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Research article

An Aitken method for accelerating the solution of the non–symmetric algebraic Riccati equation in transport theory

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Abstract: In this paper, we study the accelerated solution of a class of non–symmetric algebraic Riccati equations arising from transport theory. Based on the simple iterative method, the Aitken method is introduced to solve the above special type of Riccati equation quickly, which improves the convergence speed to some extent. However, in some cases $(\alpha, c) = (0, 1)$, the convergence speed decreases. Therefore we construct new methods with Aitken acceleration structures to preserve or improve the convergence speed based on the shift technique. Theoretical analysis and numerical experiments show the effectiveness of the method. Finally, we conclude that the generality of Aitken's method is that it can improve the convergence speed of iterative methods for other fixed-point equations.

Keywords: non–symmetric algebraic Riccati equation; Aitken method; Shift technique
Mathematics Subject Classification: 15A24, 65F10

1. Introduction

In transport theory, the scattering function for half–space particle transport can be derived from the single-group neutron transport equation [1]. This scattering function satisfies the following integral differential equation:

$$\left(\frac{1}{\mu+\alpha}+\frac{1}{\nu-\alpha}\right)X(\mu,\nu)=c\left(1+\frac{1}{2}\int_{-\alpha}^{1}\frac{X(\omega,\nu)}{\omega+\alpha}d\omega\right)\left(1+\frac{1}{2}\int_{\alpha}^{1}\frac{X(\mu,\omega)}{w-\alpha}d\omega\right) \tag{1.1}$$

where $X(\mu,\nu)$ is the scattering function and $(\mu,\nu)\in[-\alpha,1]\times[\alpha,1]$. When $c=0$ or $\alpha=1$, the integral differential Eq (1.1) has a trivial solution. When $0<c\leq 1$ and $0\leq\alpha<1$, Juang [2] has shown that Eq (1.1) has a unique positive solution. A discretization of the integral differential Eq (1.1) transforms the problem of solving the scattering function $X(\mu,\nu)$ for particle transport into the problem of solving

the solution of the nonsymmetric algebraic Riccati Eq (1.2) (NARE) [3]. In this paper, we will study the accelerated solution of (1.2) by:

$$XCX - XD - AX + B = 0 \tag{1.2}$$

with

$$A = \Delta - eq^T,\ B = ee^T,\ C = qq^T,\ D = \Gamma - qe^T,$$

where

$$\begin{aligned}
&e = (1, ..., 1)^T \in \mathbb{R}^n \\
&q = (q_1, ..., q_n)^T \in \mathbb{R}^n,\ q_i = \frac{c_i}{2\omega_i} \\
&\Delta = \mathrm{diag}\,(\delta_1, ..., \delta_n),\ \delta_i = \frac{1}{c\omega_i\,(1+\alpha)} \\
&\Gamma = \mathrm{diag}\,(\gamma_1, ..., \gamma_n),\ \gamma_i = \frac{1}{c\omega_i\,(1-\alpha)}.
\end{aligned}$$

The parameters α, c require $0 \le \alpha < 1$ and $0 < c \le 1$. The sequences $\{\omega_i\}, \{c_i\}$ satisfy $0 < \omega_n < ... < \omega_1 < 1$, $0 < c_i$ and $\sum_{i=1}^{n} c_i = 1$, $i = 1, ..., n$.

The numerical solution of the NARE problem is a topical problem of wide interest in computational mathematics. It appears in many engineering applications, such as transport theory [2–4], Wiener-Hopf decomposition of Markov chains [5, 6], spectral decomposition of rational matrix functions [7], etc.

In these applications, researchers have mainly focused on the minimal positive solution of NARE (1.2) [8]. For Eq (1.2), the existence of its minimal positive solution and the algorithm has been discussed and studied by many scholars, and a series of results have been obtained. In Ref. [9], Juang proved the existence of non-negative solutions to Eq (1.2) by using the theory of congruences and solved the convergence problem of some iterative formats for solving such equations. Further, they investigate the existence of positive solutions through invariant subspace theory [3].

The Gauss-Jacobi iterative method was proposed by Shimizu and Aoki in the literature [10], and Nelson proved its monotonic convergence in the literature [11]; In Ref. [12], Juang and Lin proposed the Gauss-seidel iterative method and proved its monotonic convergence; To simplify the calculation, Lu et al. [13] reformulated NARE (1.2) in vector form according to the special of the solution of the Eq (1.2) of a pair of immovable point equations, constructed a simple iterative method (SI) for finding its minimal positive solution, and obtained upper and lower bounds for the minimal positive solution; Bao [14] constructed the modified simple iteration (MSI); Based on Lu's work, Bai and Gao [15] improved the fixed-point iteration method even further by proposing two implicit acceleration variants, the nonlinear block Jacobi (NBJ) and the nonlinear block Gauss-Seidel iteration method (NBGS), and obtaining more accurate upper and lower bounds for the equation's minimal positive solution. In recent years, the Newton method [16] and the alternating-directional doubling method (ADD) [17] have been used to solve this equation, and both theoretical and numerical tests have yielded more satisfactory results.

When $\alpha \neq 0$ and $c \neq 1$, the SI, NBJ, and NBGS methods converge linearly; When $\alpha = 0$ and $c = 1$, the convergence rate of these methods degrades or does not converge [18]. We call this situation the "critical case" of the NARE problem. The critical case is another challenging problem for solving (1.2). Many scholars have studied how to speed up convergence in the critical case. For example, Guo and

Lin [19] proposed a shift method for solving the NARE problem in the critical case with a minimal positive solution; Lin and Chiang [20] based on Guo's work by removing all singularities to obtain the double-shift method and verified that the convergence speed of the double-shift is much faster than that of the single-shift or no-shift method.

The requisite precision can be attained for the fixed-point iteration equation $x = \varphi(x)$ with a sufficient number of iterations. A straightforward and effective extrapolation technique is the Aitken approach [21, 22]. In this study, we are interested in improving the deterioration of an iterative technique's convergence speed in the critical case and accelerating an iterative method like SI with the Aitken method in the non-critical case. Additionally, we use the single-shift methodology in the crucial case and use the Aitken method to confirm faster convergence.

The remainder of the paper is structured as follows. Section 2 introduces the simple iteration method and single-shift technique. Section 3 applies the Aitken method to simple iterative and single-shift techniques. In addition, the relevant theoretical analysis is also given. Section 4 includes some numerical experiments to show that our proposed speed-up strategy is both effective and practical. Section 5 contains the final evaluation and related generalizations.

2. Preliminaries and some results

The simple iterative method proposed by Lu [13] and the shift technique proposed by Guo [19] are all discussed in this section. We also discuss their advantages and disadvantages.

2.1. Simple iterative

In [9], Juang brings the specific form of the NARE coefficient matrix into (1.2) to obtain the fixed–point Eq (2.1) and gives the expression for the solution X^* (the Gauss-Jacobi method):

$$N \circ X^* = (X^*q + e)\left(q^T X^* + e^T\right), \quad X^* = T \circ (X^*q + e)\left(q^T X^* + e^T\right) \tag{2.1}$$

where

$$N = \left(n_{ij}\right) = \delta_i + \gamma_j, \;\; T = \left(t_{ij}\right) = \frac{1}{\delta_i + \gamma_j}$$

and $\circ$ is the Hadamard product symbol.

Let $u = X^*q + e, v = q^T X^* + e^T$. On the results of Juang's work, it can be seen that the minimal positive solution of NARE (1.2) requires finding suitable positive vectors u and v. From this point of view, Lu rewrite the system of vector Eq (2.1) in the equivalent form:

$$\begin{cases} u = u \circ (Pv) + e \\ v = v \circ (Qu) + e \end{cases} \tag{2.2}$$

with

$$P = \left(p_{ij}\right) = \frac{q_j}{\delta_i + \gamma_j}, \quad Q = \left(q_{ij}\right) = \frac{q_j}{\delta_j + \gamma_i}. \tag{2.3}$$

Based on the vector Eq (2.2), Lu constructed the simple iterative method (SI) for solving Eq (1.2):

Algorithm 1 The simple iteration method (SI) [13]

Input: Parameters: vectors e and q; parameters α, c; sequences w_i and c_i $(i = 1, ..., n)$. The error ϵ, and the maximum number of iterations K_{max}.

Output: u^*, v^*.

1: Calculate the matrices P and Q according to (2.3).
2: **Initalization:** u^1, v^1.
3: **for** k = 1,2,... **do**
4: $u^{k+1} = u^k \circ \left(Pv^k\right) + e$
5: $v^{k+1} = v^k \circ \left(Qu^k\right) + e$
6: $\mathrm{Re}s = \frac{\|u^{k+1}-u^k\|_\infty + \|v^{k+1}-v^k\|_\infty}{\|u^{k+1}\|_\infty + \|v^{k+1}\|_\infty}$
7: if $Res \leq \epsilon$ or $k \geq K_{max}$, break;
8: **endfor**

The simple iterative method does not essentially change the structure of the nonsymmetric algebraic Riccati equation, but merely reformulates it as a pair of immovable point equations in vector form, depending on the particular form of the solution. It is not difficult to see that the vector equations are simpler and easier to solve than NARE.

Lemma 2.1. *[13, Lemma 1] For* $0 \leq \alpha < 1$ *and* $0 < c \leq 1$*, the sequence of iterations* $\left\{u^{(k)}, v^{(k)}\right\}$ *generated by Algorithm 1 is strictly monotonically increasing and bounded above.*

Lemma 2.1 shows that Algorithm 1 is convergent. Detailed proof of this lemma can be found in the literature [13, Lemma 1]. The simple iterative method converges to a pair of positive vectors and Theorem 2.1 will show that the minimal positive solution to Eq (1.2) can be obtained from Algorithm 1.

Theorem 2.1. *[13, Theorem 2] The minimal positive solution* X^* *for NARE (1.2) can be obtained by the following formula:*

$$X^* = T \circ (u^*(v^*)^T)$$

where (u^*, v^*) *is the limit point at which the iterations of Algorithm 1 converge.*

Comparing the simple iteration and Gauss-Jacobi iteration methods, although they have the same procedure for solving the minimal positive solution of NARE, they have different computational costs, requiring $6n^2$ flops per step for the Gauss-Jacobi iteration and $4n^2$ flops per step for the simple iteration. However, in the critical case $(\alpha, c) = (0, 1)$, the speed of these iterations degrades or does not converge. We will verify this result in Section 4.

2.2. Shift technique

He et al. [23] first suggested using the shifting method to compute the discrete-time quasi-birth-death (QBD) Markov chains random matrix G, and Guo later used it to solve the Riccati equation [19].

Let

$$H = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}$$

where A, B, C, D are as defined as in (1.2). When NARE (1.2) is in the critical case, the eigenvalues of the matrix H have the following result:

Lemma 2.2. *[3, Lemma 2.1] When $\alpha = 0$ and $c = 1$, the matrix H has only real eigenvalues $\{-\lambda_n, ..., -\lambda_1, \lambda_1, ..., \lambda_n\}$, which are arranged in ascending order, satisfying the following inequality:*

$$-\delta_n < -\lambda_n < -\delta_{n-1} < ... < -\delta_1 < -\lambda_1 = 0 = \lambda_1 < \gamma_1 < \lambda_2 < ... < \lambda_n < \gamma_n.$$

The proof of Lemma 2.2 can be found in Ref. [3]. It shows that the matrix H has two zero eigenvalues, which is essentially the reason for the degradation of the convergence rate of the SI, NBI, and NBGS methods. The purpose of the shift technique is to apply a first-order correction to the matrix H by replacing one of the zero eigenvalues with an appropriate non-zero eigenvalue. Guo and Li used the shift technique to change the zero eigenvalues of the matrix H and solve the problem of the degradation of the convergence rate of the critical case. The implementation of the shift technique is to be based on the following results:

Lemma 2.3. *[13, Lemma 3.1] Let T be a singular matrix and $Tw = 0$ for a nonzero vector w. Assume that r is a vector with $r^T w = 1$ and η is a scalar. Then the eigenvalues of the matrix*

$$\widetilde{T} = T + \eta w r^T$$

are those of T except that one zero eigenvalue of T is replaced by η.

Remark 2.1. *The parameter η in Lemma 2.3 must satisfy $0 < \eta < \gamma_1$, where $\gamma_1 = \frac{1}{c\omega_1(1-\alpha)}$ is the smallest positive eigenvalue of T. This condition ensures that the shifted matrix $\tilde{T}$ remains nonsingular and that the zero eigenvalue is replaced by a positive value $\eta\eta$ lying strictly within the spectral gap. The value of η directly influences the convergence rate: a larger η (closer to γ_1) reduces the spectral radius of the iteration matrix, thereby accelerating convergence, but η must not reach or exceed γ_1 as this would destroy the spectral gap and may lead to divergence or severe instability.*

In the critical case, the left and right eigenvectors of the matrix H corresponding to the 0 eigenvalues are

$$\mu^T = \left[\mu_1^T \ \mu_2^T\right] = \left[\left(\Gamma^{-1}e\right)^T \ \left(-\Delta^{-1}q\right)^T\right]$$

and

$$\upsilon = \left[\upsilon_1^T \ \upsilon_2^T\right]^T = \left[\left(\Gamma^{-1}q\right)^T \ \left(-\Delta^{-1}e\right)^T\right]^T.$$

It can be shown that $\mu^T\upsilon = 0$ and let

$$p^T = \left[e^T \ q^T\right],$$

then we have $p^T\upsilon = 1$. By Lemma 2.3, they then construct the rank one correction to the matrix H:

$$\widetilde{H} = H + \eta v p^T$$

where $0 < \eta < \gamma_1$ is a scalar. One of the zero eigenvalues of the modified $\widetilde{H}$ is replaced by η. The new Riccati equation defined by the matrix $\widetilde{H}$ is then as follows:

$$X\widetilde{C}X - X\widetilde{D} - \widetilde{A}X + \widetilde{B} = 0 \tag{2.4}$$

where

$$\widetilde{A} = A - \eta v_2 q^T,\ \widetilde{B} = B + \eta v_2 e^T,\ \widetilde{C} = C - \eta v_1 q^T,\ \widetilde{D} = D + \eta v_1 e^T.$$

The literature [13, Theorem 3.1] shows that NARE (2.4) and the NARE (1.2) are equations with the same solution. When $\alpha = 0$ and $c = 0$, there is no critical case for NARE (2.4). Therefore NARE (2.4) can be solved using simple iterations. A simple iterative method introducing the shift technique is given about the construction of Algorithm 2:

Algorithm 2 The Shift technique applied to simple iterations (SSI) [19]

Input: Parameters: vectors e and q; parameters α, c and η; sequences w_i and c_i (i=1,...,n). The error ϵ , and the maximum number of iterations K_{max}.

Output: u^*, v^*.

1: Calculate the vectors $\widetilde{e} = e + \eta v_2$, $\widetilde{q} = q - \eta v_1$ and the matrices Q and $\widetilde{P} = \frac{\widetilde{q}_j}{\delta_i + \gamma_j}$.
2: **Initalization:** u^1, v^1.
3: **for** k = 1,2,... **do**
4: $uk + 1 = v^k \circ \left(\widetilde{P} v^k\right) + \widetilde{e}$
5: $v^{k+1} = v^k \circ \left(Q u^k\right) + \widetilde{e}$
6: $\mathrm{Re}s = \frac{\|u^{k+1} - u^k\|_\infty + \|v^{k+1} - v^k\|_\infty}{\|u^{k+1}\|_\infty + \|v^{k+1}\|_\infty}$
7: if $Res \le \epsilon$ or $k \ge K_{max}$, break;
8: **endfor**

As can be seen from Algorithm 2, the shift technique can also be introduced into other immobile point iteration methods. Later some scholars studied the shift technique in depth to propose another double-shift technique that effectively solves the critical case [18].

3. Application of the Aitken method

In this section, we introduce the Aitken method on top of the simple iterative method to speed up, while improving the SI in the critical case where the convergence speed degrades. In addition, we also introduce the Aitken method in the shift technique, while analyzing the advantages of the new method in terms of iteration speed.

Aitken method is a method for accelerating the convergence of sequences in numerical analysis, popularised by the mathematician Aitken. Due to its simplicity and effectiveness, it is one of the most popular convergence acceleration algorithms available. the Aitken method converts a known iterative sequence $\left\{s^{(k)}\right\}$ into another sequence $\left\{t^{(k)}\right\}$ defined by: $T : s^k \mapsto t^k$.

The aim is to make the iterative sequence $\left\{t^{(k)}\right\}$ converge more quickly to the limit of the initial iterative sequence $\left\{s^{(k)}\right\}$. The exact construction procedure is as follows:

Assume that the iterative formula for the immobile point is $x_{k+1} = \varphi(x_k)$ and that x^* is the exact solution. For the known x_k, make two corrections using the iterative formula: $x_{k+1} = \varphi(x_k)$ and $x_{k+2} = \varphi(x_{k+1})$.

Using the differential median theorem in the calculation of the two correction step errors gives the

following results:

$$x_{k+1} - x^* = \varphi(x_k) - \varphi(x^*) = \varphi'(t)(x_k - x^*)$$
$$x_{k+2} - x^* = \varphi(x_{k+1}) - \varphi(x^*) = \varphi'(t)(x_{k+1} - x^*).$$

When

$$k \to \infty, \frac{x_{k+2} - x^*}{x_{k+1} - x^*} \approx \frac{x_{k+1} - x^*}{x_k - x^*}.$$

It can be solved for

$$x^* = \frac{x_k x_{k+2} - x_{k+1}^2}{x_{k+2} - 2x_{k+1} + x_k}.$$

The formula can be written as an iteration as follows, which gives the Aitken algorithm.

$$x_{k+1} = \phi(x_k) = x_k - \frac{(\varphi(x_k) - x_k)^2}{\varphi(\varphi(x_k)) - 2\varphi(x_k) + x_k}. \tag{3.1}$$

Algorithm 3 The Aitken method

Input: Parameters: x_k.
Output: x_{k+1}.

1: Calculate $\hat{x}_{k+1} = \varphi(x_k)$.
2: Calculate $\hat{x}_{k+2} = \varphi(\hat{x}_{k+1})$.
3: Calculate $x_{k+1} = x_k - \frac{(\hat{x}_{k+1} - x_k)^2}{\hat{x}_{k+2} - 2\hat{x}_{k+1} + x_k}$

In the simple iterative method, the sequence u^k, v^k obtained by iteration of Algorithm 1 is transformed into the sequence generated by iteration of Algorithm 3 . This gives a new iterative algorithm:

Algorithm 4 Simple iterative method of applying Aitken method (ASI)

Input: Parameters: vectors e and q; parameters α, c; sequences w_i and c_i (i=1,...,n). The error ϵ , and the maximum number of iterations K_{max}.
Output: u^*, v^*.

1: Calculate the matrices P and Q according to (2.3).
2: **Initalization:** u^1, v^1.
3: **for** k = 1,2,... **do**
4: $u^{k+1} = Algorithm\ 3\ (u^k, v^k, P, e)$
5: $v^{k+1} = Algorithm\ 3\ (u^k, v^k, Q, e)$
6: $Res = \frac{\|u^{k+1} - u^k\|_\infty + \|v^{k+1} - v^k\|_\infty}{\|u^{k+1}\|_\infty + \|v^{k+1}\|_\infty}$
7: if $Res \le \epsilon$ or $k \ge K_{max}$, break;
8: **endfor**

The fourth step of Algorithm 4 calls the Aitken method with the immobility iteration

$$\varphi(u) = u \circ (Pv) + e;$$

the fifth step calls the Aitken method with the immobility iteration

$$\varphi(v) = v \circ (Qv) + e.$$

Compared to the original simple iteration method, Aitken uses two more prediction steps of information.

In the shift technique, the same transformation of the iterative sequence yields another new iterative algorithm. The iterative equation for the immobile point in the fourth step of Algorithm 5 is $\varphi(u) = u \circ \left(\widetilde{P}v\right) + \tilde{e}$; the iterative equation for the immobile point in the fifth step is $\varphi(v) = v \circ (Qu) + \tilde{e}$.

Algorithm 5 The Aitken and Shift technique applied to simple iterations (ASSI)

Input: Parameters: vectors e and q; parameters α, c and η; sequences w_i and c_i (i=1,...,n). The error ϵ , and the maximum number of iterations K_{max}.

Output: u^*, v^*.

1: Calculate the vectors $\widetilde{e} = e + \eta v_2$, $\widetilde{q} = q - \eta v_1$ and the matrices Q and $\widetilde{P} = \frac{\widetilde{q}_j}{\delta_i + \gamma_j}$.
2: **Initalization:** u^1, v^1.
3: **for** k = 1,2,... **do**
4: $u^{k+1} = Algorithm\ 3\ (u^k, v^k, \widetilde{P}, \widetilde{e})$
5: $v^{k+1} = Algorithm\ 3\ (u^k, v^k, Q, \widetilde{e})$
6: $\mathrm{Re}s = \frac{\|u^{k+1} - u^k\|_\infty + \|v^{k+1} - v^k\|_\infty}{\|u^{k+1}\|_\infty + \|v^{k+1}\|_\infty}$
7: if $Res \leq \epsilon$ or $k \geq K_{max}$, break;
8: **endfor**

Algorithms 4 and 5 are essentially equivalent in that they both substitute the Aitken method-generated sequence for the initial repetitive series of immobile points. As a result, the Aitken approach ought to be compatible in terms of acceleration for iterative sequences and the convergence theory. Theorem 3.1 provides the Aitken method's convergence theory.

Theorem 3.1. *Let $\phi : R \to R$ be a fixed-point iteration function with a fixed point x^*, i.e., $\phi(x^*) = x^*$. Assume that ϕ is differentiable at x^* and that $\phi'(x^*) \neq 1$. Define the Aitken-accelerated iteration function*

$$\mathcal{A}_\phi(x) = x - \frac{(\phi(x) - x)^2}{\phi(\phi(x)) - 2\phi(x) + x},$$

whenever the denominator is nonzero. Then x^* is a fixed point of $\mathcal{A}_\phi$ in the sense that

$$\lim_{x \to x^*} \mathcal{A}_\phi(x) = x^*.$$

Proof. Let $\lambda = \phi'(x^*)$. Since $\lambda \neq 1$, we can write the Taylor expansion of ϕ around x^*:

$$\phi(x) = x^* + \lambda(x - x^*) + o(|x - x^*|).$$

Consequently,

$$\phi(x) - x = (\lambda - 1)(x - x^*) + o(|x - x^*|),$$

and

$$\phi(\phi(x)) = \phi(x^* + \lambda(x - x^*) + o(|x - x^*|)) = x^* + \lambda^2(x - x^*) + o(|x - x^*|).$$

Thus,

$$\phi(\phi(x)) - 2\phi(x) + x = (\lambda^2 - 2\lambda + 1)(x - x^*) + o(|x - x^*|) = (\lambda - 1)^2(x - x^*) + o(|x - x^*|).$$

Substituting these into the definition of $\mathcal{A}_\phi(x)$, we obtain

$$\mathcal{A}_\phi(x) = x - (\lambda - 1)^2(x - x^*)^2 + o(|x - x^*|^2)(\lambda - 1)^2(|x - x^*|) + o(|x - x^*|).$$

Since $\lambda \neq 1$, the denominator is nonzero for xx sufficiently close to x^*. Hence,

$$\mathcal{A}_\phi(x) = x - (x - x^*) + o(|x - x^*|) = x^* + o(|x - x^*|),$$

which implies $\lim_{x \to x^*} \mathcal{A}_\phi(x) = x^*$. □

Theorem 3.1 shows that, provided the original method converges, the new sequence produced by Aitken is convergent to the limit point of the original sequence. Therefore, both Algorithm 4 and Algorithm 5, which introduce Aitken, are convergent as long as the convergence conditions of the SI and SSI methods are satisfied. Next, we discuss the speed of convergence.

Lemma 3.1. *If sequence $\{y_k\}$ is convergent and sequence $\{x_k\}$ converges to x^*. Sequence $\{y_k\}$ and sequence $\{x_k\}$ satisfy*

$$\lim_{k \to \infty} \frac{y_{k+1} - x^*}{x_k - x^*} = 0$$

then sequence $\{y_k\}$ converges to the limit point faster than sequence $\{x_k\}$.

Theorem 3.2. *Let $\{x_k\}$ be a sequence converging to x^* such that there exists a constant $c \in (0, 1)$ with*

$$\lim_{k \to \infty} \frac{|x_{k+1} - x^*|}{|x_k k - x^*|} = c, \ c \neq 1.$$

Define the Aitken-accelerated sequence $\{y_k\}$ by

$$y_{k+1} = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k},$$

where $x_{k+1} = \phi(x_k)$, $x_{k+2} = \phi(x_{k+1})$. Then

$$\lim_{k \to \infty} \frac{|y_{k+1} - x^*|}{|x_k - x^*|} = 0,$$

i.e., $\{y_k\}$ converges superlinearly to x^*. In fact, the convergence is asymptotically at least quadratic.

Proof. Let $e_k = x_k - x^*$. By the linear convergence assumption, we can write

$$e_{k+1} = (c + \epsilon_k)e_k,$$

where $\epsilon_k \to 0$ as $k \to \infty$. Consequently,

$$e_{k+2} = (c + \epsilon_k)(c + \epsilon_{k+1})e_k.$$

The Aitken error formula (see (3.2) in the paper) gives

$$y_{k+1} - x^* = \frac{e_{k+2}e_k - e_{k+1}^2}{e_{k+2} - 2e_{k+1} + e_k}.$$

Substituting the expressions above yields

$$\frac{y_{k+1} - x^*}{e_k} = \frac{(c + \epsilon_k)(c + \epsilon_{k+1}) - (c + \epsilon_k)^2}{(c + \epsilon_k)(c + \epsilon_{k+1}) - 2(c + \epsilon_k) + 1}.$$

Taking the limit $k \to \infty$(so $\epsilon_k, \epsilon_{k+1} \to 0$) gives

$$\lim_{k\to\infty} \frac{y_{k+1} - x^*}{e_k} = \frac{c^2 - c^2}{c^2 - 2c + 1} = 0,$$

because $c \neq 1$. Thus the ratio tends to zero, which proves superlinear convergence. The quadratic nature follows from the classical Aitken acceleration theory (see, e.g., Henrici [1964]). □

4. Experiments

In this section, we take a comparative experiment and solve two numerical cases by using the existing SI [13], MSI [14], and SSI [19] methods and a new algorithm introducing the Aitken technique, respectively. All numerical experiments are run in MATLAB-R2016 software, and the configuration of personal computer is as follows: Intel(R) Core(TM) I5-7200U CPU @ 2.50GHZ and 8.00GB RAM.

We use iterative steps (denoted as "IT"), CPU time in seconds (denoted as "CPU"), and error (denoted as "Res"), where Res is defined as

$$\text{Res} = \frac{\|u^{k+1} - u^k\|_\infty + \|v^{k+1} - v^k\|_\infty}{\|u^{k+1}\|_\infty + \|v^{k+1}\|_\infty}.$$

Because of the variability in parameter settings from experiment to experiment, the remaining parameters will be given separately in the corresponding sections.

Example 1. *The parameters of the NARE (1.2) coefficient matrix are taken with the following rules:*

$$w_i = \frac{n + 1 - i}{n + 1}, \quad c_i = \frac{1}{n}$$

where $i = 1, ..., n$ and n is the number of dimensions of NARE (1.2).

First, we solve the equation using the ASI, MSI, and SI methods to verify the validity and superiority of the ASI method. The parameters are set: $\alpha = 0.1, c = 0.9, \epsilon = 1e^{-12}, n = 1024$. Figure 1 shows that ASI is much faster than SI and MSI, and also shows that the introduction of Aitken is effective. Afterward, we take different parameters α, and c to illustrate the generalisability of the Aitken method. Tables 1–3 illustrate that the acceleration properties of Aitken for the original method do not change as the parameters are changed and are universal.

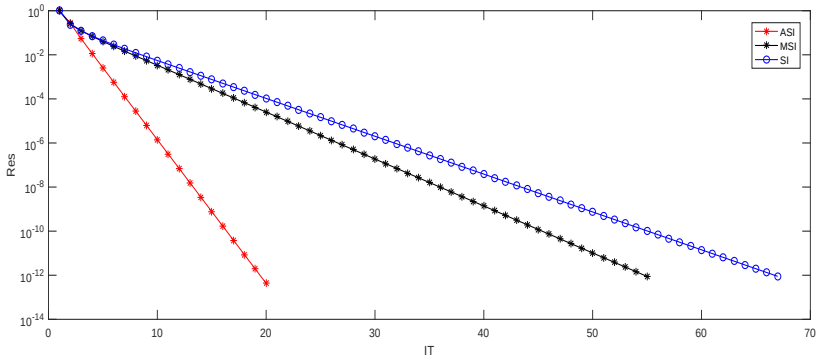

Figure 1. Convergence comparison of three methods in non-critical case.

Table 1. Iteration data of the SI method under dynamic parameters.

Parameters	IT	CPU	Res
$(\alpha, c) = (0.1,\ 0.9)$	67	0.1315	$8.926e^{-13}$
$(\alpha, c) = (0.2,\ 0.8)$	43	0.1040	$7.342e^{-13}$
$(\alpha, c) = (0.3,\ 0.7)$	32	0.0993	$5.590e^{-13}$
$(\alpha, c) = (0.4,\ 0.6)$	25	0.0911	$5.508e^{-13}$

Table 2. Iteration data of the MSI method under dynamic parameters.

Parameters	IT	CPU	Res
$(\alpha, c) = (0.1,\ 0.9)$	55	0.1258	$8.212e^{-13}$
$(\alpha, c) = (0.2,\ 0.8)$	36	0.1080	$7.501e^{-13}$
$(\alpha, c) = (0.3,\ 0.7)$	27	0.0948	$8.767e^{-13}$
$(\alpha, c) = (0.4,\ 0.6)$	22	0.0881	$5.387e^{-13}$

Table 3. Iteration data of the ASI method under dynamic parameters.

Parameters	IT	CPU	Res
$(\alpha, c) = (0.1,\ 0.9)$	20	0.0976	$4.282e^{-13}$
$(\alpha, c) = (0.2,\ 0.8)$	14	0.0926	$4.124e^{-13}$
$(\alpha, c) = (0.3,\ 0.7)$	11	0.0821	$5.747e^{-13}$
$(\alpha, c) = (0.4,\ 0.6)$	10	0.0707	$1.099e^{-13}$

It has been shown that the speed of the SI and MSI methods degrades or does not converge in the critical case of $(\alpha, c) = (0, 1)$. We use a parametric best approximation to show the convergence of the SI, MSI, and ASI methods for different stopping conditions. From Table 4 it can be seen that the convergence rate of all three methods degrades as the problem gets closer to the critical case or the stopping condition becomes exact. However, it is also seen that the ASI method is almost twice as fast as the other two methods. In summary, Aitken can improve the convergence speed of SI in all cases.

Table 4. Velocity degradation records of three methods in the critical case (IT).

Method		SI	MSI	ASI
$\epsilon = 1e^{-6}$	$(\alpha, c) = (0.1,\ 0.9)$	32	27	**11**
	$(\alpha, c) = (0.001,\ 0.999)$	251	204	**68**
	$(\alpha, c) = (0.00001,\ 0.99999)$	1132	945	**350**
	$(\alpha, c) = (0,\ 1)$	1854	1643	**867**
$\epsilon = 1e^{-7}$	$(\alpha, c) = (0.1,\ 0.9)$	38	32	**12**
	$(\alpha, c) = (0.001,\ 0.999)$	323	260	**84**
	$(\alpha, c) = (0.00001,\ 0.99999)$	1843	1505	**507**
	$(\alpha, c) = (0,\ 1)$	5876	5203	**2740**
$\epsilon = 1e^{-8}$	$(\alpha, c) = (0.1,\ 0.9)$	44	36	**14**
	$(\alpha, c) = (0.001,\ 0.999)$	394	316	**99**
	$(\alpha, c) = (0.00001,\ 0.99999)$	2568	2074	**665**
	$(\alpha, c) = (0,\ 1)$	18596	16463	**8660**

Example 2. *The coefficients w_i in NARE (1.2) are taken from the first n terms of an equivariant series in the interval* $(0, 1)$ *and are assigned in reverse order; $c_i = \frac{1}{n}, i = 1, ..., n$.*

First, we test the effect of parameter η on ASSI and SSI. Parameter settings: $n = 2048$, $\alpha = 0$, $c = 1$, $\epsilon = 1e^{-8}$. In Figure 2, we show the effect of the parameter η on SSI and ASSI. We can see that the larger η is, the faster the convergence of Algorithms 2 and 5, indicating that the introduction of Aitken on SSI is also effective in improving the convergence speed.

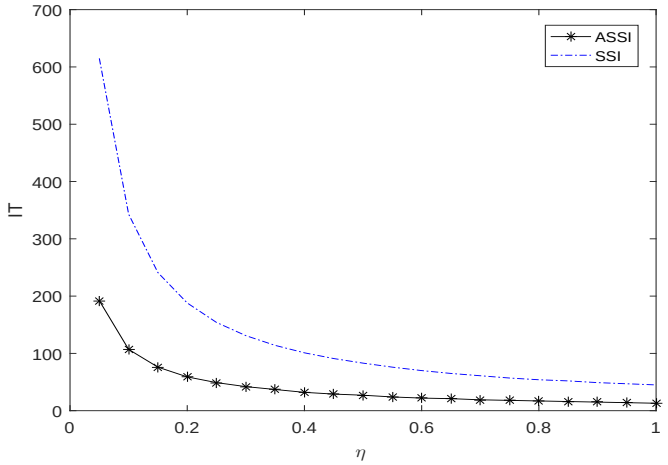

Figure 2. Influence of parameter η on ASSI and SSI in critical case

As shown in Figure 2, within the admissible range $0 < \eta < \gamma_1$, a larger η yields faster convergence for both SSI and ASSI. This is consistent with the theoretical analysis: increasing η reduces the dominant eigenvalue of the shifted iteration matrix, thus improving the linear convergence factor. However, η cannot be chosen arbitrarily large; it must remain strictly below γ_1. Approaching this upper bound may lead to numerical instability or violation of the spectral conditions, which could

degrade or even destroy convergence. In our experiments, γ_1 is approximately 1.5 (for the given parameters), and we chose $\eta = 1.1$ as a safe and effective value within this bound.

In this experiment, $n = 2048$, $\alpha = 0$, $c = 1$, $\epsilon = 1e^{-6}$, $\eta = 1.1$. In Figure 3, we show the convergence speed of the five methods. We observe that ASI converges faster than MSI and SI; ASSI and SSI are significantly faster than the other methods, and ASSI is the fastest. Therefore, in the critical case, better speeds than SI, MSI, and SSI can be obtained using Aitken.

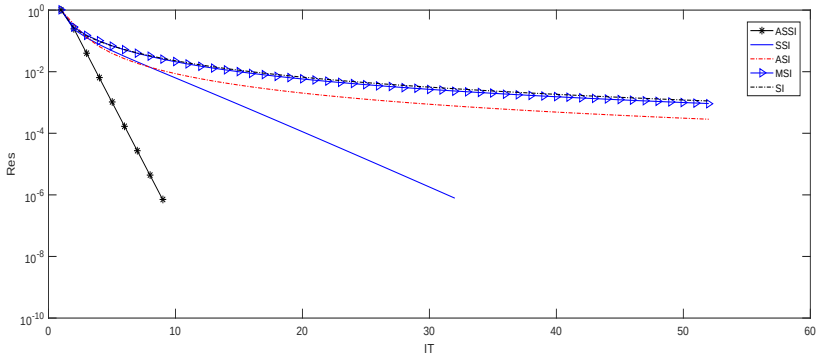

Figure 3. The effect comparison of five methods in critical case

5. Conclusions and discussion

In this paper, we have systematically investigated acceleration strategies for solving the nonsymmetric algebraic Riccati equation (NARE) arising in transport theory, with particular emphasis on the critical case $(\alpha, c) = (0, 1)$ where standard iterative methods suffer from severe convergence degradation.

Our main contribution is twofold. First, we introduced the Aitken extrapolation technique into the simple iteration (SI) framework, yielding the ASI method (Algorithm 4). Theoretical analysis (Theorem 3.2) rigorously proves that, whenever the underlying fixed-point iteration is linearly convergent, the Aitken-transformed sequence converges to the same limit point but with a strictly faster asymptotic rate. This result is not limited to the SI method; it holds for any fixed-point iteration of the form $x_{k+1} = \phi(x_k)$ that satisfies the linear convergence condition, thereby establishing the generality of the Aitken acceleration.

Second, to handle the critical case, we combined the single-shift technique (SSI) with Aitken acceleration, resulting in the ASSI method (Algorithm 5). The shift technique removes the zero eigenvalues of the associated Hamiltonian matrix, restoring linear convergence, while the subsequent Aitken step further boosts the convergence speed. Numerical experiments in Section 4 confirm that ASSI consistently outperforms SI, MSI (modified simple iteration), and SSI in both non-critical and critical settings. In particular, Table 4 clearly demonstrates that ASI reduces the iteration count by nearly half compared to SI and MSI as the problem approaches the critical case, and Figure 3 shows that ASSI is the fastest among all five compared methods.

A notable practical observation is the influence of the shift parameter η. As illustrated in Figure 2, both SSI and ASSI benefit from larger η values (within the admissible range $0 < \eta < \gamma_1$), yielding faster convergence. However, a theoretical upper bound for $\eta\eta$ beyond the existing constraint remains elusive; our current choice of $\eta = 1.1$ in Example 2 is empirically optimal for the tested configurations.

This suggests that an adaptive or optimized selection of η could be an interesting topic for future investigation.

Despite the clear advantages, our methods have some limitations. First, the Aitken transformation requires two extra evaluations of the fixed-point map per iteration, which increases the per-step computational cost—though this overhead is usually offset by the significant reduction in iteration number, especially for large-scale problems. Second, the convergence proof for Aitken (Theorem 3.1) assumes the original iteration converges; it does not address the case where the original method diverges, which is not encountered in our NARE setting. Third, our analysis and experiments are confined to the SI-type framework; extension to other iterative schemes (e.g., Newton or alternating-direction doubling) would require additional theoretical treatment.

Looking ahead, several promising directions emerge. (i) The Aitken technique can be naturally embedded into other existing methods, such as Bao's MSI, Bai's nonlinear block Jacobi (NBJ) and Gauss–Seidel (NBGS), and Lin's double-shift (DSSI) method. Preliminary tests suggest similar acceleration effects, and a unified convergence theory for Aitken-accelerated block iterations would be valuable. (ii) More advanced extrapolation schemes, such as the vector Aitken or Steffensen-type methods, could be explored to further reduce iterations. (iii) The optimal choice of the shift parameter η-potentially depending on the dimension nn and the specific quadrature nodes-deserves a systematic study, possibly via eigenvalue sensitivity analysis. (iv) Finally, extending the current acceleration framework to other classes of algebraic Riccati equations, such as those arising in optimal control or filtering problems, would test the broader applicability of our approach.

In summary, this work not only provides effective and practical algorithms for the transport-theory NARE, but also demonstrates the universal utility of the Aitken method as a simple, black-box accelerator for fixed-point iterations in computational mathematics. We believe that the insights and techniques presented here will stimulate further research on hybrid acceleration strategies for nonlinear matrix equations.

Author contributions

Zhichun Xie: Methodology, Validation, Resources, Funding acquisition; Chanffeng Ma: Methodology, Data curation, Writing–review and editing. Shihai Li: Conceptualization, Methodology, Writing original draft. All authors reviewed and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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