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Research article

Local convergence of a fifth-order Newton-type iterative method for solving systems of nonlinear equations

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Abstract: Based on the Lipschitz continuity condition, we study the local convergence analysis of a fifth-order Newton-type iterative method to solve nonlinear equations in Banach spaces. Lipschitz continuity condition on the first derivative is assumed to broaden the applicability of the method. Our study provides a thorough local convergence analysis in Banach spaces, including radii of convergence balls, computable error bounds, and the uniqueness of the solution. Finally, numerical experiments on various nonlinear models confirm the theoretical convergence criteria.

Keywords: local convergence; Banach spaces; nonlinear equations; radius of convergence; iterative method
Mathematics Subject Classification: 65B99, 65H05

1. Introduction

In both engineering and scientific fields, numerous problems necessitate the solution of nonlinear equations. For instance, the boundary value problem in the gas dynamics theory, the integral equation associated with radiation transfer theory, optimization problems, and various other issues can all be reduced to solving nonlinear equations. Therefore, finding a solution ς to the nonlinear equation

$$\Lambda(\varrho) = 0 \tag{1.1}$$

represents a crucial research issue. Here, $\Lambda : \Omega \subset S \to S$ denotes a sufficiently differentiable nonlinear operator defined on an open convex subset of S, where S is a Banach space. An analytical solution to (1.1) is generally hard to obtain, so iterative methods are typically employed to find approximate solutions.

In Banach spaces, a variety of iterative methods have been developed and explored to analyze their convergence properties [1–5]. The most famous and fundamental iterative method is Newton's method [6], which is formulated as follows:

$$\varrho^{(n+1)} = \varrho^{(n)} - \Lambda'\left(\varrho^{(n)}\right)^{-1}\Lambda\left(\varrho^{(n)}\right), \quad n = 0, 1, 2, \ldots,$$

where $\varrho^{(0)} \in \Omega$ stands for the initial approximation, and $\Lambda' : \Omega \to \mathcal{L}(S, S)$ denotes the first Fréchet derivative of the operator Λ. In this context, $\mathcal{L}(S, S)$ denotes the collection of all bounded linear mappings acting on the space S. In order to further improve the convergence order, numerous researchers have put forward multi-step iterative methods based on Newton's method, which are referred to as Newton-type methods [7–10].

Weerakoon and Fernando [11] proposed a third-order Newton-type method, known as the Arithmetic-Mean Newton's method. The specific definition of this method is given below:

$$\begin{cases} \vartheta^{(n)} = \varrho^{(n)} - \Lambda'\left(\varrho^{(n)}\right)^{-1}\Lambda\left(\varrho^{(n)}\right), \\ \varrho^{(n+1)} = \varrho^{(n)} - 2\left[\Lambda'\left(\vartheta^{(n)}\right) + \Lambda'\left(\varrho^{(n)}\right)\right]^{-1}\Lambda\left(\varrho^{(n)}\right), \quad n = 0, 1, 2, \ldots \end{cases} \tag{1.2}$$

They studied this method using the trapezoidal approximation method of interpolation quadrature formulas. Frontini and Sormani [12] showed that among all third-order Newton-type methods derived from quadrature formulas, the trapezoidal (Arithmetic-Mean) is superior to other algorithms in terms of the efficiency of function evaluation counts. Later, based on Method (1.2), Cordero et al. [13] proposed a fifth-order method by adding one matrix inversion, as follows:

$$\begin{cases} \vartheta^{(n)} = \varrho^{(n)} - \Lambda'\left(\varrho^{(n)}\right)^{-1}\Lambda\left(\varrho^{(n)}\right), \\ \tau^{(n)} = \varrho^{(n)} - 2\left[\Lambda'\left(\vartheta^{(n)}\right) + \Lambda'\left(\varrho^{(n)}\right)\right]^{-1}\Lambda\left(\varrho^{(n)}\right), \\ \varrho^{(n+1)} = \tau^{(n)} - \Lambda'\left(\vartheta^{(n)}\right)^{-1}\Lambda\left(\tau^{(n)}\right), \end{cases} \tag{1.3}$$

where $\Lambda : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$. However, the convergence of this method is verified under the premise that the operator has sixth-order derivatives, which restricts the scope of application of the method.

Recently, based on Method (1.2), Xiao [14] proposed a new fifth-order method featuring a three-step iterative structure:

$$\begin{cases} \vartheta^{(n)} = \varrho^{(n)} - \Lambda'\left(\varrho^{(n)}\right)^{-1}\Lambda\left(\varrho^{(n)}\right), \\ \tau^{(n)} = \varrho^{(n)} - 2\left[\Lambda'\left(\vartheta^{(n)}\right) + \Lambda'\left(\varrho^{(n)}\right)\right]^{-1}\Lambda\left(\varrho^{(n)}\right), \\ \varrho^{(n+1)} = \tau^{(n)} - \left[2I - \Lambda'\left(\varrho^{(n)}\right)^{-1}\Lambda'\left(\vartheta^{(n)}\right)\right]\Lambda'\left(\varrho^{(n)}\right)^{-1}\Lambda\left(\tau^{(n)}\right). \end{cases} \tag{1.4}$$

Compared to Method (1.3), Method (1.4) eliminates one matrix inversion, resulting in lower computational cost.

For many other iterative methods, numerous authors have also studied their local and semi-local convergence properties [15–20]. With respect to the local convergence behavior under consideration, the core goal lies in evaluating the convergence radius by leveraging the available data within the neighborhood of the solution. George et al. [21] adopted the following conditions in their research on local convergence analyses:

$$\left\|\Lambda'(\varsigma)^{-1}\left(\Lambda'(\varrho) - \Lambda'(\varsigma)\right)\right\| \leq k_0 \left\|\varrho - \varsigma\right\|, \quad \forall \varrho \in \Omega, \tag{1.5}$$

and

$$\left\|\Lambda'(\varsigma)^{-1}\left(\Lambda''(\varrho)-\Lambda''(\vartheta)\right)\right\| \leq k\,\|\varrho-\vartheta\|, \quad \forall \varrho, \vartheta \in \Omega. \tag{1.6}$$

The local convergence analysis based on (1.6) relies on second-order derivative assumptions, which inevitably narrows the applicable range of the iterative method.

We establish the local convergence of Method (1.4) under weaker Lipschitz conditions only involving the first-order derivative:

$$\left\|\Lambda'(\varsigma)^{-1}\left(\Lambda'(\varrho)-\Lambda'(\varsigma)\right)\right\| \leq k_0\,\|\varrho-\varsigma\|, \quad \forall \varrho \in \Omega, \tag{1.7}$$

and

$$\left\|\Lambda'(\varsigma)^{-1}\left(\Lambda'(\varrho)-\Lambda'(\vartheta)\right)\right\| \leq k\,\|\varrho-\vartheta\|, \quad \forall \varrho, \vartheta \in \Omega. \tag{1.8}$$

Such assumptions avoid extra restrictions on second-order derivatives and greatly expand the universality of the convergence results. In Formulas (1.7) and (1.8), the norm on the left-hand side of the inequalities is the operator norm defined by $\|A\|_{\text{op}} = \sup_{\|\rho\|=1} \|A\rho\|$, and the notation $\|\cdot\|$ is adopted on the right-hand side to stand for the norm on Banach space S. For all subsequent contents of this paper, we will not distinguish these two norms by separate symbols either; the notation $\|\cdot\|$ will be interpreted based on the context.

The rest of this work is organized in the subsequent manner. Section 2 focuses on investigating the local convergence of Method (1.4) by adopting assumptions based on the first derivative; additionally, we conduct a detailed analysis of both the uniqueness of the solution and the radii of the convergence balls. Section 3 investigates the computational efficiency of iterative formulas Method (1.2), Method (1.3), and Method (1.4) under different system dimensions, and quantitative comparisons are carried out via the computational efficiency index. In Section 4.1, we get the corresponding local convergence radius based on the application effect of Method (1.4) on specific nonlinear systems. In Section 4.2, the iterative Method (1.4) is utilized to address ordinary differential equations. Meanwhile, a comparative analysis is conducted between this method and other relevant iterative approaches. Ultimately, the concluding remarks are presented in Section 5.

2. Convergence analysis

This section focuses on the analysis of the local convergence property of the iterative Method (1.4). Let the open ball and closed ball centered at c with a radius of $\rho > 0$ be referred to as $U(c,\rho) = \{u \in S : \|u-c\| < \rho\}$ and $\overline{U}(c,\rho) = \{u \in S : \|u-c\| \leq \rho\}$, respectively. We take parameters $k_0 > 0$ and $k > 0$ satisfying $k_0 \leq k$. Then, we introduce the function P_1 that is defined on the interval $[0\ ,\frac{1}{k_0})$ as follows:

$$P_1(v) = \frac{kv}{2(1-k_0 v)} \tag{2.1}$$

and the parameter

$$R_1 = \frac{2}{2k_0+k} < \frac{1}{k_0}. \tag{2.2}$$

Notice that $P_1(R_1) = 1$. Again, we define P_2, P_3, and K_3 on $[0\ , R_1]$ by the following:

$$P_2(v) = \frac{k_0 v}{2} + \frac{k_0}{2} P_1(v)\, v, \tag{2.3}$$

$$P_3(v) = P_1(v) + \frac{k(1+k_0v)^3 v}{2(1-k_0v)^3(1-P_2(v))} \tag{2.4}$$

and

$$K_3(v) = P_3(v) - 1.$$

Now, $K_3(0) = -1 < 0$, $K_3(v) \to +\infty$ as $v \to {R_1}^-$. Accordingly, the interval $(0\ ,R_1)$ encloses the roots of the function $K_3(v)$, and we assume that the smallest zero of $K_3(v)$ within the interval $(0\ ,R_1)$ is R_2. Finally, let us define P_4 and K_4 on $[0\ ,R_2)$ by

$$P_4(v) = \left(1 + \frac{(2k_0v + k_0P_1(v)\,v + 1)(1 + k_0P_3(v)\,v)}{(1-k_0v)^2}\right)P_3(v) \tag{2.5}$$

and

$$K_4(v) = P_4(v) - 1.$$

Now, $K_4(0) = -1 < 0$ and $K_4(v) \to +\infty$ as $v \to {R_2}^-$. Additionally, the interval $(0\ ,R_2)$ encloses the roots of the function $K_4(v)$, and we assume that the smallest zero of $K_4(v)$ within the interval $(0\ ,R_2)$ is R_3. Let us take the following:

$$R = R_3 < R_2 < R_1. \tag{2.6}$$

Now, we have

$$0 \le P_1(v) < 1, \tag{2.7}$$

$$0 \le P_2(v) < 1, \tag{2.8}$$

$$0 \le P_3(v) < 1 \tag{2.9}$$

and

$$0 \le P_4(v) < 1 \tag{2.10}$$

holds for all $v \in [0\ ,R)$. We make the assumption that the differentiable mapping $\Lambda : \Omega \subset S \to S$ satisfies the following conditions:

$$\Lambda(\varsigma) = 0, \quad \Lambda'(\varsigma)^{-1} \in \mathcal{L}(S\ ,S), \tag{2.11}$$

$$\left\|\Lambda'(\varsigma)^{-1}(\Lambda'(\varrho) - \Lambda'(\varsigma))\right\| \le k_0\,\|\varrho - \varsigma\|, \quad \forall \varrho \in \Omega, \tag{2.12}$$

and

$$\left\|\Lambda'(\varsigma)^{-1}(\Lambda'(\varrho) - \Lambda'(\vartheta))\right\| \le k\,\|\varrho - \vartheta\|, \quad \forall \varrho, \vartheta \in \Omega, \tag{2.13}$$

Lemma 2.1. *If Λ obeys Conditions (2.11), (2.12), and (2.13), then $\forall \varrho \in \Omega$, and we get the following:*

$$\left\|\Lambda'(\varsigma)^{-1}\Lambda'(\varrho)\right\| \le 1 + k_0\,\|\varrho - \varsigma\| \tag{2.14}$$

and

$$\left\|\Lambda'(\varsigma)^{-1}\Lambda(\varrho)\right\| \le (1 + k_0\,\|\varrho - \varsigma\|)\,\|\varrho - \varsigma\|. \tag{2.15}$$

Proof. By applying (2.12), we obtain the following:

$$\left\|\Lambda'(\varsigma)^{-1}\Lambda'(\varrho)\right\| \leq 1+\left\|\Lambda'(\varsigma)^{-1}\left(\Lambda'(\varrho)-\Lambda'(\varsigma)\right)\right\| \leq 1+k_0\|\varrho-\varsigma\|.$$

For $t \in [0\,,1]$,

$$\left\|\Lambda'(\varsigma)^{-1}\Lambda'\left(\varsigma+t\left(\varrho-\varsigma\right)\right)\right\| \leq 1+k_0 t\|\varrho-\varsigma\| \leq 1+k_0\|\varrho-\varsigma\|.$$

The fundamental theorem of calculus is used to prove (2.15). We have the following:

$$\begin{aligned}
\left\|\Lambda'(\varsigma)^{-1}(\Lambda(\varrho)-\Lambda(\varsigma))\right\| &= \left\|\int_0^1 \Lambda'(\varsigma)^{-1}\frac{d}{dt}\Lambda(\varsigma+t(\varrho-\varsigma))\mathrm{d}t\right\| \\
&= \left\|\int_0^1 \Lambda'(\varsigma)^{-1}\Lambda'(\varsigma+t(\varrho-\varsigma))(\varrho-\varsigma)\mathrm{d}t\right\| \\
&\leq \int_0^1 \left\|\Lambda'(\varsigma)^{-1}\Lambda'(\varsigma+t(\varrho-\varsigma))(\varrho-\varsigma)\right\|\mathrm{d}t \\
&\leq \int_0^1 (1+k_0\|\varrho-\varsigma\|)\|\varrho-\varsigma\|\mathrm{d}t \\
&= (1+k_0\|\varrho-\varsigma\|)\|\varrho-\varsigma\|.
\end{aligned}$$

□

Lemma 2.2. *(Banach lemma [22]) Let A be a bounded linear operator on a Banach space S. If*

$$\|I-A\| < 1, \tag{2.16}$$

then A is invertible, and

$$\|A^{-1}\| \leq \frac{1}{1-\|I-A\|}. \tag{2.17}$$

This lemma is crucial to establish the existence of inverse operators in the convergence analysis.

In the following Theorem, we conduct the analysis of the local convergence of Method (1.4).

Theorem 1. *Suppose that the differentiable function $\Lambda : \Omega \subseteq S \to S$ satisfies Conditions (2.11)-(2.13). Let $\varsigma \in \Omega$ and*

$$\overline{U}(\varsigma,\, R) \subseteq \Omega, \tag{2.18}$$

with R specified in Equation (2.6). Then, starting from an initial value $\varrho^{(0)} \in U(\varsigma\,, R) - \{\varsigma\}$, the family of Methods (1.4) produces $\left\{\varrho^{(n)}\right\}_{n\geq 0}$ which is well-defined, resides within the open ball $U(\varsigma\,, R)$, and converges to the solution ς of (1.1). Additionally, the following conditions hold for $\forall n \geq 0$:

$$\left\|\vartheta^{(n)}-\varsigma\right\| \leq P_1\left(\left\|\varrho^{(n)}-\varsigma\right\|\right)\left\|\varrho^{(n)}-\varsigma\right\| < \left\|\varrho^{(n)}-\varsigma\right\| < R, \tag{2.19}$$

$$\left\|\left(\Lambda'\left(\varrho^{(n)}\right)+\Lambda'\left(\vartheta^{(n)}\right)\right)^{-1}\Lambda'(\varsigma)\right\| \leq \frac{1}{2\left(1-P_2\left(\left\|\varrho^{(n)}-\varsigma\right\|\right)\right)}, \tag{2.20}$$

$$\left\|\tau^{(n)}-\varsigma\right\| \leq P_3\left(\left\|\varrho^{(n)}-\varsigma\right\|\right)\left\|\varrho^{(n)}-\varsigma\right\| < \left\|\varrho^{(n)}-\varsigma\right\| < R \tag{2.21}$$

and

$$\left\|\varrho^{(n+1)}-\varsigma\right\| \leq P_4\left(\left\|\varrho^{(n)}-\varsigma\right\|\right)\left\|\varrho^{(n)}-\varsigma\right\| < \left\|\varrho^{(n)}-\varsigma\right\| < R, \tag{2.22}$$

where the 'P' functions are given in (2.1), (2.3), (2.4), and (2.5). For $\Delta \in [R\,, \frac{2}{k_0})$, the solution ς of $\Lambda(\varrho) = 0$ is unique in $\overline{U}(\varsigma,\, \Delta) \cap \Omega$.

Proof. By employing the hypothesis $\varrho^{(0)} \in U(\varsigma, R) - \{\varsigma\}$ along with Equations (2.6) and (2.12), we derive that

$$\left\|\Lambda'(\varsigma)^{-1}\left(\Lambda'\left(\varrho^{(0)}\right) - \Lambda'(\varsigma)\right)\right\| \leq k_0 \left\|\varrho^{(0)} - \varsigma\right\| < k_0 R < 1. \tag{2.23}$$

By applying the Banach Lemma on invertible operators, we get $\Lambda'\left(\varrho^{(0)}\right)^{-1} \in \mathcal{L}(S, S)$ and

$$\left\|\Lambda'\left(\varrho^{(0)}\right)^{-1}\Lambda'(\varsigma)\right\| \leq \frac{1}{1 - k_0\left\|\varrho^{(0)} - \varsigma\right\|} < \frac{1}{1 - k_0 R}. \tag{2.24}$$

Hence, $\vartheta^{(0)}$ is well-defined. Using the first substep of Method (1.4) for $n = 0$ and (2.11), we can have the following:

$$\begin{aligned}
\vartheta^{(0)} - \varsigma =& \varrho^{(0)} - \varsigma - \Lambda'\left(\varrho^{(0)}\right)^{-1}\Lambda\left(\varrho^{(0)}\right)\\
=& -\left[\Lambda'\left(\varrho^{(0)}\right)^{-1}\Lambda'(\varsigma)\right]\left[\int_0^1 \Lambda'(\varsigma)^{-1}\left(\Lambda'\left(\varsigma + t\left(\varrho^{(0)} - \varsigma\right)\right) - \Lambda'\left(\varrho^{(0)}\right)\right)\left(\varrho^{(0)} - \varsigma\right)dt\right]
\end{aligned} \tag{2.25}$$

Using (2.1), (2.6), (2.7), (2.13), (2.24), and (2.25), we find

$$\begin{aligned}
\left\|\vartheta^{(0)} - \varsigma\right\| \leq& \left\|\Lambda'\left(\varrho^{(0)}\right)^{-1}\Lambda'(\varsigma)\right\|\left\|\int_0^1 \Lambda'(\varsigma)^{-1}\left(\Lambda'\left(\varsigma + t\left(\varrho^{(0)} - \varsigma\right)\right) - \Lambda'\left(\varrho^{(0)}\right)\right)\left(\varrho^{(0)} - \varsigma\right)dt\right\|\\
\leq& \frac{k\left\|\varrho^{(0)} - \varsigma\right\|^2}{2\left(1 - k_0\left\|\varrho^{(0)} - \varsigma\right\|\right)}\\
=& P_1\left(\left\|\varrho^{(0)} - \varsigma\right\|\right)\left\|\varrho^{(0)} - \varsigma\right\| < \left\|\varrho^{(0)} - \varsigma\right\| < R
\end{aligned} \tag{2.26}$$

which shows Equation (2.19) for $n = 0$ and $\vartheta^{(0)} \in U(\varsigma, R)$.

Now, our claim is $\left(\Lambda'\left(\vartheta^{(0)}\right) + \Lambda'\left(\varrho^{(0)}\right)\right)^{-1} \in \mathcal{L}(S, S)$. Using (2.4), (2.8), (2.12), and (2.26), we have the following:

$$\begin{aligned}
\left\|(2\Lambda'(\varsigma))^{-1}\left(\Lambda'\left(\vartheta^{(0)}\right) + \Lambda'\left(\varrho^{(0)}\right) - 2\Lambda'(\varsigma)\right)\right\| \leq& \frac{1}{2}\left\|\Lambda'(\varsigma)^{-1}\left(\Lambda'\left(\vartheta^{(0)}\right) - \Lambda'(\varsigma)\right)\right\|\\
&+ \frac{1}{2}\left\|\Lambda'(\varsigma)^{-1}\left(\Lambda'\left(\varrho^{(0)}\right) - \Lambda'(\varsigma)\right)\right\|\\
\leq& \frac{k_0}{2}\left(\left\|\vartheta^{(0)} - \varsigma\right\| + \left\|\varrho^{(0)} - \varsigma\right\|\right)\\
\leq& \frac{k_0}{2}\left(P_1\left(\left\|\varrho^{(0)} - \varsigma\right\|\right)\left\|\varrho^{(0)} - \varsigma\right\| + \left\|\varrho^{(0)} - \varsigma\right\|\right)\\
=& P_2\left(\left\|\varrho^{(0)} - \varsigma\right\|\right) < P_2(R) < 1.
\end{aligned} \tag{2.27}$$

By applying the Banach Lemma on invertible operators, we can conclude that $\left(\Lambda'\left(\vartheta^{(0)}\right) + \Lambda'\left(\varrho^{(0)}\right)\right)^{-1} \in \mathcal{L}(S, S)$ with

$$\left\|\left(\Lambda'\left(\vartheta^{(0)}\right) + \Lambda'\left(\varrho^{(0)}\right)\right)^{-1}\Lambda'(\varsigma)\right\| \leq \frac{1}{2\left(1 - P_2\left(\left\|\varrho^{(0)} - \varsigma\right\|\right)\right)}. \tag{2.28}$$

Thus, we establish (2.20) for $n = 0$. From (2.13) and (2.24), we also have the following:

$$\begin{aligned}\left\|\Lambda'\left(\varrho^{(0)}\right)^{-1}\left(\Lambda'\left(\vartheta^{(0)}\right)-\Lambda'\left(\varrho^{(0)}\right)\right)\right\| &\leq \left\|\Lambda'\left(\varrho^{(0)}\right)^{-1}\Lambda'(\varsigma)\right\|\left\|\Lambda'(\varsigma)^{-1}\left(\Lambda'\left(\vartheta^{(0)}\right)-\Lambda'\left(\varrho^{(0)}\right)\right)\right\| \\ &\leq \frac{1}{1-k_0\left\|\varrho^{(0)}-\varsigma\right\|}\frac{k\left(1+k_0\left\|\varrho^{(0)}-\varsigma\right\|\right)\left\|\varrho^{(0)}-\varsigma\right\|}{1-k_0\left\|\varrho^{(0)}-\varsigma\right\|} \\ &\leq \frac{k\left(1+k_0\left\|\varrho^{(0)}-\varsigma\right\|\right)\left\|\varrho^{(0)}-\varsigma\right\|}{\left(1-k_0\left\|\varrho^{(0)}-\varsigma\right\|\right)^2}\end{aligned} \tag{2.29}$$

Now, using (1.4), (2.4), (2.6), (2.9), (2.14), (2.15), (2.24), (2.26), (2.28), and (2.29), we get the following:

$$\begin{aligned}\left\|\tau^{(0)}-\varsigma\right\| &= \left\|\varrho^{(0)}-\varsigma-2\left(\Lambda'\left(\vartheta^{(0)}\right)+\Lambda'\left(\varrho^{(0)}\right)\right)^{-1}\Lambda\left(\varrho^{(0)}\right)\right\| \\ &= \left\|\vartheta^{(0)}-\varsigma+\Lambda'\left(\varrho^{(0)}\right)^{-1}\Lambda\left(\varrho^{(0)}\right)-2\left(\Lambda'\left(\vartheta^{(0)}\right)+\Lambda'\left(\varrho^{(0)}\right)\right)^{-1}\Lambda\left(\varrho^{(0)}\right)\right\| \\ &\leq \left\|\vartheta^{(0)}-\varsigma\right\|+\left\|\Lambda'\left(\varrho^{(0)}\right)^{-1}-2\left(\Lambda'\left(\vartheta^{(0)}\right)+\Lambda'\left(\varrho^{(0)}\right)\right)^{-1}\right\|\left\|\Lambda\left(\varrho^{(0)}\right)\right\| \\ &\leq \left\|\vartheta^{(0)}-\varsigma\right\|+\left\|\Lambda'\left(\varrho^{(0)}\right)^{-1}\left(\Lambda'\left(\varrho^{(0)}\right)+\Lambda'\left(\vartheta^{(0)}\right)\right)\left(\Lambda'\left(\varrho^{(0)}\right)+\Lambda'\left(\vartheta^{(0)}\right)\right)^{-1}-2\left(\Lambda'\left(\varrho^{(0)}\right)+\Lambda'\left(\vartheta^{(0)}\right)\right)^{-1}\right\| \\ &\quad \left\|\Lambda\left(\varrho^{(0)}\right)\right\| \\ &\leq \left\|\vartheta^{(0)}-\varsigma\right\|+\left\|\left[\Lambda'\left(\varrho^{(0)}\right)^{-1}\left(\Lambda'\left(\varrho^{(0)}\right)+\Lambda'\left(\vartheta^{(0)}\right)\right)-2I\right]\left(\Lambda'\left(\varrho^{(0)}\right)+\Lambda'\left(\vartheta^{(0)}\right)\right)^{-1}\right\|\left\|\Lambda\left(\varrho^{(0)}\right)\right\| \\ &\leq \left\|\vartheta^{(0)}-\varsigma\right\|+\left\|\left(I+\Lambda'\left(\varrho^{(0)}\right)^{-1}\Lambda'\left(\vartheta^{(0)}\right)-2I\right)\left(\Lambda'\left(\varrho^{(0)}\right)+\Lambda'\left(\vartheta^{(0)}\right)\right)^{-1}\right\|\left\|\Lambda\left(\varrho^{(0)}\right)\right\| \\ &\leq \left\|\vartheta^{(0)}-\varsigma\right\|+\left\|\left(\Lambda'\left(\varrho^{(0)}\right)^{-1}\Lambda'\left(\vartheta^{(0)}\right)-I\right)\left(\Lambda'\left(\varrho^{(0)}\right)+\Lambda'\left(\vartheta^{(0)}\right)\right)^{-1}\right\|\left\|\Lambda\left(\varrho^{(0)}\right)\right\| \\ &\leq \left\|\vartheta^{(0)}-\varsigma\right\|+\left\|\Lambda'\left(\varrho^{(0)}\right)^{-1}\left(\Lambda'\left(\vartheta^{(0)}\right)-\Lambda'\left(\varrho^{(0)}\right)\right)\left(\Lambda'\left(\varrho^{(0)}\right)+\Lambda'\left(\vartheta^{(0)}\right)\right)^{-1}\right\|\left\|\Lambda\left(\varrho^{(0)}\right)\right\| \\ &\leq P_1\left(\left\|\varrho^{(0)}-\varsigma\right\|\right)\left\|\varrho^{(0)}-\varsigma\right\|+\left\|\Lambda'\left(\varrho^{(0)}\right)^{-1}\left(\Lambda'\left(\vartheta^{(0)}\right)-\Lambda'\left(\varrho^{(0)}\right)\right)\right\| \\ &\quad \left\|\left(\Lambda'\left(\vartheta^{(0)}\right)+\Lambda'\left(\varrho^{(0)}\right)\right)^{-1}\Lambda'\left(\varrho^{(0)}\right)\left(\vartheta^{(0)}-\varrho^{(0)}\right)\right\| \\ &\leq P_1\left(\left\|\varrho^{(0)}-\varsigma\right\|\right)\left\|\varrho^{(0)}-\varsigma\right\|+\frac{k\left(1+k_0\left\|\varrho^{(0)}-\varsigma\right\|\right)^2\left\|\varrho^{(0)}-\varsigma\right\|^2}{\left(1-k_0\left\|\varrho^{(0)}-\varsigma\right\|\right)^3} \\ &\quad \frac{1+k_0\left\|\varrho^{(0)}-\varsigma\right\|}{2\left(1-P_2\left(\left\|\varrho^{(0)}-\varsigma\right\|\right)\right)} \\ &\leq \left(P_1\left(\left\|\varrho^{(0)}-\varsigma\right\|\right)+\frac{k\left(1+k_0\left\|\varrho^{(0)}-\varsigma\right\|\right)^2\left\|\varrho^{(0)}-\varsigma\right\|}{\left(1-k_0\left\|\varrho^{(0)}-\varsigma\right\|\right)^3}\right. \\ &\quad \left.\frac{1+k_0\left\|\varrho^{(0)}-\varsigma\right\|}{2\left(1-P_2\left(\left\|\varrho^{(0)}-\varsigma\right\|\right)\right)}\right)\left\|\varrho^{(0)}-\varsigma\right\|\end{aligned}$$

$$
\begin{aligned}
&\le \left(P_1 \left(\left\| \varrho^{(0)} - \varsigma \right\| \right) + \frac{k \left(1 + k_0 \left\| \varrho^{(0)} - \varsigma \right\| \right)^3 \left\| \varrho^{(0)} - \varsigma \right\|}{2 \left(1 - k_0 \left\| \varrho^{(0)} - \varsigma \right\| \right)^3} \left(1 - P_2 \left(\left\| \varrho^{(0)} - \varsigma \right\| \right) \right) \right) \\
&\quad \left\| \varrho^{(0)} - \varsigma \right\| \\
&= P_3 \left(\left\| \varrho^{(0)} - \varsigma \right\| \right) \left\| \varrho^{(0)} - \varsigma \right\| < \left\| \varrho^{(0)} - \varsigma \right\| < R. \qquad (2.30)
\end{aligned}
$$

Hence, we show the estimate (2.21) for $n = 0$ and $\tau^{(0)} \in U(\varsigma, R)$.

Now, using (2.5), (2.6), (2.10), (2.12), (2.15), (2.21), (2.24), (2.26), and (2.30), we find

$$
\begin{aligned}
\left\| \varrho^{(1)} - \varsigma \right\| &= \left\| \tau^{(0)} - \varsigma \right\| + \left\| 2I - \Lambda' \left(\varrho^{(0)} \right)^{-1} \Lambda' \left(\vartheta^{(0)} \right) \right\| \left\| \Lambda' \left(\varrho^{(0)} \right)^{-1} \Lambda \left(\tau^{(0)} \right) \right\| \\
&= \left\| \tau^{(0)} - \varsigma \right\| + \left\| 2\Lambda' \left(\varrho^{(0)} \right)^{-1} \Lambda' \left(\varrho^{(0)} \right) - \Lambda' \left(\varrho^{(0)} \right)^{-1} \Lambda' \left(\vartheta^{(0)} \right) \right\| \left\| \Lambda' \left(\varrho^{(0)} \right)^{-1} \Lambda \left(\tau^{(0)} \right) \right\| \\
&= \left\| \tau^{(0)} - \varsigma \right\| + \left\| \Lambda' \left(\varrho^{(0)} \right)^{-1} \left(2\Lambda' \left(\varrho^{(0)} \right) - \Lambda' \left(\vartheta^{(0)} \right) \right) \right\| \left\| \Lambda' \left(\varrho^{(0)} \right)^{-1} \Lambda \left(\tau^{(0)} \right) \right\| \\
&= \left\| \tau^{(0)} - \varsigma \right\| + \left\| \Lambda' \left(\varrho^{(0)} \right)^{-1} \Lambda' (\varsigma) \Lambda' (\varsigma)^{-1} \left(2\Lambda' \left(\varrho^{(0)} \right) - \Lambda' \left(\vartheta^{(0)} \right) \right) \right\| \left\| \Lambda' \left(\varrho^{(0)} \right)^{-1} \Lambda \left(\tau^{(0)} \right) \right\| \\
&\le P_3 \left(\left\| \varrho^{(0)} - \varsigma \right\| \right) \left\| \varrho^{(0)} - \varsigma \right\| + \left\| \Lambda' \left(\varrho^{(0)} \right)^{-1} \Lambda' (\varsigma) \right\| \left\| \Lambda' \left(\varrho^{(0)} \right)^{-1} \Lambda \left(\tau^{(0)} \right) \right\| \\
&\quad \left\| \left(\Lambda' (\varsigma)^{-1} \left(2\Lambda' \left(\varrho^{(0)} \right) - 2\Lambda' (\varsigma) \right) - \Lambda' (\varsigma)^{-1} \left(\left(\Lambda' \left(\vartheta^{(0)} \right) - \Lambda' (\varsigma) \right) - \Lambda' (\varsigma) \right) \right) \right\| \\
&\le P_3 \left(\left\| \varrho^{(0)} - \varsigma \right\| \right) \left\| \varrho^{(0)} - \varsigma \right\| + \frac{2k_0 \left\| \varrho^{(0)} - \varsigma \right\| + k_0 \left\| \vartheta^{(0)} - \varsigma \right\| + 1}{1 - k_0 \left\| \varrho^{(0)} - \varsigma \right\|} \\
&\quad \cdot \frac{\left(1 + k_0 \left\| \tau^{(0)} - \varsigma \right\| \right) \left\| \tau^{(0)} - \varsigma \right\|}{1 - k_0 \left\| \varrho^{(0)} - \varsigma \right\|} \\
&\le P_3 \left(\left\| \varrho^{(0)} - \varsigma \right\| \right) \left\| \varrho^{(0)} - \varsigma \right\| + \frac{\left(2k_0 \left\| \varrho^{(0)} - \varsigma \right\| + k_0 P_1 \left(\left\| \varrho^{(0)} - \varsigma \right\| \right) \left\| \varrho^{(0)} - \varsigma \right\| + 1 \right)}{\left(1 - k_0 \left\| \varrho^{(0)} - \varsigma \right\| \right)^2} \\
&\quad \cdot \left(1 + k_0 P_3 \left(\left\| \varrho^{(0)} - \varsigma \right\| \right) \left\| \varrho^{(0)} - \varsigma \right\| \right) P_3 \left(\left\| \varrho^{(0)} - \varsigma \right\| \right) \left\| \varrho^{(0)} - \varsigma \right\| \\
&\le P_3(\left\| \varrho^{(0)} - \varsigma \right\|) \left\| \varrho^{(0)} - \varsigma \right\| + \frac{(2k_0 \left\| \varrho^{(0)} - \varsigma \right\| + k_0 P_1(\left\| \varrho^{(0)} - \varsigma \right\|) \left\| \varrho^{(0)} - \varsigma \right\| + 1)}{(1 - k_0 \left\| \varrho^{(0)} - \varsigma \right\|)^2} \\
&\quad \cdot (1 + k_0 P_3(\left\| \varrho^{(0)} - \varsigma \right\|) \left\| \varrho^{(0)} - \varsigma \right\|) P_3(\left\| \varrho^{(0)} - \varsigma \right\|) \left\| \varrho^{(0)} - \varsigma \right\| \\
&\le \left(1 + \frac{(2k_0 \left\| \varrho^{(0)} - \varsigma \right\| + k_0 P_1(\left\| \varrho^{(0)} - \varsigma \right\|) \left\| \varrho^{(0)} - \varsigma \right\| + 1)}{(1 - k_0 \left\| \varrho^{(0)} - \varsigma \right\|)^2} \right. \\
&\quad \left. \cdot (1 + k_0 P_3(\left\| \varrho^{(0)} - \varsigma \right\|) \left\| \varrho^{(0)} - \varsigma \right\|) \right) P_3(\left\| \varrho^{(0)} - \varrho^* \right\|) \left\| \varrho^{(0)} - \varsigma \right\| \\
&= P_4(\left\| \varrho^{(0)} - \varsigma \right\|) \left\| \varrho^{(0)} - \varsigma \right\| < \left\| \varrho^{(0)} - \varsigma \right\| < R.
\end{aligned}
$$

Hence, we show the estimate (2.22) for $n = 0$ and $\varrho^{(1)} \in U(\varsigma, R)$.

By substituting $\varrho^{(n)}$, $\vartheta^{(n)}$, $\tau^{(n)}$, and $\varrho^{(n+1)}$ to replace $\varrho^{(0)}$, $\vartheta^{(0)}$, $\tau^{(0)}$, and $\varrho^{(1)}$ within the preceding estimations, we derive the bounds (2.19)-(2.22). Employing the result

$$
\left\| \varrho^{(n+1)} - \varsigma \right\| \le P_4(R) \left\| \varrho^{(n)} - \varsigma \right\| < R,
$$

we conclude that $\varrho^{(n+1)} - \varsigma \in U(\varsigma, R)$ and $\lim_{n\to\infty} \varrho^{(n)} = \varsigma$.

Next, we address the uniqueness component. Suppose there exists a point $\wp \in \overline{U}(\varsigma, \Delta)$ with $\wp \neq \varsigma$ such that $\Lambda(\wp) = 0$. Define $\Pi = \int_0^1 \Lambda'(\wp + x(\varsigma - \wp))dx$. From Equation (2.12), we obtain the following:

$$\begin{aligned}\left\|\Lambda'(\varsigma)^{-1}(\Pi - \Lambda'(\varsigma))\right\| \leq &\int_0^1 k_0 \|\wp + x(\varsigma - \wp) - \varsigma\| dx \\ \leq &k_0 \int_0^1 (1-x)\|\varsigma - \wp\| dx \\ \leq &\frac{k_0\Delta}{2} < 1.\end{aligned}$$

By invoking the Banach Lemma for invertible operators, we can guarantee that $\Pi^{-1} \in \mathcal{L}(S, S)$. Next, employing the identity $0 = \Lambda(\varsigma) - \Lambda(\wp) = \Pi(\varsigma - \wp)$, we conclude that $\varsigma = \wp$. □

3. Computational efficiency index

This subsection conducts computational efficiency assessments on the WF_3, CM_5, and XM_5 iterative approaches to solve nonlinear systems. Here WF_3 represents Method (1.2), CM_5 represents Method (1.3), and XM_5 represents Method (1.4). We will use $CE = \rho^{\frac{1}{d+op}}$ [23, 24] as the computational efficiency index for comparison and analysis, where ρ represents the convergence order of the method, d represents the number of new function evaluations required by the method during each iteration, and op represents the number of operations (including multiplication and division) in each iteration process.

The LU decomposition is used to solve linear systems in the processing of iteration. We must add n products for the scalar product and n^2 products for matrix-vector multiplication. In order to compute an inverse linear operator, we need $n(n-1)(2n-1)/6$ products and $n(n-1)/2$ divisions in the LU decomposition and $n(n-1)$ products and n divisions to solve two triangular linear systems. Specifically, each method requires n evaluations of Λ and n^2 evaluations of Λ'. Table 1 provides the computational cost and computational efficiency of the different iteration methods.

Table 1. Efficiency comparison of iterative methods.

Methods	ρ	d	op	CE
WF_3	3	$2n^2+n$	$\frac{2}{3}n^3+2n^2-\frac{2}{3}n$	$3^{\frac{1}{\frac{2}{3}n^3+4n^2+\frac{n}{3}}}$
CM_5	5	$2n^2+2n$	n^3+3n^2-n	$5^{\frac{1}{n^3+5n^2+n}}$
XM_5	5	$2n^2+2n$	$\frac{2}{3}n^3+5n^2-\frac{2n}{3}$	$5^{\frac{1}{\frac{2}{3}n^3+7n^2+\frac{4}{3}n}}$

Table 1 compares iterative methods WF_3 and CM_5 with the method XM_5 investigated in this paper. To visually compare the computational performance of the above methods, Figure 1 is used to display their computational efficiency index CE under a different system dimension n. As illustrated in Figure 1, the two fifth-order methods CM_5 and XM_5 outperform the third-order WF_3 in terms of computational

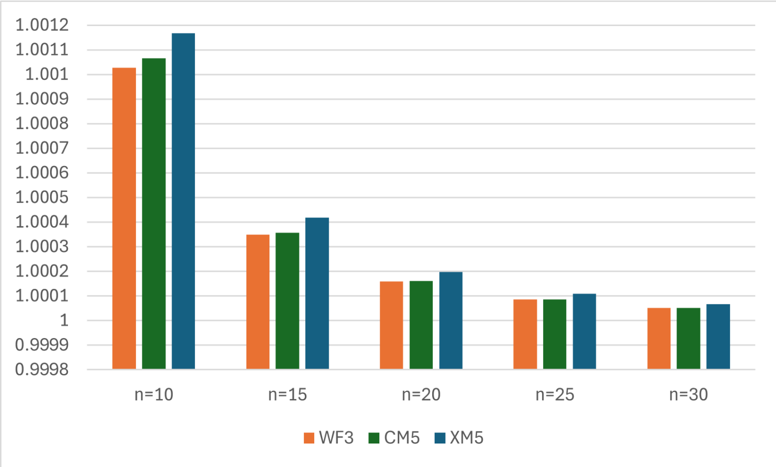

Figure 1. Comparison of computational efficiency index (CE) for WF_3, CM_5, and XM_5 under different dimension n.

efficiency for all tested n. Among all three iterative methods, XM_5 achieves the highest CE value, which indicates that XM_5 exhibits a slightly better computational efficiency under the adopted metrics.

4. Applications

4.1. Radius of convergence ball

To validate the practical utility of the aforementioned theoretical findings, this work conducts an investigation into the subsequent nonlinear problems. Furthermore, we derive the corresponding local convergence radius according to the practical performance of Method (1.4) on particular nonlinear systems.

Example 4.1. *The Van der Waals equation is given by the following:*

$$(L + \frac{cn^2}{V^2})(V - nd) = nRT. \tag{4.1}$$

To compute the volume V of a gas via other related parameters, it is necessary to address the following nonlinear equation:

$$LV^3 - (ndP - nRT)V^2 + cn^2V - cn^2d = 0. \tag{4.2}$$

Once the values of constants c and d are determined, the corresponding values of the amount of substance n, pressure L, and temperature T can be determined accordingly. By choosing specific values, one has that

$$\Lambda_1(\varrho) = 0.986\varrho^3 - 5.181\varrho^2 + 9.067\varrho - 5.289. \tag{4.3}$$

Evidently, the mapping $\Lambda_1(\varrho)$ is continuous. Furthermore, we can derive that $\Lambda_1(1) \approx -0.4170$ and $\Lambda_1(2) \approx 0.0090$. According to the Intermediate Value Theorem, $\Lambda_1(\varrho)$ has at least one zero in the interval $(1, 2)$. We get $\varsigma \approx 1.929846$ and $\Omega = [1, 2]$.

Then, in accordance with Conditions (2.11)-(2.13), we obtain the following:

$$k_0 \approx 18.2477, k \approx 17.2222.$$

Based on the definitions of the aforementioned functions P_i $(i = 1, 2, 3)$, one has that

$$R_1 \approx 0.037232, R_2 \approx 0.014954, R = R_3 \approx 0.009088.$$

Example 4.2. *We introduce the mapping Λ over $\Omega = [-1\ ,\ 1]$ as*

$$\Lambda(\varrho) = \sin(\varrho).$$

We obtain $\varsigma = 0$, where the parameters k_0 and k are both equal to 1. Additionally, the relevant parameters for Method (1.4) are given as follows:

$$R_1 = 0.666667, R_2 = 0.266923, R_3 = 0.161686 = R.$$

Example 4.3. *Consider $X = Y = \mathbb{R}^3$, $\Omega = U[0, 1]$. Define function $\Lambda(\varrho)$ on Ω for $\varrho = (d_1, d_2, d_3)^T$ by*

$$\Lambda(\varrho) = \left(e^{d_1} - 1, \frac{e-1}{2}{d_2}^2 + d_2, d_3\right)^T.$$

Using the definition, the Fréchet-derivative is defined by

$$\Lambda'(\varrho) = \begin{pmatrix} e^{d_1} & 0 & 0 \\ 0 & (e-1)d_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We observe that $\varsigma = (0, 0, 0)^T$, $\Lambda'(\varsigma)^{-1} = diag\{1, 1, 1\}$, $k_0 = e - 1$, $k = e^{1/k_0}$. The corresponding parameters for Method (1.4) read as follows:

$$R_1 = 0.382692, R_2 = 0.152918, R_3 = 0.092412 = R.$$

Example 4.4. *Define Λ on $\Omega = [-\frac{1}{2}, \frac{5}{2}]$ by*

$$\Lambda(\varrho) = \begin{cases} \varrho^3 \ln \varrho^2 - \varrho^5 + \varrho^4, & \varrho \neq 0 \\ 0, & \varrho = 0. \end{cases}$$

We obtain $\varsigma = 1$. Additionally, we have $k_0 = k = 96.6628$. Corresponding parameters for Method (1.4) read

$$R_1 = 0.006897, R_2 = 0.002761, R_3 = 0.001673 = R.$$

4.2. Applications to nonlinear systems

The computing environment for the numerical experiments in this section is a computer equipped with an AMD Ryzen 5 4600U CPU and 8 GB RAM running 64-bit Windows 11. The calculations are conducted via Maple 2024 with `Digits:=2048`, and the iteration stopping criterion is $\left\|\varrho^{(n+1)} - \varrho^{(n)}\right\| < 10^{-260}$.

Within this section, we evaluate the efficacy of relevant methods via conducting mathematical experiments. Here, WF_3 represents Method (1.2), CM_5 represents Method (1.3), and XM_5 represents Method (1.4).

In Tables 2 to 4, $\mathcal{J}(x^{(n)})$ denotes the error of the function value obtained in the last iteration, $\|\varrho^{(n)} - \varrho^{(n-1)}\|$ represents the iteration error, and ACOC stands for the approximate convergence order of the iterative method.

Example 4.5. *Nonlinear system:*

$$\sum_{\substack{s=1\\ s\neq t}}^{n} \varrho_s - \exp(-\varrho_t) = 0, \quad t = 1, \ldots, n.$$

When $n = 15$, we select the starting guess $\varrho^{(0)} = (0.4, 0, 4, 0, 4, \ldots, 0, 4)^T$ to solve the root $\varsigma = (0.06681, 0.06681, 0.06681, \ldots, 0.06681)^T$. Computational outcomes are presented in Table 2.

Example 4.6. *Next, we address the subsequent large-scale nonlinear systems:*

$$\begin{cases} \varrho_j \varrho_{j+1} - 1 = 0, & 1 \leq j \leq n-1 \\ \varrho_n \varrho_1 - 1 = 0. \end{cases}$$

The starting vector is fixed as $\varrho^{(0)} = (1.5, 1.5, 1.5, \ldots, 1.5)^T$, which corresponds to the target solution $\varsigma = (1, 1, 1, \ldots, 1)^T$. Computational outcomes are presented in Table 3.

Example 4.7. *Originating from the theory of radiative transfer, the Chandrasekhar integral equation is defined as folllows:*

$$\Lambda(Q, e) = 0; \quad Q : [0, 1] \to \mathbb{R}$$

where the operator Λ and scalar parameter e are given by the following:

$$\Lambda(Q, e)(v) = Q(v) - \left(1 - \frac{e}{2} \int_0^1 \frac{vQ(s)}{v+s}\, ds\right)^{-1}.$$

The composite midpoint rule is applied to approximate the integrals as follows:

$$\int_0^1 f(g)\, dg = \frac{1}{m} \sum_{j=1}^{m} f(g_j)$$

where $g_j = \frac{(j-1/2)}{m}$, with $1 \leq j \leq m$. The resulting discrete problem is given by the following:

$$\Lambda_i(v, e) = v_i - \left(1 - \frac{e}{2m} \sum_{j=1}^{m} \frac{g_i v_j}{g_i + g_j}\right)^{-1}, \quad 1 \leq i \leq m.$$

The starting vector is $\varrho^{(0)} = (0.4, 0.4, 0.4, \ldots, 0.4)^T$, which corresponds to the solution $\varsigma = (1.07000, 1.07000, 1.07000, \ldots, 1.07000)^T$. Computational outcomes are presented in Table 4.

Table 2. Numerical results for Example 4.5.

Methods	*iter*	$\|\varrho^{(n)} - \varrho^{(n-1)}\|$	$\mathcal{J}(\varrho^{(n)})$	ACOC	Time(s)
WF_3	6	1.734e−389	4.827e−1168	3.00000	0.937
CM_5	5	1.527e−1083	6.000e−2048	5.00000	0.781
XM_5	5	9.545e−1013	2.200e−2048	5.00000	0.687

Table 3. Numerical results for Example 4.6.

Methods	*iter*	$\|\varrho^{(n)} - \varrho^{(n-1)}\|$	$\mathcal{J}(\varrho^{(n)})$	ACOC	Time(s)
WF_3	7	5.648e−510	9.009e−1529	3.00000	0.093
CM_5	5	1.424e−397	1.462e−1985	5.00000	0.078
XM_5	5	2.562e−344	8.286e−1719	5.00000	0.078

Table 4. Numerical results for Example 4.7.

Methods	*iter*	$\|\varrho^{(n)} - \varrho^{(n-1)}\|$	$\mathcal{J}(\varrho^{(n)})$	ACOC	Time(s)
WF_3	7	1.405e−718	2.000e−2047	3.00000	3.609
CM_5	5	2.485e−479	2.000e−2047	5.00000	2.609
XM_5	5	5.498e−437	1.000e−2047	5.00003	2.343

From Table 2 to Table 4, it can be observed that the method studied in this paper has a shorter time, higher accuracy, and good convergence compared to other methods.

5. Conclusions

We conducted an investigation on the local convergence analysis for Method (1.4) to solve the locally unique solution of nonlinear equations in Banach spaces. The Lipschitz continuity condition imposed upon the first derivative was adopted to confirm the applicability of this iterative method. This study addresses the limitations of previous research that relied on higher-order derivatives and often resulted in failures. Additionally, we were able to compute the precise radii of the convergence balls. Concurrently, the uniqueness of the solution has been thoroughly confirmed. Moreover, this paper conducted a comparative analysis of the computational efficiency between the studied Method (1.4) and other iterative methods mentioned herein. The results demonstrated that Method (1.4) achieved a superior computational efficiency. Finally, we applied Method (1.4) to solve specific physical and nonlinear system problems, thereby obtaining the solution ς of the equation and the radius R of the convergence sphere. The feasibility of our suggested method was further validated through numerical experiments.

Author contributions

Le Zhang and Xiaofeng Wang designed the research. Xiaofeng Wang provided the methodology; Le Zhang wrote the paper. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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