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Research article

On the nonexistence of periodic orbits and instability in nonlinear fourth-order multi-delay vector systems

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Abstract: This study establishes a new set of sufficient criteria for guaranteeing the nonexistence of nontrivial periodic solutions and the instability of the trivial solution for a class of nonlinear fourth-order vector differential equations with multiple delays by constructing an appropriate Lyapunov–Krasovskii (LK) functional specifically tailored to the system's dynamics. The proposed approach provides a consistent basis that supports and builds on findings in the literature by incorporating multiple delay terms in a vector structure. Furthermore, two examples illustrate the applicability of the theoretical results. These findings provide a robust basis for the analyses of nonexistence and instability of higher-order systems in engineering and applied mathematics.

Keywords: Lyapunov–Krasovskii functional; nonexistence of periodic solutions; instability of solutions; fourth-order nonlinear systems; vector differential equations
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1. Introduction

The qualitative analysis of higher-order nonlinear differential equations is a highly active research area with broad applications in mechanics, control theory, and structural engineering [1,2]. One of the central tools in this field is Lyapunov's second method, including its extensions to systems with delay [3]. A closely related theme is identifying conditions that exclude nonconstant periodic solutions, thereby ensuring either long-term stability or predictable divergence; foundational results for scalar fourth-order equations were established by Ezeilo and Tejumola [4–6], while Tiryaki [7,8] subsequently extended these investigations to more complex models.

Fourth-order dynamics in coupled elastic structures and feedback systems are often governed by matrix operators. Time delay is a crucial factor, often a primary source of instability. Refined oscillation and stability criteria for fourth-order delay equations have been established for various deviating-argument and delay settings [9–11], and more recently [12–14]. Other important related results have been provided [15–17]. While the vectorial case without delays was addressed by Tunç [18], introducing multiple delays poses additional difficulties, especially in constructing suitable LK functionals. Torres [19] demonstrated nonexistence of periodic solutions for the relativistic pendulum with friction. Several works have also established stability, boundedness, and nonexistence results for fourth-order vector equations with and without delays [20–22], with additional contributions on general framework for fourth and fifth-order systems [23,24]. Instability and multi-delay extensions have been developed [25–27]. More recently, LK-based techniques have produced delay-dependent stability and boundedness criteria for nonlinear vector delay systems and multi-delay models [28,29].

Beyond the integer-order setting, the stability of fractional-order time-delay systems has also been examined through delay-dependent criteria based on covering sets and filters [30] and delay-independent necessary and sufficient conditions [31]. Moreover, Razumikhin-based stability analyses for nonlinear multi-delay models [32–34], descriptor/linear-matrix-inequality (LMI) formulations for delay systems [35], and Popov-type frequency-domain criteria for robust stability of delayed vector systems [36–38] indicate that modern techniques are widely used in the qualitative study of time-delay dynamics. However, these contributions are predominantly stability-oriented.

Such fourth-order multi-delay vector models arise in applications including elastic beam dynamics with delayed feedback, high-order mechanical and control systems, and coupled dynamical networks; in these settings, excluding sustained (nonconstant periodic) oscillations and characterizing the instability of equilibria are physically meaningful and practically important.

Motivated by these developments, this study investigates a generalized class of fourth-order nonlinear vector differential equations with multiple delays. By constructing a single, suitable LK functional, sufficient conditions are obtained for the nonexistence of nonconstant periodic solutions and the instability of the trivial solution.

To illustrate the motivating progression of this research, the gradual generalization of some models investigated in the literatüre is highlighted. In 2008, Tunç [22] obtained a nonexistence result for the following vector differential equation without delays:

$$X^{(4)} + \Omega(\ddot{X})\dddot{X} + \Phi(X,\dot{X},\ddot{X},\dddot{X})\ddot{X} + \Theta(\dot{X}) + F(X) = 0.$$

In 2015, Tunç [27] established an instability result for a fourth-order vector differential equation with multiple constant deviating arguments for the following model:

$$X^{(4)} + A\dddot{X} + H(\dot{X})\ddot{X} + G(X)\dot{X} + F(X) + \sum_{i=1}^{n} E_i(X(t-\tau_i)) = 0.$$

Building upon these foundational works, this study focuses on a more general class of fourth-order vector differential equations with multiple constant delays to derive some nonexistence and instability criteria:

$$X^{(4)} + \Psi(\ddot{X})\dddot{X} + G(X,\dot{X},\ddot{X},\dddot{X})\ddot{X} + H(X)\dot{X} + \Theta(\dot{X}) + \textstyle\sum_{i=1}^{m} F_i(X(t-\tau_i)) = 0. \qquad (1)$$

Here $X \in R^n$, τ_i are certain positive constants, $t-\tau_i > 0$. Ψ, G and H are continuous $n \times n-$ symmetric matrix functions for the components shown in (1) ; $\Theta : R^n \to R^n$, $F_i : R^n \to R^n$ and $\Theta(0) = F_i(0) = 0$ for $i = 1,2,...,m$, and F_i and Θ are continuous for all $X, \dot{X} \in R^n$, respectively.

Remark 1. It is worth noting that Eq (1) serves as a direct generalization of the model in [22]. Specifically, by considering a single delay $(m=1)$ and setting the time-lag to zero $(\tau_1=0)$, the term $F_i(X(t-\tau_i))$ reduces to $F(X)$ in [22]. Furthermore, under the assumption $H(X)\equiv 0$, our model fully encapsulates the structure of the delay-free system investigated by Tunç [22]. Additionally, my model enhances the framework of Tunç [27], which focuses primarily on instability results by introducing operators that depend on higher-order derivatives, thereby providing a more generalized structural setting.

2. Materials and methods

Now, (1) can be rearranged in the equivalent first-order vector form:

$$\dot{X} = Y$$

$$\dot{Y} = Z$$

$$\dot{Z} = W$$

$$\dot{W} = -\Psi(Z)W - G(X,Y,Z,W)Z - H(X)Y - \Theta(Y)$$

$$-\sum_{i=1}^{m} F_i(X) + \sum_{i=1}^{m} \int_{t-\tau_i}^{t} J_{F_i}(X(s))Y(s)ds, \tag{2}$$

where the fundamental theorem of calculus yields the following relation:

$$\sum_{i=1}^{m} F_i(X(t-\tau_i)) = \sum_{i=1}^{m} F_i(X) - \sum_{i=1}^{m} \int_{t-\tau_i}^{t} J_{F_i}(X(s))Y(s)ds.$$

Let $J_{F_i}(X), J_{\Theta}(Y)$ denote the linear operators from the vectors $F_i(X)$, $\Theta(Y)$ to the matrices

$$J_{F_i}(X) = \frac{\partial f_{(i)j}}{\partial x_k}, \ J_{\Theta}(Y) = \frac{\partial \theta_j}{\partial y_k}, \quad i=1,2,...,m, \ j,k=1,2,...,n,$$

respectively, where $(x_1,x_2,...,x_n)$, $(y_1,y_2,...,y_n)$, $(z_1,z_2,...,z_n)$, $(w_1,w_2,...,w_n)$, $(f_{(i)1},f_{(i)2},...,f_{(i)n})$, $(\theta_1,\theta_2,...,\theta_n)$ are components of $X,Y,Z,W,F_i,\Theta.$ Moreover, it is presumed that both $F_i(X)$ and $\Theta(Y)$ represent gradient vector fields. In other words, there exist scalar functions f_i and θ such that $F_i=\nabla f_i$ and $\Theta=\nabla\theta$, and the matrices $J_{F_i}(X)$ and $J_{\Theta}(Y)$ exist and are continuous.

For any X,Y in R^n, $\langle X,Y\rangle$ represents the standard inner product, that is, $\langle X,Y\rangle=\sum_{i=1}^{n}x_iy_i$, $\|X\|^2=\langle X,X\rangle$ and $\lambda_j(M)$, $(j=1,2,...,n)$, denote the eigenvalues of the $n\times n-$ matrix M. Furthermore, recall that the positive definiteness of a given quadratic form X^TMX is necessary and sufficient for the positivity of the real symmetric $n\times n-$matrix $M=(m_{jk})$; here X^T defines the transpose of X.

The next lemmas, which are necessary for the proofs of the main theorems, are presented below.
Lemma 1. Let M be a real symmetric $n\times n$-matrix and for all $j=1,2,...n$

$$m' \geq \lambda_j(M) \geq m > 0,$$

where m' and m are constants. Then

$$m'\langle X,X\rangle \geq \langle MX,X\rangle \geq m\langle X,X\rangle$$

and

$$m'^2\langle X,X\rangle \geq \langle MX,MX\rangle \geq m^2\langle X,X\rangle.$$

For the proof of the lemma, see [1].

Lemma 2. Assume $\Theta(0)=0$ and $\Psi(Z)Z=\nabla\varphi(Z)$, $\varphi:\Re^n\to\Re$, then the following hold:

$$\frac{d}{dt}\int_0^1\langle\Theta(\sigma Y),Y\rangle d\sigma=\langle\Theta(Y),Z\rangle,$$

$$\frac{d}{dt}\int_0^1\langle\sigma\Psi(\sigma Z)Z,Z\rangle d\sigma=\langle\Psi(Z)W,Z\rangle.$$

Proof. It is clear that

$$\frac{d}{dt}\int_0^1\langle\Theta(\sigma Y),Y\rangle d\sigma=\int_0^1\sigma\langle J_\Theta(\sigma Y)Z,Y\rangle d\sigma+\int_0^1\langle\Theta(\sigma Y),Z\rangle d\sigma$$

$$=\int_0^1\sigma\langle J_\Theta(\sigma Y)Y,Z\rangle d\sigma+\int_0^1\langle\Theta(\sigma Y),Z\rangle d\sigma$$

$$=\int_0^1\sigma\frac{\partial}{\partial\sigma}\langle\Theta(\sigma Y),Z\rangle d\sigma+\int_0^1\langle\Theta(\sigma Y),Z\rangle d\sigma$$

$$=\sigma\langle\Theta(\sigma Y),Z\rangle\Big|_0^1$$

$$=\langle\Theta(Y),Z\rangle.$$

Since $\Psi(Z)Z=\nabla\varphi(Z)$, evaluating at σZ gives

$$\nabla\varphi(\sigma Z)=\Psi(\sigma Z)(\sigma Z)=\sigma\Psi(\sigma Z)Z.$$

Hence,

$$\int_0^1\langle\sigma\Psi(\sigma Z)Z,Z\rangle d\sigma=\int_0^1\langle\nabla\varphi(\sigma Z),Z\rangle d\sigma=\int_0^1\frac{d}{d\sigma}\varphi(\sigma Z)d\sigma=[\varphi(\sigma Z)]\Big|_0^1=\varphi(Z)-\varphi(0).$$

Using $\dot{Z}=W$ and the symmetry of Ψ,

$$\frac{d}{dt}\int_0^1\langle\sigma\Psi(\sigma Z)Z,Z\rangle d\sigma=\frac{d}{dt}\varphi(Z)=\langle\nabla\varphi(Z),W\rangle=\langle\Psi(Z)Z,W\rangle=\langle\Psi(Z)W,Z\rangle.$$

Lemma 3 (Krasovskii's instability theorem). Suppose there exists a continuous LK functional $V=V(X(t),Y(t),Z(t),W(t))$ which possesses the following Krasovskii properties:

(K_1) in every neighborhood of $(0,0,0,0)$ there exists a point $(\xi_1,\xi_2,\xi_3,\xi_4)$ such that $V(\xi_1,\xi_2,\xi_3,\xi_4)>0$;

(K_2) the derivative $\frac{d}{dt}V$ along the solutions of system (2) satisfies $\frac{d}{dt}V \geq 0$;

(K_3) the only solution $(X,Y,Z,W)=(X(t),Y(t),Z(t),W(t))$ of (2) for which $\frac{d}{dt}V=0\,(t\geq 0)$ is the trivial solution $(0,0,0,0)$.

Then the trivial solution of system (2) is unstable in the sense of Lyapunov. For the proof of Lemma 3, see [3].

3. Results

Theorem 1 (Nonexistence of periodic solutions). Let Ψ, G, Θ, H and F_i, mentioned in system (2), be continuous and continuously differentiable functions, and there exist constants a_H, a_i, a_i' and a_0, (with $a_0>0$, $a_H>0$), independent of the state variables (X,Y,Z,W) and of the eigenvalue index j, such that the following sufficient conditions hold uniformly for all $i=1,2,...,m,\ j=1,2,...,n$:

(i) there exist scalar functions f_i, θ and $\varphi:\Re^n \to \Re$ such that $F_i=\nabla f_i$, $\Theta=\nabla\theta$, $\Psi(Z)Z=\nabla\varphi(Z)$ and Ψ, G, H are symmetric,

(ii) $F_i(0)=0$, $F_i(X)\neq 0$ if $X\neq 0$,

(iii) $\lambda_j(G)\leq a_0$, for all $(X,Y,Z,W)\in\Re^{4n}$,

(iv) $a_i' \geq \lambda_j(J_{F_i}(X))\geq a_i$, $\left|\lambda_j(H(X))\right|\leq a_H$ for all $X\in\Re^n$, for $\forall i,j$,

(v) $$\tau_* < \min\{\frac{2a_0-a_H\varepsilon^{-1}}{ma^*}, \frac{2(\sum_{i=1}^{m}a_i - a_0^2)-a_H\varepsilon}{ma^*}\},\ a^*=\max_i\{|a_i|,|a_i'|\},\ \tau_*=\max_i\{\tau_i\},\qquad \gamma=\frac{1}{2}a^*,$$

$$a_H^2 < 4a_0(\sum_{i=1}^{m}a_i - a_0^2),\qquad \sum_{i=1}^{m}a_i > a_0^2,\qquad \varepsilon\in(\frac{a_H}{2a_0}, \frac{2(\sum_{i=1}^{m}a_i - a_0^2)}{a_H}),\quad \varepsilon>0, \varepsilon\in\Re,$$

in which $\lambda_j(J_{F_i}(X))$, $\lambda_j(G(X,Y,Z,W))$ and $\lambda_j(H(X))$ indicate the eigenvalues of $J_{F_i}(X)$, $G(X,Y,Z,W)$ and $H(X)$, respectively. Then, equation (1) has no periodic solution other than the trivial solution $X\equiv 0$.

Remark 2. The trivial solution refers to the zero equilibrium $X(t)\equiv 0$ (equivalently, $(X,Y,Z,W)=(0,0,0,0)$ for system (2)). Constant solutions are regarded as trivial (degenerate) periodic orbits; in fact, by assumption (ii) together with the strict monotonicity of $\sum_{i=1}^{m}F_i$ established in the proof of Theorem 1, $X\equiv 0$ is the only constant solution of (1). Accordingly, the terms "nontrivial" and "nonconstant" periodic solution are used synonymously, and Theorem 1 asserts that (1) admits no such solution.

Proof. Assume that $(X,Y,Z,W)=(X(t),Y(t),Z(t),W(t))$ is a nontrivial ω-periodic solution of the given system (2) $(\omega>0)$. To analyze its behavior, the LK functional is introduced as in the following definition:

$$V = V(t) = V(X(t), Y(t), Z(t), W(t))$$

$$= \langle Y, 2a_0 Z\rangle + \langle W, Z\rangle + \sum_{i=1}^{m}\langle F_i(X), Y\rangle + \int_0^1 \langle \Theta(\sigma Y), Y\rangle d\sigma$$

$$+\int_0^1 \langle \sigma\Psi(\sigma Z)Z, Z\rangle d\sigma - \gamma\sum_{i=1}^{m}\int_{-\tau_i}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2 d\theta ds, \tag{3}$$

where $\gamma > 0$, $\gamma \in \Re$ which will be chosen later.

The functional combines two ingredients: Cross-product terms that capture the coupling among the state variables induced by the fourth-order vector structure, and integral terms that account for the past states and compensate for the delayed nonlinearities. Upon differentiation, these absorb the delay contributions and yield a tractable estimate suitable for both the nonexistence and the instability results. In particular, each delay τ_i is matched by its own weighted integral term, with $\gamma = (1/2)\,a^*$ chosen so that the history contributions of all m delays cancel in $\dot{V}$.

The differentiation of the functional (3) throughout the system (2) results in the following:

$$\dot{V} = \frac{d}{dt}V(t)$$

$$= \langle Z, 2a_0 Z\rangle + \langle Y, 2a_0 W\rangle + \langle W, W\rangle + \langle \dot{W}, Z\rangle + \sum_{i=1}^{m}\langle J_{F_i}(X)Y, Y\rangle + \sum_{i=1}^{m}\langle F_i(X), Z\rangle$$

$$+\frac{d}{dt}\int_0^1 \langle \Theta(\sigma Y), Y\rangle d\sigma + \frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Z)Z, Z\rangle d\sigma - \gamma\frac{d}{dt}\sum_{i=1}^{m}\int_{-\tau_i}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2 d\theta ds. \tag{4}$$

where

$$\langle \dot{W}, Z\rangle = -\langle \Psi(Z)W, Z\rangle - \langle G(X,Y,Z,W)Z, Z\rangle - \langle H(X)Y, Z\rangle - \langle \Theta(Y), Z\rangle$$

$$-\sum_{i=1}^{m}\langle F_i(X), Z\rangle + \sum_{i=1}^{m}\langle \int_{t-\tau_i}^{t} J_{F_i}(X(s))Y(s)ds, Z\rangle.$$

From condition (i) and Lemma 2, recall that

$$\frac{d}{dt}\int_0^1 \langle \Theta(\sigma Y), Y\rangle d\sigma = \langle \Theta(Y), Z\rangle, \tag{5}$$

$$\frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma Z)Z, Z\rangle d\sigma = \langle \Psi(Z)W, Z\rangle \tag{6}$$

and by Young's inequality

$$-\langle H(X)Y, Z\rangle \geq -a_H\|Y\|\|Z\| \geq -\frac{a_H}{2}(\varepsilon\|Y\|^2 + \varepsilon^{-1}\|Z\|^2). \tag{7}$$

Substituting equalities (5), (6) into (4) to eliminate some terms involving matrix operators, the following is obtained:

$$\dot{V} = \langle Z, 2a_0 Z\rangle + \langle Y, 2a_0 W\rangle + \langle W, W\rangle - \langle G(X,Y,Z,W)Z, Z\rangle - \langle H(X)Y, Z\rangle$$

$$+\sum_{i=1}^{m}\left\langle J_{F_i}(X)Y,Y\right\rangle+\sum_{i=1}^{m}\langle\int_{t-\tau_i}^{t}J_{F_i}(X(s))Y(s)ds,Z\rangle-\gamma\frac{d}{dt}\sum_{i=1}^{m}\int_{-\tau_i}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2\,d\theta ds. \tag{8}$$

Now, examine the term:

$$\left\langle Z,2a_0IZ\right\rangle-\left\langle G(X,Y,Z,W)Z,Z\right\rangle$$

included in (8). In consideration of the aforementioned assumption (iii) of Theorem 1 and from the Lemma 1, the following holds:

$$\left\langle Z,2a_0Z\right\rangle-\left\langle G(X,Y,Z,W)Z,Z\right\rangle\geq 2a_0\left\langle Z,Z\right\rangle-a_0\left\langle Z,Z\right\rangle=a_0\left\langle Z,Z\right\rangle. \tag{9}$$

By replacing (7) and (9) in (8), it results in

$$\dot{V}\geq a_0\langle Z,Z\rangle+\|W+a_0Y\|^2-a_0^2\langle Y,Y\rangle-\frac{a_H}{2}(\varepsilon\|Y\|^2+\varepsilon^{-1}\|Z\|^2)+\sum_{i=1}^{m}\left\langle J_{F_i}(X)Y,Y\right\rangle$$

$$+\sum_{i=1}^{m}\langle\int_{t-\tau_i}^{t}J_{F_i}(X(s))Y(s)ds,Z\rangle-\gamma\frac{d}{dt}\sum_{i=1}^{m}\int_{-\tau_i}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2\,d\theta ds.$$

Hence, it follows:

$$\dot{V}\geq a_0\langle Z,Z\rangle-a_0^2\langle Y,Y\rangle-\frac{a_H}{2}(\varepsilon\|Y\|^2+\varepsilon^{-1}\|Z\|^2)+\sum_{i=1}^{m}\left\langle J_{F_i}(X)Y,Y\right\rangle$$

$$+\sum_{i=1}^{m}\langle\int_{t-\tau_i}^{t}J_{F_i}(X(s))Y(s)ds,Z\rangle-\gamma\frac{d}{dt}\sum_{i=1}^{m}\int_{-\tau_i}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2\,d\theta ds. \tag{10}$$

Recall that

$$\frac{d}{dt}\int_{-\tau}^{0}\int_{t+s}^{t}\|Y(\theta)\|^2\,d\theta ds=\tau\|Y\|^2-\int_{t-\tau}^{t}\|Y(s)\|^2\,ds$$

and

$$\langle\int_{t-\tau_i}^{t}J_{F_i}(X(s))Y(s)ds,Z\rangle\geq-\|Z\|\int_{t-\tau_i}^{t}\|J_{F_i}(X(s))\|\|Y(s)\|ds.$$

By the Cauchy-Schwarz inequality in $\Re^n$ and condition (iv), the following inequality holds:

$$\sum_{i=1}^{m}\langle\int_{t-\tau_i}^{t}J_{F_i}(X(s))Y(s)ds,Z\rangle\geq-\|Z\|\sum_{i=1}^{m}a^*\int_{t-\tau_i}^{t}\|Y(s)\|ds$$

$$\geq-\frac{1}{2}\sum_{i=1}^{m}a^*\tau_i\|Z\|^2-\frac{1}{2}\sum_{i=1}^{m}a^*\int_{t-\tau_i}^{t}\|Y(s)\|^2\,ds, \tag{11}$$

where $a^*=\max_i\{|a_i|,|a_i'|\}$. Applying (11) and assumption (iv) in (10), it follows

$$\dot{V} \geq a_0\langle Z,Z\rangle - a_0^2\langle Y,Y\rangle + \sum_{i=1}^{m} a_i\langle Y,Y\rangle - \frac{a_H}{2}(\varepsilon\|Y\|^2 + \varepsilon^{-1}\|Z\|^2)$$

$$-\frac{1}{2}\sum_{i=1}^{m} a^*\tau_i\|Z\|^2 - \frac{1}{2}\sum_{i=1}^{m} a^*\int_{t-\tau_i}^{t}\|Y(s)\|^2 ds - \gamma\sum_{i=1}^{m}\tau_i\|Y\|^2 + \gamma\sum_{i=1}^{m}\int_{t-\tau_i}^{t}\|Y(s)\|^2 ds.$$

Therefore, by choosing, $\tau_* = \max_i\{\tau_i\}$, it yields:

$$\dot{V} \geq \left(a_0 - \frac{1}{2}ma^*\tau_* - \frac{a_H\varepsilon^{-1}}{2}\right)\|Z\|^2 + \left(\sum_{i=1}^{m} a_i - \gamma m\tau_* - a_0^2 - \frac{a_H\varepsilon}{2}\right)\|Y\|^2$$

$$+(\gamma - \tfrac{1}{2}a^*)\textstyle\sum_{i=1}^{m}\int_{t-\tau_i}^{t}\|Y(s)\|^2 ds. \tag{12}$$

Taking $\gamma = \frac{1}{2}a^*$, the coefficient $(\gamma - \frac{1}{2}a^*)$ of the history integral terms vanishes; hence these terms drop out of the estimate and (12) reduces to

$$\dot{V} \geq (a_0 - \tfrac{1}{2}ma^*\tau_* - \tfrac{a_H\varepsilon^{-1}}{2})\|Z\|^2 + (\textstyle\sum_{i=1}^{m} a_i - \tfrac{1}{2}ma^*\tau_* - a_0^2 - \tfrac{a_H\varepsilon}{2})\|Y\|^2. \tag{13}$$

By condition (v), the constants ε and τ_* are chosen so that both coefficients on the right-hand side are positive; indeed, $a_0 - \frac{1}{2}ma^*\tau_* - \frac{a_H\varepsilon^{-1}}{2} > 0$ and $\sum_{i=1}^{m} a_i - \frac{1}{2}ma^*\tau_* - a_0^2 - \frac{a_H\varepsilon}{2} > 0$ are equivalent to $\tau_* < \frac{2a_0 - a_H\varepsilon^{-1}}{ma^*}$ and $\tau_* < \frac{2(\sum_{i=1}^{m} a_i - a_0^2) - a_H\varepsilon}{ma^*}$, respectively, which together constitute condition (v). Consequently,

$$\dot{V} \geq 0 \text{ for all t,}$$

with equality only when $Y = Z = 0$.

It can be deduced from the evidence presented that $V(t)$ is monotone in t. Given the continuity of V and the periodicity of the solution (X,Y,Z,W) in t, it can be concluded that $V(t)$ is bounded. Therefore, there exists a constant V_0 such that

$$\lim_{t\to\infty} V(t) = V_0\,. \tag{14}$$

From (14) and the equality $V(t) = V(t + s\omega)$ arising from periodicity for any integer s, it can be demonstrated that

$$V(t) \equiv V_0\,,\ \ \forall t\,. \tag{15}$$

Evidently, from (15):

$$\dot{V}(t) = 0\,,\ \forall t. \tag{16}$$

From (13) and (16) together, it is concluded that $Y = 0$ for $\forall t \geq 0$, and hence also that $X = \zeta$ (a constant vector). That is:

$$X = \zeta\,, \qquad Y = \dot{X} = 0, \qquad Z = \dot{Y} = 0, \qquad W = \dot{Z} = 0, \tag{17}$$

for $\forall t \geq 0.$ Replacing the estimate (17) in (2) leads to the result $\sum_{i=1}^{m} F_i(\zeta) = 0.$ Since $\sum_{i=1}^{m} a_i > 0,$ it holds:

$$\lambda_{\min}(\sum_{i=1}^{m} J_{F_i}(X)) \geq \sum_{i=1}^{m} \lambda_{\min}(J_{F_i}(X)) \geq \sum_{i=1}^{m} a_i > 0.$$

So $\sum_{i=1}^{m} F_i(\zeta) = 0$ is strictly monotone, hence injective. Together with $\sum_{i=1}^{m} F_i(\zeta) = 0$ (from $F_i(0) = 0$), $\sum_{i=1}^{m} F_i(\zeta) = 0$ yields $\zeta = 0.$ Consequently, the preceding argument on $\dot{V} = 0\,, t \geq 0$ results in the following conclusion:

$$X = Y = Z = W = 0, \tag{18}$$

for $\forall t \geq 0.$ Because of this result, the proof is complete. □

Theorem 2 (Instability). Assume that all assumptions and hypotheses (i)–(v) of Theorem 1 hold. Then the trivial solution of system (2) is unstable in the sense of Lyapunov.

Proof. Consider the functional V in (3); Krasovskii's instability conditions are verified. $V(0,0,0,0) = 0,$ and V assumes positive values in every neighborhood of the origin: At $X = Y = 0,\ Z = \xi,\ W = c\xi,\ \ \xi \neq 0,\ \xi \in \Re^n,$

$$V = c\|\xi\|^2 + \int_0^1 \langle \sigma\Psi(\sigma\xi)\xi, \xi\rangle\, d\sigma$$

which is positive for c sufficiently large and attained arbitrarily close to the origin as $\|\xi\| \to 0$ (condition (K_1) of Lemma 3 is satisfied).

Let $(X,Y,Z,W) = (X(t),Y(t),Z(t),W(t))$ be an arbitrary nontrivial solution of system (2). From inequality (13) and condition (v) of Theorem 1, there exist $k_1 > 0,\ k_2 > 0$ with

$$\dot{V} \geq k_1\|Y\|^2 + k_2\|Z\|^2 > 0 \text{ for all } (Y,Z) \neq (0,0),$$

and $\dot{V} = 0$ only when $Y = Z = 0;$ as in (17)–(18) of Theorem 1; this forces $X = Y = Z = W = 0.$ Hence $\dot{V}$ is positive along every nontrivial solution. Since $V(0) = 0$, V is positive arbitrarily near the origin (K_1), and $\dot{V}$ is positive along nontrivial solutions (conditions (K_2) and (K_3) of Lemma 3 are satisfied). By Lemma 3 [Krasovskii's instability theorem], the trivial solution of (2) is unstable in the sense of Lyapunov; that is, trajectories starting arbitrarily near the origin cannot remain in any fixed neighborhood of it. This completes the proof.

Example 1. Let $n = 2$ and $m = 2$ in equation (1). The matrices G, Ψ, H and vectors Θ, F_1, F_2 are chosen as follows:

$$G(X,Y,Z,W)=\begin{bmatrix} \dfrac{e^{-x_1^2}}{1+y_1^{\,2}} & 0 \\ 0 & \dfrac{1}{1+z_2^2+w_2^2} \end{bmatrix},$$

$$\Psi(Z)=\begin{bmatrix} 1+z_1^{\,2} & 0 \\ 0 & 1+z_2^{\,2} \end{bmatrix},\ H(X)=\begin{bmatrix} 0.2+0.1e^{-x_1^{\,2}} & 0 \\ 0 & 0.2+0.1e^{-x_2^{\,2}} \end{bmatrix},\ \Theta(Y)=\begin{bmatrix} y_1^{\,5} \\ y_2^{\,5} \end{bmatrix}.$$

$$F_1(X(t-\tau_1))=\begin{bmatrix} 2.5x_1(t-\tau_1)+\arctan(x_1(t-\tau_1)) \\ 2.5x_2(t-\tau_1)+\arctan(x_2(t-\tau_1)) \end{bmatrix},$$

$$F_2(X(t-\tau_2))=\begin{bmatrix} -0.8x_1(t-\tau_2)-0.2\sin(x_1(t-\tau_2)) \\ -0.8x_2(t-\tau_2)-0.2\sin(x_2(t-\tau_2)) \end{bmatrix}.$$

It can be verified that $\lambda_j(G)\le 1=a_0$ for all states, so condition (iii) holds. Moreover, $F_i=\nabla f_i$, $\Theta=\nabla\theta$ with $\theta=\frac{1}{6}(y_1^6+y_2^6)$ and $\Psi(Z)Z=\nabla\varphi$ with $\varphi=\frac{1}{2}(z_1^2+z_2^2)+\frac{1}{4}(z_1^4+z_2^4)$; Ψ,G,H are symmetric, and $F_i(0)=0,\ F_i(X)\ne 0$ for $X\ne 0,$ so (i) and (ii) hold. From

$$\lambda_j(J_{F_1}(X))=2.5+\frac{1}{1+x_j^{\,2}},\ \ 2.5\le\lambda_j(J_{F_1}(X))\le 3.5,\ \ a_1=2.5,\ \ a_1'=3.5,$$

$$\lambda_j(J_{F_2}(X))=-0.8-0.2\cos x_j,\ \ -1\le\lambda_j(J_{F_2}(X))\le -0.6,\ \ a_2=-1,\ \ a_2'=-0.6,$$

and H is symmetric with

$$\lambda_1(H(X))=0.2+0.1e^{-x_1^{\,2}},\ \ \lambda_2(H(X))=0.2+0.1e^{-x_2^{\,2}},\ \ \left|\lambda_j(H(X))\right|\le 0.3=a_H;$$

hence (iv) holds. Therefore, $\sum_{i=1}^{2}a_i=2.5+(-1)=1.5>a_0^{\,2}=1,$ $a^*=\max_i\{|a_i|,|a_i'|\}=3.5,$ $a_H^{\,2}=0.09<4a_0(\sum_{i=1}^{m}a_i-a_0^2)=2,$ and $\varepsilon=0.3\in(\frac{a_H}{2a_0},\frac{2(\sum_{i=1}^{m}a_i-a_0^2)}{a_H})=(0.15,\ 3.33).$

Using condition (v),

$$\tau_*<\min\{\frac{2a_0-a_H\varepsilon^{-1}}{ma^*},\frac{2(\sum_{i=1}^{m}a_i-a_0^2)-a_H\varepsilon}{ma^*}\}=\min\{\frac{1}{7},\frac{0.91}{7}\}=0.13.$$

If $\tau_*=\max\{\tau_1,\tau_2\}<0.13,$ all hypotheses of Theorem 1 hold, then the equation has no nonconstant periodic solution. Furthermore, taking $\tau_1=0.10,\ \tau_2=0.08$ (so $\tau_*=0.10$),

$$k_1 = \sum_{i=1}^{m} a_i - \frac{1}{2} m a^* \tau_* - a_0^2 - \frac{a_H \varepsilon}{2} = 1.5 - 0.35 - 1^2 - 0.045 = 0.105 > 0,$$

$$k_2 = a_0 - \frac{1}{2} m a^* \tau_* - \frac{a_H \varepsilon^{-1}}{2} = 1 - (3.5)(0.1) - 0.5 = 0.15 > 0,$$

such that

$$\dot{V} \geq 0.105\|Y\|^2 + 0.15\|Z\|^2 > 0$$

for all $(Y, Z) \neq (0,0)$. Therefore, the conditions of Theorem 2 are satisfied, and the trivial solution is unstable in the sense of Lyapunov.

For the numerical simulation, the initial functions were taken as the constant histories $X(s) = (0.1,\ 0.1)^T$ and $\dot{X}(s) = \ddot{X}(s) = \dddot{X}(s) = (0,0)^T$ for $s \in [-0.10, 0]$. The corresponding trajectories, computed by a fourth-order Runge–Kutta scheme (implemented in Python, using NumPy and Matplotlib) with fixed step size $h = 5 \times 10^{-4}$, the delay terms being handled by the method of steps with linear interpolation of the history, are displayed in Figures 1 and 2: Both the first component $x_1(t)$ and the norm $\|X(t)\|$ grow away from the origin, in agreement with the instability established in Theorem 2.

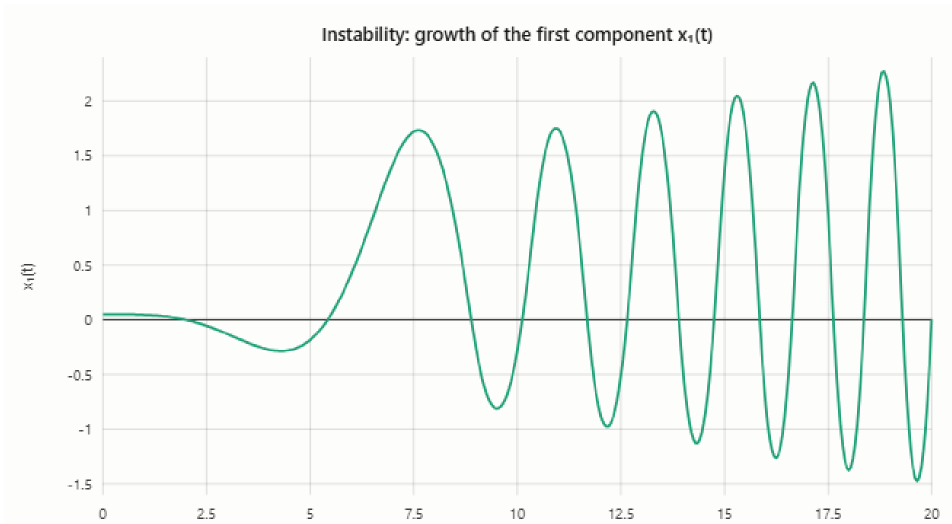

Figure 1. The time response of the first component $x_1(t)$ of the solution vector.

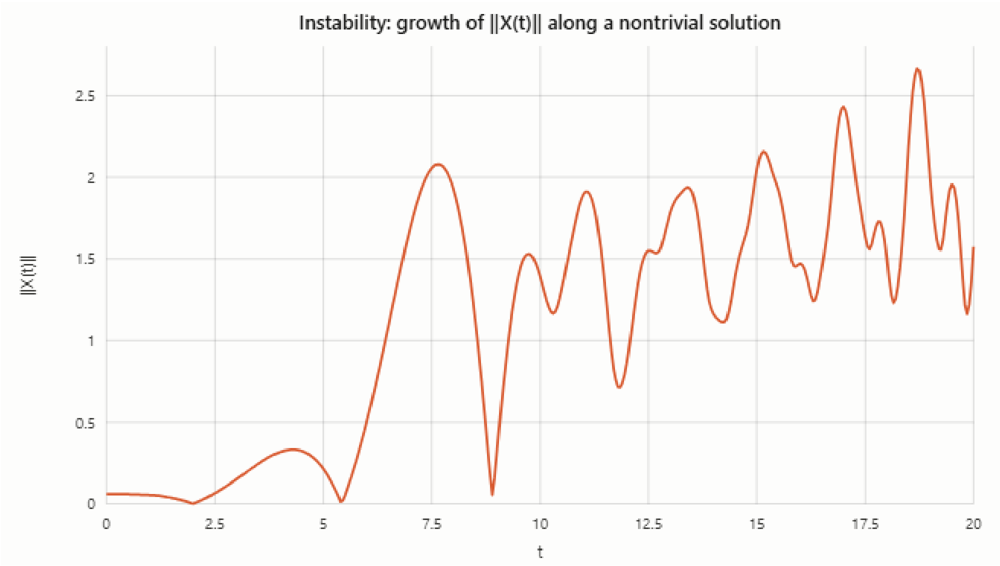

Figure 2. Growth of $\|X(t)\|$ along the nontrivial solution.

Example 2: Let $n=2$ and $m=2$ in equation (1). The matrices G,Ψ,H and vectors Θ,F_1,F_2 are chosen as follows:

$$G(X,Y,Z,W)=\begin{bmatrix}0.5+0.3\tanh(z_1^2) & 0.2\cos(w_1+x_2)\\ 0.2\cos(w_1+x_2) & 0.5+0.3\tanh(z_2^2)\end{bmatrix},\ H(X)=\begin{bmatrix}0.2\tanh x_1 & 0.1\sin(x_1+x_2)\\ 0.1\sin(x_1+x_2) & 0.2\tanh x_2\end{bmatrix},$$

$$\Psi(Z)=\begin{bmatrix}1.3+0.5e^{-(z_1^2+z_2^2)} & 0.1\\ 0.1 & 1.3+0.5e^{-(z_1^2+z_2^2)}\end{bmatrix},\ \Theta(Y)=\begin{bmatrix}y_1+0.1\sin(y_1+y_2)\\ y_2+0.1\sin(y_1+y_2)\end{bmatrix}.$$

$$F_1(X(t-\tau_1))=\begin{bmatrix}2.4x_1(t-\tau_1)+0.2x_2(t-\tau_1)+0.1\sin(x_1(t-\tau_1)+x_2(t-\tau_1))\\ 2.4x_2(t-\tau_1)+0.2x_1(t-\tau_1)+0.1\sin(x_1(t-\tau_1)+x_2(t-\tau_1))\end{bmatrix},$$

$$F_2(X(t-\tau_2))=\begin{bmatrix}1.875x_1(t-\tau_2)+0.125x_2(t-\tau_2)+0.125\sin(x_1(t-\tau_2)-x_2(t-\tau_2))\\ 1.875x_2(t-\tau_2)+0.125x_1(t-\tau_2)-0.125\sin(x_1(t-\tau_2)-x_2(t-\tau_2))\end{bmatrix}.$$

It can be verified that $\lambda_j(G)\le 1=a_0$ for all states, so condition (iii) holds. Moreover, $F_i=\nabla f_i$, $\Theta=\nabla\theta$ with $\theta(Y)=\frac{1}{2}(y_1^2+y_2^2)-0.1\cos(y_1+y_2)$, $f_1(X)=1.2(x_1^2+x_2^2)+0.2x_1x_2-0.1\cos(x_1+x_2)$, $f_2(X)=0.9375(x_1^2+x_2^2)+0.125x_1x_2-0.125\cos(x_1-x_2)$, and $\Psi(Z)Z=\nabla\varphi$ with $\varphi(Z)=\frac{1}{2}Z^TMZ+\frac{1}{2}(z_1^2+z_2^2)-\frac{1}{4}e^{-(z_1^2+z_2^2)}$, $M=\begin{bmatrix}0.3 & 0.1\\ 0.1 & 0.3\end{bmatrix}$, Ψ,G,H are symmetric and $F_i(0)=0$, $F_i(X)\ne 0$ for $X\ne 0$, so (i) - (ii) hold. Writing $c_1=0.1\cos(x_1+x_2)$, and $c_2=0.125\cos(x_1-x_2)$, the Jacobians $J_{F_1}(X)=\begin{bmatrix}2.4+c_1 & 0.2+c_1\\ 0.2+c_1 & 2.4+c_1\end{bmatrix}$, $J_{F_2}(X)=\begin{bmatrix}1.875+c_2 & 0.125-c_2\\ 0.125-c_2 & 1.875+c_2\end{bmatrix}$ have eigenvalues

$\lambda_j(J_{F_1}(X))\in\{2.2,\ 2.6+0.2\cos(x_1+x_2)\}$ $2.2\le\lambda_j(J_{F_1}(X))\le 2.8$, $a_1=2.2$, $a_1'=2.8$,

$\lambda_j(J_{F_2}(X))\in\{2.0,\ 1.75+0.25\cos(x_1-x_2)\}$ $1.5\le\lambda_j(J_{F_2}(X))\le 2.0$, $a_2=1.5$, $a_2'=2.0$, and H is symmetric with

$$\lambda_j(H(X)) = 0.1(\tanh x_1 + \tanh x_2) \mp 0.1\sqrt{(\tanh x_1 - \tanh x_2)^2 + \sin^2(x_1 + x_2)},$$

$\left|\lambda_j(H(X))\right| \le 0.3 = a_H$; hence (iv) holds. Therefore,

$$\sum_{i=1}^{2} a_i = 2.2 + 1.5 = 3.7 > a_0{}^2 = 1,\quad a^* = \max_i \{|a_i|,|a_i'|\} = 2.8,\quad a_H{}^2 = 0.09 < 4a_0(\sum_{i=1}^{m} a_i - a_0^2) = 10.8,$$

and $\varepsilon = 1.0 \in (\frac{a_H}{2a_0}, \frac{2(\sum_{i=1}^{m} a_i - a_0^2)}{a_H}) = (0.15,\ 18.0)$. Using condition (v),

$$\tau_* < \min\{\frac{2a_0 - a_H\varepsilon^{-1}}{ma^*}, \frac{2(\sum_{i=1}^{m} a_i - a_0^2) - a_H\varepsilon}{ma^*}\} = \min\{0.3036, 0.9107\} = 0.3036.$$

If $\tau_* = \max\{\tau_1, \tau_2\} < 0.3036$, all hypotheses of Theorem 1 hold, then the equation has no nonconstant periodic solution. Furthermore, taking $\tau_1 = 0.20$, $\tau_2 = 0.25$ (so $\tau_* = 0.25$, and $\gamma = (1/2)a^* = 1.4$),

$$k_1 = \sum_{i=1}^{m} a_i - \frac{1}{2}ma^*\tau_* - a_0^2 - \frac{a_H\varepsilon}{2} = 3.7 - 0.7 - 1^2 - 0.15 = 1.85 > 0,$$

$$k_2 = a_0 - \frac{1}{2}ma^*\tau_* - \frac{a_H\varepsilon^{-1}}{2} = 1 - 0.7 - 0.15 = 0.15 > 0,$$

such that $\dot{V} \ge 1.85\|Y\|^2 + 0.15\|Z\|^2 > 0$ for all $(Y, Z) \ne (0,0)$. Therefore, the conditions of Theorem 2 are satisfied, and the trivial solution is unstable in the sense of Lyapunov.

4. Conclusions

In this study, a class of nonlinear fourth-order vector differential systems has been investigated qualitatively. The proposed model incorporates multiple nonlinear terms that act on different derivatives of the state variable, enabling a unified treatment of a wide range of higher-order dynamical behaviors. By constructing an LK functional tailored to the fourth-order structure of the model under consideration and exploiting the structural properties of symmetric matrix operators and gradient-type nonlinearities, sufficient conditions have been established that govern the long-term behavior of solutions.

First, sufficient conditions ensuring the nonexistence of nonconstant periodic solutions have been derived. By showing that the LK functional is nondecreasing along every solution, any periodic solution is forced to reduce to the trivial equilibrium, thereby excluding nonconstant periodic orbits and sustained oscillations. Second, under the same structural hypotheses, complementary conditions guaranteeing the instability of the trivial solution have been obtained: The time derivative of the LK functional is nonnegative and is positive along every nontrivial solution, while the functional attains positive values arbitrarily close to the origin, so that, by Krasovskii's instability theorem, the trivial solution is unstable in the sense of Lyapunov. Additionally, two illustrative examples are included to demonstrate the feasibility and effectiveness of the obtained results.

The results highlight the effectiveness of Lyapunov-based methods for higher-order differential systems and contribute to the qualitative theory of fourth-order nonlinear equations. The analyzed equation class can model elastic beam dynamics, high-order mechanical systems, and coupled dynamical networks. From this perspective, the nonexistence of periodic solutions corresponds to the exclusion of sustained oscillatory or resonant behavior, a property that is highly desirable in mechanical design and control applications. Unlike many existing studies that focus on stability or boundedness, the present analysis emphasizes the exclusion of periodic behavior and the instability of the trivial equilibrium, thereby enriching the understanding of fourth-order systems and providing a framework for further investigations into more complex nonlinear models.

A natural further direction is the fractional-order setting, which generalizes the integer-order case considered here. Extending the present results (the nonexistence of nonconstant periodic solutions and the instability of the trivial solution) to fractional-order multi-delay systems, by adapting the LK approach within the Mittag–Leffler stability framework and drawing on recent techniques for fractional-order time-delay systems such as covering-set and filter-based methods [30] and delay-independent criteria [31], is left as an interesting problem for future work.

Use of Generative-AI tools declaration

The author declares the use of Claude (a generative AI tool developed by Anthropic) for English language editing and to assist with the illustrative examples, including the simulation code used to generate the figures. All mathematical content and the design of the examples are the author's own and were verified by the author, who takes full responsibility for the content of this publication.

Conflict of interest

The author declares no conflict of interest.

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