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Research article

Lagrange inversion of Taylor polynomials using Kronecker product notation

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Abstract: This paper was concerned with the multivariate Lagrange-Good inversion of power series when written using Kronecker products and inverted with respect to a subset of the variables. This problem arose in the estimation of macroeconomic nonlinear dynamic stochastic general equilibrium (DSGE) models. It was shown that the problem of finding the coefficients of the inverse series became much simpler when using Kronecker products. An explicit recursive formula for the solution was derived from first principles, and, regardless of the number of variables, the formula had the same simplicity as in the univariate case. In this Kronecker product formulation, all the coefficients for a given order were contained in one matrix, and this matrix could be calculated with simple operations on other matrices of coefficients. The results of this paper provided a useful tool to estimate nonlinear DSGE models solved with a higher order approximation.

Keywords: DSGE; reversion of series; higher order inverse derivatives; perturbation methods; multinomial coefficient
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1. Introduction

Lagrange [1] derived a formula to calculate the higher order derivatives of the inverse function in the univariate case. This formula can be used to obtain the coefficients of the Taylor approximation of the (local) inverse function (e.g., [1], [2, §189-190], [3, §107]). The formula requires recursive differentiation of an auxiliary function, and several authors have provided alternative formulas which give the coefficients without such differentiation and in a more direct convenient manner (e.g., [4–6]).

In a groundbreaking paper, Good in [7] generalized the Lagrange formula to any number of variables. Several authors have provided alternative proofs of this important multivariate Lagrange-Good inversion formula (e.g., [8–10]), which requires differentiation of a multivariate auxiliary function containing a determinant. Noting the difficulty in dealing with the determinant, authors in [11, 12] use graph theory to provide a tree based formula, which avoids the determinant in the auxiliary function. Authors in [13] used a multivariate version of Faà Di Bruno's formula, and introducing symmetric tensors of degree r and the symmetric product of tensors, derived a linear recurrence procedure to obtain the r-order derivatives of the inverse mapping. In contrast, this paper, by using Kronecker products, derives a formula which has the same simplicity as in the univariate case, and can be derived using just simple summations and the mixed product property of the Kronecker product (e.g., [14]).

We focus on multivariate power series expressed via Kronecker products, which yields a compact and convenient form. This particular formulation is utilized, for instance, within the popular software Dynare for addressing optimal control problems ([15]) in the context of dynamic stochastic general equilibrium (DSGE) models. Furthermore, we would like to invert the series only with respect to a subset of variables, a case which has not seemed to have been analyzed in the literature. We show that the coefficients of the inverse power series can be obtained by solving a linear system of equations, and we find an explicit solution of this system by solving it recursively.

To set the context, let Y be a $n\times 1$ vector, and let Y be a function of two vectors ε and X of dimensions $n_o \times 1$ and $n_x \times 1$, respectively. A multivariate formal power series can be conveniently written using Kronecker products as follows:

$$Y = \sum_{a=0}^{\infty} \sum_{\alpha_1=0}^{a} G^{\alpha_1, a-\alpha_1}(\varepsilon^{[\alpha_1]} \otimes X^{[a-\alpha_1]}), \tag{1.1}$$

where

$$\varepsilon^{[\alpha_1]} = \varepsilon \otimes \varepsilon^{[\alpha_1-1]} = \bigotimes_{i=1}^{\alpha_1} \varepsilon$$

and $G^{\alpha_1,a-\alpha_1}$ is a matrix with n rows and number of columns equal to $n_o^{\alpha_1} n_x^{a-\alpha_1}$.

Writing the vector Y as $Y = ((Y^o)', (Y^n)')'$, such that Y^o and Y^n are two vectors of dimensions $n_o \times 1$ and $n_n \times 1$, respectively, (1.1) can be written as:

$$\begin{aligned} \tilde{Y}_o &= Y_o - G_o^{0,0} = g_o(\varepsilon, X) = \sum_{a=1}^{\infty} \sum_{\alpha_1=0}^{a} G_o^{\alpha_1, a-\alpha_1}(\varepsilon^{[\alpha_1]} \otimes X^{[a-\alpha_1]}), \\ Y_n &= g_n(\varepsilon, X) = \sum_{a=0}^{\infty} \sum_{\alpha_1=0}^{a} G_n^{\alpha_1, a-\alpha_1}(\varepsilon^{[\alpha_1]} \otimes X^{[a-\alpha_1]}). \end{aligned} \tag{1.2}$$

Note that Y_o and ε have the same dimensions. We assume that the Jacobian ($|G_o^{1,0}|$) is not zero at the origin. The inverse of (1.2) can be written by swapping the positions of $\tilde{Y}_o$ and ε as follows:

$$\begin{aligned} \varepsilon &= f_o(\tilde{Y}_o, X) = \sum_{a=0}^{\infty} \sum_{\alpha_1=0}^{a} \tilde{G}_o^{\alpha_1, a-\alpha_1}(\tilde{Y}_o^{[\alpha_1]} \otimes X^{[a-\alpha_1]}), \\ Y_n &= f_n(\tilde{Y}_o, X) = \sum_{a=0}^{\infty} \sum_{\alpha_1=0}^{a} \tilde{G}_n^{\alpha_1, a-\alpha_1}(\tilde{Y}_o^{[\alpha_1]} \otimes X^{[a-\alpha_1]}). \end{aligned} \tag{1.3}$$

We say that (1.3) is the inverse of (1.2) if $f_o(g_o(\varepsilon, X), X) = \varepsilon$ and $f_n(g_o(\varepsilon, X), X) = g_n(\varepsilon, X)$. This paper finds that the matrices $\tilde{G}_j^{\alpha_1,a-\alpha_1}$, for $j = o, n$, can be obtained from the matrices $G_j^{\alpha_1,a-\alpha_1}$ by solving a linear system of equations, and an explicit solution is provided.

Univariate and multivariate Lagrange inversions have been widely used in applied research. For example, in economics, a finite order version of Eq (1.2) is used to approximate, using perturbation methods (e.g., [16]), the optimal control law conditional on the realization of some stochastic shocks ε. Therefore, this equation can be used to simulate the system by generating random values for ε. The vector X represents predetermined variables, such as lagged values of Y. The inverse in Eq (1.3) can be used to estimate the parameters of the system given data on Y_o (e.g., [17]).

The mathematical problem of recovering shocks or structural parameters from indirect observations has a rich parallel literature in the theory of inverse problems for partial differential equations. Across various physical sciences, researchers frequently need to estimate hidden variables or system parameters that cannot be measured directly, relying instead on simpler, observable data. For example, in geophysics, analytical equations are used to calculate hidden subsurface fault structures and slip parameters from easily observed ground surface measurements (e.g., [18]). More generally, these methodologies belong to the broader framework of inverse theory, where the goal is to reconstruct the internal state of a system from its external outputs ([19]).

When (1.2) represents a convergent Taylor expansion, the finite order Taylor expansion of the inverse function will be given by the truncated version of (1.3). We do not discuss conditions for the convergence and uniqueness of (1.3), which have been given in [7]. Our objective is to find a formula to calculate the matrices in (1.3) from those in (1.2) that can be implemented with computer code, and does not require differentiation.

Section 2 gives the solution for general order, whereas Sections 3 and 4 give the solutions for order 2 and 3, respectively. The solution for a higher order q can be computed with a loop that, loosely speaking, sums over ordered partitions of q into c parts, for $c \leq q$. Section 5 obtains a simplified expression for the matrices $\tilde{G}_j^{\alpha_1,0}$, which completely defines the inverse series in the case that there is no vector X. In this simplified expression, the summation is over non-ordered partitions of q, which represent a smaller set than the ordered partitions. This formula has the same simplicity as in the univariate case. Section 6 presents the discussion and conclusions.

2. General order

For a $c \times 2$ matrix $\beta = (\beta_{ij})$ of nonnegative integers, define C_c^β as a Kronecker product of c matrices of the type $G_o^{\beta_{i1},\beta_{i2}}$, such that

$$C_c^\beta = \bigotimes_{i=1}^{c} G_o^{\beta_{i1},\beta_{i2}} = (G_o^{\beta_{11},\beta_{12}} \otimes G_o^{\beta_{21},\beta_{22}} \otimes ... \otimes G_o^{\beta_{c1},\beta_{c2}}).$$

This definition implies, for example, the following:

$$\begin{aligned} C_1^\beta &= G_o^{\beta_{11},\beta_{12}}, \\ C_2^\beta &= G_o^{\beta_{11},\beta_{12}} \otimes G_o^{\beta_{21},\beta_{22}}. \end{aligned}$$

Let us define $(s_1(\beta), s_2(\beta))$ as the column sums of β, such that $s_1(\beta) = \sum_{i=1}^{c} \beta_{i1}$ and $s_2(\beta) = \sum_{i=1}^{c} \beta_{i2}$, where c is the number of rows of β.

Let us define $B_c^{\alpha_1,\alpha_2}$ as the set of matrices β with c rows and 2 columns that verify the following conditions:

- The elements of β are nonnegative integers.
- None of the rows in β are equal to $(0,0)$.
- The column sums of β are such that $s_1(\beta)=\alpha_1$ and $s_2(\beta)=\alpha_2$.

We define $B_0^{0,0}$ as containing only an empty matrix β of dimension 0×2 with the convention that $s_1(\beta)=s_2(\beta)=0$ and $C_0^{\beta}=1$. However, $B_0^{\alpha_1,\alpha_2}$ does not contain any matrix if either α_1 or α_2 is strictly greater than 0.

For each matrix $\beta\in B_c^{\alpha_1,\alpha_2}$, let P_β be a permutation matrix that has the following property:

$$P_\beta\left(\bigotimes_{i=1}^{c}(\varepsilon^{[\beta_{i1}]}\otimes X^{[\beta_{i2}]})\right)=P_\beta\left((\varepsilon^{[\beta_{11}]}\otimes X^{[\beta_{12}]})\otimes ...\otimes(\varepsilon^{[\beta_{c1}]}\otimes X^{[\beta_{c2}]})\right)=\varepsilon^{[\alpha_1]}\otimes X^{[\alpha_2]}.$$

Although for a particular β there might be several permutation matrices P_β with that property, the results are invariant to the choice among candidates with the mentioned property. Define $J_{c,l-c}^{\alpha_1,\alpha_2-(l-c)}$ as the following matrix:

$$J_{c,l-c}^{\alpha_1,\alpha_2-(l-c)}=\sum_{\beta\in B_c^{\alpha_1,\alpha_2-(l-c)}}(C_c^{\beta}P'_\beta)\otimes(I_{n_x})^{[l-c]},\tag{2.1}$$

where I_{n_x} is the identity matrix of order n_x (where n_x is the dimension of X, as defined earlier). The summation in (2.1) is over all the matrices β belonging to $B_c^{\alpha_1,\alpha_2-(l-c)}$, so that if $B_c^{\alpha_1,\alpha_2-(l-c)}$ contains no matrices (e.g., when $c=0$ and $\alpha_1>0$), then the summation is 0. The transpose of P_β is denoted as P'_β and it is noted that $P'_\beta P_\beta$ is equal to the identity matrix. Using concepts from combinatorics (e.g., [20, Sec. 3.6]), if matrix β belongs to $B_c^{\alpha_1,\alpha_2-(l-c)}$, then its first (or second) column is an ordered weak partition into c parts of α_1 (or $\alpha_2-(l-c)$).

Proposition 2.1. *The values of the matrices $\tilde{G}^{\alpha_1,\alpha_2}$ in Eq (1.3) can be obtained from the matrices G^{α_1,α_2} in (1.2) by solving the following linear system of equations:*

$$F_j^{0,0}=\tilde{G}_j^{0,0}\qquad \textit{for}\quad j=o\ \ \textit{or}\ \ j=n,\tag{2.2}$$
$$F_j^{\alpha_1,\alpha_2}=\sum_{l=1}^{\alpha_1+\alpha_2}\sum_{c=\max(0,l-\alpha_2)}^{l}\tilde{G}_j^{c,l-c}J_{c,l-c}^{\alpha_1,\alpha_2-(l-c)},\ \ \textit{for}\quad j=o,n,\quad(\alpha_1,\alpha_2)\neq(0,0),$$

where

$$\begin{aligned}F_o^{1,0}&=I_{n_o},\\ F_o^{\alpha_1,\alpha_2}&=0G_o^{\alpha_1,\alpha_2}\ \ \textit{for}\ \ (\alpha_1,\alpha_2)\neq(1,0),\\ F_n^{\alpha_1,\alpha_2}&=G_n^{\alpha_1,\alpha_2}\ \ \textit{for all}\ \ (\alpha_1,\alpha_2).\end{aligned}\tag{2.3}$$

The solution to (2.2)-(2.3) is as follows:

$$\tilde{G}_j^{0,0}=F_j^{0,0},\tag{2.4}$$
$$\tilde{G}_j^{1,0}=\left(F_j^{1,0}\right)\left(G_o^{1,0}\right)^{-1},\tag{2.5}$$

$$\tilde{G}_j^{0,1} = F_j^{0,1} - \left(\tilde{G}_j^{1,0} G_o^{0,1}\right), \tag{2.6}$$

$$\tilde{G}_j^{\alpha_1,0} = \left(F_j^{\alpha_1,0} - \sum_{l=1}^{\alpha_1-1} \tilde{G}_j^{l,0} \sum_{\beta \in B_l^{\alpha_1,0}} C_l^{\beta} \right) \left(\left(G_o^{1,0}\right)^{-1} \right)^{[\alpha_1]} \quad for \quad \alpha_1 \geq 2, \tag{2.7}$$

$$\tilde{G}_j^{\alpha_1,1} = \left(F_j^{\alpha_1,1} - \sum_{l=1}^{\alpha_1} \tilde{G}_j^{l-1,1} J_{l-1,1}^{\alpha_1,0} - \sum_{l=1}^{\alpha_1+1} \tilde{G}_j^{l,0} J_{l,0}^{\alpha_1,1} \right) \left((G_o^{1,0})^{-1} \right)^{[\alpha_1]} \otimes I_{n_x} \quad for \quad \alpha_1 \geq 1, \tag{2.8}$$

$$\begin{aligned}\tilde{G}_j^{\alpha_1,\alpha_2} = &\left(F_j^{\alpha_1,\alpha_2} - \sum_{l=\max(1,\alpha_2)}^{\alpha_1+\alpha_2-1} \tilde{G}_j^{l-\alpha_2,\alpha_2} J_{l-\alpha_2,\alpha_2}^{\alpha_1,0} \right.\\ &\left. - \sum_{b=0}^{\alpha_2-1} \sum_{l=\max(1,b)}^{\alpha_1+\alpha_2} \tilde{G}_j^{l-b,b} J_{l-b,b}^{\alpha_1,\alpha_2-b} \right) \left(\left(\left(G_o^{1,0}\right)^{-1} \right)^{[\alpha_1]} \otimes I_{n_x}^{[\alpha_2]} \right) \quad for \quad \alpha_2 \geq 1.\end{aligned} \tag{2.9}$$

Proof. We want to find the coefficients of (1.3) such that by inserting the expression for $\tilde{Y}_o$ in (1.2) into the equation for ε in (1.3), we obtain the identity function

$$\varepsilon = \varepsilon = \sum_{a=0}^{\infty} \sum_{\alpha_1=0}^{a} F_o^{\alpha_1,a-\alpha_1} (\varepsilon^{[\alpha_1]} \otimes X^{[a-\alpha_1]}),$$

such that $F_o^{1,0} = I_{n_o}$, and $F_o^{\alpha_1,\alpha_2} = 0 G_o^{\alpha_1,\alpha_2}$ for $(\alpha_1, \alpha_2) \neq (1,0)$. Similarly, inserting the expression for $\tilde{Y}_o$ in (1.2) into the equation for Y_n in (1.3), we obtain Y_n in terms of ε and X, which denoted by

$$\sum_{a=0}^{q} \sum_{\alpha_1=0}^{a} F_n^{\alpha_1,a-\alpha_1} (\varepsilon^{[\alpha_1]} \otimes X^{[a-\alpha_1]}),$$

implies that $F_n^{\alpha_1,\alpha_2} = G_n^{\alpha_1,\alpha_2}$. This gives us (2.3). To prove (2.2), we now find the expressions for the coefficients $F_j^{\alpha_1,\alpha_2}$.

Using (1.2), the term $(\tilde{Y}_o)^{[\alpha_1]}$ in (1.3) can be written as:

$$\begin{aligned}(\tilde{Y}_o)^{[\alpha_1]} &= \left(\sum_{a=1}^{\infty} \sum_{b=0}^{a} G_o^{b,a-b} (\varepsilon^{[b]} \otimes X^{[a-b]})) \right)^{[\alpha_1]} = \bigotimes_{i=1}^{\alpha_1} \left(\sum_{a_i=1}^{\infty} \sum_{b_i=0}^{a_i} G_o^{b_i,a_i-b_i} (\varepsilon^{[b_i]} \otimes X^{[a_i-b_i]})) \right)\\ &= \sum_{\substack{a_i \geq 1\\ 0 \leq b_i \leq a_i}} \left(\bigotimes_{i=1}^{\alpha_1} \left(G_o^{b_i,a_i-b_i} (\varepsilon^{[b_i]} \otimes X^{[a_i-b_i]}) \right) \right)\\ &= \sum_{\substack{a_i \geq 1\\ 0 \leq b_i \leq a_i}} \left(\bigotimes_{i=1}^{\alpha_1} \left(G_o^{b_i,a_i-b_i} \right) \right) \left(\bigotimes_{i=1}^{\alpha_1} (\varepsilon^{[b_i]} \otimes X^{[a_i-b_i]}) \right),\end{aligned} \tag{2.10}$$

where in the last step we used the mixed product property of the Kronecker product. Let

$$\widetilde{\alpha}_1 = \sum_{i=1}^{\alpha_1} b_i \quad \text{and} \quad \widetilde{\alpha}_2 = \sum_{i=1}^{\alpha_1} (a_i - b_i).$$

Using the previously defined set $B_{\alpha_1}^{\tilde{\alpha}_1,\tilde{\alpha}_2}$, permutation matrix P_β, and matrix C_c^β, Eq (2.10) can be written as:

$$\begin{aligned}
&\sum_{\tilde{\alpha}_1=0}^{\infty}\sum_{\tilde{\alpha}_2=0}^{\infty}\sum_{\beta\in B_{\alpha_1}^{\tilde{\alpha}_1,\tilde{\alpha}_2}}\left(\bigotimes_{i=1}^{\alpha_1}\left(G_o^{\beta_{i1},\beta_{i2}}\right)\right)\left(\bigotimes_{i=1}^{\alpha_1}(\varepsilon^{[\beta_{i1}]}\otimes X^{[\beta_{i2}]})\right)\\
&=\sum_{\tilde{\alpha}_1=0}^{\infty}\sum_{\tilde{\alpha}_2=0}^{\infty}\sum_{\beta\in B_{\alpha_1}^{\tilde{\alpha}_1,\tilde{\alpha}_2}} C_{\alpha_1}^{\beta}\left(P'_\beta P_\beta\right)\left(\bigotimes_{i=1}^{\alpha_1}(\varepsilon^{[\beta_{i1}]}\otimes X^{[\beta_{i2}]})\right)\\
&=\sum_{\tilde{\alpha}_1=0}^{\infty}\sum_{\tilde{\alpha}_2=0}^{\infty}\sum_{\beta\in B_{\alpha_1}^{\tilde{\alpha}_1,\tilde{\alpha}_2}}\left(C_{\alpha_1}^{\beta}P'_\beta\right)\left(\varepsilon^{[\tilde{\alpha}_1]}\otimes X^{[\tilde{\alpha}_2]}\right).
\end{aligned}\tag{2.11}$$

Therefore, replacing $(\tilde{Y}_o)^{[\alpha_1]}$ with (2.11), the equation for ε in (1.3) becomes as follows:

$$\begin{aligned}
&\sum_{a=0}^{\infty}\sum_{\alpha_1=0}^{a}\tilde{G}_o^{\alpha_1,a-\alpha_1}\left(\sum_{\tilde{\alpha}_1=0}^{\infty}\sum_{\tilde{\alpha}_2=0}^{\infty}\sum_{\beta\in B_{\alpha_1}^{\tilde{\alpha}_1,\tilde{\alpha}_2}}\left(C_{\alpha_1}^{\beta}P'_\beta\right)\left(\varepsilon^{[\tilde{\alpha}_1]}\otimes X^{[\tilde{\alpha}_2]}\right)\right)\otimes X^{[a-\alpha_1]}\\
&=\sum_{a=0}^{\infty}\sum_{\alpha_1=0}^{a}\tilde{G}_o^{\alpha_1,a-\alpha_1}\left(\sum_{\tilde{\alpha}_1=0}^{\infty}\sum_{\tilde{\alpha}_2=0}^{\infty}\sum_{\beta\in B_{\alpha_1}^{\tilde{\alpha}_1,\tilde{\alpha}_2}}\left(\left(C_{\alpha_1}^{\beta}P'_\beta\right)\otimes(I_{n_x})^{[a-\alpha_1]}\right)\left(\varepsilon^{[\tilde{\alpha}_1]}\otimes X^{[\tilde{\alpha}_2+a-\alpha_1]}\right)\right)\\
&=\sum_{a=0}^{\infty}\sum_{\alpha_1=0}^{a}\tilde{G}_o^{\alpha_1,a-\alpha_1}\left(\sum_{\tilde{\alpha}_1=0}^{\infty}\sum_{\tilde{\alpha}_2=0}^{\infty}J_{\alpha_1,a-\alpha_1}^{\tilde{\alpha}_1,\tilde{\alpha}_2}\left(\varepsilon^{[\tilde{\alpha}_1]}\otimes X^{[\tilde{\alpha}_2+a-\alpha_1]}\right)\right).
\end{aligned}\tag{2.12}$$

Making a change of variables, $\alpha_2=\tilde{\alpha}_2+a-\alpha_1$, $c=\alpha_1$, $l=a$, and $\alpha_1=\tilde{\alpha}_1$, Eq (2.12) can be written as:

$$\sum_{l=0}^{\infty}\sum_{c=0}^{l}\tilde{G}_o^{c,l-c}\left(\sum_{\alpha_1=0}^{\infty}\sum_{\alpha_2=l-c}^{\infty}J_{c,l-c}^{\alpha_1,\alpha_2-(l-c)}\left(\varepsilon^{[\alpha_1]}\otimes X^{[\alpha_2]}\right)\right).\tag{2.13}$$

Therefore, for a fixed value of (α_1,α_2), the coefficient of $\varepsilon^{[\alpha_1]}\otimes X^{[\alpha_2]}$ is:

$$F_o^{\alpha_1,\alpha_2}=\sum_{l=0}^{\infty}\sum_{c=\max(0,l-\alpha_2)}^{l}\tilde{G}_o^{c,l-c}J_{c,l-c}^{\alpha_1,\alpha_2-(l-c)},\tag{2.14}$$

where we have used that in (2.13) $\alpha_2\geq l-c$, and therefore in (2.14) we have $c\geq l-\alpha_2$. Note that it is not possible to partition α_1 into more than α_1 nonzero parts. In addition, because $(0,0)$ is not a row of $\beta\in B_c^{\alpha_1,\alpha_2}$, we have that $J_{c,l-c}^{\alpha_1,\alpha_2-(l-c)}=0$ when $c>\alpha_1+\alpha_2-(l-c)$. This implies that the upper limit of the first summation in (2.14) can be truncated to $l\leq\alpha_1+\alpha_2$. When $(\alpha_1,\alpha_2)\neq(0,0)$, the summation in (2.14) effectively starts from $l=1$ because in that case $J_{0,0}^{\alpha_1,\alpha_2}=0$. This proves (2.2) for $j=o$. The case $j=n$ is obtained analogously, by replacing $(\tilde{Y}_o)^{[\alpha_1]}$ with (2.11) in the equation for Y_n in (1.3). The solution to (2.2) and (2.3) can be obtained by solving for $\tilde{G}_j^{\alpha_1,\alpha_2}$ in the equation for $F_j^{\alpha_1,\alpha_2}$. It should be noted that

$$J_{\alpha_1,\alpha_2}^{\alpha_1,0}=\sum_{\beta\in B_{\alpha_1}^{\alpha_1,0}}(C_{\alpha_1}^{\beta}P'_\beta)\otimes(I_{n_x})^{[\alpha_2]}=(G_o^{1,0})^{[\alpha_1]}\otimes(I_{n_x})^{[\alpha_2]}.$$

In addition,

$$J_{l,0}^{\alpha_1,0} = \sum_{\beta\in B_l^{\alpha_1,0}} (C_{\alpha_1}^{\beta} P'_{\beta}) \otimes (I_{n_x})^{[0]} = \sum_{\beta\in B_l^{\alpha_1,0}} C_{\alpha_1}^{\beta}$$

for $l \leq \alpha_1$ and 0 otherwise.

Furthermore, by making a change of variables $b = l - c$, when $(\alpha_1, \alpha_2) \neq (0, 0)$, Eq (2.14) can be written as:

$$F_j^{\alpha_1,\alpha_2} = \sum_{l=1}^{\alpha_1+\alpha_2} \sum_{b=0}^{\min(l,\alpha_2)} \tilde{G}_j^{l-b,b} J_{l-b,b}^{\alpha_1,\alpha_2-b} \tag{2.15}$$

$$= \sum_{b=0}^{\alpha_2} \sum_{l=\max(1,b)}^{\alpha_1+\alpha_2} \tilde{G}_j^{l-b,b} J_{l-b,b}^{\alpha_1,\alpha_2-b} \tag{2.16}$$

and solving for $\tilde{G}_j^{\alpha_1,\alpha_2}$ in Eq (2.16) gives (2.5)–(2.9). □

3. Terms of second order

Truncating the series in Eq (1.2) up to terms of order 2, we obtain:

$$\begin{aligned} Y_o &= G_o^{0,0} + G_o^{1,0}\varepsilon + G_o^{0,1}X + G_o^{2,0}\varepsilon^{[2]} + G_o^{0,2}X^{[2]} + G_o^{1,1}(\varepsilon \otimes X), \\ Y_n &= G_n^{0,0} + G_n^{1,0}\varepsilon + G_n^{0,1}X + G_n^{2,0}\varepsilon^{[2]} + G_n^{0,2}X^{[2]} + G_n^{1,1}(\varepsilon \otimes X), \end{aligned} \tag{3.1}$$

and the truncated version of Eq (1.3) becomes:

$$\begin{aligned} \varepsilon &= \tilde{G}_o^{0,0} + \tilde{G}_o^{1,0}\tilde{Y}_o + \tilde{G}_o^{0,1}X + \tilde{G}_o^{2,0}\tilde{Y}_o^{[2]} + \tilde{G}_o^{0,2}X^{[2]} + \tilde{G}_o^{1,1}(\tilde{Y}_o \otimes X), \\ Y_n &= \tilde{G}_n^{0,0} + \tilde{G}_n^{1,0}\tilde{Y}^{o} + \tilde{G}_n^{0,1}X + \tilde{G}_n^{2,0}\tilde{Y}_o^{[2]} + \tilde{G}_n^{0,2}X^{[2]} + \tilde{G}_n^{1,1}(\tilde{Y}_o \otimes X). \end{aligned} \tag{3.2}$$

We assume that $\tilde{G}_j^{\alpha_1,\alpha_2}$ and $G_j^{\alpha_1,\alpha_2}$ are formed by the derivatives of a function with the equality of mixed partial derivatives property, which implies, for example, that

$$\tilde{G}_j^{2,0}(\varepsilon_1 \otimes \varepsilon_2) = \tilde{G}_j^{2,0}(\varepsilon_2 \otimes \varepsilon_1),$$

for any two vectors ε_1 and ε_2. For this reason, we have that

$$\tilde{G}_j^{2,0}(G_o^{0,1}X \otimes G_o^{1,0}\varepsilon) = \tilde{G}_j^{2,0}(G_o^{1,0}\varepsilon \otimes G_o^{0,1}X) = \tilde{G}_j^{2,0}(G_o^{1,0} \otimes G_o^{0,1})(\varepsilon \otimes X).$$

Note that each row of $\tilde{G}_j^{2,0}$ is a vectorized symmetric matrix (a Hessian multiplied by a scalar). Using this, the solution in (2.4)–(2.9) implies that the matrices in (3.2) can be obtained from those in (3.1) as follows:

$$\begin{cases} \tilde{G}_o^{0,0} = 0G_o^{0,0}, \\ \tilde{G}_n^{0,0} = G_n^{0,0}, \end{cases} \tag{3.3}$$

$$\begin{cases} \tilde{G}_o^{1,0} = \left(G_o^{1,0}\right)^{-1}, \\ \tilde{G}_n^{1,0} = \left(G_n^{1,0}\right)\tilde{G}_o^{1,0}, \end{cases} \tag{3.4}$$

$$\begin{cases}\tilde{G}_o^{0,1} = -\left(\tilde{G}_o^{1,0} G_o^{0,1}\right),\\ \tilde{G}_n^{0,1} = G_n^{0,1} - \left(\tilde{G}_n^{1,0} G_o^{0,1}\right),\end{cases} \tag{3.5}$$

$$\begin{cases}\tilde{G}_o^{2,0} = -\tilde{G}_o^{1,0} G_o^{2,0}\left(G_o^{1,0} \otimes G_o^{1,0}\right)^{-1},\\ \tilde{G}_n^{2,0} = \left(G_n^{2,0} - \tilde{G}_n^{1,0} G_o^{2,0}\right)\left(G_o^{1,0} \otimes G_o^{1,0}\right)^{-1},\end{cases} \tag{3.6}$$

$$\begin{cases}\tilde{G}_o^{1,1} = -\left(\tilde{G}_o^{1,0} G_o^{1,1} + 2\tilde{G}_o^{2,0}(G_o^{1,0} \otimes G_o^{0,1})\right)\left((G_o^{1,0})^{-1} \otimes I_{n_x}\right),\\ \tilde{G}_n^{1,1} = \left(G_n^{1,1} - \left(\tilde{G}_n^{1,0} G_o^{1,1} + 2\tilde{G}_n^{2,0}(G_o^{1,0} \otimes G_o^{0,1})\right)\right)\left((G_o^{1,0})^{-1} \otimes I_{n_x}\right),\end{cases} \tag{3.7}$$

$$\begin{cases}\tilde{G}_o^{0,2} = -\left(\tilde{G}_o^{1,0} G_o^{0,2} + \tilde{G}_o^{2,0}(G_o^{0,1} \otimes G_o^{0,1}) + \tilde{G}_o^{1,1}(G_o^{0,1} \otimes I_{n_x})\right),\\ \tilde{G}_n^{0,2} = G_n^{0,2} - \left(\tilde{G}_n^{1,0} G_o^{0,2} + \tilde{G}_n^{2,0}(G_o^{0,1} \otimes G_o^{0,1}) + \tilde{G}_n^{1,1}(G_o^{0,1} \otimes I_{n_x})\right).\end{cases} \tag{3.8}$$

4. Third order

When the truncation is up to terms of order 3, Eq (1.2) becomes:

$$\begin{aligned} Y_o &= G_o^{0,0} + G_o^{1,0}\varepsilon + G_o^{0,1}X + G_o^{2,0}\varepsilon^{[2]} + G_o^{0,2}X^{[2]} + G_o^{1,1}(\varepsilon \otimes X)\\ &\quad + G_o^{3,0}\varepsilon^{[3]} + G_o^{2,1}(\varepsilon^{[2]} \otimes X) + G_o^{1,2}(\varepsilon \otimes X^{[2]}) + G_o^{0,3}X^{[3]},\\ Y_n &= G_n^{0,0} + G_n^{1,0}\varepsilon + G_n^{0,1}X + G_n^{2,0}\varepsilon^{[2]} + G_n^{0,2}X^{[2]} + G_n^{1,1}(\varepsilon \otimes X)\\ &\quad + G_n^{3,0}\varepsilon^{[3]} + G_n^{2,1}(\varepsilon^{[2]} \otimes X) + G_n^{1,2}(\varepsilon \otimes X^{[2]}) + G_n^{0,3}X^{[3]}, \end{aligned} \tag{4.1}$$

and the truncated version of (1.3) is:

$$\begin{aligned} \varepsilon &= \tilde{G}_o^{0,0} + \tilde{G}_o^{1,0}\tilde{Y}_o + \tilde{G}_o^{0,1}X + \tilde{G}_o^{2,0}\tilde{Y}_o^{[2]} + \tilde{G}_o^{0,2}X^{[2]} + \tilde{G}_o^{1,1}(\tilde{Y}_o \otimes X)\\ &\quad + \tilde{G}_o^{3,0}\tilde{Y}_o^{[3]} + \tilde{G}_o^{2,1}(\tilde{Y}_o^{[2]} \otimes X) + \tilde{G}_o^{1,2}(\tilde{Y}_o \otimes X^{[2]}) + \tilde{G}_o^{0,3}X^{[3]},\\ Y_n &= \tilde{G}_n^{0,0} + \tilde{G}_n^{1,0}\tilde{Y}_o + \tilde{G}_n^{0,1}X + \tilde{G}_n^{2,0}\tilde{Y}_o^{[2]} + \tilde{G}_n^{0,2}X^{[2]} + \tilde{G}_n^{1,1}(\tilde{Y}_o \otimes X)\\ &\quad + \tilde{G}_n^{3,0}\tilde{Y}_o^{[3]} + \tilde{G}_n^{2,1}(\tilde{Y}_o^{[2]} \otimes X) + \tilde{G}_n^{1,2}(\tilde{Y}_o \otimes X^{[2]}) + \tilde{G}_n^{0,3}X^{[3]}. \end{aligned} \tag{4.2}$$

As in the previous section, we use the equality of mixed partial derivatives property, which for order 3 further implies that, for example,

$$\tilde{G}_j^{3,0}(\varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_3) = \tilde{G}_j^{3,0}(\varepsilon_l \otimes \varepsilon_k \otimes \varepsilon_m),$$

for any three vectors ε_1, ε_2, and ε_3, and any permutation (l, k, m). Also, we have that

$$\tilde{G}_j^{2,1}(\varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_3) = \tilde{G}_j^{2,1}(\varepsilon_2 \otimes \varepsilon_1 \otimes \varepsilon_3).$$

For this reason, we have, for example, that

$$\tilde{G}_j^{3,0}(G_o^{0,1}X \otimes G_o^{1,0}\varepsilon \otimes G_o^{1,0}\varepsilon) = \tilde{G}_j^{3,0}(G_o^{1,0}\varepsilon \otimes G_o^{1,0}\varepsilon \otimes G_o^{0,1}X) = \tilde{G}_j^{3,0}(G_o^{1,0} \otimes G_o^{1,0} \otimes G_o^{0,1})(\epsilon^{[2]} \otimes X)$$

and

$$\tilde{G}_j^{2,1}(G_o^{0,1}X \otimes G_o^{1,0}\varepsilon \otimes X) = \tilde{G}_j^{2,1}(G_o^{1,0}\varepsilon \otimes G_o^{0,1}X \otimes X) = \tilde{G}_j^{2,1}(G_o^{1,0} \otimes G_o^{0,1} \otimes I_{n_x})(\varepsilon \otimes X^{[2]}).$$

The terms up to order 2 were given in the previous section, and the terms of order 3 implied by Eqs (2.4)–(2.9) are as follows:

$$\begin{aligned}
\tilde{G}_j^{3,0} &= \left(F_j^{3,0} - \left(\tilde{G}_j^{1,0}G_o^{3,0} + 2\tilde{G}_j^{2,0}(G_o^{1,0} \otimes G_o^{2,0})\right)\right)\left(\left(G_o^{1,0}\right)^{-1}\right)^{[3]}, \\
\tilde{G}_j^{2,1} &= \left(F_j^{2,1} - A_j^{2,1}\right)\left(((G_o^{1,0})^{-1})^{[2]} \otimes I_{n_x}\right), \\
\tilde{G}_j^{1,2} &= \left(F_j^{1,2} - A_j^{1,2}\right)\left(((G_o^{1,0})^{-1}) \otimes I_{n_x}^{[2]}\right), \\
\tilde{G}_j^{0,3} &= \left(F_j^{0,3} - A_j^{0,3}\right),
\end{aligned} \tag{4.3}$$

where the matrices $A_j^{2,1}$, $A_j^{1,2}$, and $A_j^{0,3}$ are defined as follows:

$$\begin{aligned}
A_j^{2,1} &= \tilde{G}_j^{1,0}G_o^{2,1} + 2\tilde{G}_j^{2,0}(G_o^{1,0} \otimes G_o^{1,1} + G_o^{2,0} \otimes G_o^{0,1}) + \tilde{G}_j^{1,1}(G_o^{2,0} \otimes I_{n_x}) + 3\tilde{G}_j^{3,0}((G_o^{1,0})^{[2]} \otimes G_o^{0,1}), \\
A_j^{1,2} &= \tilde{G}_j^{1,0}G_o^{1,2} + 2\tilde{G}_j^{2,0}(G_o^{1,1} \otimes G_o^{0,1} + G_o^{1,0} \otimes G_o^{0,2}) + \tilde{G}_j^{1,1}(G_o^{1,1} \otimes I_{n_x}) + 3\tilde{G}_j^{3,0}(G_o^{1,0} \otimes (G_o^{0,1})^{[2]}) \\
&\quad + 2\tilde{G}_j^{2,1}(G_o^{1,0} \otimes G_o^{0,1} \otimes I_{n_x}), \\
A_j^{0,3} &= \tilde{G}_j^{1,0}G_o^{0,3} + 2\tilde{G}_j^{2,0}(G_o^{0,1} \otimes G_o^{0,2}) + \tilde{G}_j^{1,1}(G_o^{0,2} \otimes I_{n_x}) + \tilde{G}_j^{3,0}((G_o^{0,1})^{[3]}), \\
&\quad + \tilde{G}_j^{2,1}(G_o^{0,1} \otimes G_o^{0,1} \otimes I_{n_x}) + \tilde{G}_j^{1,2}(G_o^{0,1} \otimes I_{n_x} \otimes I_{n_x}),
\end{aligned} \tag{4.4}$$

and where $F_j^{\alpha_1,\alpha_2}$ are given in (2.3), for $j = o$ or $j = n$.

5. A simplified expression for $\tilde{G}_j^{\alpha_1,0}$

The expression for $\tilde{G}_j^{\alpha_1,0}$ in (2.7) can be simplified by using partitions of α_1. The coefficients $\tilde{G}_j^{\alpha_1,0}$ completely define the inverse series when there is no vector X, which is the standard case in the Lagrange inversion literature. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_c)$ be a partition of α_1 into c parts, such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_c \geq 1$ and $\sum_{i=1}^{c} \lambda_i = \alpha_1$. Let $P_c(\alpha_1)$ be the set of all such partitions. For a partition $\lambda \in P_c(\alpha_1)$, define C_c^λ as the following Kronecker product:

$$C_c^\lambda = \bigotimes_{i=1}^{c} G_o^{\lambda_i,0} = (G_o^{\lambda_1,0} \otimes G_o^{\lambda_2,0} \otimes ... \otimes G_o^{\lambda_c,0}). \tag{5.1}$$

Let κ_λ be the number of distinct parts in λ, and let $m_\lambda = (m_1, ..., m_{\kappa_\lambda})$ be their multiplicities. We assume the equality of mixed partial derivatives property, which, for example, for order 4 implies that

$$\tilde{G}_j^{4,0}(\varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_3, \varepsilon_4) = \tilde{G}_j^{4,0}(\varepsilon_l \otimes \varepsilon_k \otimes \varepsilon_m \otimes \varepsilon_n),$$

for any four vectors ε_1–ε_4, and any permutation (l, k, m, n). Let us define the following multinomial coefficient:

$$s_{\lambda,c} = \binom{c}{m_1, m_2, \dots, m_{\kappa_\lambda}} = \frac{c!}{m_1! m_2! \cdots m_{\kappa_\lambda}!}.$$

With these definitions, a simplified expression for $\tilde{G}_j^{\alpha_1,0}$ can be obtained as the following corollary of Proposition 2.1.

Corollary 5.1. *Assuming that $\tilde{G}_j^{\alpha_1,0}$ is formed by derivatives of a function with equal mixed partial derivatives, the coefficients $\tilde{G}_j^{\alpha_1,0}$ in Eq (2.7) can be equivalently written as:*

$$\tilde{G}_j^{\alpha_1,0} = \left(F_j^{\alpha_1,0} - \sum_{l=1}^{\alpha_1-1} \tilde{G}_j^{l,0} \sum_{\lambda \in P_l(\alpha_1)} s_{\lambda,l} C_l^{\lambda} \right) \left(\left(G_o^{1,0} \right)^{-1} \right)^{[\alpha_1]} \quad for \quad \alpha_1 \geq 2. \tag{5.2}$$

Proof. For a given partition $\lambda \in P_l(\alpha_1)$, the number of different ways to reorder the terms in the product C_l^{λ} defined in (5.1) is given by the multinomial coefficient $s_{\lambda,l}$ (e.g., [20]). Together with the mixed partial derivatives property, and the mixed product property of the Kronecker product, we have:

$$\tilde{G}_j^{l,0} \left(\sum_{\beta \in B_l^{\alpha_1,0}} C_l^{\beta} \right) \left(\bigotimes \varepsilon_i \right) = \tilde{G}_j^{l,0} \left(\sum_{\lambda \in P_l(\alpha_1)} s_{\lambda,l} C_l^{\lambda} \right) \left(\bigotimes \varepsilon_i \right).$$

Therefore, in Eq (2.7) we can write the summation over the partitions, as in (5.2). □

6. Discussion and conclusions

For computational efficiency, we recommend using expression (5.2) instead of (2.7) to calculate $\tilde{G}_j^{\alpha_1,0}$ because the number of unordered partitions is much smaller than the number of ordered partitions. Therefore, the number of terms in the summation is smaller when expression (5.2) is used. For example, for $\alpha_1 = 8$, expression (5.2) requires adding up 21 terms (the number of unordered partitions of α_1 minus 1), whereas (2.7) requires 127 terms (the number of ordered partitions minus 1), so there is a 6 fold reduction. As α_1 gets larger, the number of unordered partitions can be approximated with the Hardy-Ramanujan ([21]) asymptotic formula as $\exp\left(\frac{\pi}{k}\sqrt{\frac{2\alpha_1}{3}}\right)$, so the number of terms in the sum grows sub-exponentially. In contrast, the number of terms in (2.7) grows exponentially according to (2^{α_1-1}), so the relative efficiency increases very rapidly with α_1. Regarding algebraic operations, Eq (5.2) requires multiplying matrices of dimensions $(n_o \times n_o^{\alpha_1})$ and $(n_o^{\alpha_1} \times n_o^{\alpha_1})$, as well as inverting a matrix of dimension $(n_o \times n_o)$.

However, the computational cost to calculate $\tilde{G}_j^{\alpha_1,\alpha_2}$ for $\alpha_2 > 0$ is larger because the summation is over *ordered* partitions of α_1 and α_2. However, in the macroeconomic literature on DSGE models, the order of approximation ($q = \alpha_1 + \alpha_2$) is usually 2 or 3 and never larger than 7 (e.g., [22–24]), implying that all the formulas given in this paper can be calculated in a practical and fast manner. The partitions can be obtained with efficient algorithms such as those in [25].

Using the formulas in this paper for the case of $q = 2$, authors in [17] obtained the stochastic shocks as a function of the observed data, providing a new method of estimating the parameters in a DSGE model. The results in this paper provide formulas to use this method of estimation with higher degrees of approximation. The formulas given in this paper are given in an explicit manner, using Kronecker product notation, and can therefore be directly written in computer code.

Author contributions

Roberto Leon-Gonzalez: contributed in the conception and design, the drafting of the paper and revising it critically for intellectual content; Elnura Baiaman kyzy: contributed in the conception and design, the drafting of the paper and revising it critically for intellectual content. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors used Gemini (Google) for language editing and assistance with LaTeX document formatting. All output was thoroughly verified and edited by the authors, who take full responsibility for the manuscript's integrity and content.

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Conflict of interest

The authors declare no conflicts of interest.

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