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Research article

Neimark-Sacker bifurcation and dynamical behavior in a discrete predator-prey system with exponential terms

Bingyan Li[1,2] **and Qianhong Zhang**[1,2,*]

[1] School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China

[2] School of Big Data Statistics, Guizhou University of Finance and Economics, Guiyang 550025, China

* **Correspondence:** Email: zqianhong68@163.com.cn; Tel: 085188510734.

Abstract: The present research explores the complex dynamics within a two-dimensional discrete-time predator-prey framework incorporating exponential nonlinearities. The interactions between species are formulated through the corresponding difference equations

$$y_{n+1} = \frac{az_n}{1+py_n}e^{-y_n}, z_{n+1} = \frac{bz_n}{1+qz_n}e^{-\alpha y_n},$$

where the discrete time step is $n \geq 0$. All associated parameters a, b, p, q, α represent strictly positive real numbers operating under nonnegative initial states. Essential mathematical features of this formulated model undergo rigorous examination. These analytical steps include verifying the boundedness of all trajectories alongside determining the local asymptotic stability (LAS) criteria for the positive fixed point and estimating its convergence speed. A major objective of this work involves deriving the exact parameter thresholds required for the emergence of an Neimark-Sacker (NS) bifurcation. Applying normal form theory for discrete-time systems allows us to ascertain both the bifurcation direction and the topological stability of the newly generated invariant closed curve. Extensive numerical experiments subsequently corroborate all derived theoretical results.

Keywords: predator-prey system; Neimark-Sacker bifurcation; local asymptotic stability; rate of convergence; bifurcation theory
Mathematics Subject Classification: 39A10

1. Introduction

The study of nonlinear difference equations has increasingly captivated researchers across mathematics and biology. When modeling species with distinct reproductive seasons, such as specific insects and vegetation, discrete frameworks provide a more precise representation than continuous-time differential equations, including periodic solutions, bifurcations, and chaos [1–3]. Driven by these fascinating dynamical properties, the intrinsic mathematical structures and stability of various complex difference equations have been extensively explored in recent literature. For instance, recent works have deeply analyzed the trajectory rules of high-order rational difference equations [4], as well as the global asymptotic behaviors of exponential-type difference systems under uncertain environments [5].

Among numerous biomathematical models, predator-prey and host-parasitoid systems have been hotspots of research. Stability analysis and bifurcation theory are key tools for understanding the evolutionary laws of these ecosystems. In particular, the NS bifurcation, which describes the transition of a system from a stable fixed point to an invariant curve or quasi-periodic solutions, is of great significance for understanding the periodic oscillation of populations.

Many scholars have conducted in-depth studies on the NS bifurcation of such models. For instance, Din et al. detailed the stability and bifurcation phenomena in host-parasitoid models [6, 7] and plant-herbivore models [8, 9], and explored chaos control strategies [10]. For predator-prey systems, literature [11–14] established the conditions for the occurrence of NS bifurcation under different scenarios, such as considering prey refuge effects or Leslie-type functional response functions. These studies demonstrated that when parameters vary and cross critical values, the system exhibits an attracting invariant circle. Recently, a diversity of refined mathematical schemes and ecological factors have been introduced to further explore discrete population interactions. For instance, Gümüş [15] investigated the dynamics of a discrete-time predator-prey system using a nonstandard finite difference scheme, while Aldosary and Ahmed [16] examined the stability and bifurcation of a discrete Leslie predator-prey system via the piecewise constant argument method. Concurrently, behavioral mechanisms such as fear effects and spatial refuges have gained prominent focus. Liu et al. [17] scrutinized the bifurcation and optimal harvesting of a discrete predator-prey model with fear and prey refuge effects, and Almaslokh et al. [18] expanded this to a discrete fractional-order Leslie-Gower framework incorporating predator fear and the Allee effect. Furthermore, advanced bifurcation analyses in multi-dimensional or complex functional structures have been developed. Liu et al. [19] investigated the analysis and anti-control of bifurcations in two-dimensional discrete systems with cubic terms, whereas Abdelaziz et al. [20] explored the rich dynamics and bifurcation control within a discrete multi-team predator-prey configuration. These contemporary studies collectively demonstrate that non-linear difference equations can exhibit highly sensitive and rich quasi-periodic behaviors under parametric variations.

In addition to biological models, the intrinsic mathematical structures of difference equations naturally command rigorous investigation. Rational difference equations and those incorporating exponential terms are especially notable in this regard. Elsadany et al. [21] and Din et al. [10] investigated the global stability and bifurcation behavior of difference equations with exponential terms. Meanwhile, rational difference equations have been widely studied due to their rich dynamic properties. For example, Shareef et al. [22] and Khyat et al. [23] explored the NS bifurcation and the approximation of invariant curves for third-order rational difference equations and perturbed Beverton-

Holt equations, respectively.

Traditionally, discrete population models tend to rely on either rational functions like the Beverton-Holt equation [24] to represent resource limits or exponential functions such as the Ricker model [25, 26] to capture density-dependent mortality.

Recent advancements in purely exponential models have yielded profound insights into complex ecological phenomena. For instance, Rocha et al. [27] recently proposed a generalized gamma-Ricker discrete predator-prey model, comprehensively detailing how the incorporation of a double Allee effect in the prey population triggers rich dynamical transitions, including both transcritical and NS bifurcations. While models with exponential terms are indeed classically associated with such gamma-Ricker frameworks, our study diverges from purely exponential regulation or Allee effect assumptions.

Recently, Ouyang et al. [28] demonstrated that merging these two mechanisms into a hybrid Ricker and Beverton-Holt difference equation offers a more realistic representation of population dynamics, particularly for commercial fisheries. Inspired by their structural framework, we extend this concept to a two-dimensional predator-prey environment. In this paper, we construct a discrete-time model that directly couples rational density dependence with exponential nonlinearities to describe the interactions between two species, governed by the following system of difference equations:

$$\begin{cases} y_{n+1} = \frac{az_n}{1+py_n}e^{-y_n}, \\ z_{n+1} = \frac{bz_n}{1+qz_n}e^{-\alpha y_n}, \end{cases}$$

with parameters $a, b, p, q, \alpha > 0$ and initial values bounded in the nonnegative real domain.

Biologically, the formulated system integrates multiple density-dependent mechanisms to capture the complex interactions between the prey population z_n and the predator population y_n under the assumption of non-overlapping generations. It is crucial to emphasize that the parameters across the two equations possess fundamentally distinct biological dimensions. Within the prey dynamics, the term bz_n represents the intrinsic demographic growth. This growth is subsequently regulated by a rational intra-specific competition factor $\frac{1}{1+qz_n}$ that reflects the environmental carrying capacity, while the exponential term $e^{-\alpha y_n}$ functions as a survival probability denoting the fraction of prey that successfully evades predation. Conversely, the abundance of the subsequent predator population y_{n+1} is driven by the numerical response to the available prey az_n. Although mathematically one could hypothesize parameter states such as $\alpha = 1$ and $a = b$, doing so confounds distinct biological processes (intrinsic prey reproduction versus predator numerical response). Furthermore, the discrete growth of the predator is not unbounded; it is strictly constrained by two distinct mechanisms. First, the rational fraction $\frac{1}{1+py_n}$ accounts for interference competition such as territorial disputes. Second, the exponential term e^{-y_n} captures density-dependent mortality caused by severe overcrowding.

By coupling these rational (Beverton-Holt type) and exponential (Ricker type) mechanisms, the mathematical model avoids contradicting classical theories; instead, it provides a rigorous hybrid framework for analyzing discrete population dynamics. Compared to existing literature on discrete ecodynamics, the original contributions and novelties of this work are twofold. First, from a modeling perspective, we extend the hybrid Ricker and Beverton-Holt formulation, which was previously confined to single-species commercial fisheries, into a comprehensive two-dimensional predator-prey framework. This model uniquely and simultaneously characterizes rational interference competition among predators alongside exponential prey survival probabilities, capturing multi-dimensional density-dependent regularities that single-mechanism frameworks fail to represent. Second, from an analytical standpoint, we provide an exhaustive mathematical treatment that bridges algebraic conditions with biological trajectories. This includes proving the mathematical boundedness of all solutions, calculating the precise asymptotic convergence speed to the positive fixed point, and establishing exact parametric thresholds required for the emergence of an NS bifurcation through center manifold and normal form reductions. Using this formulated system as our mathematical foundation, the main objective of this study is to rigorously analyze its local dynamics and bifurcation behavior. We apply bifurcation analysis and normal form theory to evaluate the stability of the interior positive fixed point. Beyond standard stability analysis, we explicitly establish the parametric conditions that trigger an NS bifurcation. We then determine the direction of this bifurcation and verify the topological stability of the emerging invariant closed curve, corroborating all theoretical derivations with numerical simulations.

The remainder of this article is organized sequentially. The initial section surveys prior research concerning discrete-time frameworks and exponential difference equations, paying particular attention to bifurcation phenomena and stability analysis. Section 2 establishes the lemmas regarding the local dynamical behavior around the fixed points. Section 3 proves the boundedness of the system's solutions, while Section 4 analyzes the rate of convergence toward the unique positive fixed point. In Section 5, we explore the precise conditions for the NS bifurcation. The penultimate section offers extensive numerical experiments that visually corroborate all analytical findings. Finally, Section 7 concludes the paper.

2. Local stability of the fixed point

Now, we determine the fixed points of Eq (2.1) and evaluate their LAS.

Consider the discrete dynamical system

$$\begin{cases} y_{n+1} = \dfrac{az_n}{1+py_n}e^{-y_n}, \\ z_{n+1} = \dfrac{bz_n}{1+qz_n}e^{-\alpha y_n}, \end{cases} \tag{2.1}$$

where $a, b, p, q, \alpha > 0$ and the initial conditions are nonnegative.

A fixed point (y, z) of Eq (2.1) satisfies

$$\begin{cases} y = \dfrac{az}{1+py}e^{-y}, \\ z = \dfrac{bz}{1+qz}e^{-\alpha y}. \end{cases} \tag{2.2}$$

It is straightforward to verify that Eq (2.1) always admits the trivial fixed point $E_0(0, 0)$.

If $z > 0$, the second equation of (2.2) yields

$$1 + qz = be^{-\alpha y},$$

which implies

$$z = \frac{be^{-\alpha y} - 1}{q}.$$

Substituting this expression into the first equation, one arrives at the following relation:

$$y(1+py)=\frac{a}{q}(be^{-\alpha y}-1)e^{-y}.$$

Theorem 2.1. The discrete dynamical system possesses a unique positive fixed point E_1 with coordinates y^* and z^* if and only if the intrinsic growth parameter satisfies $b>1$.

Proof. We define two continuous auxiliary functions $H(y)$ and $K(y)$ to analyze the roots of the fixed point condition. Let the first function be

$$H(y)=y(1+py)e^{y}, \tag{2.3}$$

and let the second function be

$$K(y)=\frac{a}{q}(be^{-\alpha y}-1). \tag{2.4}$$

From (2.3), we have

$$H'(y)=(1+2py+y+py^2)e^{y}. \tag{2.5}$$

Since all system parameters and the population variable y are strictly positive, we conclude that $H'(y)$ is strictly greater than zero. This indicates that $H(y)$ is monotonically increasing on the nonnegative real line with an initial value $H(0)=0$.

Conversely, from (2.4), we have

$$K'(y)=-\frac{a\alpha b}{q}e^{-\alpha y}. \tag{2.6}$$

This derivative is strictly negative, implying that $K(y)$ is a monotonically decreasing function. For a unique positive intersection to exist between a monotonically increasing function starting at the origin and a monotonically decreasing function, we strictly require the initial value of the decreasing function to be greater than zero. Evaluating the initial boundary provides $K(0)=\frac{a}{q}(b-1)$. Enforcing the requirement that this initial value must be strictly positive yields the precise mathematical threshold $b>1$. Under this specific parameter condition, there exists a unique positive root y^* satisfying $H(y^*)=K(y^*)$. Substituting this unique root into the fixed point relation guarantees that the corresponding population abundance z^* is also strictly positive.

This confirms the existence and uniqueness of the interior fixed point and completes the proof.

Evaluating the Jacobian matrix at $E_0(0,0)$, we obtain

$$J(0,0)=\begin{pmatrix}0 & a\\ 0 & b\end{pmatrix}.$$

The corresponding characteristic equation is

$$\lambda^2-b\lambda=0,$$

which has eigenvalues

$$\lambda_1=0,\qquad \lambda_2=b.$$

Therefore, the following result holds.

Lemma 2.1 The trivial fixed point $E_0(0,0)$ of Eq (2.1) satisfies the following:

(1) E_0 is LAS if $0 < b < 1$;
(2) E_0 is a saddle point if $b > 1$.

The Jacobian matrix at the positive fixed point $E_1(y^*, z^*)$ is given by

$$J(E_1) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$A_{11} = -\frac{az^*}{1+py^*}e^{-y^*} - \frac{apz^*}{(1+py^*)^2}e^{-y^*}, \qquad A_{12} = \frac{a}{1+py^*}e^{-y^*},$$

$$A_{21} = -\alpha\frac{bz^*}{1+qz^*}e^{-\alpha y^*}, \qquad A_{22} = \frac{b}{(1+qz^*)^2}e^{-\alpha y^*}.$$

To simplify this matrix for subsequent stability and bifurcation analyses, we utilize the fixed point conditions satisfied at E_1: $y^* = \dfrac{az^*}{1+py^*}e^{-y^*}$ and $z^* = \dfrac{bz^*}{1+qz^*}e^{-\alpha y^*}$. Substituting these identities into the partial derivatives, the matrix elements can be elegantly reduced to

$$A_{11} = -y^* - \frac{py^*}{1+py^*} = -y^*\left(1 + \frac{p}{1+py^*}\right), \quad A_{12} = \frac{y^*}{z^*},$$

$$A_{21} = -\alpha z^*, \qquad A_{22} = \frac{1}{1+qz^*}.$$

Thus, the strictly simplified Jacobian matrix evaluated at the positive fixed point, which will be consistently used throughout the bifurcation analysis, is

$$J(E_1) = \begin{pmatrix} -y^*\left(1 + \dfrac{p}{1+py^*}\right) & \dfrac{y^*}{z^*} \\ -\alpha z^* & \dfrac{1}{1+qz^*} \end{pmatrix}. \tag{2.7}$$

The characteristic equation of $J(E_1)$ is

$$\lambda^2 - \mathrm{tr}(J)\lambda + \det(J) = 0, \tag{2.8}$$

where

$$\mathrm{tr}(J) = A_{11} + A_{22}, \qquad \det(J) = A_{11}A_{22} - A_{12}A_{21}.$$

According to the standard Jury stability criterion for two-dimensional discrete systems, the positive fixed point E_1 is LAS if the conditions $1 - Det(J) > 0$ and $1 \pm Tr(J) + Det(J) > 0$ are simultaneously satisfied. If $|Tr(J)| < |1 + Det(J)|$ and $Det(J) > 1$, the fixed point becomes a repeller. Alternatively, if $|Tr(J)| > |1 + Det(J)|$, it behaves as a saddle point.

3. Boundedness of solutions

Next, we investigate the boundedness of the solutions to Eq (2.1).

Lemma 3.1 If $y_0 \geq 0$ and $z_0 \geq 0$, then

$$y_n \geq 0, \qquad z_n \geq 0, \qquad \forall n \geq 0.$$

Proof. From Eq (2.1), it is clear that

$$\frac{az_n}{1+py_n}e^{-y_n} \geq 0, \qquad \frac{bz_n}{1+qz_n}e^{-\alpha y_n} \geq 0$$

for all $y_n, z_n \geq 0$. Hence, by induction, the solution remains nonnegative for all $n \geq 0$. We first establish the boundedness of the sequence $\{z_n\}$.

From the second equation of Eq (2.1), we have

$$z_{n+1} = \frac{bz_n}{1+qz_n}e^{-\alpha y_n} \leq \frac{bz_n}{1+qz_n}, \tag{3.1}$$

since $e^{-\alpha y_n} \leq 1$ for all $y_n \geq 0$.

Consider the auxiliary scalar difference equation

$$w_{n+1} = \frac{bw_n}{1+qw_n}, \tag{3.2}$$

with $w_0 = z_0 \geq 0$.

To determine the precise upper bound of the sequence, we define a continuous mapping function

$$f(x) = \frac{bx}{1+qx}$$

for $x \geq 0$. Evaluating the first derivative yields

$$f'(x) = \frac{b}{(1+qx)^2}.$$

Since all system parameters are strictly positive, the derivative $f'(x)$ is strictly greater than zero, indicating that $f(x)$ is a monotonically increasing function on the nonnegative real domain. The nontrivial fixed point of this mapping is determined by solving $f(x) = x$, which gives

$$w^* = \frac{b-1}{q}.$$

For the parameter range $b > 1$, this fixed point is strictly positive. Due to the strict monotonicity of the function $f(x)$, the asymptotic behavior of the sequence w_n depends solely on the initial condition w_0. If $0 \leq w_0 \leq w^*$, the sequence is monotonically increasing and rigorously bounded above by the fixed point w^*. Conversely, if $w_0 > w^*$, the sequence is monotonically decreasing and bounded above by its initial value w_0. Combining these two exhaustive scenarios guarantees that all solutions of the auxiliary equation are bounded and strictly satisfy

$$0 \leq w_n \leq \max\left\{w_0, \frac{b-1}{q}\right\} \quad \text{for } b > 1,$$

and

$$\lim_{n\to\infty} w_n = 0 \quad \text{for } 0 < b \leq 1.$$

By comparison principle and inequality (3.1), it follows that

$$0 \leq z_n \leq w_n, \qquad \forall n \geq 0,$$

and hence $\{z_n\}$ is bounded.

Using the boundedness of $\{z_n\}$, we now show that $\{y_n\}$ is also bounded.

From the first equation of Eq (2.1), we have

$$y_{n+1} = \frac{az_n}{1+py_n}e^{-y_n} \leq az_n e^{-y_n} \leq az_n, \tag{3.3}$$

since $e^{-y_n} \leq 1$ and $1 + py_n \geq 1$.

Since $\{z_n\}$ is bounded, there exists a constant $M > 0$ such that

$$z_n \leq M, \qquad \forall n \geq 0.$$

Therefore,

$$y_{n+1} \leq aM, \qquad \forall n \geq 0,$$

which implies that $\{y_n\}$ is bounded.

So, every solution $\{(y_n, z_n)\}_{n\geq 0}$ of Eq (2.1) with nonnegative initial conditions is bounded.

4. Rate of convergence analysis

The current section focuses on quantifying the speed at which system trajectories approach the positive fixed point $E_1(y^*, z^*)$. We proceed under the premise that the model possesses a unique interior fixed point characterized by LAS.

Let λ_1 and λ_2 denote the eigenvalues of the Jacobian matrix $J(E_1)$ evaluated at the positive fixed point. Since E_1 is LAS, it follows that $|\lambda_1| < 1$ and $|\lambda_2| < 1$. We define the spectral radius of $J(E_1)$ as

$$\rho = \max\{|\lambda_1|, |\lambda_2|\}.$$

Clearly, $\rho < 1$. To establish the rate at which solutions converge to the fixed point E_1, we introduce the perturbation variables $\varphi_n = y_n - y^*$ and $\psi_n = z_n - z^*$. By shifting the fixed point to the origin, the system can be expressed in its linearized form as

$$\begin{pmatrix}\varphi_{n+1}\\ \psi_{n+1}\end{pmatrix} = J(E_1)\begin{pmatrix}\varphi_n\\ \psi_n\end{pmatrix} + \begin{pmatrix}f(\varphi_n, \psi_n)\\ g(\varphi_n, \psi_n)\end{pmatrix}, \tag{4.1}$$

where the functions f and g represent the higher-order nonlinear remainder terms satisfying

$$\lim_{\|(\varphi_n,\psi_n)\|\to 0} \frac{\|(f(\varphi_n,\psi_n), g(\varphi_n,\psi_n))\|}{\|(\varphi_n,\psi_n)\|} = 0. \tag{4.2}$$

Based on this linearization, the following result holds.

Theorem 4.1. Let $\{(y_n, z_n)\}_{n=0}^{\infty}$ be a solution of the system converging to the positive fixed point $E_1(y^*, z^*)$. Then the error sequence satisfies

$$\lim_{n\to\infty} \frac{\|(y_{n+1}, z_{n+1}) - (y^*, z^*)\|}{\|(y_n, z_n) - (y^*, z^*)\|} = \rho.$$

Moreover, the exact rate of convergence is given by

$$\limsup_{n\to\infty} \|(y_n, z_n) - (y^*, z^*)\|^{1/n} = \rho.$$

Proof. From the Taylor expansion (4.1), the nonlinear remainder terms are of higher order and hence negligible in the asymptotic regime as the solution approaches the fixed point. Therefore, the asymptotic behavior of (φ_n, ψ_n) is dominated by the linear part governed by $J(E_1)$.

According to the classical results on the asymptotic behavior of linear difference equations, the rate of convergence of a perturbed linear system is determined by the maximum modulus of the eigenvalues of its coefficient matrix. Consequently, as $n \to \infty$, the norm of the perturbation vector $\|(\varphi_n, \psi_n)\|$ decays at a rate proportional to ρ^n. Thus, we have $\|(\varphi_n, \psi_n)\| \sim C\rho^n$ for some constant $C > 0$. Substituting $\varphi_n = y_n - y^*$ and $\psi_n = z_n - z^*$ back into the asymptotic estimates immediately yields the limits stated in the theorem.

This completes the proof.

5. NS bifurcation of Eq (2.1)

This section analyzes the NS bifurcation of Eq (2.1) at a fixed point with complex conjugate eigenvalues. We base our study on the standard NS bifurcation theorem, also known as the discrete Hopf theorem; see [29–31].

5.1. Existence condition of NS bifurcation

This subsection outlines the mathematical prerequisites for the emergence of an NS bifurcation within our established Eq (2.1). To construct this analytical framework, we initially introduce the foundational NS bifurcation theorem, which is frequently documented in the literature as the Poincaré-Andronov-Hopf bifurcation theorem applied to discrete maps.

Theorem 5.1. [21] Consider a C^4 mapping $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ with $x \mapsto F(\mu, x)$, where μ is a real parameter. Suppose the following criteria are met:

(i) The origin remains a fixed point in the vicinity of a critical value μ_0 such that $F(\mu, 0) = 0$;
(ii) The Jacobian matrix $DF(\mu, 0)$ possesses a pair of complex conjugate eigenvalues, denoted as $\lambda(\mu)$ and $\bar{\lambda}(\mu)$, which intersect the unit circle at $\mu = \mu_0$ (i.e., $|\lambda(\mu_0)| = 1$);
(iii) The eigenvalues cross the unit circle with a non-zero velocity, satisfying the transversality condition $\frac{d}{d\mu}|\lambda(\mu)|_{\mu=\mu_0} = d(\mu_0) \neq 0$;

(iv) The strong non-resonance requirement holds, meaning $\lambda^k(\mu_0) \neq 1$ for $k \in \{1, 2, 3, 4\}$.

Under these assumptions, there exists a smooth, parameter-dependent coordinate transformation that reduces the system to its normal form, resulting in the emergence of an invariant closed curve bifurcating from the origin.

In order to apply Theorem 5.1, we analyze the local stability of the unique positive fixed point $E_1(y^*, z^*)$ of the system. The Jacobian matrix $J(E_1)$ is given by

$$J(E_1) = \begin{pmatrix} -y^*\left(1 + \frac{p}{1+py^*}\right) & \frac{y^*}{z^*} \\ -\alpha z^* & \frac{1}{1+qz^*} \end{pmatrix}. \tag{5.1}$$

The characteristic polynomial of $J(E_1)$ is computed as follows:

$$P(\lambda) = \lambda^2 - \mathrm{Tr}(J)\lambda + \mathrm{Det}(J) = 0, \tag{5.2}$$

where

$$\mathrm{Tr}(J) = -y^*\left(1 + \frac{p}{1+py^*}\right) + \frac{1}{1+qz^*}, \tag{5.3}$$

$$\mathrm{Det}(J) = \frac{-y^*\left(1 + \frac{p}{1+py^*}\right)}{1+qz^*} + \alpha y^*. \tag{5.4}$$

We choose a as the bifurcation parameter. The eigenvalues of $J(E_1)$ are $\lambda_{1,2}(a)$, where

$$\lambda_{1,2}(a) = \frac{\mathrm{Tr}(J) \pm i\sqrt{4\mathrm{Det}(J) - \mathrm{Tr}(J)^2}}{2}. \tag{5.5}$$

For the emergence of NS bifurcation around the positive fixed point E_1, the roots of $P(\lambda)$ must be complex conjugates with a unit modulus. This requires the conditions $\Delta = \mathrm{Tr}(J)^2 - 4\mathrm{Det}(J) < 0$ and $|\lambda| = 1$. Since $|\lambda| = \sqrt{\mathrm{Det}(J)}$, the condition $|\lambda| = 1$ is equivalent to

$$\mathrm{Det}(J) = \alpha y^* - \frac{y^*\left(1 + \frac{p}{1+py^*}\right)}{1+qz^*} = 1. \tag{5.6}$$

Let a_{NS} be the critical value of the parameter a satisfying Eq (5.6). At $a = a_{NS}$, we have

$$\lambda(a_{NS}) = e^{i\theta}, \quad \text{where } \theta = \arccos\left(\frac{\mathrm{Tr}(J)}{2}\right). \tag{5.7}$$

We verify the transversality condition. Taking the derivative of the modulus of the eigenvalues with respect to the parameter a,

$$\frac{d}{da}|\lambda(a)|\bigg|_{a=a_{NS}} = \frac{d}{da}\sqrt{\mathrm{Det}(J)}\bigg|_{a=a_{NS}} = \frac{1}{2\sqrt{\mathrm{Det}(J)}}\frac{d\mathrm{Det}(J)}{da}\bigg|_{a=a_{NS}}. \tag{5.8}$$

Since $\mathrm{Det}(J) = 1$ at the critical point, we obtain

$$\frac{d}{da}|\lambda(a)|\bigg|_{a=a_{NS}} = \frac{1}{2}\frac{d\mathrm{Det}(J)}{da}. \tag{5.9}$$

Applying the chain rule and noting that y^* and z^* are functions of a,

$$\frac{d\mathrm{Det}(J)}{da} = \frac{\partial \mathrm{Det}}{\partial y^*}\frac{dy^*}{da} + \frac{\partial \mathrm{Det}}{\partial z^*}\frac{dz^*}{da}. \tag{5.10}$$

Generally, for generic values of parameters b, p, q, α, the term $\frac{d\mathrm{Det}(J)}{da} \neq 0$. Thus, the eigenvalues cross the unit circle with non-zero speed, satisfying the transversality condition

$$\frac{d}{da}|\lambda(a)|\bigg|_{a=a_{NS}} \neq 0. \tag{5.11}$$

Finally, we require the non-resonance conditions $\lambda^k(a_{NS}) \neq 1$ for $k = 1, 2, 3, 4$. This is equivalent to requiring that $\mathrm{Tr}(J)$ avoids specific values at the bifurcation point

- $k = 1 \implies \lambda \neq 1 \implies \mathrm{Tr}(J) \neq 2.$
- $k = 2 \implies \lambda^2 \neq 1 \implies \mathrm{Tr}(J) \neq -2.$
- $k = 3 \implies \lambda^3 \neq 1 \implies \mathrm{Tr}(J) \neq -1.$
- $k = 4 \implies \lambda^4 \neq 1 \implies \mathrm{Tr}(J) \neq 0.$

Based on the preceding derivation, we present the subsequent lemma to summarize the necessary conditions for bifurcation emergence.

Lemma 5.1. If conditions (i)–(iv) of Theorem 5.1 are satisfied and $\mathrm{Tr}(J) \notin \{0, -1, -2, 2\}$ at $a = a_{NS}$, then the system undergoes an NS bifurcation at its positive fixed point E_1 when a passes through the critical value a_{NS}.

Remark 5.1. While the primary focus of this section is the NS bifurcation, it is mathematically pertinent to note that Eq (2.1) may also undergo a flip (period-doubling) bifurcation. A flip bifurcation occurs if one of the eigenvalues of the Jacobian matrix evaluated at E_1 equals -1, while the other satisfies $|\lambda| \neq 1$. Analytically, this threshold corresponds to the parametric condition $1 + Tr(J) + Det(J) = 0$ alongside $Tr(J) \neq 0$ and $Tr(J) \neq -2$. An exhaustive analysis of the flip bifurcation boundaries and the subsequent period-doubling cascades is slated for future investigation.

5.2. The direction of NS bifurcation

We consider the translations $u_n = y_n - y^*$ and $v_n = z_n - z^*$ for shifting the fixed point $E_1(y^*, z^*)$ to the origin. Through calculating, we obtain the following map:

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = J(E_1)\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} F(u_n, v_n) \\ G(u_n, v_n) \end{pmatrix}, \tag{5.12}$$

where $J(E_1)$ is the Jacobian matrix at the fixed point. Expanding the system up to the third order at the origin using a Taylor series, we obtain

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ b_{11} & b_{12} \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} F(u_n, v_n) \\ G(u_n, v_n) \end{pmatrix}, \tag{5.13}$$

where the nonlinear parts F and G are given by

$$
\begin{aligned}
F(u_n, v_n) = {} & a_{20}u_n^2 + a_{11}u_nv_n + a_{02}v_n^2 + a_{30}u_n^3 \\
& + a_{21}u_n^2v_n + a_{12}u_nv_n^2 + a_{03}v_n^3 + O((|u_n| + |v_n|)^4), \qquad (5.14) \\
G(u_n, v_n) = {} & b_{20}u_n^2 + b_{11}u_nv_n + b_{02}v_n^2 + b_{30}u_n^3 \\
& + b_{21}u_n^2v_n + b_{12}u_nv_n^2 + b_{03}v_n^3 + O((|u_n| + |v_n|)^4). \qquad (5.15)
\end{aligned}
$$

Based on the derivatives calculated at the fixed point E_1, the coefficients of the Taylor expansion are detailed as follows (note that terms involving derivatives with respect to z for the first equation are zero due to linearity):

$$
\begin{aligned}
a_{20} &= \frac{1}{2}\frac{\partial^2 y_{n+1}}{\partial y_n^2}\bigg|_{E_1} = \frac{y^*}{2}\left[\left(1 + \frac{p}{1+py^*}\right)^2 + \frac{p^2}{(1+py^*)^2}\right], \\
a_{11} &= \frac{\partial^2 y_{n+1}}{\partial y_n \partial z_n}\bigg|_{E_1} = -\frac{y^*}{z^*}\left(1 + \frac{p}{1+py^*}\right), \\
a_{02} &= \frac{1}{2}\frac{\partial^2 y_{n+1}}{\partial z_n^2}\bigg|_{E_1} = 0, \\
a_{30} &= \frac{1}{6}\frac{\partial^3 y_{n+1}}{\partial y_n^3}\bigg|_{E_1} = -\frac{y^*}{6}\left[\left(1 + \frac{p}{1+py^*}\right)^3 + \frac{3p^2(1+py^*) + 3p^3}{(1+py^*)^3}\right], \\
a_{21} &= \frac{1}{2}\frac{\partial^3 y_{n+1}}{\partial y_n^2 \partial z_n}\bigg|_{E_1} = \frac{1}{2z^*}\left[y^*\left(1 + \frac{p}{1+py^*}\right)^2 + \frac{y^*p^2}{(1+py^*)^2}\right], \\
a_{12} &= 0, \quad a_{03} = 0.
\end{aligned}
$$

$$
\begin{aligned}
b_{20} &= \frac{1}{2}\frac{\partial^2 z_{n+1}}{\partial y_n^2}\bigg|_{E_1} = \frac{\alpha^2 z^*}{2}, \\
b_{11} &= \frac{\partial^2 z_{n+1}}{\partial y_n \partial z_n}\bigg|_{E_1} = -\frac{\alpha}{1+qz^*}, \\
b_{02} &= \frac{1}{2}\frac{\partial^2 z_{n+1}}{\partial z_n^2}\bigg|_{E_1} = -\frac{q}{(1+qz^*)^2}, \\
b_{30} &= \frac{1}{6}\frac{\partial^3 z_{n+1}}{\partial y_n^3}\bigg|_{E_1} = -\frac{\alpha^3 z^*}{6}, \\
b_{21} &= \frac{1}{2}\frac{\partial^3 z_{n+1}}{\partial y_n^2 \partial z_n}\bigg|_{E_1} = \frac{\alpha^2}{2(1+qz^*)}, \\
b_{03} &= \frac{1}{6}\frac{\partial^3 z_{n+1}}{\partial z_n^3}\bigg|_{E_1} = \frac{q^2}{(1+qz^*)^3}.
\end{aligned}
$$

Assume that the Jacobian matrix evaluated at the bifurcation threshold possesses a pair of complex conjugate eigenvalues denoted by $\lambda_{1,2} = \mu \pm i\omega$, satisfying the unit modulus condition $|\lambda_{1,2}| = 1$. To determine the properties of the NS bifurcation around the fixed point E_1, we apply normal form reduction techniques for two-dimensional discrete maps. Consequently, we introduce the following non-singular transformation matrix:

$$
T = \begin{pmatrix} a_{12} & 0 \\ \mu - a_{11} & -\omega \end{pmatrix}. \qquad (5.16)
$$

Using the translation $\binom{u_n}{v_n} = T\binom{X_n}{Y_n}$, the system is transformed into

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} \mathcal{P}(X_n, Y_n) \\ \mathcal{Q}(X_n, Y_n) \end{pmatrix}, \tag{5.17}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are nonlinear functions determined by the coefficients a_{ij} and b_{ij} derived above.

According to the normal form theories related to bifurcation analysis, we require the following quantity (the first Lyapunov coefficient) to determine the stability:

$$\Psi = -\mathrm{Re}\left(\frac{(1 - 2\lambda_1)\bar{\lambda}_2^2}{1 - \lambda_1} \xi_{11}\xi_{20} \right) - \frac{1}{2}|\xi_{11}|^2 - |\xi_{02}|^2 + \mathrm{Re}(\bar{\lambda}_2 \xi_{21}), \tag{5.18}$$

where the parameters $\xi_{11}, \xi_{20}, \xi_{02}, \xi_{21}$ (often denoted as θ_{ij} in literature) are derived from the transformed nonlinear functions,

$$\xi_{20} = \frac{1}{8}[(\mathcal{P}_{XX} - \mathcal{P}_{YY} + 2\mathcal{Q}_{XY}) + i(\mathcal{Q}_{XX} - \mathcal{Q}_{YY} - 2\mathcal{P}_{XY})], \tag{5.19}$$

$$\xi_{11} = \frac{1}{4}[(\mathcal{P}_{XX} + \mathcal{P}_{YY}) + i(\mathcal{Q}_{XX} + \mathcal{Q}_{YY})], \tag{5.20}$$

$$\xi_{02} = \frac{1}{8}[(\mathcal{P}_{XX} - \mathcal{P}_{YY} - 2\mathcal{Q}_{XY}) + i(\mathcal{Q}_{XX} - \mathcal{Q}_{YY} + 2\mathcal{P}_{XY})], \tag{5.21}$$

$$\begin{aligned} \xi_{21} = \frac{1}{16}[&(\mathcal{P}_{XXX} + \mathcal{P}_{XYY} + \mathcal{Q}_{XXY} + \mathcal{Q}_{YYY}) \\ &+ i(\mathcal{Q}_{XXX} + \mathcal{Q}_{XYY} - \mathcal{P}_{XXY} - \mathcal{P}_{YYY})]. \end{aligned} \tag{5.22}$$

The partial derivatives of $\mathcal{P}$ and $\mathcal{Q}$ are evaluated at the origin $(0, 0)$.

Theorem 5.1. The following results hold for the direction and stability of the NS bifurcation,

- If $\Psi < 0$, the NS bifurcation is supercritical, and the unique positive fixed point E_1 bifurcates into a stable invariant closed curve (attracting limit cycle).
- If $\Psi > 0$, the NS bifurcation is subcritical, and the bifurcated invariant closed curve is unstable (repelling).

The results in Theorem 5.1 classify the NS bifurcation into supercritical ($\Psi < 0$) and subcritical ($\Psi > 0$) types. However, the borderline case $\Psi = 0$ corresponds to a degenerate scenario, mathematically identified as a generalized NS (or discrete Bautin/Chenciner) bifurcation [32, 33]. In this critical state, the standard non-degeneracy condition fails because the first Lyapunov coefficient vanishes. Consequently, a rigorous determination of the dynamical behavior can no longer rely on Ψ and instead strictly requires higher-order normal form computations, such as deriving the second Lyapunov coefficient [32, 33]. From a dynamical systems perspective, this degeneracy implies a codimension-two bifurcation. In the context of discrete population models, such mechanisms can lead to highly complex phenomena, including the coexistence of multiple invariant closed curves (e.g., one stable and one unstable) [34, 35]. Furthermore, because the deterministic system is structurally highly sensitive at this borderline state, even infinitesimal perturbations—such as environmental noise—could dramatically alter the population dynamics. Investigating these stochastic bifurcation mechanisms driven by environmental fluctuations remains a compelling direction for future studies.

6. Numerical examples

In this section, we provide numerical simulations to verify the theoretical analysis regarding the local stability and the NS bifurcation of the discrete predator-prey system. The phase portraits and bifurcation diagrams are generated to demonstrate the dynamical behavior of the populations. We consider the system with the following fixed parameters:

$$b = 13.445, \quad p = 1, \quad q = 1, \quad \alpha = 1.5.$$

We choose a as the bifurcation parameter. According to the bifurcation conditions established in Section 5, specifically the critical modulus condition $Det(J) = 1$, the critical value for the NS bifurcation is calculated as

$$a_{NS} = e \approx 2.7183.$$

At this critical value, the unique positive fixed point is precisely $E_1 \approx (1.0, 2.0)$. The Jacobian matrix $J(E_1)$ at this point yields the characteristic equation

$$\lambda^2 - Tr(J)\lambda + Det(J) = 0.$$

Substituting the parameter values, we obtain $Det(J) = 1$ and $Tr(J) \approx -1.1667$. The eigenvalues are a pair of complex conjugates

$$\lambda_{1,2} \approx -0.5833 \pm 0.8124i.$$

The modulus of the eigenvalues is $|\lambda_{1,2}| = \sqrt{(-0.5833)^2 + (0.8124)^2} \approx 1$. Furthermore, the rotation number $\theta = \arccos(Tr(J)/2)$ satisfies the non-resonance conditions, ensuring the occurrence of the bifurcation. We observe the following dynamical behaviors as the parameter a varies.

Case 1: Stable fixed point ($a < a_{NS}$)

For $a = 2.5$, which is less than the critical value a_{NS}, the positive fixed point E_1 is LAS.

As shown in Figure 1, to clearly illustrate the asymptotic stability of the positive fixed point, we set the initial condition closer to the equilibrium at $(y_0, z_0) = (0.8, 1.7)$ and extend the simulation to 300 iterations. The trajectory prominently spirals inward and strictly converges to the stable fixed point. This visually confirms that the predator and prey populations eventually settle at a constant level over time.

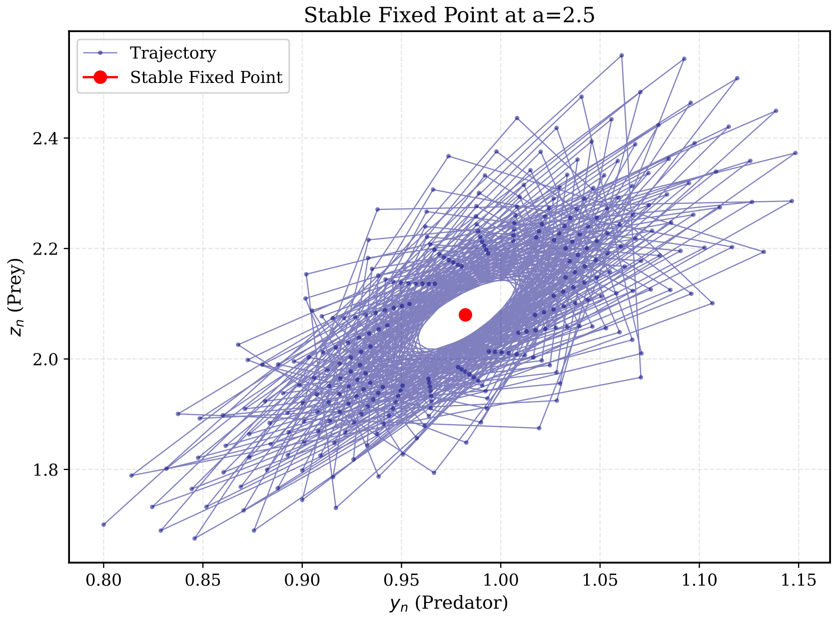
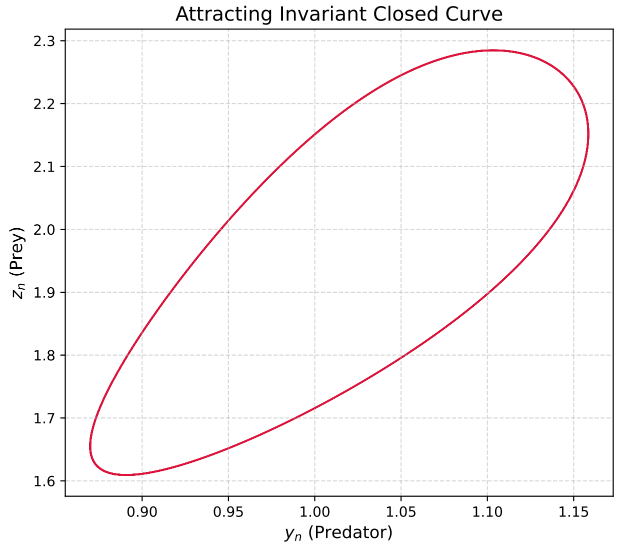


Figure 1. Phase portrait of Eq (2.1) with $a = 2.5$. The system is initialized at $(y_0, z_0) = (0.8, 1.7)$ and run for 300 iterations, explicitly demonstrating that the trajectory converges to the LAS positive fixed point E_1.

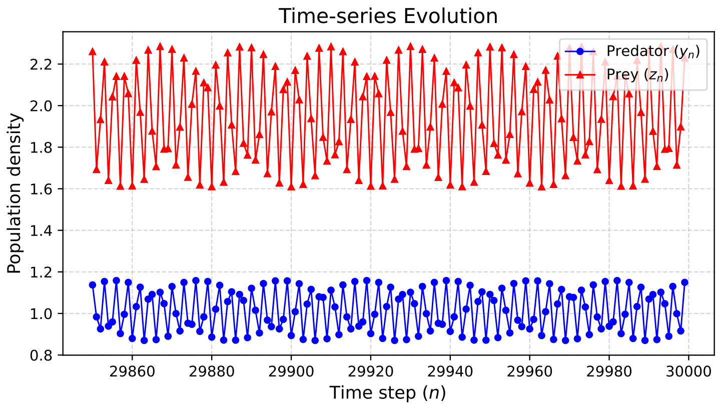
Case 2: NS bifurcation ($a > a_{NS}$)

When the bifurcation parameter is incrementally increased beyond the critical threshold $a_{NS} \approx 2.7183$ to $a = 2.9$, the complex conjugate eigenvalues of the Jacobian matrix cross the unit circle, causing the positive fixed point $E_1(y^*, z^*)$ to lose its LAS. Driven by the NS bifurcation, the system transitions from a stable coexistence equilibrium to a dynamic regime characterized by quasi-periodic oscillations. To strictly illustrate this discrete dynamical behavior, we establish the initial condition at $(y_0, z_0) = (1.0, 2.0)$ in the vicinity of the unstable fixed point. In accordance with the rigorous visualization standards for discrete maps, we perform $30,000$ iterations and deliberately discard the initial $10,000$ transient steps to isolate the pure asymptotic attractor. As depicted in Figure 2(a), the resulting trajectory, plotted exclusively as discrete scatters without continuous connecting lines, maps out a dense and topologically unbroken attracting invariant closed curve in the phase plane. Furthermore, the corresponding discrete time-series evolution in Figure 2(b) confirms that both predator and prey populations exhibit sustained, bounded periodic fluctuations. Biologically, this signifies that a higher numerical response rate of the predator triggers persistent cyclical dynamics between the interacting species, precluding a static equilibrium.

To further validate the analytical conditions derived for the NS bifurcation, we present the bifurcation diagram and the corresponding MLE spectrum with respect to parameter a in Figure 3. Figure 3(a) illustrates that as the parameter a increases, the system undergoes a transition from a stable interior fixed point E_1 to an invariant closed curve via an NS bifurcation. As a continues to increase, the system exhibits more complex dynamical behaviors, including periodic and chaotic oscillations. The quantitative dynamic transition is further corroborated by the MLE spectrum shown in Figure 3(b). The analysis reveals the following regimes: When $a < a_{NS}$, the MLE < 0, confirming the LAS of the interior fixed point E_1. At the critical value $a_{NS} \approx 2.7183$, the MLE approaches zero, providing strong numerical evidence for the onset of the NS bifurcation, where the fixed point loses its stability and generates an invariant closed curve. In the regime where $a > a_{NS}$, the presence of positive MLE values indicates the emergence of chaotic dynamics, demonstrating the high sensitivity of the system to parametric variations. These numerical results are in excellent agreement with our theoretical derivations, confirming that the hybrid Ricker and Beverton-Holt framework effectively captures the intricate nonlinear dynamics characteristic of population interactions in discrete-time ecosystems.

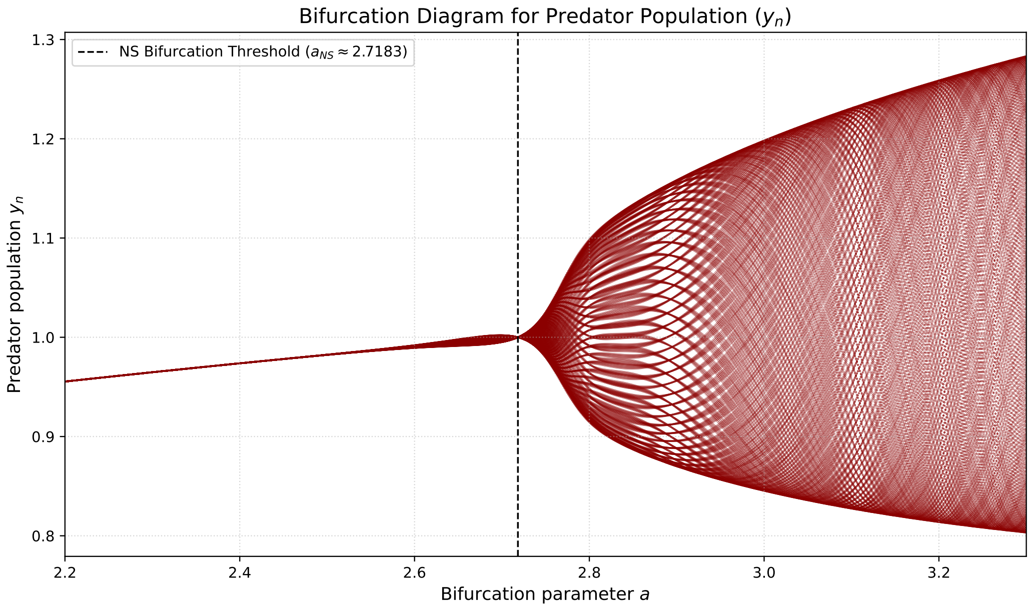


(a)

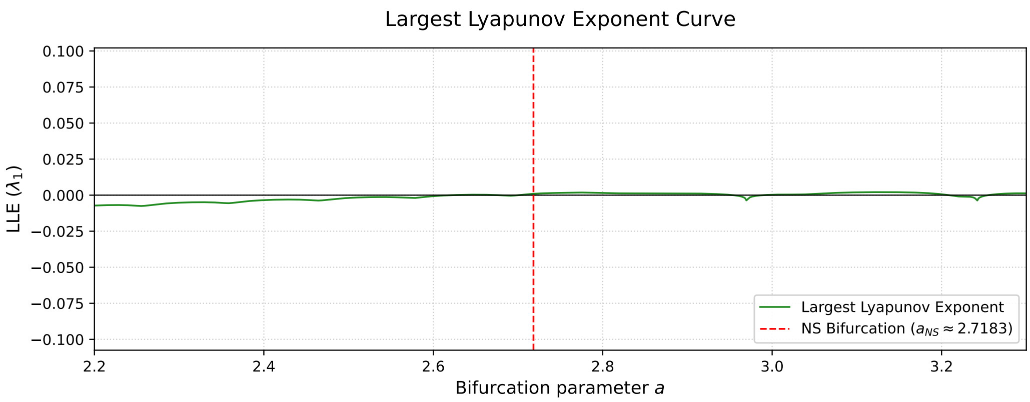


(b)

Figure 2. The emergence of the NS bifurcation at $a = 2.9$ with initial conditions $(y_0, z_0) = (1.0, 2.0)$. (a) The phase portrait demonstrates a dense, attracting invariant closed curve formed by discrete trajectory points after discarding 10,000 transient iterations. (b) The corresponding time-series evolution for the subsequent 150 iterations, illustrating the sustained quasi-periodic oscillations of the predator (y_n) and prey (z_n) populations.



(a)



(b)

Figure 3. Bifurcation diagram (a) and the corresponding maximal Lyapunov exponent (MLE) (b) for system (2.1) with respect to the parameter a. The system parameters are fixed at $b = 13.445, p = 1, q = 1, \alpha = 1.5$. The vertical dashed line indicates the NS bifurcation threshold at $a_{NS} \approx 2.7183$.

7. Conclusions

In this paper, we systematically analyzed the complex dynamic behaviors of a class of two-dimensional discrete dynamical systems containing exponential nonlinear terms. Our study first established the boundedness of the solutions and the existence of a positive fixed point, which are fundamental prerequisites for the biological relevance of population models. Subsequently, utilizing eigenvalue analysis and stability theory, we determined the parametric ranges for the LAS of the fixed point and investigated the rate of convergence of the solutions to the positive fixed point.

Building on this foundation, we focused on the NS bifurcation of the system. By strictly applying the center manifold theorem and normal form theory, we derived the critical conditions for the occurrence of the bifurcation and computed the key coefficients that determine the direction of the bifurcation and the stability of the closed invariant curve.

The theoretical predictions are well supported by our numerical simulations. We observed that changing the bifurcation parameter causes the system to shift from a stable fixed point to periodic or quasi-periodic oscillations, creating an attracting invariant closed curve. This demonstrates how discrete models capture a broader range of complex behaviors for populations with non-overlapping generations compared to continuous-time models. Subsequent studies could explore chaos control techniques to stabilize erratic population sizes or analyze how random environmental noise affects bifurcation patterns. Ultimately, these insights can offer practical theoretical tools for ecosystem regulation.

Although this paper has made substantial progress in revealing the discrete dynamical characteristics of the exponential predator-prey model, numerous issues of both theoretical and practical ecological significance warrant further investigation. For instance, building upon the current findings, future studies could comprehensively analyze period-doubling bifurcations and even higher-codimension bifurcation phenomena that may emerge under large-scale parameter fluctuations. Furthermore, given that real-world ecosystems are inevitably subjected to environmental perturbations, extending the current deterministic difference equations into stochastic discrete models to investigate noise-driven stochastic bifurcation mechanisms will further bridge the gap between theoretical modeling and realistic ecological systems.

Author contributions

Bingyan Li: Writing-original draft, Revising and Editing; Qianhong Zhang: Writing-review, Funding acquisition and Guiding the revision of the paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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