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Research article

A mathematical approach to ABC fractional modeling of the Klein–Gordon equation

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Abstract: This work investigated a fractional Klein–Gordon model formulated with the Atangana–Baleanu (AB) fractional derivative. The physical motivation for employing this fractional operator was discussed, emphasizing its ability to describe time-dependent processes with memory effects. The mathematical model was established, and the existence and uniqueness of its solution were examined using fixed-point theory. In addition, the Laplace multistage telescoping decomposition method (LMTDM) was introduced and applied to obtain approximate analytical solutions. The effectiveness of the proposed approach was evaluated through a comparison of absolute errors with those obtained using the Laplace Adomian decomposition method (LADM). The numerical results indicated that the LMTDM provides accurate approximations and exhibits good convergence properties, making it a reliable technique for solving fractional Klein–Gordon equations.

Keywords: fractional derivative; mathematical model; fractional Klein–Gordon formulation; LMTDM analytical procedure; AB-type Mittag–Leffler kernel derivative; fixed point approach; existence and uniqueness
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1. Introduction

The modeling of complex natural and engineered systems increasingly requires mathematical tools capable of accurately reflecting their intricate behaviors. In this context, fractional partial differential equations (FPDEs) have emerged as essential analytical instruments in contemporary science. Compared to traditional integer-order models, which often fail to capture the inherent memory and hereditary features of many real-world phenomena, fractional models provide a richer and more faithful framework for describing both temporal and spatial dynamics.

From this perspective, fractional calculus has developed into a highly adaptable and sophisticated mathematical field, bridging theoretical constructs and empirical observations. It enables the systematic modeling of nonlocal interactions and long-range dependencies, phenomena commonly observed in disciplines such as physics, biology, engineering, economics, and materials science.

Initially, the development of fractional calculus faced challenges due to the existence of multiple, non-equivalent definitions of fractional derivatives, which posed significant conceptual and theoretical hurdles. Nevertheless, over the last two decades, substantial progress has been achieved, leading to the unification of several formulations and reinforcing the effectiveness of fractional order operators as reliable tools for analyzing complex dynamical systems.

The past few decades have also witnessed the emergence of a wide range of advanced analytical and numerical techniques designed to solve fractional differential equations (FDEs), meeting the growing demand for accurate modeling of intricate dynamical behaviors across scientific disciplines. Among the most prominent are the Adomian decomposition method (ADM) [3] compared with the Elzaki-transform-based MTDEM, the Laplace transform is more naturally suited to the Atangana–Baleanu–Caputo (ABC) operator because of its closed-form transform representation. This enables a direct derivation of the LMTDM recursive scheme without requiring an intermediate Elzaki–Laplace duality transformation, the homotopy perturbation method (HPM) [18, 41], and the variational iteration method (VIM) [1], each providing a distinct computational framework to obtain approximate analytical solutions for nonlinear fractional systems.

In recent years, data-driven and intelligent computational paradigms have also attracted considerable attention for solving nonlinear partial differential equations. Physics-informed neural networks (PINNs), reinforcement-learning-assisted PINNs, neural-symbolic reasoning, and structure-preserving numerical schemes have demonstrated promising performance in nonlinear wave propagation, soliton dynamics, parameter inversion, and inverse problems [25, 26, 35–37, 39]. Likewise, Riemann–Hilbert-based analytical techniques continue to provide powerful mathematical tools for investigating integrable nonlinear wave equations and their exact solutions [34]. These emerging approaches complement traditional semi-analytical decomposition methods and provide additional perspectives for solving complex nonlinear systems.

To improve convergence rates and computational stability, these semi-analytical techniques are frequently coupled with integral transform operators. Commonly used transforms include the Laplace transform [14, 17, 19], Sumudu transform [1, 21, 28], Elzaki transform (ET) [12, 13, 15, 16], ZZ transform [5, 22, 29], Aboodh transform [40], and ARA transform [27, 30]. Such hybrid strategies combine decomposition or perturbation techniques with transform methods. This combination markedly enhances accuracy, robustness, and computational efficiency, and provides a solid foundation for studying fractional-order systems in which memory-dependent and nonlocal effects are significant.

Driven by the need to address nonlinearities and memory effects efficiently, researchers have developed hybrid analytical approaches that transform complex fractional models into sequences of simpler subproblems. The telescoping decomposition method (TDM) [5] represents a fundamental strategy, employing Taylor series expansions to construct polynomial approximations without the computational burden of Adomian polynomials. This approach was extended to fractional systems as the multistage telescoping decomposition method (MTDM) [7], enhancing flexibility, stability, and convergence for a broader class of nonlinear equations [10].

Further advancement was achieved by integrating the MTDM with the Elzaki transform (ET), producing the multistage telescoping decomposition Elzaki method (MTDEM) [11]. This fusion harnesses the advantages of decomposition methods and the stability of integral transform theory, significantly improving convergence, numerical reliability, and computational efficiency. Extensive numerical experiments demonstrate that the MTDEM provides high accuracy, robustness, and ease of implementation, making it well-suited for high-dimensional nonlinear fractional systems.

Meanwhile, recent advances have also explored artificial intelligence (AI)-assisted frameworks to improve computational efficiency and numerical accuracy in nonlinear wave problems. These developments include physics-informed neural networks (PINNs) [25, 26, 37], reinforcement-learning-assisted PINNs, neural-symbolic reasoning, and AI-enhanced mathematical derivation techniques for solving nonlinear evolution equations [36, 38, 39]. Although these approaches are fundamentally different from the present semi-analytical decomposition framework, they provide valuable complementary perspectives for future developments of fractional-order numerical methods.

In [8], the authors applied these hybrid techniques to solve systems of nonlinear fractional differential equations (SFDEs) governed by Caputo fractional derivatives (CFDs), achieving superior computational efficiency and solution stability. The MTDEM framework successfully combined the Elzaki transform with the MTDM, offering a versatile, robust, and precise approach for nonlinear fractional systems across mathematical physics, engineering, and applied sciences. A notable advancement in modern fractional calculus is the Atangana–Baleanu–Caputo (ABC) fractional derivative, which addresses the limitations of classical derivatives like Caputo and Riemann–Liouville. Its non-singular Mittag–Leffler kernel allows realistic modeling of systems with long-term memory and nonlocal interactions, while providing a flexible framework applicable to both linear and nonlinear models [6].

The ABC fractional Klein–Gordon equation has recently attracted considerable attention due to its capability of describing nonlinear wave phenomena with memory effects. Several analytical and numerical investigations have been reported for fractional Klein–Gordon-type models. Recent contributions include analytical and optimization-based approaches [2], spectral polynomial techniques [23], modified Atangana–Baleanu–Caputo formulations for nonlinear wave and diffusion problems [32], and computational studies of wave propagation using modified ABC operators [20]. Furthermore, recent studies have investigated ABC-type fractional models combined with Laplace-based decomposition techniques, demonstrating their effectiveness in stability analysis and nonlinear dynamical systems [4, 31]. However, despite these advances, comprehensive analytical frameworks combining ABC fractional Klein–Gordon equations with efficient decomposition-transform methods and rigorous convergence analysis remain limited.

The physical relevance of the ABC fractional derivative in the considered nonlinear Klein–Gordon-type model is associated with memory and hereditary effects described through the non-singular Mittag–Leffler kernel. In particular, the present evolution of the nonlinear field depends not only on its current state, but also on its previous states, which allows the model to describe fading memory phenomena in complex media. Consequently, the ABC fractional operator provides a more realistic mathematical framework for describing nonlinear wave propagation involving relaxation phenomena and nonlocal temporal effects.

In this work, we utilize a new method called the Laplace–multistage telescoping decomposition method (LMTDM) as a refined semi-analytical framework to obtain precise solutions of the Klein–Gordon equation (KGE) under well-defined initial conditions. The paper is structured to ensure logical progression and facilitate a deep understanding of both theoretical formulation and practical implementation of the methodology.

Section 2 elaborates on the principles of fractional calculus, describing the Atangana–Baleanu (AB) derivative, associated integral operators, the Laplace transform, and the Laplace multistage telescoping decomposition method (LMTDM), establishing the analytical groundwork. Section 3 proves the existence and uniqueness of the derived solutions, providing rigorous justification for the mathematical validity and robustness of the approach. Section 4 details the algorithmic implementation, including error analysis and convergence studies for the fractional Klein–Gordon equation with ABC derivatives. Section 5 presents extensive numerical experiments, demonstrating the method's precision, robustness, and computational efficiency across various fractional-order models.

Finally, the conclusion summarizes the key findings, emphasizing the method's effectiveness, computational advantages, and broad applicability, while outlining future research directions for enhancing the accuracy, stability, and efficiency of hybrid analytical-numerical techniques in fractional calculus.

2. Preliminaries

In this section, we discuss several important definitions, notations, and basic results regarding fractional derivatives.

Definition 2.1. The Caputo fractional derivative is defined as follows [24]:

$$({}^{C}D_{0^+}^{\gamma}\varpi)(\varsigma) = \begin{cases} \dfrac{1}{\Gamma(m-\gamma)} \displaystyle\int_0^{\varsigma} (\varsigma - \varkappa)^{m-\gamma-1} \dfrac{\partial^m \varpi(\varkappa)}{\partial \varkappa^m}\, d\varkappa, & m-1 < \gamma < m, \\ \dfrac{\partial^m \varpi(\varsigma)}{\partial \varsigma^m}, & \gamma = m, \end{cases} \tag{2.1}$$

where $m \in \mathbb{N}^*$.

Definition 2.2. Let $s \in \mathbb{N}_0$ and Θ be an open subset of $\mathbb{R}$. The Sobolev space ($H^s(\Theta)$) is defined as

$$H^s(\Theta) = \left\{ \varpi \in L^2(\Theta) \ : \ D^k \varpi \in L^2(\Theta), \text{ for all } 0 \le k \le s \right\}.$$

Definition 2.3. The Atangana–Baleanu–Caputo derivative is expressed as [6]: Let $\varpi \in H^1(\xi_1, \xi_2)$, $\xi_2 > \xi_1$. The ABC sense is

$$ {}^{ABC}D_{\varsigma}^{\gamma}\varpi(\varsigma) = \frac{M(\gamma)}{1-\gamma}\int_0^{\varsigma} E_{\gamma}\Big(\frac{-\gamma(\varsigma-\varkappa)^{\gamma}}{1-\gamma}\Big)\varpi'(\varkappa)d\varkappa, \varsigma \geqslant 0, \tag{2.2} $$

such that $0 < \gamma < 1$, $M(\gamma)$ is a normalizing function that ensures $M(0) = M(1) = 1$, $\varpi^{'}$ is the derivative of ϖ, and E_{γ} represents the Mittag–Leffler function in Eq (2.2).

The properties of (2.2) are specified as follows:

1) ${}^{ABC}D_{\varsigma}^{\gamma}c = 0,\ c$ is a constant.
2) $\left\{ {}^{ABC}D_{\varsigma}^{\gamma}\varpi(\varsigma)\right\} = \dfrac{p^{\gamma}L\varpi(\varsigma)}{p^{\gamma}(1-\gamma)+\gamma} - \dfrac{p^{\gamma-1}\varpi(0)}{p^{\gamma}(1-\gamma)+\gamma}.$

Definition 2.4. The fractional integral corresponding to the Atangana–Baleanu fractional operator is defined as [6]:

$$ {}^{AB}I_{\varsigma}^{\gamma}\varpi(\varsigma) = \frac{1-\gamma}{M(\gamma)}\varpi(\varsigma) + \frac{\gamma}{M(\gamma)\Gamma(\gamma)}\int_0^{\varsigma}(\varsigma-\varkappa)^{\gamma-1}\varpi(\varkappa)\,d\varkappa, \tag{2.3} $$

where $0 < \gamma < 1$ and $M(\gamma)$ is a normalization function with $M(0) = M(1) = 1$.

Some properties of (2.3) are given by:

1) ${}^{AB}I_{\varsigma}^{\gamma}\,{}^{ABC}D_{\varsigma}^{\gamma}\varpi(\varsigma) = \varpi(\varsigma) - \varpi(0).$
2) ${}^{AB}I_{\varsigma}^{\gamma}c = \dfrac{c}{M(\gamma)}\left(1-\gamma+\dfrac{\varsigma^{\gamma}}{\Gamma(\gamma)}\right).$
3) ${}^{AB}I_{\varsigma}^{\gamma}\varsigma^{k} = \dfrac{\varsigma^{k}}{M(\gamma)}\left(1-\gamma+\dfrac{\gamma\,\Gamma(k+1)\varsigma^{\gamma}}{\Gamma(\gamma+k+1)}\right).$

Definition 2.5. The Laplace transform of a given function $\varpi(\varkappa, \varsigma)$ is defined as follows [24]:

$$ \mathcal{L}\{\varpi(\varkappa,\varsigma)\} = \widehat{\varpi}(\varkappa, p) = \int_0^{\infty} e^{-p\varsigma}\,\varpi(\varkappa,\varsigma)\,d\varsigma, \tag{2.4} $$

where p is a complex variable and $\Re(p) > 0$ ensures the convergence of the integral.

Theorem 2.6. *For the convolution theorem of the Laplace transform, the following equality holds [9]:*

$$ \mathcal{L}\{\varpi * v\} = \mathcal{L}\{\varpi\}\,\mathcal{L}\{v\}, \tag{2.5} $$

where $(\varpi * v)(\varsigma) = \int_0^{\varsigma}\varpi(\varsigma-\varkappa)\,v(\varkappa)\,d\varkappa.$

Definition 2.7. The Laplace transform of the Caputo fractional derivative of order γ $(0 < \gamma < 1)$ is given by [24]

$$ \mathcal{L}\left\{{}^{C}D_{\varsigma}^{\gamma}\varpi(\varkappa,\varsigma)\right\} = p^{\gamma}\widehat{\varpi}(\varkappa,p) - p^{\gamma-1}\varpi(\varkappa,0), \tag{2.6} $$

where ${}^{C}D_{\varsigma}^{\gamma}$ denotes the Caputo fractional derivative with respect to ς, $\widehat{\varpi}(\varkappa, p)$ is the Laplace transform of $\varpi(\varkappa, \varsigma)$, and $\varpi(\varkappa, 0)$ is the initial value at $\varsigma = 0$.

Theorem 2.8. *The Laplace transform of the Atangana–Baleanu–Caputo (ABC) fractional derivative of order $\gamma \in (0,1)$ is defined as*

$$\mathcal{L}\left\{ {}^{ABC}D_{\varsigma}^{\gamma}\varpi(\varsigma)\right\}(p) = \frac{M(\gamma)}{1-\gamma} \cdot \frac{p^{\gamma}F(p) - p^{\gamma-1}\varpi(0)}{p^{\gamma} + \dfrac{\gamma}{1-\gamma}}, \tag{2.7}$$

where $F(p) = \mathcal{L}\{\varpi(\varsigma)\}(p)$ is the Laplace transform of $\varpi(\varsigma)$, $M(\gamma)$ is the normalization constant satisfying $M(0) = M(1) = 1$, and ${}^{ABC}D_{\varsigma}^{\gamma}\varpi(\varsigma)$ denotes the Atangana–Baleanu–Caputo fractional derivative. This formulation originates from the work of Atangana and Baleanu [6].

3. Fractional mathematical model

Consider the Klein–Gordon equations involving the Atangana–Baleanu operator in the Caputo sense:

$${}^{ABC}D_{\varsigma}^{\gamma}\varpi(\varkappa,\varsigma) = \varpi_{\varkappa\varkappa}(\varkappa,\varsigma) + \lambda_1\varpi(\varkappa,\varsigma) + \lambda_2\varpi^2(\varkappa,\varsigma) + \lambda_3\varpi^3(\varkappa,\varsigma), \quad \varsigma > 0, \tag{3.1}$$

subject to the initial conditions:

$$\varpi(\varkappa,0) = K_1(\varkappa), \varpi_{\varsigma}(\varkappa,0) = K_2(\varkappa),\ \varkappa \in \mathbb{R}, \tag{3.2}$$

where ${}^{ABC}D_{\varsigma}^{\gamma}$ is the Atangana–Baleanu derivative in the Caputo sense and λ_i are constants.

Taking the ABC integral operator, we obtain:

$$\begin{aligned}\varpi(\varkappa,\varsigma) - \varpi(\varkappa,0) - \varsigma\varpi_{\varsigma}(\varkappa,0) = {} & \frac{(1-\gamma)}{M(\gamma)}(\varpi_{\varkappa\varkappa}(\varkappa,\varsigma) + \lambda_1\varpi(\varkappa,\varsigma) + \lambda_2\varpi^2(\varkappa,\varsigma) + \lambda_3\varpi^3(\varkappa,\varsigma)) \\ & + \frac{\gamma}{M(\gamma)} \cdot \frac{1}{\Gamma(\gamma)}\int_0^{\varsigma}(\varsigma-\xi)^{\gamma-1}(\varpi_{\varkappa\varkappa}(\varkappa,\varsigma) + \lambda_1\varpi(\varkappa,\varsigma) + \lambda_2\varpi^2(\varkappa,\varsigma) \\ & + \lambda_3\varpi^3(\varkappa,\varsigma))d\xi,\end{aligned} \tag{3.3}$$

and we let

$$\Upsilon(\varkappa,\varsigma,\varpi) = \varpi_{\varkappa\varkappa}(\varkappa,\varsigma) + \lambda_1\varpi(\varkappa,\varsigma) + \lambda_2\varpi^2(\varkappa,\varsigma) + \lambda_3\varpi^3(\varkappa,\varsigma). \tag{3.4}$$

Then, Eq (3.3) becomes

$$\varpi(\varkappa,\varsigma) = \varpi(\varkappa,0) + \varsigma\varpi_{\varsigma}(\varkappa,0) + \frac{1-\gamma}{M(\gamma)}\Upsilon(\varkappa,\varsigma,\varpi) + \frac{\gamma}{M(\gamma)}\frac{1}{\Gamma(\gamma)}\int_0^{\varsigma}(\varsigma-\xi)^{\gamma-1}\cdot\Upsilon(\varkappa,\xi,\varpi)d\xi, \tag{3.5}$$

for all $\varsigma \in [0,T]$.

3.1. Physical motivation of the ABC fractional Klein–Gordon model

The considered equation is a nonlinear Klein–Gordon-type equation with polynomial nonlinearities. The terms $\lambda_1\varpi, \lambda_2\varpi^2, \lambda_3\varpi^3$ represent linear and nonlinear restoring forces or self-interaction effects of the field. In classical models, the time evolution of the field is assumed to be local, meaning that the future state depends only on the present state. However, in complex media, nonlinear wave propagation may exhibit memory effects due to relaxation, damping, hereditary behavior, or interactions with the internal structure of the medium. To incorporate these effects, the classical time derivative is replaced by the Atangana–Baleanu derivative in the Caputo sense. The Mittag–Leffler kernel associated with the ABC derivative introduces a fading memory effect, where the past states of $\varpi(\varkappa, \varsigma)$ contribute to its current evolution. Moreover, unlike classical singular kernels, the ABC kernel is non-singular at the initial time, which provides a smoother and more physically realistic description of the transition from the initial state $K(\varkappa)$ to the subsequent nonlinear wave evolution. The use of the Atangana–Baleanu–Caputo (ABC) fractional derivative in the present study is motivated by several mathematical and computational advantages compared with the classical Caputo operator. The ABC operator is characterized by a non-singular and non-local Mittag–Leffler kernel, which remains smooth and bounded over the entire integration domain. In particular, when the integration variable approaches the present time, the Mittag–Leffler function satisfies $E_\gamma(0) = 1$, showing that the kernel does not exhibit singular behavior near the present time. Consequently, the ABC operator provides a smooth fading memory effect and a more physically realistic description of memory-dependent nonlinear wave propagation phenomena associated with the Klein–Gordon equation.

Moreover, within the framework of the Laplace multistage telescoping decomposition method (LMTDM), the Laplace transform of the ABC operator introduces a Mittag–Leffler memory structure into the analytical formulation of the problem. After applying the inverse Laplace transform, the obtained solution components naturally involve Mittag–Leffler functions, which allows the proposed framework to capture nonlocal memory effects more effectively during the iterative decomposition procedure.

In addition, the smoothness and boundedness of the ABC kernel improve the numerical stability of the recursive integrations associated with the nonlinear terms of the considered Klein–Gordon model. The absence of singularity also reduces the accumulation of numerical errors and enhances the convergence behavior of the obtained solution series. Consequently, the proposed LMTDM generates highly accurate approximate solutions with rapid convergence while requiring only a limited number of iterative terms, thereby reducing computational complexity and improving computational efficiency.

Theorem 3.1. *Let $\Theta \subset \mathbb{R}$ be a bounded open interval, let $\gamma \in (1, 2]$, and let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Consider the Atangana–Baleanu–Caputo fractional Klein–Gordon problem*

$$^{ABC}D_\varsigma^\gamma \varpi(\varkappa, \varsigma) = \varpi_{\varkappa\varkappa}(\varkappa, \varsigma) + \lambda_1 \varpi(\varkappa, \varsigma) + \lambda_2 \varpi^2(\varkappa, \varsigma) + \lambda_3 \varpi^3(\varkappa, \varsigma), \qquad \varkappa \in \Theta,\ \varsigma \in (0, \varsigma^*], \tag{3.6}$$

subject to the initial conditions

$$\varpi(\varkappa, 0) = K(\varkappa), \qquad \varpi_\varsigma(\varkappa, 0) = K'(\varkappa), \qquad \varkappa \in \Theta, \tag{3.7}$$

where

$$K \in H^2(\Theta), \qquad K' \in H^1(\Theta). \tag{3.8}$$

Equip $H^2(\Theta)$ with the norm

$$\|u\|_{H^2(\Theta)} := \left(\|u\|^2_{L^2(\Theta)} + \|u_\varkappa\|^2_{L^2(\Theta)} + \|u_{\varkappa\varkappa}\|^2_{L^2(\Theta)} \right)^{1/2}, \qquad u \in H^2(\Theta), \tag{3.9}$$

and let

$$\Upsilon(\varpi) := \varpi_{\varkappa\varkappa} + \lambda_1 \varpi + \lambda_2 \varpi^2 + \lambda_3 \varpi^3 \tag{3.10}$$

denote the nonlinear operator associated with the right-hand side of the equation above. Since Θ *is bounded, there exists a finite constant* $C_E = C_E(\Theta) > 0$ *such that*

$$\|u\|_{L^\infty(\Theta)} \leq C_E \|u\|_{H^2(\Theta)} \qquad \text{for every } u \in H^2(\Theta). \tag{3.11}$$

Then

$$\Upsilon : H^2(\Theta) \longrightarrow L^2(\Theta) \tag{3.12}$$

is well-defined. Moreover, for every $M, M' > 0$ *and every* $\varpi, \varrho \in H^2(\Theta)$ *satisfying*

$$\|\varpi\|_{H^2(\Theta)} \leq M, \qquad \|\varrho\|_{H^2(\Theta)} \leq M', \tag{3.13}$$

there exists a constant

$$L_\Upsilon = L_\Upsilon(M, M', \lambda_1, \lambda_2, \lambda_3, C_E) = 1 + |\lambda_1| + |\lambda_2|\, C_E(M + M') + |\lambda_3|\, C_E^2(M^2 + MM' + (M')^2) \tag{3.14}$$

such that

$$\|\Upsilon(\varpi) - \Upsilon(\varrho)\|_{L^2(\Theta)} \leq L_\Upsilon\, \|\varpi - \varrho\|_{H^2(\Theta)}. \tag{3.15}$$

In particular, Υ *is locally Lipschitz continuous from* $H^2(\Theta)$ *into* $L^2(\Theta)$.

Proof. For $u \in H^2(\Theta)$, we use the standard norm

$$\|u\|_{H^2(\Theta)} := \left(\|u\|^2_{L^2(\Theta)} + \|u_\varkappa\|^2_{L^2(\Theta)} + \|u_{\varkappa\varkappa}\|^2_{L^2(\Theta)}\right)^{1/2}, \tag{3.16}$$

where $u_\varkappa$ and $u_{\varkappa\varkappa}$ denote the first and second weak derivatives of u with respect to $\varkappa$. From (3.16), it follows immediately, since each summand under the square root is nonnegative, that

$$\|u\|_{L^2(\Theta)} \leq \|u\|_{H^2(\Theta)} \qquad \text{and} \qquad \|u_{\varkappa\varkappa}\|_{L^2(\Theta)} \leq \|u\|_{H^2(\Theta)} \qquad \text{for every } u \in H^2(\Theta). \tag{3.17}$$

$H^2(\Theta)$, being a Sobolev space, is a vector space over $\mathbb{R}$; hence if $u, v \in H^2(\Theta)$, then $u \pm v \in H^2(\Theta)$, a fact used repeatedly below without further comment.

By hypothesis, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. The absolute value $|\cdot| : \mathbb{R} \to [0, \infty)$ is, by construction of the real number field, a well-defined, finite-valued map on all of $\mathbb{R}$ (the extended value $+\infty$ is not an element of $\mathbb{R}$). Hence

$$0 \leq |\lambda_i| < \infty, \qquad i = 1, 2, 3, \tag{3.18}$$

and each λ_i is in particular independent of ϖ, ϱ, M, M' and of the domain Θ; this is what allows $|\lambda_i|$ to appear as fixed, finite multiplicative constants in every estimate below.

Since $\Theta \subset \mathbb{R}$ is a bounded open interval, the Sobolev embedding theorem applies. and we obtain the continuous embedding

$$H^2(\Theta) \hookrightarrow L^\infty(\Theta). \tag{3.19}$$

Continuity of the embedding (3.19) means precisely that there exists a finite constant $C_E = C_E(\Theta) > 0$, such that

$$\|u\|_{L^\infty(\Theta)} \leq C_E \|u\|_{H^2(\Theta)} \qquad \text{for every } u \in H^2(\Theta). \tag{3.20}$$

Define

$$\Upsilon(u) := u_{\varkappa\varkappa} + \lambda_1 u + \lambda_2 u^2 + \lambda_3 u^3, \qquad u \in H^2(\Theta). \tag{3.21}$$

Let $u \in H^2(\Theta)$ be arbitrary. By the definition of $H^2(\Theta)$, $u_{\varkappa\varkappa} \in L^2(\Theta)$. By (3.20), $u \in L^\infty(\Theta)$ with $\|u\|_{L^\infty(\Theta)} \leq C_E \|u\|_{H^2(\Theta)} < \infty$. Since Θ is bounded, $|\Theta| < \infty$, and for each $k = 1, 2, 3$,

$$\|u^k\|^2_{L^2(\Theta)} = \int_\Theta |u(\varkappa)|^{2k}\, d\varkappa \leq \|u\|^{2k}_{L^\infty(\Theta)} |\Theta| < \infty, \tag{3.22}$$

so $u, u^2, u^3 \in L^2(\Theta)$. By Step 2, $\lambda_1, \lambda_2, \lambda_3$ are finite real numbers, so $\lambda_1 u, \lambda_2 u^2, \lambda_3 u^3 \in L^2(\Theta)$ (scalar multiples of $L^2(\Theta)$ functions by finite scalars remain in $L^2(\Theta)$, since $L^2(\Theta)$ is a vector space). Consequently $\Upsilon(u)$, being a finite sum of elements of $L^2(\Theta)$, satisfies $\Upsilon(u) \in L^2(\Theta)$. This proves that $\Upsilon : H^2(\Theta) \to L^2(\Theta)$ is well-defined. Let $\varpi, \varrho \in H^2(\Theta)$ satisfy

$$\|\varpi\|_{H^2(\Theta)} \leq M, \qquad \|\varrho\|_{H^2(\Theta)} \leq M' \tag{3.23}$$

for some $M, M' > 0$. By the linearity of $\partial_{\varkappa\varkappa}$ and direct subtraction in (3.21),

$$\Upsilon(\varpi) - \Upsilon(\varrho) = (\varpi_{\varkappa\varkappa} - \varrho_{\varkappa\varkappa}) + \lambda_1(\varpi - \varrho) + \lambda_2(\varpi^2 - \varrho^2) + \lambda_3(\varpi^3 - \varrho^3). \tag{3.24}$$

By the triangle inequality in $L^2(\Theta)$ applied to (3.24),

$$\begin{aligned} \|\Upsilon(\varpi) - \Upsilon(\varrho)\|_{L^2(\Theta)} \leq{}& \|\varpi_{\varkappa\varkappa} - \varrho_{\varkappa\varkappa}\|_{L^2(\Theta)} + |\lambda_1|\, \|\varpi - \varrho\|_{L^2(\Theta)} \\ &+ |\lambda_2|\, \|\varpi^2 - \varrho^2\|_{L^2(\Theta)} + |\lambda_3|\, \|\varpi^3 - \varrho^3\|_{L^2(\Theta)}. \end{aligned} \tag{3.25}$$

Since $H^2(\Theta)$ is a vector space, $\varpi - \varrho \in H^2(\Theta)$, and by (3.17) applied to $u = \varpi - \varrho$,

$$\|\varpi_{\varkappa\varkappa} - \varrho_{\varkappa\varkappa}\|_{L^2(\Theta)} = \|(\varpi - \varrho)_{\varkappa\varkappa}\|_{L^2(\Theta)} \leq \|\varpi - \varrho\|_{H^2(\Theta)}. \tag{3.26}$$

By (3.17) applied to $u = \varpi - \varrho$,

$$\|\varpi - \varrho\|_{L^2(\Theta)} \leq \|\varpi - \varrho\|_{H^2(\Theta)}, \tag{3.27}$$

hence

$$|\lambda_1|\, \|\varpi - \varrho\|_{L^2(\Theta)} \leq |\lambda_1|\, \|\varpi - \varrho\|_{H^2(\Theta)}. \tag{3.28}$$

The algebraic identity

$$\varpi^2 - \varrho^2 = (\varpi - \varrho)(\varpi + \varrho) \tag{3.29}$$

holds pointwise almost everywhere on Θ, since for a.e. $\varkappa \in \Theta$, $\varpi(\varkappa)^2 - \varrho(\varkappa)^2 = (\varpi(\varkappa) - \varrho(\varkappa))(\varpi(\varkappa) + \varrho(\varkappa))$ is an identity of real numbers. We now invoke Hölder's inequality explicitly: For $f \in L^2(\Theta)$ and $g \in L^\infty(\Theta)$,

$$\|fg\|_{L^2(\Theta)} = \left(\int_\Theta |f(\varkappa)g(\varkappa)|^2\, d\varkappa\right)^{1/2} \leq \|g\|_{L^\infty(\Theta)} \left(\int_\Theta |f(\varkappa)|^2\, d\varkappa\right)^{1/2} = \|f\|_{L^2(\Theta)} \|g\|_{L^\infty(\Theta)}, \tag{3.30}$$

which is the case where $p = 2, q = \infty, r = 2$ (with $1/r = 1/p + 1/q$) of the general Hölder inequality $\|fg\|_{L^r(\Theta)} \leq \|f\|_{L^p(\Theta)} \|g\|_{L^q(\Theta)}$. Applying (3.29) and (3.30) with $f = \varpi - \varrho \in L^2(\Theta)$ and $g = \varpi + \varrho \in L^\infty(\Theta)$ (the latter by (3.20), since $\varpi + \varrho \in H^2(\Theta)$),

$$\|\varpi^2 - \varrho^2\|_{L^2(\Theta)} = \|(\varpi - \varrho)(\varpi + \varrho)\|_{L^2(\Theta)} \leq \|\varpi - \varrho\|_{L^2(\Theta)}\, \|\varpi + \varrho\|_{L^\infty(\Theta)}. \tag{3.31}$$

By the Sobolev embedding (3.20) applied to $u = \varpi + \varrho \in H^2(\Theta)$,

$$\|\varpi + \varrho\|_{L^\infty(\Theta)} \le C_E \|\varpi + \varrho\|_{H^2(\Theta)}. \tag{3.32}$$

By the triangle inequality in the normed space $H^2(\Theta)$,

$$\|\varpi + \varrho\|_{H^2(\Theta)} \le \|\varpi\|_{H^2(\Theta)} + \|\varrho\|_{H^2(\Theta)} \le M + M', \tag{3.33}$$

where the last step is determined by (3.23). Substituting (3.27), (3.32), and (3.33) into (3.31) in turn,

$$\begin{aligned} \|\varpi^2 - \varrho^2\|_{L^2(\Theta)} &\le \|\varpi - \varrho\|_{L^2(\Theta)} \, \|\varpi + \varrho\|_{L^\infty(\Theta)} \\ &\le \|\varpi - \varrho\|_{H^2(\Theta)} \cdot C_E \|\varpi + \varrho\|_{H^2(\Theta)} \\ &\le C_E(M + M') \, \|\varpi - \varrho\|_{H^2(\Theta)}. \end{aligned} \tag{3.34}$$

Set $d_1 := C_E(M + M') > 0$ and then $\|\varpi^2 - \varrho^2\|_{L^2(\Theta)} \le d_1 \|\varpi - \varrho\|_{H^2(\Theta)}$. The algebraic identity

$$\varpi^3 - \varrho^3 = (\varpi - \varrho)(\varpi^2 + \varpi\varrho + \varrho^2) \tag{3.35}$$

holds pointwise a.e. on Θ, by the factorization of the difference of cubes of real numbers. Applying Hölder's inequality (3.30) with $f = \varpi - \varrho \in L^2(\Theta)$ and $g = \varpi^2 + \varpi\varrho + \varrho^2 \in L^\infty(\Theta)$ (the latter since, by (3.20), $\varpi, \varrho \in L^\infty(\Theta)$ and $L^\infty(\Theta)$ is closed under products and sums),

$$\|\varpi^3 - \varrho^3\|_{L^2(\Theta)} = \|(\varpi - \varrho)(\varpi^2 + \varpi\varrho + \varrho^2)\|_{L^2(\Theta)} \le \|\varpi - \varrho\|_{L^2(\Theta)} \, \|\varpi^2 + \varpi\varrho + \varrho^2\|_{L^\infty(\Theta)}. \tag{3.36}$$

By the triangle inequality in $L^\infty(\Theta)$,

$$\|\varpi^2 + \varpi\varrho + \varrho^2\|_{L^\infty(\Theta)} \le \|\varpi^2\|_{L^\infty(\Theta)} + \|\varpi\varrho\|_{L^\infty(\Theta)} + \|\varrho^2\|_{L^\infty(\Theta)}. \tag{3.37}$$

By the submultiplicativity of the $L^\infty(\Theta)$-norm,

$$\|uv\|_{L^\infty(\Theta)} = \operatorname*{ess\,sup}_{\varkappa \in \Theta} |u(\varkappa)v(\varkappa)| \le \Big(\operatorname*{ess\,sup}_{\varkappa \in \Theta} |u(\varkappa)|\Big)\Big(\operatorname*{ess\,sup}_{\varkappa \in \Theta} |v(\varkappa)|\Big) = \|u\|_{L^\infty(\Theta)} \|v\|_{L^\infty(\Theta)} \tag{3.38}$$

for all $u, v \in L^\infty(\Theta)$. Applying (3.38) to each term in (3.37) (with $(u, v) = (\varpi, \varpi)$, (ϖ, ϱ), (ϱ, ϱ)),

$$\|\varpi^2 + \varpi\varrho + \varrho^2\|_{L^\infty(\Theta)} \le \|\varpi\|^2_{L^\infty(\Theta)} + \|\varpi\|_{L^\infty(\Theta)} \|\varrho\|_{L^\infty(\Theta)} + \|\varrho\|^2_{L^\infty(\Theta)}. \tag{3.39}$$

By the Sobolev embedding (3.20) applied separately to ϖ and ϱ, together with (3.23),

$$\|\varpi\|_{L^\infty(\Theta)} \le C_E \|\varpi\|_{H^2(\Theta)} \le C_E M, \qquad \|\varrho\|_{L^\infty(\Theta)} \le C_E \|\varrho\|_{H^2(\Theta)} \le C_E M'. \tag{3.40}$$

Substituting (3.40) into (3.39),

$$\|\varpi^2 + \varpi\varrho + \varrho^2\|_{L^\infty(\Theta)} \le C_E^2 M^2 + C_E^2 M M' + C_E^2 (M')^2 = C_E^2 (M^2 + M M' + (M')^2). \tag{3.41}$$

Combining (3.27), (3.36), and (3.41),

$$\|\varpi^3 - \varrho^3\|_{L^2(\Theta)} \le C_E^2 (M^2 + M M' + (M')^2) \, \|\varpi - \varrho\|_{H^2(\Theta)}. \tag{3.42}$$

Set $d_2 := C_E^2(M^2 + MM' + (M')^2) > 0$ and then $\|\varpi^3 - \varrho^3\|_{L^2(\Theta)} \leq d_2\|\varpi - \varrho\|_{H^2(\Theta)}$. Substituting (3.26), (3.28), (3.34), and (3.42) into (3.25),

$$\begin{aligned}\|\Upsilon(\varpi) - \Upsilon(\varrho)\|_{L^2(\Theta)} &\leq \|\varpi - \varrho\|_{H^2(\Theta)} + |\lambda_1|\,\|\varpi - \varrho\|_{H^2(\Theta)} \\ &\quad + |\lambda_2|\,d_1\,\|\varpi - \varrho\|_{H^2(\Theta)} + |\lambda_3|\,d_2\,\|\varpi - \varrho\|_{H^2(\Theta)} \\ &= \Big(1 + |\lambda_1| + |\lambda_2|d_1 + |\lambda_3|d_2\Big)\|\varpi - \varrho\|_{H^2(\Theta)}.\end{aligned} \tag{3.43}$$

Define

$$L_\Upsilon := 1 + |\lambda_1| + |\lambda_2|\,C_E(M + M') + |\lambda_3|\,C_E^2(M^2 + MM' + (M')^2). \tag{3.44}$$

We have $|\lambda_1|, |\lambda_2|, |\lambda_3| < \infty$; $C_E < \infty$; and $M, M' < \infty$ by hypothesis (3.23). Hence $0 < L_\Upsilon < \infty$, and L_Υ depends only on $M, M', \lambda_1, \lambda_2, \lambda_3, C_E$ — and not on ϖ or ϱ individually, nor on γ, K, or K', since none of these appear in (3.44). Substituting (3.44) into (3.43) gives

$$\|\Upsilon(\varpi) - \Upsilon(\varrho)\|_{L^2(\Theta)} \leq L_\Upsilon\,\|\varpi - \varrho\|_{H^2(\Theta)}. \tag{3.45}$$

Since (3.45) holds for *every* pair $\varpi, \varrho \in H^2(\Theta)$ satisfying (3.23), and since Υ was shown to be well-defined from $H^2(\Theta)$ to $L^2(\Theta)$, we conclude that Υ is Lipschitz continuous, with Lipschitz constant L_Υ given explicitly by (3.44), on the bounded subset $\{u \in H^2(\Theta) : \|u\|_{H^2(\Theta)} \leq \max(M, M')\}$ of $H^2(\Theta)$. As $M, M' > 0$ were arbitrary, Υ is Lipschitz continuous on every bounded subset of $H^2(\Theta)$, i.e., Υ is locally Lipschitz from $H^2(\Theta)$ to $L^2(\Theta)$. □

3.2. Existence and uniqueness results

Theorem 3.2. *Let $1 < \gamma \leq 2$, let $\Theta \subset \mathbb{R}$ be a bounded open interval, let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, and let*

$$K_1(\varkappa) \in H^2(\Theta), \qquad K_2(\varkappa) \in H^1(\Theta). \tag{3.46}$$

Define the nonlinear operator

$$\Upsilon(\varpi(\varkappa, \varsigma)) = \varpi(\varkappa, \varsigma)_{\varkappa\varkappa} + \lambda_1\varpi(\varkappa, \varsigma) + \lambda_2\varpi(\varkappa, \varsigma)^2 + \lambda_3\varpi(\varkappa, \varsigma)^3. \tag{3.47}$$

Then the following hold.
(i) *$\Upsilon : H^2(\Theta) \to L^2(\Theta)$ is well-defined. Moreover, there exists a constant $C_E = C_E(\Theta) > 0$, depending only on Θ, such that for every $M > 0$ and every $\varpi(\varkappa, \varsigma) \in H^2(\Theta)$ with $\|\varpi(\varkappa, \varsigma)\|_{H^2(\Theta)} \leq M$,*

$$\|\Upsilon(\varpi(\varkappa, \varsigma))\|_{L^2(\Theta)} \leq C(M) := (1 + |\lambda_1|)M + |\lambda_2|C_E M^2 + |\lambda_3|C_E^2 M^3. \tag{3.48}$$

(ii) *For all $\varpi, \varrho \in H^2(\Theta)$ with $\|\varpi\|_{H^2(\Theta)} \leq M_1$ and $\|\varrho\|_{H^2(\Theta)} \leq M_2$,*

$$\|\Upsilon(\varpi) - \Upsilon(\varrho)\|_{L^2(\Theta)} \leq L_\Upsilon\|\varpi - \varrho\|_{H^2(\Theta)}, \tag{3.49}$$

where

$$L_\Upsilon = 1 + |\lambda_1| + |\lambda_2|\,C_E(M_1 + M_2) + |\lambda_3|\,C_E^2(M_1^2 + M_1M_2 + M_2^2). \tag{3.50}$$

(iii) *If, in addition, there exists $T > 0$ such that*

$$\left(\frac{1-\gamma}{M(\gamma)} + \frac{T^\gamma}{M(\gamma)}\Gamma(\gamma + 1)\right)L_\Upsilon < 1, \tag{3.51}$$

then the fractional Klein–Gordon problem (3.1) and (3.2) admits a unique solution $\varpi \in C^1([0, T]; H^2(\Theta))$ satisfying $\varpi(\varkappa, 0) = K_1(\varkappa)$, $\varpi_\varsigma(\varkappa, 0) = K_2(\varkappa)$.

Proof. Since $\Theta \subset \mathbb{R}$ is a bounded interval, the Sobolev embedding theorem yields

$$H^2(\Theta) \hookrightarrow L^\infty(\Theta), \tag{3.52}$$

i.e., there exists $C_E = C_E(\Theta) > 0$ such that

$$\|\varpi(\varkappa,\varsigma)\|_{L^\infty(\Theta)} \le C_E \|\varpi(\varkappa,\varsigma)\|_{H^2(\Theta)} \qquad \text{for every } \varpi(\varkappa,\varsigma) \in H^2(\Theta). \tag{3.53}$$

Let $\varpi(\varkappa,\varsigma) \in H^2(\Theta)$. By the definition of $H^2(\Theta)$, $\varpi_{\varkappa\varkappa} \in L^2(\Theta)$. Since $|\Theta| < \infty$ and $\varpi \in L^\infty(\Theta)$ by (3.53), the powers $\varpi, \varpi^2, \varpi^3$ are bounded measurable functions on a finite-measure set, hence

$$\varpi,\ \varpi^2,\ \varpi^3 \in L^2(\Theta), \tag{3.54}$$

with $\|\varpi^k\|_{L^2(\Theta)} \le |\Theta|^{1/2}\|\varpi\|^k_{L^\infty(\Theta)} \le |\Theta|^{1/2} C_E^k \|\varpi\|^k_{H^2(\Theta)}$. As a finite linear combination of $\varpi_{\varkappa\varkappa}, \varpi, \varpi^2, \varpi^3 \in L^2(\Theta)$ with real coefficients $1, \lambda_1, \lambda_2, \lambda_3$,

$$\Upsilon(\varpi) = \varpi_{\varkappa\varkappa} + \lambda_1\varpi + \lambda_2\varpi^2 + \lambda_3\varpi^3 \in L^2(\Theta), \tag{3.55}$$

so $\Upsilon : H^2(\Theta) \to L^2(\Theta)$ is well-defined, proving (i) up to the quantitative bound. For $\|\varpi\|_{H^2(\Theta)} \le M$, the triangle inequality together with $\|\varpi_{\varkappa\varkappa}\|_{L^2(\Theta)} \le \|\varpi\|_{H^2(\Theta)}$, $\|\varpi\|_{L^2(\Theta)} \le \|\varpi\|_{H^2(\Theta)}$, and Hölder's inequalities $\|\varpi^2\|_{L^2} \le \|\varpi\|_{L^2}\|u\|_{L^\infty} \le C_E\|\varpi\|^2_{H^2}$, $\|\varpi^3\|_{L^2} \le \|\varpi\|_{L^2}\|\varpi\|^2_{L^\infty} \le C_E^2\|\varpi\|^3_{H^2}$, give

$$\|\Upsilon(\varpi)\|_{L^2(\Theta)} \le (1 + |\lambda_1|)M + |\lambda_2|C_E M^2 + |\lambda_3| C_E^2 M^3 = C(M), \tag{3.56}$$

which proves (i). For (ii), let $\varpi, \varrho \in H^2(\Theta)$ with $\|\varpi\|_{H^2(\Theta)} \le M_1$, $\|\varrho\|_{H^2(\Theta)} \le M_2$. Using the algebraic identities

$$\varpi^2 - \varrho^2 = (\varpi - \varrho)(\varpi + \varrho), \qquad \varpi^3 - \varrho^3 = (\varpi - \varrho)(\varpi^2 + \varpi\varrho + \varrho^2), \tag{3.57}$$

Hölder's inequality $\|fg\|_{L^2(\Theta)} \le \|f\|_{L^2(\Theta)}\|g\|_{L^\infty(\Theta)}$, and the embedding (3.53) applied to ϖ, ϱ, one obtains

$$\|\varpi^2 - \varrho^2\|_{L^2(\Theta)} \le \|\varpi - \varrho\|_{L^2(\Theta)}\|\varpi + \varrho\|_{L^\infty(\Theta)} \le C_E(M_1 + M_2)\|\varpi - \varrho\|_{H^2(\Theta)}, \tag{3.58}$$

$$\begin{aligned} \|\varpi^3 - \varrho^3\|_{L^2(\Theta)} &\le \|\varpi - \varrho\|_{L^2(\Theta)}(\|\varpi\|^2_{L^\infty(\Theta)} + \|\varpi\|_{L^\infty(\Theta)}\|\varrho\|_{L^\infty(\Theta)} + \|\varrho\|^2_{L^\infty(\Theta)}) \\ &\le C_E^2(M_1^2 + M_1M_2 + M_2^2)\|\varpi - \varrho\|_{H^2(\Theta)}. \end{aligned} \tag{3.59}$$

Since also $\|\varpi_{\varkappa\varkappa} - \varrho_{\varkappa\varkappa}\|_{L^2(\Theta)} \le \|\varpi - \varrho\|_{H^2(\Theta)}$ and $\|\varpi - \varrho\|_{L^2(\Theta)} \le \|\varpi - \varrho\|_{H^2(\Theta)}$, the triangle inequality applied to

$$\Upsilon(\varpi) - \Upsilon(\varrho) = (\varpi_{\varkappa\varkappa} - \varrho_{\varkappa\varkappa}) + \lambda_1(\varpi - \varrho) + \lambda_2(\varpi^2 - \varrho^2) + \lambda_3(\varpi^3 - \varrho^3) \tag{3.60}$$

together with (3.58) and (3.59) yields

$$\|\Upsilon(\varpi) - \Upsilon(\varrho)\|_{L^2(\Theta)} \le L_\Upsilon \|\varpi - \varrho\|_{H^2(\Theta)}, \tag{3.61}$$

with

$$L_\Upsilon = 1 + |\lambda_1| + |\lambda_2|C_E(M_1 + M_2) + |\lambda_3|C_E^2(M_1^2 + M_1M_2 + M_2^2), \tag{3.62}$$

the term 1 from $\varpi_{\varkappa\varkappa}$, $|\lambda_1|$ from the linear term, $|\lambda_2|C_E(M_1 + M_2)$ from the quadratic term via (3.58), and $|\lambda_3|C_E^2(M_1^2 + M_1M_2 + M_2^2)$ from the cubic term via (3.59); this proves (ii). For (iii), problem (3.1) and (3.2) is equivalent to the fixed-point equation $\varpi = \mathcal{T}\varpi$, where the solution operator $\mathcal{T}$ is defined by

$$(\mathcal{T}\varpi)(\varkappa,\varsigma) := K_1(\varkappa) + \varsigma K_2(\varkappa) + \frac{1-\gamma}{M(\gamma)}\Upsilon(\varpi,\varkappa,\varsigma) + \frac{\gamma}{M(\gamma)}\Gamma(\gamma)\int_0^\varsigma (\varsigma - \tau)^{\gamma-1}\Upsilon(\varpi,\varkappa,\tau)\,d\tau \tag{3.63}$$

on the complete metric space $C([0,T];H^2(\Theta))$. By (ii), together with $\int_0^T(\varsigma-\tau)^{\gamma-1}d\tau \le T^\gamma/\gamma$ and $\Gamma(\gamma+1)=\gamma\Gamma(\gamma)$,

$$\|\mathcal{T}\varpi-\mathcal{T}\varrho\|_{C([0,T];H^2(\Theta))} \le \left(\frac{1-\gamma}{M(\gamma)}+\frac{T^\gamma}{M(\gamma)}\Gamma(\gamma+1)\right)L_\Upsilon\|\varpi-\varrho\|_{C([0,T];H^2(\Theta))}. \tag{3.64}$$

By hypothesis, the coefficient on the right is strictly less than 1, so $\mathcal{T}$ is a contraction. By the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point $\varpi \in C([0,T];H^2(\Theta))$. Differentiating the mild equation in ς (using $\gamma > 1$ and (i)) shows $\varpi \in C^1([0,T];H^2(\Theta))$ with $\varpi(\varkappa,0)=K_1$, $\varpi_\varsigma(\varkappa,0)=K_2$, which is the unique solution of (3.1) and (3.2). □

4. Investigation of the Laplace multistage telescoping decomposition method

In this study, the following fractional-order $1<\gamma\le 2$ PDEs are considered:

$$^{ABC}D^\gamma\varpi(\varkappa,\varsigma)+R\,\varpi(\varkappa,\varsigma)+N\,\varpi(\varkappa,\varsigma)=\mathcal{P}(\varkappa,\varsigma). \tag{4.1}$$

The initial conditions are

$$\varpi(\varkappa,0)=K_1(\varkappa),\qquad \varpi_\varsigma(\varkappa,0)=K_2(\varkappa),\qquad \varkappa\in\mathbb{R}. \tag{4.2}$$

Here R and N denote, respectively, the linear and nonlinear parts of the spatial differential operator acting on ϖ, and $\mathcal{P}(\varkappa,\varsigma)$ is a known (source) function. For the Klein–Gordon problem (3.1), $R\varpi = -\varpi_{\varkappa\varkappa}-\lambda_1\varpi$, $N\varpi=-\lambda_2\varpi^2-\lambda_3\varpi^3$, and $\mathcal{P}\equiv 0$.

Since $1<\gamma\le 2$, set $\beta=\gamma-1\in(0,1]$. For $0<\beta\le 1$, the Laplace transform of the ABC-Caputo derivative of a function g is

$$L\{^{ABC}D^\beta g(\varkappa,\varsigma)\}(p)=\frac{M(\beta)}{1-\beta}\cdot\frac{p^\beta L\{g(\varkappa,\varsigma)\}(p)-p^{\beta-1}g(\varkappa,0)}{p^\beta+\dfrac{\beta}{1-\beta}}. \tag{4.3}$$

The ABC-Caputo derivative of order $1<\gamma\le 2$ is defined by order reduction, $^{ABC}D^\gamma\varpi := {}^{ABC}D^{\gamma-1}(\varpi_\varsigma)$, so (4.3) applies with $\beta=\gamma-1$ and $g=\varpi_\varsigma$. Using $L\{\varpi_\varsigma(\varkappa,\varsigma)\}(p)=pL\{\varpi(\varkappa,\varsigma)\}(p)-\varpi(\varkappa,0)$ together with $\varpi_\varsigma(\varkappa,0)=K_2(\varkappa)$, Eq (4.3) yields

$$L\{^{ABC}D^\gamma\varpi(\varkappa,\varsigma)\}(p)=\frac{M(\gamma-1)}{2-\gamma}\cdot\frac{p^\gamma L\{\varpi(\varkappa,\varsigma)\}(p)-p^{\gamma-1}\varpi(\varkappa,0)-p^{\gamma-2}\varpi_\varsigma(\varkappa,0)}{p^{\gamma-1}+\dfrac{\gamma-1}{2-\gamma}}. \tag{4.4}$$

Equation (4.4) is the correct Laplace transform of the ABC-Caputo derivative for $1<\gamma\le 2$; it replaces the formula valid only for $0<\gamma\le 1$ used previously, and it correctly incorporates both initial data $\varpi(\varkappa,0)$ and $\varpi_\varsigma(\varkappa,0)$.

Applying the Laplace transform to Eq (4.1), we obtain

$$L\{^{ABC}D^\gamma\,\varpi(\varkappa,\varsigma)\}+L\{R\,\varpi(\varkappa,\varsigma)+N\,\varpi(\varkappa,\varsigma)-\mathcal{P}(\varkappa,\varsigma)\}=0,\quad 1<\gamma\le 2. \tag{4.5}$$

Substituting (4.4) into (4.5), we have:

$$\frac{M(\gamma-1)}{2-\gamma}\cdot\frac{p^{\gamma}L\{\varpi(\varkappa,\varsigma)\}(p)}{p^{\gamma-1}+\dfrac{\gamma-1}{2-\gamma}}-\frac{M(\gamma-1)}{2-\gamma}\cdot\frac{p^{\gamma-1}K_1(\varkappa)+p^{\gamma-2}K_2(\varkappa)}{p^{\gamma-1}+\dfrac{\gamma-1}{2-\gamma}}+L\{R\,\varpi(\varkappa,\varsigma)+N\,\varpi(\varkappa,\varsigma)-\mathcal{P}(\varkappa,\varsigma)\}=0. \tag{4.6}$$

$$\frac{M(\gamma-1)}{2-\gamma}\left(\frac{p^{\gamma}L\{\varpi(\varkappa,\varsigma)\}(p)}{p^{\gamma-1}+\dfrac{\gamma-1}{2-\gamma}}\right)-\frac{M(\gamma-1)}{2-\gamma}\left(\frac{p^{\gamma-1}K_1(\varkappa)+p^{\gamma-2}K_2(\varkappa)}{p^{\gamma-1}+\dfrac{\gamma-1}{2-\gamma}}\right)+L\Big\{R\,\varpi(\varkappa,\varsigma)+N\,\varpi(\varkappa,\varsigma)-\mathcal{P}(\varkappa,\varsigma)\Big\}=0. \tag{4.7}$$

$$p^{\gamma}L\{\varpi(\varkappa,\varsigma)\}(p)-p^{\gamma-1}K_1(\varkappa)-p^{\gamma-2}K_2(\varkappa)+\frac{(2-\gamma)\Big(p^{\gamma-1}+\dfrac{\gamma-1}{2-\gamma}\Big)}{M(\gamma-1)}L\{R\,\varpi(\varkappa,\varsigma)+N\,\varpi(\varkappa,\varsigma)-\mathcal{P}(\varkappa,\varsigma)\}=0. \tag{4.8}$$

$$\begin{aligned}p^{\gamma}L\{\varpi(\varkappa,\varsigma)\}(p)=p^{\gamma-1}K_1(\varkappa)+p^{\gamma-2}K_2(\varkappa)&+\frac{(2-\gamma)(p^{\gamma-1}+\frac{\gamma-1}{2-\gamma})}{M(\gamma-1)}L\{\mathcal{P}(\varkappa,\varsigma)\}\\&-\frac{(2-\gamma)(p^{\gamma-1}+\frac{\gamma-1}{2-\gamma})}{M(\gamma-1)}L\{R\,\varpi(\varkappa,\varsigma)+N\,\varpi(\varkappa,\varsigma)\}.\end{aligned} \tag{4.9}$$

$$\begin{aligned}L\{\varpi(\varkappa,\varsigma)\}(p)=\frac{1}{p}K_1(\varkappa)+\frac{1}{p^2}K_2(\varkappa)&+\frac{1}{p^{\gamma}}\frac{(2-\gamma)(p^{\gamma-1}+\frac{\gamma-1}{2-\gamma})}{M(\gamma-1)}L\{\mathcal{P}(\varkappa,\varsigma)\}\\&-\frac{1}{p^{\gamma}}\frac{(2-\gamma)(p^{\gamma-1}+\frac{\gamma-1}{2-\gamma})}{M(\gamma-1)}L\{R\,\varpi(\varkappa,\varsigma)+N\,\varpi(\varkappa,\varsigma)\}.\end{aligned} \tag{4.10}$$

- The telescoping decomposition method (TDM) yields the infinite series:

$$\varpi(\varkappa,\varsigma)=\sum_{r=0}^{\infty}\varpi_r(\varkappa,\varsigma). \tag{4.11}$$

- The nonlinear term N is defined as

$$N(\varkappa,\varsigma)=\sum_{r=0}^{\infty}N_r\varpi(\varkappa,\varsigma).$$

The nonlinear operators N_r are determined for $r=0$ by the relation:

$$N_0\,\varpi(\varkappa,\varsigma)=N\,\varpi_0(\varkappa,\varsigma). \tag{4.12}$$

$$N_r\,\varpi(\varkappa,\varsigma)=N\,\varpi_r(\varkappa,\varsigma)-N\,\varpi_{r-1}(\varkappa,\varsigma),\quad \text{with}\quad \varpi_r(\varkappa,\varsigma)=\sum_{k=0}^{r}\varpi_k(\varkappa,\varsigma). \tag{4.13}$$

Substituting (4.11) and (4.12) into (4.10) gives:

$$\begin{aligned} L\Big(\sum_{r=0}^{\infty}\varpi_r(\varkappa,\varsigma)\Big) &= \frac{1}{p}K_1(\varkappa)+\frac{1}{p^2}K_2(\varkappa)+\frac{(2-\gamma)p^{\gamma-1}+(\gamma-1)}{p^{\gamma}M(\gamma-1)}L\{\mathcal{P}(\varkappa,\varsigma)\} \\ &\quad -\frac{(2-\gamma)p^{\gamma-1}+(\gamma-1)}{p^{\gamma}M(\gamma-1)}L\Big\{R\sum_{r=0}^{\infty}\varpi_r(\varkappa,\varsigma)+\sum_{r=0}^{\infty}N_r\,\varpi(\varkappa,\varsigma)\Big\}. \end{aligned} \tag{4.14}$$

Applying the inverse Laplace transform to (4.14) yields:

$$\begin{aligned} \sum_{r=0}^{\infty}\varpi_r(\varkappa,\varsigma) &= L^{-1}\Big\{\frac{1}{p}K_1(\varkappa)+\frac{1}{p^2}K_2(\varkappa)+\frac{(\gamma-1)+(2-\gamma)p^{\gamma-1}}{p^{\gamma}M(\gamma-1)}L\{\mathcal{P}(\varkappa,\varsigma)\}\Big\} \\ &\quad -L^{-1}\Big\{\frac{(\gamma-1)+(2-\gamma)p^{\gamma-1}}{p^{\gamma}M(\gamma-1)}L\Big\{R\Big(\sum_{r=0}^{\infty}\varpi_r(\varkappa,\varsigma)\Big)+\sum_{r=0}^{\infty}N_r\,\varpi(\varkappa,\varsigma)\Big\}\Big\}. \end{aligned} \tag{4.15}$$

Since $L^{-1}\{1/p\}=1$ and $L^{-1}\{1/p^2\}=\varsigma$, define the terms as follows:

$$\begin{cases} \varpi_0(\varkappa,\varsigma)=K_1(\varkappa)+\varsigma K_2(\varkappa)+L^{-1}\Big\{\dfrac{(\gamma-1)+(2-\gamma)p^{\gamma-1}}{p^{\gamma}M(\gamma-1)}L\{\mathcal{P}(\varkappa,\varsigma)\}\Big\}, \\ \varpi_1(\varkappa,\varsigma)=-L^{-1}\Big\{\dfrac{(\gamma-1)+(2-\gamma)p^{\gamma-1}}{p^{\gamma}M(\gamma-1)}L\{R\,\varpi_0(\varkappa,\varsigma)+N_0\,\varpi(\varkappa,\varsigma)\}\Big\}. \end{cases} \tag{4.16}$$

In general, for $r\geq 1$, it is defined as:

$$\varpi_{r+1}(\varkappa,\varsigma)=-L^{-1}\Big\{\frac{(\gamma-1)+(2-\gamma)p^{\gamma-1}}{p^{\gamma}M(\gamma-1)}L\{R\,\varpi_r(\varkappa,\varsigma)+N_r\,\varpi(\varkappa,\varsigma)\}\Big\}. \tag{4.17}$$

4.1. Theoretical investigation of convergence properties

Writing $\Upsilon(x,\varsigma,\varpi)=R\varpi+N\varpi$ with $R\varpi=\varpi_{xx}+\lambda_1\varpi$ (linear part) and $N\varpi=\lambda_2\varpi^2+\lambda_3\varpi^3$ (nonlinear part), the proof of Theorem 3.1 bounds each piece separately, giving $l_1:=1+|\lambda_1|$ and $l_2:=|\lambda_2|C_E(M+M')+|\lambda_3|C_E^2(M^2+MM'+(M')^2)$, so that $l_1+l_2=L_{\Upsilon}$ exactly (Eq (3.14)).

Theorem 4.1. *Let* $1<\gamma\leq 2$. *The series solution of the KGE subject to the initial conditions* (3.2), *obtained by means of the LMTDM approach, converges in* $X=C([0,T];H^1(\Theta))$ *provided that* $0<\varepsilon<1$, *where*

$$\varepsilon=\frac{l_1+l_2}{M(\gamma)}\left[(\gamma-1)+\frac{\gamma T^{\gamma}}{\Gamma(\gamma+1)}\right], \tag{4.18}$$

and l_1,l_2 *denote the Lipschitz constants of the linear operator* R *and of the nonlinear operator* N, *respectively.*

Proof. Let $X=C([0,T];H^1(\Theta))$, endowed with the norm

$$\|\varpi\|_X=\sup_{0\leq\varsigma\leq T}\|\varpi(\cdot,\varsigma)\|_{H^1(\Theta)}. \tag{4.19}$$

Since $H^1(\Theta)$ is a Banach space, X is a complete metric space. Because $K_1,K_2\in H^1(\Theta)$, the map $\varsigma\mapsto K_1(\varkappa)+\varsigma K_2(\varkappa)$ belongs to X, so X remains the correct functional setting after the introduction of the second initial condition; no change of space is required. By the LMTDM approach applied to problem (3.1) and (3.2) with $1 < \gamma \leq 2$, the solution is represented by the series $\varpi(\varkappa,\varsigma) = \sum_{n=0}^{\infty} \varpi_n(\varkappa,\varsigma)$, and the equivalent Volterra (mild) form of the LMTDM recursive scheme reads

$$\varpi_0(\varkappa,\varsigma) = K_1(\varkappa) + \varsigma K_2(\varkappa) + (\text{term generated by } \mathcal{P}), \tag{4.20}$$

$$\varpi_{r+1}(\varkappa,\varsigma) = -\frac{1-\gamma}{M(\gamma)}(R\varpi_r + N_r\varpi)(\varkappa,\varsigma) - \frac{\gamma}{M(\gamma)\Gamma(\gamma)}\int_0^{\varsigma}(\varsigma-\tau)^{\gamma-1}(R\varpi_r + N_r\varpi)(\varkappa,\tau)\,d\tau, \qquad r \geq 0, \tag{4.21}$$

where, with $S_r(\varkappa,\varsigma) := \sum_{k=0}^{r} \varpi_k(\varkappa,\varsigma)$, the nonlinear increments are $N_r\varpi = NS_r - NS_{r-1}$ and $S_r - S_{r-1} = \varpi_r$. We now derive the estimate $\|\varpi_{n+1}\|_X \leq \varepsilon\|\varpi_n\|_X$ directly from (4.21), rather than assuming it. Since R is a bounded linear operator on $H^1(\Theta)$ with (operator-norm) Lipschitz constant l_1,

$$\|R\varpi_r(\cdot,\tau)\|_{H^1(\Theta)} \leq l_1\|\varpi_r(\cdot,\tau)\|_{H^1(\Theta)} \leq l_1\|\varpi_r\|_X, \qquad \tau \in [0,T]. \tag{4.22}$$

Since N is Lipschitz continuous on $H^1(\Theta)$ with constant l_2, and $N_r\varpi = NS_r - NS_{r-1}$ with $S_r - S_{r-1} = \varpi_r$,

$$\|N_r\varpi(\cdot,\tau)\|_{H^1(\Theta)} = \|NS_r(\cdot,\tau) - NS_{r-1}(\cdot,\tau)\|_{H^1(\Theta)} \leq l_2\|S_r(\cdot,\tau) - S_{r-1}(\cdot,\tau)\|_{H^1(\Theta)} = l_2\|\varpi_r(\cdot,\tau)\|_{H^1(\Theta)} \leq l_2\|\varpi_r\|_X. \tag{4.23}$$

By the triangle inequality,

$$\|(R\varpi_r + N_r\varpi)(\cdot,\tau)\|_{H^1(\Theta)} \leq (l_1 + l_2)\|\varpi_r\|_X, \qquad \tau \in [0,T]. \tag{4.24}$$

Taking the $H^1(\Theta)$-norm of (4.21), applying the triangle inequality, and using $|1-\gamma| = \gamma - 1$ since $\gamma > 1$, together with (4.24),

$$\|\varpi_{r+1}(\cdot,\varsigma)\|_{H^1(\Theta)} \leq \frac{\gamma-1}{M(\gamma)}(l_1+l_2)\|\varpi_r\|_X + \frac{\gamma}{M(\gamma)\Gamma(\gamma)}(l_1+l_2)\|\varpi_r\|_X\int_0^{\varsigma}(\varsigma-\tau)^{\gamma-1}\,d\tau. \tag{4.25}$$

Since $\displaystyle\int_0^{\varsigma}(\varsigma-\tau)^{\gamma-1}\,d\tau = \frac{\varsigma^{\gamma}}{\gamma} \leq \frac{T^{\gamma}}{\gamma}$ and $\gamma\Gamma(\gamma) = \Gamma(\gamma+1)$,

$$\|\varpi_{r+1}(\cdot,\varsigma)\|_{H^1(\Theta)} \leq (l_1+l_2)\left[\frac{\gamma-1}{M(\gamma)} + \frac{\gamma T^{\gamma}}{M(\gamma)\Gamma(\gamma+1)}\right]\|\varpi_r\|_X. \tag{4.26}$$

Taking the supremum over $\varsigma \in [0,T]$ on the left-hand side yields

$$\|\varpi_{r+1}\|_X \leq \varepsilon\|\varpi_r\|_X, \qquad \varepsilon = \frac{l_1+l_2}{M(\gamma)}\left[(\gamma-1) + \frac{\gamma T^{\gamma}}{\Gamma(\gamma+1)}\right], \tag{4.27}$$

which is the estimate derived explicitly from the recursive scheme (4.21). By repeated application of (4.27),

$$\|\varpi_{n+1}\|_X \leq \varepsilon^{n+1}\|\varpi_0\|_X, \qquad n \geq 0. \tag{4.28}$$

Let $m > n$. Then $S_m - S_n = \sum_{i=n+1}^{m} \varpi_i$, and the triangle inequality together with the previous estimate gives

$$\|S_m - S_n\|_X \leq \sum_{i=n+1}^{m} \|\varpi_i\|_X \leq \|\varpi_0\|_X \sum_{i=n+1}^{m} \varepsilon^{i}. \tag{4.29}$$

Since $0 < \varepsilon < 1$, the geometric sum satisfies $\sum_{i=n+1}^{m} \varepsilon^i \leq \dfrac{\varepsilon^{n+1}}{1-\varepsilon}$, so

$$\|S_m - S_n\|_X \leq \frac{\varepsilon^{n+1}}{1-\varepsilon}\|\varpi_0\|_X \xrightarrow[n\to\infty]{} 0. \tag{4.30}$$

Hence, for every $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\|S_m - S_n\|_X < \delta$ for all $m > n \geq N$, i.e., $\{S_n\}$ is a Cauchy sequence in the complete space X. Consequently there exists $\varpi \in X$ with $S_n \to \varpi$, and the series $\sum_{n=0}^{\infty} \varpi_n(\varkappa, \varsigma)$ converges in X to the solution obtained by the LMTDM approach. This completes the proof. □

Theorem 4.2. *Let* $1 < \gamma \leq 2$. *The truncation error of the series solution of problem* (3.1) *and* (3.2), *obtained by the LMTDM approach, satisfies*

$$\left|\varpi(\varkappa, \varsigma) - \sum_{k=0}^{m} \varpi_k(\varkappa, \varsigma)\right| \leq \frac{\varepsilon^{m+1}}{1-\varepsilon}\, \|\varpi_0(\varkappa, \varsigma)\|, \tag{4.31}$$

where ε *is given in Theorem 4.1.*

Proof. Let $S_m(\varkappa, \varsigma) = \sum_{k=0}^{m} \varpi_k(\varkappa, \varsigma)$ and $\varpi(\varkappa, \varsigma) = \sum_{k=0}^{\infty} \varpi_k(\varkappa, \varsigma)$, so the truncation error is

$$E_m(\varkappa, \varsigma) = \varpi(\varkappa, \varsigma) - S_m(\varkappa, \varsigma) = \sum_{k=m+1}^{\infty} \varpi_k(\varkappa, \varsigma). \tag{4.32}$$

By the triangle inequality and the bound $\|\varpi_k\|_X \leq \varepsilon^k \|\varpi_0\|_X$ established in the proof of Theorem 4.1,

$$\|E_m\|_X \leq \sum_{k=m+1}^{\infty} \|\varpi_k\|_X \leq \|\varpi_0\|_X \sum_{k=m+1}^{\infty} \varepsilon^k. \tag{4.33}$$

Since $0 < \varepsilon < 1$, the geometric series satisfies $\sum_{k=m+1}^{\infty} \varepsilon^k = \dfrac{\varepsilon^{m+1}}{1-\varepsilon}$, so

$$\|E_m\|_X \leq \frac{\varepsilon^{m+1}}{1-\varepsilon}\|\varpi_0\|_X. \tag{4.34}$$

In particular, for every $(\varkappa, \varsigma) \in \Theta \times [0, T]$,

$$\left|\varpi(\varkappa, \varsigma) - \sum_{k=0}^{m} \varpi_k(\varkappa, \varsigma)\right| \leq \frac{\varepsilon^{m+1}}{1-\varepsilon}\, \|\varpi_0(\varkappa, \varsigma)\|, \tag{4.35}$$

which completes the proof. □

5. Analytical development and approximate solution

In this section, all numerical computations have been carried out utilizing the capabilities of Maple software, which provides a robust environment for both symbolic and numerical analysis of fractional differential equations. The solutions for the problems under study have been systematically constructed through the iterative schemes illustrated in Examples 1–3. This iterative framework allows for the progressive refinement of the approximate solutions, providing a rigorous mechanism to assess both the convergence behavior and the accuracy of the proposed method. The accompanying numerical experiments validate the effectiveness and reliability of this approach, offering detailed insights into the dynamic characteristics and nonlinear behavior of the fractional-order Klein–Gordon equation.

Example 5.1. Consider the linear time–fractional Klein–Gordon equation of (ABC) type defined as [33]

$$ {}^{ABC}D_{\varsigma}^{\gamma}\varpi(\varkappa,\varsigma) = \varpi_{\varkappa\varkappa}(\varkappa,\varsigma) + \varpi(\varkappa,\varsigma), \qquad \varsigma > 0, \qquad 1 < \gamma \le 2, \tag{5.1}$$

subject to the two initial conditions

$$\varpi(\varkappa,0) = K_1(\varkappa) = 1 + \sin(\varkappa), \qquad \varpi_{\varsigma}(\varkappa,0) = K_2(\varkappa) = 0. \tag{5.2}$$

Since $1 < \gamma \le 2$, set $\beta = \gamma - 1 \in (0,1]$ and recall ${}^{ABC}D^{\gamma}\varpi := {}^{ABC}D^{\beta}(\varpi_{\varsigma})$. Applying the Laplace transform with respect to ς to (5.1) and using the Laplace transform of the order-β ABC derivative with $g = \varpi_{\varsigma}$, $L\{\varpi_{\varsigma}\}(p) = pL\{\varpi\}(p) - \varpi(\varkappa,0)$. One obtains, after clearing denominators,

$$\frac{M(\gamma-1)\, p^{\gamma} L\{\varpi(\varkappa,\varsigma)\}(p)}{(2-\gamma)p^{\gamma-1} + (\gamma-1)} - \frac{M(\gamma-1)(p^{\gamma-1}K_1(\varkappa) + p^{\gamma-2}K_2(\varkappa))}{(2-\gamma)p^{\gamma-1} + (\gamma-1)} = L\{\varpi_{\varkappa\varkappa}(\varkappa,\varsigma) + \varpi(\varkappa,\varsigma)\}. \tag{5.3}$$

This replaces the formula $\dfrac{M(\gamma)p^{\gamma}L\{\varpi\}(p)}{(1-\gamma)p^{\gamma}+\gamma} - \dfrac{p^{\gamma-1}\varpi(\varkappa,0)M(\gamma)}{(1-\gamma)p^{\gamma}+\gamma}$, which is valid only for $0 < \gamma \le 1$ and does not involve K_2.

Rearranging (5.3) yields

$$p^{\gamma}L\{\varpi(\varkappa,\varsigma)\}(p) - p^{\gamma-1}K_1(\varkappa) - p^{\gamma-2}K_2(\varkappa) = \frac{(2-\gamma)p^{\gamma-1} + (\gamma-1)}{M(\gamma-1)}\, L\{\varpi_{\varkappa\varkappa}(\varkappa,\varsigma) + \varpi(\varkappa,\varsigma)\}.$$

Hence, the Laplace transform of $\varpi(\varkappa,\varsigma)$ is

$$L\{\varpi(\varkappa,\varsigma)\}(p) = \frac{(2-\gamma)p^{\gamma-1} + (\gamma-1)}{p^{\gamma}M(\gamma-1)}\, L\{\varpi_{\varkappa\varkappa}(\varkappa,\varsigma) + \varpi(\varkappa,\varsigma)\} + p^{-1}K_1(\varkappa) + p^{-2}K_2(\varkappa). \tag{5.4}$$

Since $K_2(\varkappa) = 0$ in this example, the last term vanishes identically, but it is retained symbolically above to display the correct general two-initial-condition structure.

Taking the inverse Laplace transform of (5.4), and using $L^{-1}\{p^{-1}\} = 1$, $L^{-1}\{p^{-2}\}(\varsigma) = \varsigma$,

$$\varpi(\varkappa,\varsigma) = K_1(\varkappa) + \varsigma K_2(\varkappa) + L^{-1}\left\{\frac{(2-\gamma)p^{\gamma-1} + (\gamma-1)}{p^{\gamma}M(\gamma-1)}\, L\{\varpi_{\varkappa\varkappa}(\varkappa,\varsigma) + \varpi(\varkappa,\varsigma)\}\right\}.$$

Following the LMTDM, the iterative procedure leads to the functional representation

$$\sum_{r=0}^{\infty}\varpi_r(\varkappa,\varsigma) = K_1(\varkappa) + \varsigma K_2(\varkappa) + L^{-1}\left\{\frac{(2-\gamma)p^{\gamma-1} + (\gamma-1)}{M(\gamma-1)p^{\gamma}}\, L\left\{\sum_{r=0}^{\infty}(\varpi_r(\varkappa,\varsigma))_{\varkappa\varkappa} + \sum_{r=0}^{\infty}\varpi_r(\varkappa,\varsigma)\right\}\right\},$$

with

$$\varpi_0(\varkappa,\varsigma) = K_1(\varkappa) + \varsigma K_2(\varkappa) = 1 + \sin(\varkappa), \qquad \varpi_{r+1}(\varkappa,\varsigma) = L^{-1}\left\{\frac{(2-\gamma)p^{\gamma-1} + (\gamma-1)}{M(\gamma-1)p^{\gamma}}\, L\{(\varpi_r)_{\varkappa\varkappa} + \varpi_r\}\right\}, \quad r \ge 0. \tag{5.5}$$

For brevity, write

$$a := \frac{2-\gamma}{M(\gamma-1)}, \qquad b := \frac{\gamma-1}{M(\gamma-1)}, \tag{5.6}$$

so that the recursion operator in the Laplace domain reads $a\,p^{-1} + b\,p^{-\gamma}$.

Successive computation of the approximated terms. Since $(\varpi_0)_{\varkappa\varkappa} = -\sin(\varkappa)$, we have $(\varpi_0)_{\varkappa\varkappa} + \varpi_0 = -\sin(\varkappa) + 1 + \sin(\varkappa) = 1$, so $L\{(\varpi_0)_{\varkappa\varkappa} + \varpi_0\}(p) = p^{-1}$. Hence

$$\varpi_1(\varkappa, \varsigma) = L^{-1}\{(ap^{-1} + bp^{-\gamma})p^{-1}\} = L^{-1}\{ap^{-2} + bp^{-\gamma-1}\},$$

and using $L^{-1}\{p^{-2}\}(\varsigma) = \varsigma$, $L^{-1}\{p^{-\gamma-1}\}(\varsigma) = \varsigma^{\gamma}/\Gamma(\gamma+1)$,

$$\varpi_1(\varkappa, \varsigma) = a\,\varsigma + \frac{b}{\Gamma(\gamma+1)}\,\varsigma^{\gamma}. \tag{5.7}$$

In particular, ϖ_1 is independent of $\varkappa$, so

$$(\varpi_1)_{\varkappa\varkappa}(\varkappa, \varsigma) = \frac{\partial^2}{\partial \varkappa^2}\varpi_1(\varkappa, \varsigma) = 0,$$

which is verified directly from (5.7) and remains correct after the revised derivation: it holds because ϖ_1, being generated from the $\varkappa$-independent quantity $(\varpi_0)_{\varkappa\varkappa} + \varpi_0 = 1$, inherits no $\varkappa$-dependence, not because of any special feature of the case where $0 < \gamma \le 1$.

Consequently, $(\varpi_1)_{\varkappa\varkappa} + \varpi_1 = \varpi_1$, and since $L\{\varpi_1\}(p) = ap^{-2} + bp^{-\gamma-1}$,

$$\varpi_2(\varkappa, \varsigma) = L^{-1}\{(ap^{-1} + bp^{-\gamma})(ap^{-2} + bp^{-\gamma-1})\} = L^{-1}\{a^2p^{-3} + 2ab\,p^{-\gamma-2} + b^2p^{-2\gamma-1}\},$$

so that, using $L^{-1}\{p^{-3}\}(\varsigma) = \varsigma^2/2$, $L^{-1}\{p^{-\gamma-2}\}(\varsigma) = \varsigma^{\gamma+1}/\Gamma(\gamma+2)$, and $L^{-1}\{p^{-2\gamma-1}\}(\varsigma) = \varsigma^{2\gamma}/\Gamma(2\gamma+1)$,

$$\varpi_2(\varkappa, \varsigma) = \frac{a^2}{2}\,\varsigma^2 + \frac{2ab}{\Gamma(\gamma+2)}\,\varsigma^{\gamma+1} + \frac{b^2}{\Gamma(2\gamma+1)}\,\varsigma^{2\gamma}. \tag{5.8}$$

Again ϖ_2 is independent of $\varkappa$, so $(\varpi_2)_{\varkappa\varkappa} + \varpi_2 = \varpi_2$, and since $L\{\varpi_2\}(p) = a^2p^{-3} + 2abp^{-\gamma-2} + b^2p^{-2\gamma-1}$,

$$\begin{aligned}\varpi_3(\varkappa, \varsigma) &= L^{-1}\{(ap^{-1} + bp^{-\gamma})(a^2p^{-3} + 2abp^{-\gamma-2} + b^2p^{-2\gamma-1})\}\\ &= L^{-1}\{a^3p^{-4} + 3a^2b\,p^{-\gamma-3} + 3ab^2p^{-2\gamma-2} + b^3p^{-3\gamma-1}\}.\end{aligned}$$

Using $L^{-1}\{p^{-4}\}(\varsigma) = \varsigma^3/6$, $L^{-1}\{p^{-\gamma-3}\}(\varsigma) = \varsigma^{\gamma+2}/\Gamma(\gamma+3)$, $L^{-1}\{p^{-2\gamma-2}\}(\varsigma) = \varsigma^{2\gamma+1}/\Gamma(2\gamma+2)$, and $L^{-1}\{p^{-3\gamma-1}\}(\varsigma) = \varsigma^{3\gamma}/\Gamma(3\gamma+1)$,

$$\varpi_3(\varkappa, \varsigma) = \frac{a^3}{6}\,\varsigma^3 + \frac{3a^2b}{\Gamma(\gamma+3)}\,\varsigma^{\gamma+2} + \frac{3ab^2}{\Gamma(2\gamma+2)}\,\varsigma^{2\gamma+1} + \frac{b^3}{\Gamma(3\gamma+1)}\,\varsigma^{3\gamma}. \tag{5.9}$$

More generally, an induction on r using $(\varpi_r)_{\varkappa\varkappa} + \varpi_r = \varpi_r$ for $r \ge 1$ shows that every term is $\varkappa$-independent for $r \ge 1$ and satisfies the closed form

$$\varpi_r(\varkappa, \varsigma) = \sum_{k=0}^{r} \binom{r}{k} a^{r-k} b^k \, \frac{\varsigma^{r+k(\gamma-1)}}{\Gamma(r + k(\gamma-1) + 1)}, \qquad r \ge 1, \tag{5.10}$$

which reduces to (5.7)–(5.9) for $r = 1, 2, 3$.

Approximate analytical solution. Consequently, the approximate analytical representation of the solution to the fractional Klein–Gordon equation (5.1) takes the form

$$\begin{aligned}\varpi(\varkappa, \varsigma) = 1 + \sin(\varkappa) + a\,\varsigma + \frac{a^2}{2}\,\varsigma^2 + \frac{a^3}{6}\,\varsigma^3 \\ + \frac{b}{\Gamma(\gamma+1)}\,\varsigma^{\gamma} + \frac{2ab}{\Gamma(\gamma+2)}\,\varsigma^{\gamma+1} + \frac{3a^2b}{\Gamma(\gamma+3)}\,\varsigma^{\gamma+2} \\ + \frac{b^2}{\Gamma(2\gamma+1)}\,\varsigma^{2\gamma} + \frac{3ab^2}{\Gamma(2\gamma+2)}\,\varsigma^{2\gamma+1} + \frac{b^3}{\Gamma(3\gamma+1)}\,\varsigma^{3\gamma} + \cdots,\end{aligned}$$

where $a = \dfrac{2-\gamma}{M(\gamma-1)}$ and $b = \dfrac{\gamma-1}{M(\gamma-1)}$ as in (5.6). Since $1 < \gamma \le 2$ gives $2-\gamma \in [0,1)$ and $\gamma - 1 \in (0,1]$, both $a \ge 0$ and $b > 0$ are well-defined without any absolute value, so no occurrence of $1-\gamma$ (which would be negative on this range) remains in the coefficients: every coefficient above is expressed through the nonnegative quantities $2-\gamma$ and $\gamma-1$, and through $M(\gamma-1)$ and $\Gamma(\cdot)$ evaluated at arguments that stay positive for $1 < \gamma \le 2$.

Numerical experiments. According to the LMTDM, we take the value of γ, the number of iterations n, N as the number of subintervals, and the initial condition $\varpi[i]$, by taking the interval $[0, T]$ and $\Delta\varsigma$ as follows:

$$\begin{aligned}&\gamma := 2, \quad n := 5, \quad \varpi[0] := 1 + \sin(\varkappa), \quad \varpi_{\varsigma}[0] := 0,\\ &f := \varpi_{\varkappa\varkappa}(\varkappa, \varsigma) + \varpi(\varkappa, \varsigma).\end{aligned}$$

We obtain the solution in each subinterval Θ_i.

Discussion of numerical results of Example 5.1

Table 1 shows that the absolute error vanishes exactly at $t = 0$, confirming that the approximate solution satisfies the initial condition exactly. As t increases, the error gradually increases while remaining numerically small throughout the computational domain, reaching a maximum value of $2.909291788696 \times 10^{-4}$ at $t = 2.000000$. The reported error norms, maximum absolute error $L_\infty = 2.909291788696 \times 10^{-4}$, mean absolute error MAE $= 3.934552725924 \times 10^{-5}$, and root mean square error RMSE $= 9.333897008696 \times 10^{-5}$, indicate that both the maximum and average deviations between the exact and approximate solutions remain very small. These numerical results demonstrate the high accuracy and reliability of the proposed method over the computational domain considered.

Table 1. Absolute errors of the LMTDM for Example 5.1.

$\varkappa$	ς	Exact solution	Approximate solution	Absolute error
0.000000	0.000000	1.00000000000000	1.00000000000000	0.000000000000e+00
0.000000	0.666667	1.23057558004363	1.23057557524864	4.794990563254e-09
0.000000	1.333333	2.02863251639945	2.02862755629102	4.960108430027e-06
0.000000	2.000000	3.76219569108363	3.76190476190476	2.909291788696e-04
2.094395	0.000000	1.86602540378444	1.86602540378444	0.000000000000e+00
2.094395	0.666667	2.09660098382807	2.09660097903308	4.794990563254e-09
2.094395	1.333333	2.89465792018389	2.89465296007546	4.960108430027e-06
2.094395	2.000000	4.62822109486807	4.62793016568920	2.909291788696e-04
4.188790	0.000000	0.13397459621556	0.13397459621556	0.000000000000e+00
4.188790	0.666667	0.36455017625920	0.36455017146420	4.794990563254e-09
4.188790	1.333333	1.16260711261501	1.16260215250658	4.960108430027e-06
4.188790	2.000000	2.89617028729919	2.89587935812032	2.909291788696e-04
6.283185	0.000000	1.00000000000000	1.00000000000000	0.000000000000e+00
6.283185	0.666667	1.23057558004363	1.23057557524864	4.794990563254e-09
6.283185	1.333333	2.02863251639945	2.02862755629102	4.960108430027e-06
6.283185	2.000000	3.76219569108363	3.76190476190476	2.909291788696e-04

The surface plot demonstrates that the absolute error remains uniformly small over the computational domain, increasing only slightly as t approaches its upper bound while remaining on the order of 10^{-4}. The smooth and uniform error distribution confirms the excellent agreement between the approximate and exact solutions, demonstrating the accuracy, stability, and robustness of the proposed method (see Figure 1).

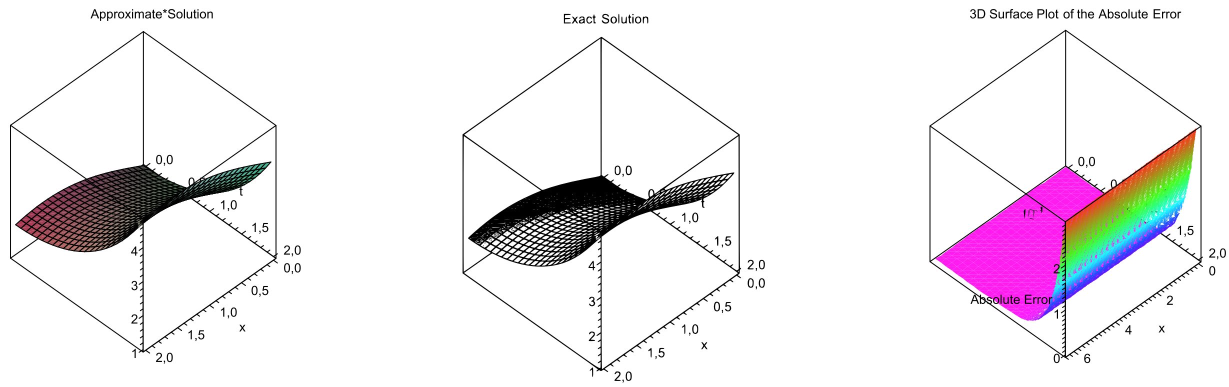

Figure 1. The far-left image shows the approximate solution of Example 5.1 for $\gamma = 2$, while the middle image corresponds to the exact solution $\gamma = 2$. The far-right image presents the absolute error.

From a physical standpoint, the fractional order γ controls the strength of the memory (hereditary) effect encoded in the Atangana–Baleanu–Caputo kernel. The classical case $\gamma = 2$ corresponds to the memoryless, non-dissipative Klein–Gordon wave equation, in which the field evolves solely from its present state. As γ decreases away from 2, the non-singular Mittag-Leffler kernel assigns increasing weight to the field's past history, which manifests as an effective damping-like correction to the wave response: the solution no longer depends only on instantaneous curvature and self-interaction, but also relaxes gradually according to its own history, analogous to wave propagation in a viscoelastic or lossy medium. This is visible in Figure 2, where decreasing γ visibly increases the amplitude of the response relative to the classical case. Physically, this suggests that the ABC fractional Klein–Gordon model is well-suited to describing wave phenomena in media exhibiting relaxation, hysteresis, or long-range temporal correlations, where the classical integer-order model ($\gamma = 2$) would predict purely oscillatory, non-attenuating propagation (Figure 3).

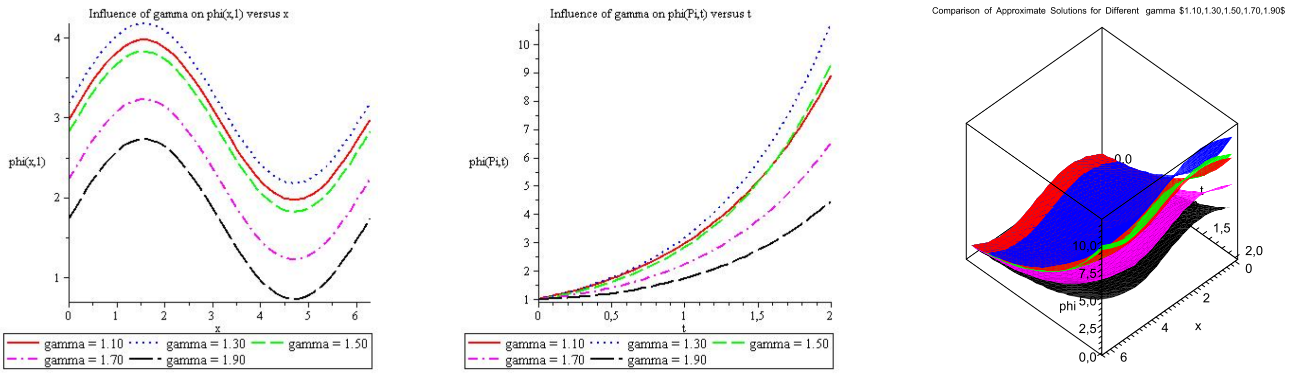

Figure 2. The far-left image shows the influence of gamma on $phi(x, 1)$ versus x of Example 5.1, while the middle image corresponds to the influence of gamma on $phi(Pi, t)$ versus t. The far-right image presents a comparison of approximate solutions for different gamma values: 1.10, 1.30, 1.50, 1.70, 1.90.

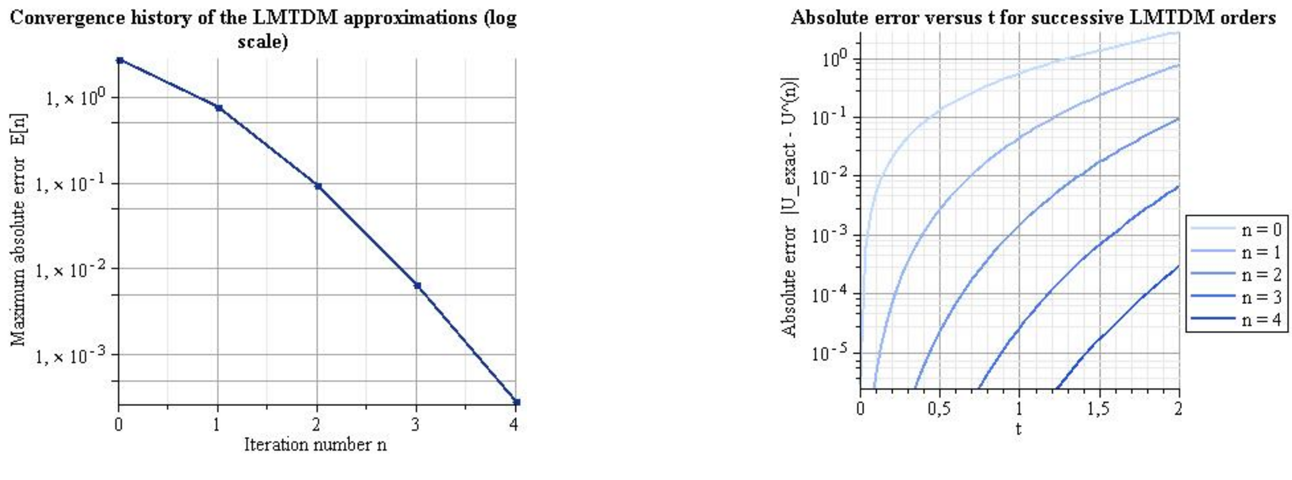

Figure 3. The left image shows the convergence history of the LMTDM approximations (log scale), while the right image corresponds to absolute error versus for LMTDM orders of Example 5.1.

Example 5.2. Let us consider the nonlinear fractional-order Klein–Gordon equation defined as [33]

$$ {}^{ABC}D_{\varsigma}^{\gamma}\varpi(\varkappa,\varsigma) = \frac{\partial^2\varpi(\varkappa,\varsigma)}{\partial\varkappa^2} - \varpi^2(\varkappa,\varsigma) + \varkappa^2\varsigma^2, \quad \varsigma \geq 0, \tag{5.11} $$

subject to the initial conditions

$$ \varpi(\varkappa,0) = 0, \varpi_{\varsigma}(\varkappa,0) = x. \tag{5.12} $$

Numerical experiments. According to the LMTDM, we take the value of γ, the number of iterations n, and the initial condition $\varpi[i]$, by taking the interval $[0,T]$ as follows:

$$ \gamma := 2, \quad n := 5, \quad \varpi[0] := 0, \varpi_{\varsigma}[0] := x, $$
$$ f := \varpi_{\varkappa\varkappa}(\varkappa,\varsigma) + \varpi^2(\varkappa,\varsigma) + \varkappa^2\varsigma^2. $$

We obtain the numerical solution in Table 2.

Table 2. Comparison between the exact solution, approximate solution, and absolute error of Example 5.2.

$\varkappa$	ς	Exact solution	Approximate solution	Absolute error
0.000000	0.000000	0.00000000000000	0.00000000000000	0.000000000000e+00
0.000000	0.666667	0.00000000000000	0.00000000000000	0.000000000000e+00
0.000000	1.333333	0.00000000000000	0.00000000000000	0.000000000000e+00
0.000000	2.000000	0.00000000000000	0.00000000000000	0.000000000000e+00
2.094395	0.000000	0.00000000000000	0.00000000000000	0.000000000000e+00
2.094395	0.666667	1.39626340159546	1.39626340159546	0.000000000000e+00
2.094395	1.333333	2.79252680319093	2.79252680319093	0.000000000000e+00
2.094395	2.000000	4.18879020478639	4.18879020478639	0.000000000000e+00
4.188790	0.000000	0.00000000000000	0.00000000000000	0.000000000000e+00
4.188790	0.666667	2.79252680319093	2.79252680319093	0.000000000000e+00
4.188790	1.333333	5.58505360638185	5.58505360638185	0.000000000000e+00
4.188790	2.000000	8.37758040957278	8.37758040957278	0.000000000000e+00
6.283185	0.000000	0.00000000000000	0.00000000000000	0.000000000000e+00
6.283185	0.666667	4.18879020478639	4.18879020478639	0.000000000000e+00
6.283185	1.333333	8.37758040957278	8.37758040957278	0.000000000000e+00
6.283185	2.000000	12.56637061435917	12.56637061435917	0.000000000000e+00

Discussion of numerical results of Example 5.2

The numerical results presented in Table 2 show that the approximate solution obtained by the proposed method coincides exactly with the analytical solution at all selected spatial and temporal grid points. Consequently, the computed absolute error is zero throughout the computational domain within the reported numerical precision. This exact agreement confirms the correctness of the recursive formulation and the accuracy of its computational implementation for the present benchmark problem.

It is worth noting that, for this particular example, the initial approximation already satisfies the governing equation, causing all subsequent correction terms generated by the decomposition procedure to vanish identically. As a result, the proposed method reproduces the analytical solution exactly, leading to zero absolute error (see Figure 4). Therefore, this example should be regarded primarily as a verification test of the proposed algorithm rather than a comprehensive assessment of its numerical performance. A more complete evaluation of the method should be supported by additional nonlinear benchmark problems, convergence studies, and comparisons with existing numerical methods.

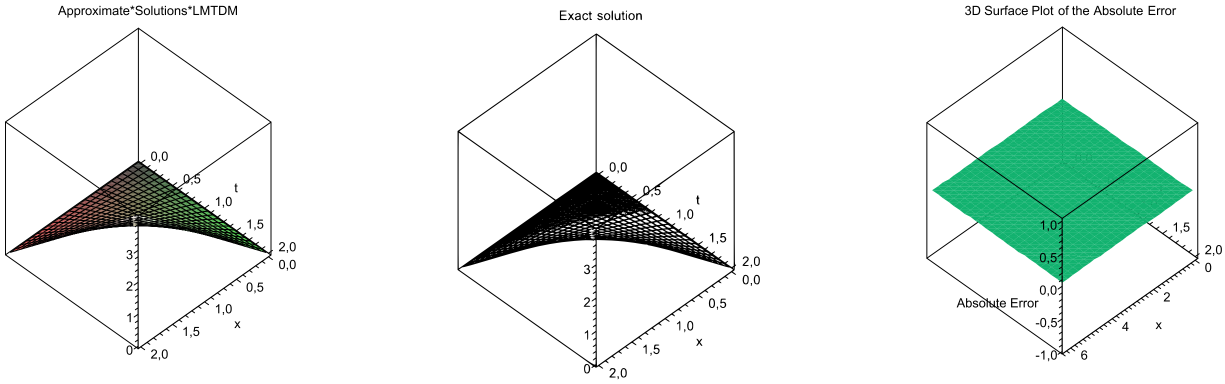

Figure 4. The far-left image shows the approximate solution of Example 5.2 for $\gamma = 2$, while the middle image corresponds to the exact solution. The far-right image presents the absolute error.

As in Example 5.1 (Figure 2), decreasing γ in Figure 5 produces the same memory-induced damping-like effect on the wave response, confirming that this behavior is not specific to the linear case but persists for the nonlinear Klein–Gordon model considered here.

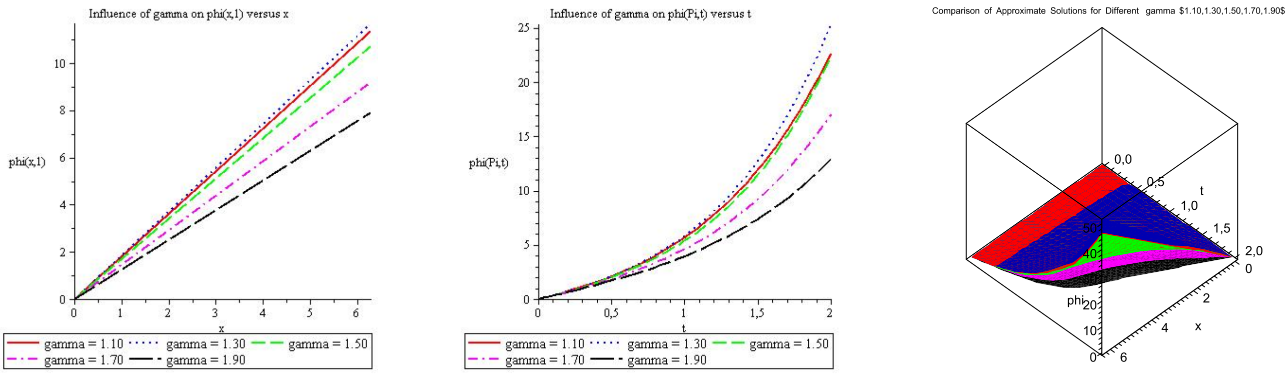

Figure 5. The far-left image shows the influence of gamma on $phi(x, 1)$ versus x of Example 5.2, while the upper-right image corresponds to Influence of gamma on $phi(Pi, t)$ versus t. The far-right image presents a comparison of approximate solutions for different gamma values: 1.10, 1.30, 1.50, 1.70, 1.90.

Example 5.3. Consider the nonlinear fractional-order KGE of the form

$$ {}^{ABC}D_{\varsigma}^{\gamma}\varpi(\varkappa,\varsigma) = \frac{\partial^2\varpi(\varkappa,\varsigma)}{\partial\varkappa^2} - \sin(\varpi(\varkappa,\varsigma)), \qquad 1 < \gamma \leq 2, \quad \varsigma > 0, \tag{5.13} $$

subject to the two initial conditions

$$ \varpi(\varkappa,0) = 0, \frac{\partial\varpi}{\partial\varsigma}(\varkappa,0) = 4sech(\varkappa). \tag{5.14} $$

The exact solution of Eq (5.13) for $\gamma = 2$ is $\varpi(\varkappa, \varsigma) = 4.arctan(\varsigma.sech(\varkappa))$. The 3-term LMTDM solution is

$$
\begin{aligned}
\varpi(\varkappa, \varsigma) = {} & 4\ \mathrm{sech}(\varkappa)\, \varsigma - \frac{2}{3}\ \mathrm{sech}(\varkappa)\, \varsigma^3 \\
& + \left(\frac{8}{15} \varkappa^3\, \mathrm{sech}^3(\varkappa) + \frac{1}{30}\ \mathrm{sech}(\varkappa) \right) \varsigma^5 \\
& + \left(-\frac{44}{315} \varkappa^3\, \mathrm{sech}^3(\varkappa) - \frac{64}{315} \varkappa^5\, \mathrm{sech}^5(\varkappa) + \frac{8}{105}\ \mathrm{sech}^3(\varkappa) \right) \varsigma^7 \\
& + \left(\frac{1}{81} \varkappa^3\, \mathrm{sech}^3(\varkappa) - \frac{32}{567} \varkappa^3\, \mathrm{sech}^5(\varkappa) + \frac{152}{945} \varkappa^5\, \mathrm{sech}^5(\varkappa) \right) \varsigma^9 \\
& + \left(-\frac{256}{7425} \varkappa^5\, \mathrm{sech}^5(\varkappa) - \frac{2}{4455} \varkappa^3\, \mathrm{sech}^3(\varkappa) - \frac{128}{1925} \varkappa^7\, \mathrm{sech}^7(\varkappa) \right) \varsigma^{11} + O(\varsigma^{13}).
\end{aligned}
\tag{5.15}
$$

Discussion of numerical results of Example 5.3

Figure 6 demonstrates that the fractional order γ has a pronounced effect on the solution behavior. As γ increases, the temporal growth of ϖ decreases noticeably, while the spatial oscillations become progressively damped, resulting in a smoother and more stable solution.

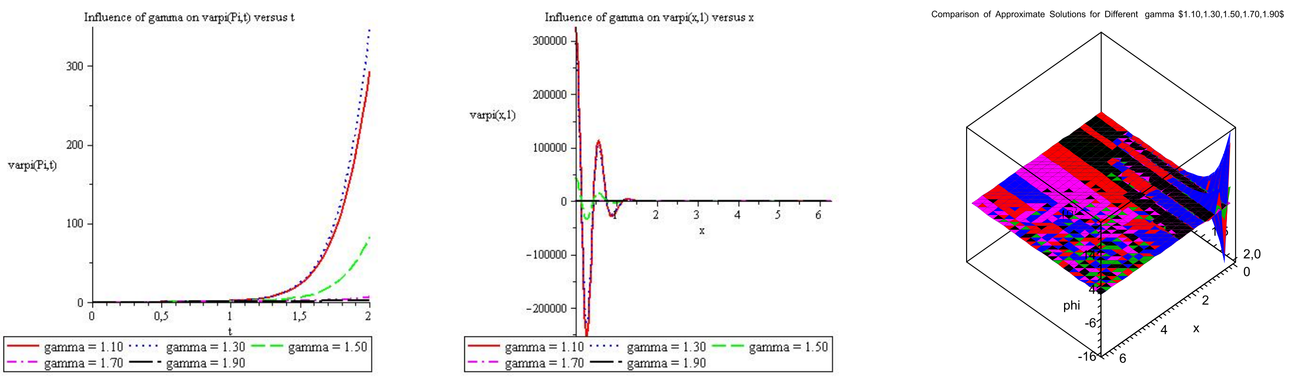

Figure 6. The far-left image shows the influence of gamma on $\varpi(\varkappa, 1)$ versus $\varkappa$ of Example 5.3, while the middle image corresponds to influence of gamma on $\varpi(Pi, \varsigma)$ versus ς. The far-right image presents a comparison of approximate solutions for different gamma values: 1.10, 1.30, 1.50, 1.70, 1.90.

6. Comparison with existing methods

Example 6.1. We finally close our analysis by studying the fractional Klein–Gordon equation

$$
{}^{ABC}D_{\varsigma}^{\gamma} \varpi(\varkappa, \varsigma) = \varpi_{\varkappa\varkappa}(\varkappa, \varsigma) - \frac{3}{4} \varpi(\varkappa, \varsigma) + \frac{3}{2} \varpi^3(\varkappa, \varsigma), \qquad \varsigma > 0, \qquad 1 < \gamma \leq 2, \tag{6.1}
$$

subject to the two initial conditions

$$
\varpi(\varkappa, 0) = -sech(\varkappa), \qquad \varpi_{\varsigma}(\varkappa, 0) = \frac{1}{2} sech(\varkappa) \tanh(\varkappa). \tag{6.2}
$$

The exact solution of Eq (6.1) for $\gamma = 2$ is $\varpi(\varkappa, \varsigma) = -sech(\varkappa + \frac{1}{2}\varsigma)$.

Comment. As shown in Table 3, the proposed LMTDM provides highly accurate numerical results and consistently outperforms the ADM. Within the considered time interval, the LMTDM and VIM solutions are nearly identical. However, for larger time intervals ($\varsigma > 1$), the implementation of the VIM approximation becomes increasingly difficult, since the approximation is constructed over the entire time domain, leading to a rapid growth in the complexity of successive correction terms. In contrast, the proposed LMTDM remains applicable because the time interval is partitioned into a sequence of smaller subintervals. The approximate solution obtained at the end of each subinterval is used as the initial condition for the next one, allowing the method to preserve its accuracy and computational efficiency over a long time simulation.

Table 3. Comparison between the absolute errors of ADM, VIM, and LMTDM for Example 6.1.

$\varkappa$	ς	ADM	VIM	$LMTDM$
1	0.1	5.201×10^{-11}	4.809×10^{-12}	4.809×10^{-12}
1	0.3	4.427×10^{-8}	3.177×10^{-8}	3.177×10^{-8}
1	0.5	2.325×10^{-5}	1.904×10^{-6}	1.904×10^{-6}
2	0.1	3.362×10^{-11}	2.607×10^{-13}	2.605×10^{-13}
2	0.3	6.142×10^{-9}	1.651×10^{-9}	1.641×10^{-9}
2	0.5	7.570×10^{-6}	9.522×10^{-8}	9.365×10^{-8}
3	0.1	2.379×10^{-11}	4.985×10^{-14}	4.985×10^{-14}
3	0.3	3.528×10^{-10}	3.210×10^{-10}	3.208×10^{-10}
3	0.5	4.958×10^{-7}	1.877×10^{-8}	1.872×10^{-8}
4	0.1	1.509×10^{-11}	2.774×10^{-15}	2.774×10^{-15}
4	0.3	2.774×10^{-11}	1.787×10^{-11}	1.786×10^{-11}
4	0.5	1.545×10^{-8}	1.0456×10^{-9}	1.043×10^{-9}
5	0.1	1.496×10^{-11}	1.291×10^{-16}	1.291×10^{-16}
5	0.3	8.682×10^{-12}	8.318×10^{-13}	8.313×10^{-13}
5	0.5	3.039×10^{-9}	4.862×10^{-11}	4.854×10^{-11}
6	0.1	2.471×10^{-12}	2.315×10^{-18}	2.315×10^{-18}
6	0.3	1.430×10^{-13}	1.741×10^{-14}	1.468×10^{-14}
6	0.5	1.554×10^{-9}	8.485×10^{-13}	8.442×10^{-13}

7. Conclusions

A computationally efficient solution for the fractional Klein–Gordon equation (KGE) has been successfully derived using the LMTDM, which represents one of the most effective semi-analytical strategies for addressing nonlinear fractional differential equations. The resulting three-dimensional (3D) graphical visualizations, generated via the Maple software environment, provide a detailed examination of the dynamic evolution of solutions across different orders of the fractional derivative.

In addition, the convergence behavior and singularity properties of the LMTDM solutions for a general fractional partial differential equation (FPDE) have been rigorously investigated using Banach's fixed-point theorem. This analysis highlights the mathematical validity, analytical robustness, and computational reliability of the proposed methodology, confirming both its stability and high accuracy in producing approximate solutions for complex fractional-order systems.

Future research will focus on extending the proposed method to practical engineering applications using real measured data, particularly in viscoelastic media and other fractional dynamic systems, in order to further evaluate its applicability and performance under realistic conditions.

Furthermore, a systematic comparison of the proposed LMTDM with other modern approaches, such as physics-informed neural networks (PINNs), Lie-group structure-preserving schemes, and Riemann–Hilbert-based methods, in terms of computational cost, long-term stability, and applicability to multidimensional problems, will also be considered in future work.

Author contributions

Each author made an equal and substantial contribution to each part of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest in this paper.

References

1. A. S. Abedl-Rady, S. Z. Rida, A. A. M. Arafa, H. R. Abedl-Rahim, Variational iteration Sumudu transform method for solving fractional nonlinear gas dynamics equation, *Int. J. Res. Stu. Sci. Eng. Techn.*, **1** (2014), 82–90.
2. K. A. Abro, A. Siyal, A. Atangana, Q. M. Al-Mdallal, Analytical solution for the dynamics and optimization of fractional Klein–Gordon equation: An application to quantum particle, *Opt. Quant. Electron.*, **55** (2023), 704. https://doi.org/10.1007/s11082-023-04919-1
3. G. Adomian, Solution of physical problems by decomposition, *Comput. Math. Appl.*, **27** (1994), 145–154. https://doi.org/10.1016/0898-1221(94)90132-5
4. M. Alhazmi, S. Saber, Glucose-insulin regulatory system: Chaos control and stability analysis via Atangana–Baleanu fractal-fractional derivatives, *Alex. Eng. J.*, **122** (2025), 77–90. https://doi.org/10.1016/j.aej.2025.02.066
5. M. Al-Refai, M. Abu-Dalu, A. Al-Rawashde, Telescoping decomposition method for solving first order nonlinear differential equations, In: *Proceedings of the International MultiConference of Engineers and Computer Scientists 2008 Vol II*, Hong Kong, 2008.

6. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and nonsingular kernel: Theory and application to heat transfer model, *Thermal Sci.*, **20** (2016), 763–769. https://doi.org/10.2298/TSCI160111018A
7. A. Bouhassoun, Multistage telescoping decomposition method for solving fractional differential equations, *Int. J. Appl. Math.*, **43** (2013), 1–7.
8. M. Boukedroun, S. Ayadi, F. Chita, M. Erden Ege, O. Ege, R. Ramaswamy, Solutions of nonlinear fractional-order differential equation systems using a numerical technique, *Axioms*, **14** (2025), 233. https://doi.org/10.3390/axioms14040233
9. J. W. Brown, R. V. Churchill, *Complex variables and applications*, Boston: McGraw-Hill, 2014.
10. F. Chita, K. Belghaba, Solving systems of nonlinear fractional differential equations using multistage telescoping decomposition method, *Asian J. Math. Comput. Res.*, **13** (2016), 203–214.
11. F. Chita, S. Ayadi, M. Erden Ege, O. Ege, R. Ramaswamy, A novel approach for solving fractional differential equations via a multistage telescoping decomposition method, *Fractal Fract.*, **9** (2025), 65. https://doi.org/10.3390/fractalfract9020065
12. T. M. Elzaki, The new integral transform Elzaki transform, *Glob. J. Pure Appl. Math.*, **7** (2011), 57–64.
13. T. M. Elzaki, S. M. Elzaki, On the Elzaki transform and ordinary differential equation with variable coefficients, *Adv. Theor. Appl. Math.*, **6** (2011), 41–46.
14. T. M. Elzaki, On the connections between Laplace and Elzaki transforms, *Adv. Theor. Appl. Math.*, **6** (2011), 1–11.
15. T. M. Elzaki, E. M. A. Hilal, Homotopy perturbation and Elzaki transform for solving nonlinear partial differential equations, *Math. Theor. Model.*, **2** (2012), 33–42.
16. T. M. Elzaki, J. Biazar, Homotopy perturbation method and Elzaki transform for solving system of nonlinear partial differential equations, *World Appl. Sci. J.*, **24** (2013), 944–948. https://doi.org/10.5829/idosi.wasj.2013.24.07.1041
17. N. Gul, S. Noor, A. M. Saeed, M. S. Aldhabani, R. Ullah, Analytical solution of the systems of nonlinear fractional partial differential equations using conformable Laplace transform iterative method, *AIMS Mathematics*, **10** (2025), 1945–1966. https://doi.org/10.3934/math.2025091
18. J. H. He, A coupling method of homotoping technique and perturbation technique for nonlinear problems, *Int. J. Nonlinear Mech.*, **35** (2000), 37–43. https://doi.org/10.1016/S0020-7462(98)00085-7
19. E. M. A. Hilal, T. M. Elzaki, Solution of nonlinear partial differential equations by new Laplace variational iteration method, *J. Funct. Spaces*, **2014** (2014), 790714. https://doi.org/10.1155/2014/790714
20. M. J. Huntul, Kamran, F. A. Shah, Computational modeling of wave propagation phenomena via a fractional Klein-Gordon equation with modified Atangana-Baleanu-Caputo derivative, *Ain Shams Eng. J.*, **17** (2026), 104039. https://doi.org/10.1016/j.asej.2026.104039
21. D. Kumar, J. Singh, S. Rathore, Sumudu decomposition method for nonlinear equations, *Int. Math. Forum*, **7** (2012), 515–521.

22. M. Nibron, A. K. Mishra, D. Kabala, Telescoping decomposition method for solving second order nonlinear differential equations, *Int. J. Eng. Res. Technol.*, **3** (2014), 590–595. https://doi.org/10.17577/IJERTV3IS20067

23. M. Partohaghighi, M. Mortezaee, A. Akgül, Numerical estimation of the fractional Klein-Gordon equation with discrete Chebyshev polynomials, *Alex. Eng. J.*, **90** (2024), 44–53. https://doi.org/10.1016/j.aej.2024.01.032

24. I. Podlubny, *Fractional differential equations*, San Diego: Academic Press, 1999.

25. W. X. Qui, K. L. Geng, B. W. Zhu, W. Liu, J. T. Li, C. Q. Dai, Data-driven forward-inverse problems of the 2-coupled mixed derivative nonlinear Schrödinger equation using deep learning, *Nonlinear Dyn.*, **112** (2024), 10215–10228. https://doi.org/10.1007/s11071-024-09605-9

26. W. X. Qiu, Z. Z. Si, D. S. Mou, C. Q. Dai, J. T. Li, W. Liu, Data-driven vector degenerate and nondegenerate solitons of coupled nonlocal nonlinear Schrödinger equation via improved PINN algorithm, *Nonlinear Dyn.*, **113** (2025), 4063–4076. https://doi.org/10.1007/s11071-024-09648-y

27. K. Raghavendar, K. Pavani, K. Aruna, N. I. Okposo, M. Inc, Application of natural transform decomposition method for solution of fractional Richards equation, *Contemp. Math.*, **5** (2024), 5881–5900. https://doi.org/10.37256/cm.5420245314

28. S. Rathore, D. Kumar, J. Singh, S. Gupta, Homotopy analysis Sumudu transform method for nonlinear equations, *Int. J. Ind. Math.*, **4** (2012), 301–314.

29. L. Riabi, K. Belghaba, M. H. Cherif, D. Ziane, Homotopy perturbation method combined with ZZ transform to solve some nonlinear fractional differential equations, *Int. J. Anal. Appl.*, **17** (2019), 406–419. https://doi.org/10.28924/2291-8639-17-2019-406

30. R. Saadeh, A. Qazza, A. Burqan, A new integral transform: ARA transform and its properties and applications, *Symmetry*, **12** (2020), 925. https://doi.org/10.3390/sym12060925

31. S. Saber, E. Solouma, R. A. Alharb, A. Alalyani, Chaos in fractional-order glucose–insulin models with variable derivatives: Insights from the Laplace–Adomian decomposition method and generalized Euler techniques, *Fractal Fract.*, **9** (2025), 149. https://doi.org/10.3390/fractalfract9030149

32. N. H. Sweilam, N. Bahaa El Din, M. K. Ammar, E. M. Abo El-Dahab, Modified Atangana-Baleanu-Caputo derivative for nonlinear thermal wave diffusion in rigid conductors: Numerical approach, *Malays. J. Math. Sci.*, **19** (2025), 1179–1196. https://doi.org/10.47836/mjms.19.3.20

33. M. Tamsir, V. K. Srivastava, Analytical study of time-fractional order Klein–Gordon equation, *Alex. Eng. J.*, **55** (2016), 561–567. https://doi.org/10.1016/j.aej.2016.01.025

34. H. Y. Wei, E. G. Fan, H. D. Guo, Riemann–Hilbert approach and nonlinear dynamics of the coupled higher-order nonlinear Schrödinger equation in the birefringent or two-mode fiber, *Nonlinear Dyn.*, **104** (2021), 649–660. https://doi.org/10.1007/s11071-021-06286-6

35. F. Yin, Z. Xu, Y. Fu, Novel high-order explicit energy-preserving schemes for NLS-type equations based on the Lie-group method, *Math. Comput. Simulat.*, **225** (2024), 570–585. https://doi.org/10.1016/j.matcom.2024.05.029

36. H. Zhang, R. Zhang, Q. Liu, A novel multi-modal neurosymbolic reasoning intelligent algorithm for BLMP equation, *Chinese Phys. Lett.*, **42** (2025), 100002. https://doi.org/10.1088/0256-307X/42/10/100002

37. R. Zhao, W. X. Qiu, Y. Zhao, J. Yang, Y. Zhu, J. Li, Deep learning prediction of fractional dissipative optical solitons via the improved quasi-residual physical-informed neural networks, *Ain Shams Eng. J.*, **17** (2026), 103901. https://doi.org/10.1016/j.asej.2025.103901

38. Z. L. Zhao, R. F. Zhang, Artificial intelligence-enhanced mathematical derivation method: exact solutions of the Benjamin–Bona–Mahony equation, *Eng. Appl. Artif. Intel.*, **173** (2026), 114464. https://doi.org/10.1016/j.engappai.2026.114464

39. T. H. Zhou, R. M. Zhao, Z. X. Yu, X. Yan, J. T. Li, C. Q. Dai, Predicting PT-symmetric solitons in competing cubic–quintic saturable media via enhanced physics-informed neural networks with quadratic residual and reinforcement learning-based sampling, *Opt. Laser Technol.*, **198** (2026), 114887. https://doi.org/10.1016/j.optlastec.2026.114887

40. D. Ziane, The combined of homotopy analysis method with new transform for nonlinear partial differential equations, *Malaya J. Math.*, **6** (2018), 34–40. https://doi.org/10.26637/MJM0601/0005

41. D. Ziane, K. Belghaba, M. H. Cherif, Fractional homotopy perturbation transform method for solving the time-fractional KdV, K(2,2) and Burgers equations, *Int. J. Open Problems Compt. Math.*, **8** (2015), 63–75. https://doi.org/10.12816/0017358

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