*

The proof is analogous to that of Claim 3.4. Let C_2' be an oriented cycle which is obtained from C_2 in Lemma 3.1, by exchanging $v_{\delta-2}$ and $v_{\delta-1}$ in C_2. Note that $v_{\delta-2}$ and v_δ are consecutive on C_2' and the longer oriented path on C_2' from v_δ to $v_{\delta-2}$ has at least $\delta-3$ forward arcs by the construction of C_2'. Consider $C = w_1 \dots w_k v_\delta v_0 v_1 \dots v_{\delta-3} v_{\delta-1} v_{\delta-2} w_{k+1} \dots w_{n-\delta-1} w_1$, which is obtained from C_1 by deleting the arc between w_k and w_{k+1}, and adding the arc between w_k and v_δ, the arc between w_{k+1} and $v_{\delta-2}$, and the longer oriented path of C_2' from v_δ to $v_{\delta-2}$. We have

$$\sigma_{\max}(C) \geq \sigma^+(C) \geq n-2\delta+2-1+\delta-3 = n-\delta-2.$$

■

If $\delta(D[W]) = n-2\delta+2$, then w_k is not adjacent to at most two vertices in $V(C_2) \setminus \{v_0\}$. By the assumptions before, w_k is not adjacent to exactly two vertices v_{i_1} and v_{i_2} in $V(C_2) \setminus \{v_0\}$, in which $i_1 < i_2$. Since w_{k+1} is adjacent to at least two vertices in $V(C_2) \setminus \{v_0\}$, we may assume that v_{j_1} and v_{j_2} are adjacent to w_{k+1} and $j_1 < j_2$. By Claims 3.1 and 3.3, we may assume that $i_1 = j_1+1 = j_2-1$ and $i_2 = j_2+1$, or $i_1 = j_1-1$ and $i_2 = j_1+1 = j_2-1$. In the former case, we may assume $j_1 = 1$ by Claim 3.1, and in the latter case, we may assume $j_2 = \delta$ by Claim 3.3. Therefore, we have that $(j_1, i_1, j_2, i_2) = (1, 2, 3, 4)$ or $(i_1, j_1, i_2, j_2) = (\delta-3, \delta-2, \delta-1, \delta)$. Then, both of w_k and w_{k+1} are adjacent to v_1 and v_3, or $v_{\delta-2}$ and v_δ. By Claims 3.4 and 3.5, we are done.

If $\delta(D[W]) \geq n - 2\delta + 3$, then assume that the vertices in $V(C_2)$ which are adjacent to w_{k+1} are $\{v_{j_1}, \ldots, v_{j_a}\}$, in which $a \geq 2$ and $j_1 < j_2 < \cdots < j_a$. By Claims 3.1 and 3.3, w_k is not adjacent to v_{j_1+1} or v_{j_1-1}. Since w_k must be adjacent to at least two vertices in $V(C_2) \setminus \{v_0\}$, we can choose v_{i_1} in $V(C_2) \setminus \{v_0\}$ that is adjacent to w_k such that $i_1 < j_1 - 1$ or $i_1 > j_1 + 1$. In both cases, v_{i_1} and v_{j_1+1} are distinct vertices, and v_{i_1} and v_{j_1} are also distinct vertices. Without loss of generality, assume $i_1 < j_1 - 1$. Let C_2' be an oriented cycle which is obtained from C_2 in Lemma 3.1, by exchanging v_{i_1} and v_{j_1+1} in C_2. By Lemma 3.1, $\sigma^+(C_2') \geq \delta - 4$. Note that the longer oriented path on C_2' from v_{i_1} to v_{j_1} has at least $\delta - 4$ forward arcs by the construction of C_2'. Consider $C' = w_1 \ldots w_k v_{i_1} v_{j_1+2} v_{j_1+3} \ldots v_\delta v_0 v_1 \ldots v_{i_1-1} v_{j_1+1} v_{i_1+1} \ldots v_{j_1} w_{k+1} \ldots w_{n-\delta-1} w_1$, which is obtained from C_1 by deleting the arc between w_k and w_{k+1}, and adding the arc between w_k and v_{i_1}, the arc between w_{k+1} and v_{j_1}, and the longer oriented path of C_2' from v_{i_1} to v_{j_1}. Thus, $\sigma_{\max}(C) \geq \sigma^+(C) \geq n-2\delta+3-1+\delta-4 = n - \delta - 2$. □

4. Proof of Theorem 1.8

The following lemma informs us of a lower bound on the size $a(D)$ of the arc set of D. For an undirected graph G, let $e(G) = |E(G)|$.

Lemma 4.1. *Let D be an oriented graph and t a nonnegative integer. If $s^*(D) \geq t$, then $a(D) \geq \frac{n(n+t)}{4}$.*

Proof. As D is an oriented graph, we only need to show that the underlying graph G of D has at least $\frac{n(n+t)}{4}$ edges. Consider the complement $\overline{G}$ of G. On the one hand, since $s^*(D) \geq t$, we have that

$$\begin{aligned} \sum_{uv \in E(\overline{G})} (d_{\overline{G}}(u) + d_{\overline{G}}(v)) &= \sum_{uv \notin E(G)} (2n - 2 - (d_G(u) + d_G(v))) \\ &\leq e(\overline{G})(n - 2 - t). \end{aligned} \tag{4.1}$$

On the other hand, by the Cauchy-Schwarz inequality, we have that

$$\sum_{uv \in E(\overline{G})} (d_{\overline{G}}(u) + d_{\overline{G}}(v)) = \sum_{u \in V(\overline{G})} d_{\overline{G}}^2(u) \geq \frac{1}{n} \left(\sum_{u \in V(\overline{G})} d_{\overline{G}}(u) \right)^2 = \frac{4e^2(\overline{G})}{n}. \tag{4.2}$$

By (4.1) and (4.2), we have $e(\overline{G}) \leq n(n - 2 - t)/4$, and therefore $e(G) = \binom{n}{2} - e(\overline{G}) \geq \frac{n(n+t)}{4}$, which completes the proof. □

We call an undirected graph G a diamond, if G is isomorphic to K_4^-, where K_4^- is obtained from K_4 by removing one edge. Let $V(G) = \{a, b, c, d\}$, $d(a) = d(b) = 3$, and $d(c) = d(d) = 2$. Given an orientation of G, it is a good diamond if the following two conditions hold: (i) either both c and d have arcs directed toward a, or a has arcs directed toward both c and d; (ii) either both c and d have arcs directed toward b, or b has arcs directed toward both c and d (see Figure 3). Let D be a good diamond. Note that there are oriented paths P_{cd} and P_{dc} of length 3 from c to d and from d to c, respectively, in D, such that $\sigma^+(P_{cd}) \geq 2$ and $\sigma^+(P_{dc}) \geq 2$.

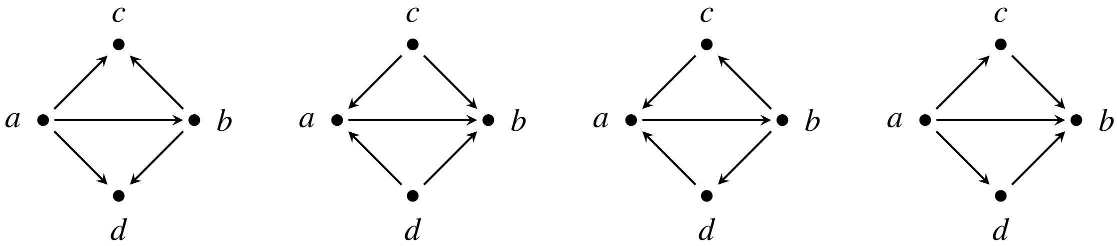


Figure 3. The four good diamonds.

A path forest is a graph in which every component is a path. The following lemmas can be found in [6]. The second lemma is a slight generalization of a result by Pósa [13]. Pósa's result was stated for graphs with $\delta(G) \geq (n+k)/2$, but a slight modification of his proof allows us to replace the minimum-degree condition with an Ore-degree condition. For the sake of completion, we provide a self-contained proof here.

Lemma 4.2. *[6] Let $k \geq 1$, and let G be a graph with $|V(G)| = n \geq 30 + 4(k-1)$ and $e(G) \geq \frac{n^2}{4} + 2(k-1)n - 4k^2 + 6k - 1$. Then, every orientation of G contains k vertex-disjoint good diamonds.*

For any n-vertex graph G, define $s^*(G) = \min\{d(u) + d(v) - n : uv \notin E(G)\}$ if G is not complete, and let $s^*(G) = n - 2$ otherwise.

Lemma 4.3. *Let $t \geq 0$ and let G be a graph with n vertices and $s^*(G) \geq t$. Let $E \subseteq E(G)$ be the edge set of a path forest of size at most t. Then, there exists a Hamilton cycle in G which uses all edges in E.*

Proof. Suppose G is a maximal counterexample to this lemma, which means by adding any new edge uv, the lemma holds for $G + uv$. Since G is a counterexample, there is a path forest $E \subseteq E(G)$ of size at most t, which is not contained in any Hamilton cycle of G. Note that G cannot be a complete graph as otherwise this lemma would have held for G. Let v_1 and v_n be a pair of non-adjacent vertices. As the lemma holds for $G + v_1v_n$ but not for G, there is a Hamilton cycle $C = v_1v_2 \ldots v_nv_1$ where $E \subseteq E(C)$. Let P be the Hamilton path $P = C - \{v_1v_n\}$ in G. Note that $E \subseteq E(P)$ as $E \subseteq E(G)$.

For a property $\mathcal{P}$, let $I(\mathcal{P})$ be the indicator of $\mathcal{P}$, i.e., $I(\mathcal{P}) = 1$ if $\mathcal{P}$ holds and $I(\mathcal{P}) = 0$, otherwise. For every $i \in [n]$, let $I(v_1v_i)$ and $I(v_nv_i)$ be the indicators for whether $v_1v_i \in E(G)$ and $v_nv_i \in E(G)$, respectively. Then, $I(v_1v_i) + I(v_nv_{i-1}) \leq 2$ for every $v_{i-1}v_i \in E(P)$. Let $S = E(P) \setminus E$. Thus, $|S| = n - 1 - |E|$. Note that for every $v_{i-1}v_i \in S$, we have that

$$I(v_1v_i) + I(v_nv_{i-1}) \leq 1, \tag{4.3}$$

as it is obvious if $i = 2$ or n (since $I(v_1v_n) = 0$); and if $i \in \{3, 4, \ldots, n-1\}$ and $I(v_1v_i) + I(v_nv_{i-1}) = 2$, then $v_1v_iv_{i+1} \ldots v_nv_{i-1}v_{i-2} \ldots v_1$ is a Hamilton cycle containing all edges in E, a contradiction. Thus, on the one hand, by (4.3), we have that

$$\sum_{v_{i-1}v_i \in E(P)} (I(v_1v_i) + I(v_nv_{i-1})) \leq |S| + 2|E| = n + |E| - 1 \leq n + t - 1. \tag{4.4}$$

On the other hand,

$$\sum_{v_{i-1}v_i \in E(P)} (I(v_1v_i) + I(v_nv_{i-1})) = \sum_{i=1}^{n-1} I(v_nv_i) + \sum_{i=2}^{n} I(v_1v_i) = d(v_n) + d(v_1) \geq n + t,$$

which contradicts (4.4). This completes the proof. $\square$

Now we give the proof of Theorem 1.8.

Proof of Theorem 1.8. Let D be an oriented graph with order $n \geq 30 + 4(k-1)$ and $s^*(D) \geq 8k$. By Lemma 4.1, we have

$$a(D) \geq \frac{n(n+8k)}{4} \geq \frac{n^2}{4} + 2(k-1)n - 4k^2 + 6k - 1.$$

Thus, by Lemma 4.2, D contains k vertex-disjoint good diamonds. Let c_i and d_i be the two vertices with degree 2 on diamond i, and let P_i and Q_i be oriented paths of length 3 from c_i to d_i and d_i to c_i, respectively, satisfying $\sigma^+(P_i) \geq 2$ and $\sigma^+(Q_i) \geq 2$. By Lemma 4.3, there is a Hamilton oriented cycle C containing all arcs in P_i (for all $i \in [k]$). Without loss of generality, we assume that most of the arcs in $A(C) \setminus (\cup_{i=1}^{k} A(P_i))$ are forward, that is, at least $\frac{n-3k}{2}$ arcs are forward. Then, if necessary, we replace some P_i with Q_i with most of the arcs forward. Thus, P_i or Q_i contributes two forward arcs to the new oriented cycle C. Thus, there are at least $\frac{n-3k}{2} + 2k \geq \frac{n+k}{2}$ forward arcs and, therefore, $\sigma_{\max}(C) \geq \frac{n+k}{2}$. □

5. Further research of digraph discrepancy

Very recently, Chang et al. [14] proved an asymptotic version of Conjecture 1.6: For every oriented graph D of sufficiently large order n and with $s^*(D) \geq 0$, we have

$$\sigma_{\max}(D) \geq \max\{n/2, (n + s^*(D))/2 - o(n)\}.$$

Also, very recently Gerke et al. [15] considered digraphs D with the independence number $\alpha(D)$ bounded from above by a constant k. They proved that if $k = 2$, then (i) each such D with connected $U(D)$ has a Hamilton oriented path with at most two backward arcs, and (ii) each such D with 2-connected $U(D)$ has a Hamilton oriented cycle with at most five backward arcs. They left it as an open question of whether there is a function f such that every digraph D with $\alpha(D) \leq k$, which has a Hamilton oriented cycle, has one with at most $f(k)$ backward arcs. Note that if such a function f does exist, then it does exist also for the similar question where instead of Hamilton oriented cycles, we consider Hamilton oriented paths.

Guo et al. [16] took the study of digraph discrepancy to a different direction: They investigated the maximum number of forward arcs in Hamilton oriented paths and cycles in digraphs in some classes of generalizations of tournaments. For semicomplete multipartite digraphs and locally semicomplete digraphs, they proved related characterizations leading to polynomial-time algorithms for finding Hamilton oriented paths and cycles with the maximum number of forward arcs. A digraph is semicomplete multipartite if it can be obtained from a complete multipartite graph by replacing every edge with an arc with the same end-vertices. A digraph is semicomplete if it is semicomplete multipartite in which there is only one vertex in each color class. A digraph is locally semicomplete if the out-neighborhood and in-neighborhood of every vertex induce semicomplete digraphs. Guo et al. [16] also observed that for several other classes of digraphs, the problems of maximizing the number of forward arcs in Hamilton oriented cycles are NP-hard, since even the problems of determining the existence of Hamilton oriented cycles in those digraphs are NP-hard.

London [17] introduced the notion of oriented anti-discrepancy and studied it, together with oriented discrepancy, for several natural families of subgraphs of oriented graphs. In particular, let G be an

Eulerian graph and let D be an orientation of G. For an Euler circuit T of G, let $\sigma_D^+(T)$ and $\sigma_D^-(T)$ denote, respectively, the numbers of edges of T traversed in the same direction as, and in the opposite direction to, their orientation in D. The Euler anti-discrepancy of D is

$$\min_T \left|\sigma_D^+(T) - \sigma_D^-(T)\right|,$$

where the minimum is taken over all Euler circuits T of G. The oriented Euler anti-discrepancy of G is the maximum of this quantity over all orientations D of G. London [17] proved that the oriented Euler anti-discrepancy of every Eulerian graph on n vertices is at most n.

6. Conclusions

In this paper, we investigated Ore-type extensions of the oriented-discrepancy version of Dirac's theorem for Hamilton oriented cycles. We formulated Conjectures 1.2 and 1.6 and gave constructions showing that their proposed lower bounds are best possible. As evidence for Conjecture 1.2, we proved its conclusion under the stronger degree-sum condition $d(u) + d(v) \geq n + \delta - 2$, which in particular establishes the conjecture when $\delta = 2$. Under the original Ore-type condition, we also obtained the bound $n-\delta-2$ when the neighborhood of a minimum-degree vertex induces a tournament. As evidence for Conjecture 1.6, we proved that if $n \geq 30 + 4(k-1)$ and $s^*(D) \geq 8k$, then D contains a Hamilton oriented cycle C satisfying

$$\sigma_{\max}(C) \geq \left\lceil \frac{n+k}{2} \right\rceil.$$

These results show that Ore-type degree-sum conditions can force not only Hamiltonicity but also a substantial directional imbalance in a Hamilton oriented cycle. The principal remaining challenge is to prove Conjectures 1.2 and 1.6 in full. In particular, it would be interesting to weaken the additional hypotheses in Theorems 1.4 and 1.5 and to reduce the factor 8 in the hypothesis of Theorem 1.8.

Author contributions

Jiangdong Ai: Validation, writing – review & editing; Qiwen Guo: Investigation, writing – original draft, writing – review & editing; Gregory Gutin: Conceptualization, supervision, writing – review & editing; Yongxin Lan: Validation, writing – review & editing; Qi Shao: Investigation, writing – original draft; Anders Yeo: Investigation, validation, writing – review & editing; Yacong Zhou: Investigation, writing – original draft, writing – review & editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

JA is partially supported by the National Natural Science Foundation of China (No.12401456, No.12522117) and the Natural Science Foundation of Tianjin (No.24JCQNJC01960). QG is partially supported by China Scholarship Council (No.202406200159). YL is partially supported by China Scholarship Council (No.202308130157).

Conflict of interest

The authors declare no conflicts of interest.

References

1. J. Matoušek, *Geometric discrepancy*, Berlin: Springer, 1999. https://doi.org/10.1007/978-3-642-03942-3
2. P. Erdős, Ramsey és Van der Waerden tételével Kapcsolatos Kombinatorikai Kérdésekről, *Mat. Lapok*, **14** (1963), 29–37.
3. J. Balogh, B. Csaba, Y. Jing, A. Pluhár, On the discrepancies of graphs, *Electron. J. Comb.*, **27** (2020), P2.12. https://doi.org/10.37236/8425
4. J. Balogh, B. Csaba, A. Pluhár, A. Treglown, A discrepancy version of the Hajnal–Szemerédi theorem, *Comb. Probab. Comput.*, **30** (2021), 444–459. https://doi.org/10.1017/S0963548320000516
5. A. Freschi, J. Hyde, J. Lada, A. Treglown, A note on colour-bias Hamilton cycles in dense graphs, *SIAM J. Discrete Math.*, **35** (2021), 970–975. https://doi.org/10.1137/20M1378983
6. L. Gishboliner, M. Krivelevich, P. Michaeli, Oriented discrepancy of Hamilton cycles, *J. Graph Theory*, **103** (2023), 780–792. https://doi.org/10.1002/jgt.22947
7. A. Freschi, A. Lo, An oriented discrepancy version of Dirac's theorem, *J. Combin. Theory Ser. B*, **169** (2024), 338–351. https://doi.org/10.1016/j.jctb.2024.06.008
8. J. Bang-Jensen, G. Z. Gutin, *Digraphs: Theory, algorithms and applications*, 2 Eds., London: Springer, 2009. https://doi.org/10.1007/978-1-84800-998-1
9. J. Bang-Jensen, G. Gutin, Basic terminology, notation and results, In: *Classes of directed graphs*, Cham: Springer, 2018, 1–34. https://doi.org/10.1007/978-3-319-71840-8_1
10. J. A. Bondy, U. S. R. Murty, *Graph theory*, London: Springer, 2008.
11. G. A. Dirac, Some theorems on abstract graphs, *Proc. Lond. Math. Soc.*, **s3-2** (1952), 69–81. https://doi.org/10.1112/plms/s3-2.1.69
12. O. Ore, Note on Hamiltonian circuits, *Am. Math. Mon.*, **67** (1960), 55. https://doi.org/10.2307/2308928
13. L. Pósa, On the circuits of finite graphs, *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, **8** (1963), 355–361.
14. Y. F. Chang, Y. Y. Cheng, Z. L. Wang, S. Wei, J. Yan, An Ore-type theorem for oriented discrepancy of Hamilton cycles, *arXiv:2603.18915*, 2026. https://doi.org/10.48550/arXiv.2603.18915

15. S. Gerke, Q. Guo, G. Gutin, Y. Hao, W. Veeranonchai, A. Yeo, Backward arcs in Hamilton oriented cycles and paths in directed graphs with independence number two, *arXiv:2603.22993*, 2026. https://doi.org/10.48550/arXiv.2603.22993

16. Q. Guo, G. Gutin, Y. Lan, Q. Shao, A. Yeo, Y. Zhou, Forward arc maximization for Hamilton oriented cycles and paths in generalizations of tournaments, *arXiv:2602.10713*, 2026. https://doi.org/10.48550/arXiv.2602.10713

17. A. London, Notes on discrepancy and anti-discrepancy in oriented graphs, *Discr. Appl. Math.*, **380** (2026), 283–289. https://doi.org/10.1016/j.dam.2025.10.019