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Research article

Generalized bi-skew Lie-type higher derivations on $*$-algebras

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Abstract: Let $\mathcal{A}$ be a unital $*$-algebra over the complex field $\mathbb{C}$, and let $G = \{G_m\}_{m\in\mathbb{N}}$ be a nonlinear generalized bi-skew Lie n-higher derivation on $\mathcal{A}$. In this paper, we prove that under certain conditions, G is an additive generalized higher $*$-derivation on $\mathcal{A}$. As applications, we characterize nonlinear generalized bi-skew Lie n-higher derivations on some classical unital $*$-algebras, such as von Neumann algebras.

Keywords: bi-skew Lie product; $*$-algebra; generalized bi-skew Lie n-higher derivation; generalized higher $*$-derivation
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1. Introduction

Let $\mathcal{A}$ be a unital $*$-algebra over the complex field $\mathbb{C}$ with involution $*$ satisfying $(xy)^* = y^*x^*$, $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \bar{\lambda}x^*(\lambda \in \mathbb{C})$, and $(x^*)^* = x$ for all $x, y \in \mathcal{A}$. For any $A, B \in \mathcal{A}$, define $[A, B]_\star = AB - BA^*$. We refer to $[A, B]_\star$ as the skew Lie product on $\mathcal{A}$. Moreover, for any $A, B \in \mathcal{A}$, define $[A, B]_\diamond = AB^* - BA^*$, which we call the bi-skew Lie product of A and B. Furthermore, for any $n \geq 2$ $(n \in \mathbb{Z})$ and $a_1, a_2, \ldots, a_n \in \mathcal{A}$, we consider a sequence of polynomials under the involution $*$ defined by $q_1(a_1) = a_1$, $q_2(a_1, a_2) = [a_1, a_2]_\diamond$, and $q_n(a_1, \ldots, a_n) = [q_{n-1}(a_1, \ldots, a_{n-1}), a_n]_\diamond$, which is referred to as the bi-skew Lie n-product of $a_1, a_2, \ldots, a_n$.

An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is called an additive $*$-derivation if it satisfies $d(a_1a_2) = d(a_1)a_2 + a_1d(a_2)$ and $d(a_1^*) = d(a_1)^*$ for all $a_1, a_2 \in \mathcal{A}$. An additive mapping $g : \mathcal{A} \to \mathcal{A}$ is called a generalized additive $*$-derivation if there exists an additive $*$-derivation $d : \mathcal{A} \to \mathcal{A}$ such that $g(a_1a_2) = g(a_1)a_2 + a_1d(a_2)$ and $g(a_1^*) = g(a_1)^*$ for all $a_1, a_2 \in \mathcal{A}$.

In recent years, the structure and characterization of $*$-derivations and related mappings on $*$-algebras have been extensively studied. In 2022, Ashraf et al. introduced the notion of bi-skew Lie-type derivations and proved that every bi-skew Lie-type derivation on a von Neumann algebra is an additive $*$-derivation [1]. In 2023, Madni et al. investigated nonlinear skew Lie-type derivations on

$*$-algebras and proved that they are additive. They also showed that such derivations are additive $*$-derivations under certain conditions [2]. In 2023, Zhang et al. generalized the above results to broader settings by proving that every nonlinear bi-skew Lie-type derivation on a $*$-algebra is an additive $*$-derivation [3]. Rehman et al. studied skew Lie n-derivations on $*$-algebras in 2025 and proved that they are additive $*$-derivations [4].

Moreover, many scholars have extended existing results to more general settings by introducing the concept of generalized derivations. In 2024, Ashraf et al. proved that every nonlinear generalized bi-skew Jordan n-derivation G on a $*$-algebra $\mathcal{A}$ can be expressed as $G(T) = \delta(T) + ZT$ for all $T \in \mathcal{A}$, where δ is an additive $*$-derivation and $Z \in \mathcal{Z}(\mathcal{A})$. Therefore, every nonlinear generalized bi-skew Jordan n-derivation on a $*$-algebra is an additive generalized $*$-derivation [5]. In the same year, they used a similar method to study nonlinear generalized bi-skew Lie n-derivations on $*$-algebras and showed that such mappings are additive generalized $*$-derivations [6]. For generalized skew mappings on other algebras, relevant studies can be found in references [7–9].

In order to extend the research results to more general cases, many authors have investigated higher derivations on various algebras. In 2024, Liang et al. investigated nonlinear bi-skew Jordan n-higher derivations ϕ on $*$-algebras and proved that ϕ is an additive higher $*$-derivation [10]. In the same year, Li et al. generalized the result from [2] and studied nonlinear skew Lie n-higher derivations on operator algebras [11]. Meanwhile, Yuan et al. characterized nonlinear generalized Lie n-higher derivations on certain triangular algebras [12]. In 2026, Ren et al. studied nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivations on $*$-algebras [13]. It is worth noting that mathematical induction is widely employed in [10–12] when dealing with higher derivations. Motivated by the above research, in this paper we will also use mathematical induction to investigate whether every nonlinear generalized bi-skew Lie n-higher derivation on $*$-algebras is an additive generalized higher $*$-derivation.

2. Preliminaries

Before presenting the verification, we first introduce some definitions. Unless otherwise specified, throughout this paper, $\mathcal{A}$ denotes a unital $*$-algebra over the complex field $\mathbb{C}$. Throughout this paper, n denotes a fixed positive integer with $n \geq 2$.

Definition 2.1. *A mapping $\phi : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is called a nonlinear bi-skew Lie n-derivation if*

$$\phi(q_n(a_1, a_2, \ldots, a_n)) = \sum_{k=1}^{n} q_n(a_1, a_2, \ldots, \phi(a_k), \ldots, a_n)$$

for all $a_1, a_2, \ldots, a_n \in \mathcal{A}$.

Definition 2.2. *Let $\mathbb{N}$ denote the set of all non-negative integers, and suppose that $\Phi = \{\phi_m\}_{m \in \mathbb{N}}$ is a family of mappings on $\mathcal{A}$ satisfying $\phi_0 = id_{\mathcal{A}}$. If*

$$\phi_m(a_1 a_2) = \sum_{i+j=m} \phi_i(a_1)\phi_j(a_2), \quad \phi_m(a_1 + a_2) = \phi_m(a_1) + \phi_m(a_2) \quad \text{and} \quad \phi_m(a_1{}^*) = \phi_m(a_1)^*$$

for all $a_1, a_2 \in \mathcal{A}$ and all $m \in \mathbb{N}$, then Φ is referred to as an additive higher $*$-derivation.

In Definition 2.2, if $m = 1$, then ϕ_1 is an additive $*$-derivation. By Definitions 2.1 and 2.2, we may introduce the notion of a nonlinear bi-skew Lie n-higher derivation.

Definition 2.3. *Let $\mathbb{N}$ be the set of all non-negative integers, and suppose that $\Phi = \{\phi_m\}_{m\in\mathbb{N}}$ is a family of mappings on $\mathcal{A}$ satisfying $\phi_0 = id_{\mathcal{A}}$. Then, Φ is called a nonlinear bi-skew Lie n-higher derivation if*

$$\phi_m(q_n(a_1,\dots,a_k,\dots,a_n)) = \sum_{i_1+\dots+i_n=m} q_n(\phi_{i_1}(a_1),\dots,\phi_{i_k}(a_k),\dots,\phi_{i_n}(a_n))$$

for all $a_1, a_2, \dots, a_n \in \mathcal{A}$ and all $m \in \mathbb{N}$.

When $m = 1$ in Definition 2.3, ϕ_1 reduces to a nonlinear bi-skew Lie n-derivation on $\mathcal{A}$. In light of the definitions presented above, we now define an additive generalized higher $*$-derivation and a nonlinear generalized bi-skew Lie n-higher derivation on $\mathcal{A}$.

Definition 2.4. *Let $\mathbb{N}$ be the set of all non-negative integers, and suppose that $G = \{G_m\}_{m\in\mathbb{N}}$ is a family of mappings on $\mathcal{A}$ with $G_0 = id_{\mathcal{A}}$. If there exists an additive higher $*$-derivation $\Phi = \{\phi_m\}_{m\in\mathbb{N}}$ ($\phi_0 = id_{\mathcal{A}}$) such that*

$$G_m(a_1a_2) = \sum_{i+j=m} G_i(a_1)\phi_j(a_2), \quad G_m(a_1 + a_2) = G_m(a_1) + G_m(a_2) \quad \text{and} \quad G_m(a_1^*) = G_m(a_1)^*$$

for all $a_1, a_2 \in \mathcal{A}$ and all $m \in \mathbb{N}$, then G is called an additive generalized higher $*$-derivation.

Similarly, setting $m = 1$ in Definition 2.4 implies that G_1 is an additive generalized $*$-derivation.

Definition 2.5. *Let $\mathbb{N}$ be the set of all non-negative integers, and suppose that $G = \{G_m\}_{m\in\mathbb{N}}$ is a family of mappings on $\mathcal{A}$ with $G_0 = id_{\mathcal{A}}$. If there exists a nonlinear bi-skew Lie n-higher derivation $\Phi = \{\phi_m\}_{m\in\mathbb{N}}$ ($\phi_0 = id_{\mathcal{A}}$) such that*

$$G_m(q_n(a_1,a_2,\dots,a_k,\dots,a_n)) = \sum_{i_1+i_2+\dots+i_n=m} q_n(G_{i_1}(a_1),\phi_{i_2}(a_2),\dots,\phi_{i_k}(a_k),\dots\phi_{i_n}(a_n))$$

for all $a_1, a_2, \dots, a_n \in \mathcal{A}$ and all $m \in \mathbb{N}$, then G is called a nonlinear generalized bi-skew Lie n-higher derivation, and Φ is called the associated nonlinear bi-skew Lie n-higher derivation of G.

Accordingly, setting $m = 1$ in Definition 2.5 yields that G_1 is a nonlinear generalized bi-skew Lie n-derivation. In other words, there exists a nonlinear bi-skew Lie n-derivation ϕ_1 such that

$$G_1(q_n(a_1,a_2,\dots,a_k,\dots,a_n)) = q_n(G_1(a_1),a_2,\dots,a_k,\dots,a_n) + \sum_{k=2}^{n} q_n(a_1,a_2,\dots,\phi_1(a_k),\dots a_n)$$

for all $a_1, a_2, \dots, a_n \in \mathcal{A}$.

Let I denote the identity element of $\mathcal{A}$, and let P_1 be a nontrivial projection on $\mathcal{A}$, that is, $P_1 = {P_1}^2 = {P_1}^*$. Define $P_2 = I - P_1$, which is also a nontrivial projection on $\mathcal{A}$. For $i, j = 1, 2$, set $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$, so that $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$. Define $\mathcal{M} = \{A \in \mathcal{A} \mid A^* = A\}$, $\mathcal{N} = \{A \in \mathcal{A} \mid A^* = -A\}$, and for $k = 1, 2$, let $\mathcal{N}_{12} = \{P_1NP_2 + P_2NP_1 \mid N \in \mathcal{N}\}$, $\mathcal{N}_{kk} = \{P_kNP_k \mid N \in \mathcal{N}\}$. Then, every $N \in \mathcal{N}$ admits a decomposition $N = N_{11} + N_{12} + N_{22}$, where $N_{11} \in \mathcal{N}_{11}$, $N_{12} \in \mathcal{N}_{12}$ and $N_{22} \in \mathcal{N}_{22}$. Unless otherwise specified, the $*$-algebra $\mathcal{A}$ in this paper is assumed to satisfy the following condition:

$$X\mathcal{A}P_k = 0 \Rightarrow X = 0 \ (k = 1, 2). \tag{2.1}$$

Several properties of a nonlinear generalized bi-skew Lie n-derivation G_1 on a $*$-algebra are established in [6]. These properties will be useful for further research.

Proposition 2.6. *[6] G_1 satisfies the following properties:*

(i) $G_1(0) = 0$,

(ii) $G_1(N) \in \mathcal{N}$ *for all* $N \in \mathcal{N}$,

(iii) $G_1(A^*) = G_1(A)^*$ *for all* $A \in \mathcal{A}$,

(iv) G_1 *is additive on* $\mathcal{A}$,

(v) $G_1(iA) = iG_1(A)$ *for all* $A \in \mathcal{A}$.

Let $\Phi = \{\phi_m\}_{m\in\mathbb{N}}$ be the associated nonlinear bi-skew Lie n-higher derivation of $G = \{G_m\}_{m\in\mathbb{N}}$. The nonlinear bi-skew Lie n-derivation ϕ_1 has the following properties.

Proposition 2.7. *[3] ϕ_1 satisfies the following properties:*

(i) $\phi_1(0) = \phi_1(I) = \phi_1(iI) = 0$,

(ii) ϕ_1 *is additive on* $\mathcal{A}$,

(iii) $\phi_1(N) \in \mathcal{N}$ *for all* $N \in \mathcal{N}$, *and* $\phi_1(M) \in \mathcal{M}$ *for all* $M \in \mathcal{M}$,

(iv) $\phi_1(iA) = i\phi_1(A)$ *and* $\phi_1(A^*) = \phi_1(A)^*$ *for all* $A \in \mathcal{A}$.

The following result can be deduced directly from [3].

Remark 2.8. *A nonlinear bi-skew Lie n-derivation ϕ_1 satisfies $\phi_1(A) = \xi(A) + \chi(A)$ for all $A \in \mathcal{A}$, where $\xi : \mathcal{A} \to \mathcal{A}$ is a $*$-derivation satisfying $P_1\xi(P_1)P_2 = P_2\xi(P_1)P_1 = 0$ and $\chi : \mathcal{A} \to \mathcal{A}$ is a $*$-inner derivation.*

Proof. As presented in [3], ϕ_1 is an additive $*$-derivation and $\phi_1(P_1)^* = \phi_1(P_1)$. Let $W = P_1\phi_1(P_1)P_2 - P_2\phi_1(P_1)P_1$. Then, $[A^*, W] = [A, W]^*$. Set $\chi(A) = [A, W]$. It is clear that χ is a $*$-inner derivation. Define $\xi(A) = \phi_1(A) - \chi(A)$. Then,

$$\begin{aligned}\xi(P_1) =&\phi_1(P_1) - [P_1, W]\\ =&P_1\phi_1(P_1)P_1 + P_1\phi_1(P_1)P_2 + P_2\phi_1(P_1)P_1 + P_2\phi_1(P_1)P_2\\ &- P_1\phi_1(P_1)P_2 - P_2\phi_1(P_1)P_1\\ =&P_1\phi_1(P_1)P_1 + P_2\phi_1(P_1)P_2.\end{aligned}$$

Obviously, $P_1\xi(P_1)P_2 = P_2\xi(P_1)P_1 = 0$. It follows that ξ is a $*$-derivation satisfying $P_1\xi(P_1)P_2 = P_2\xi(P_1)P_1 = 0$. □

3. Nonlinear generalized bi-skew Lie n-higher derivations

In this section, we first investigate the additivity of the nonlinear generalized bi-skew Lie n-higher derivation $G = \{G_m\}_{m\in\mathbb{N}}$ on the $*$-algebra $\mathcal{A}$ with respect to the subset $\mathcal{N}$ of $\mathcal{A}$. In what follows, we analyze this problem by means of mathematical induction.

According to Properties 2.6 and 2.7, we assume that for all $1 < s < m$ ($m \in \mathbb{N}$), the mappings ϕ_s and G_s satisfy the following induction hypotheses, where ϕ_s and G_s are respectively from the nonlinear bi-skew Lie n-higher derivation $\Phi = \{\phi_m\}_{m\in\mathbb{N}}$ and the nonlinear generalized bi-skew Lie n-higher derivation $G = \{G_m\}_{m\in\mathbb{N}}$,

$$\mathfrak{C}_s^{\phi}:\begin{cases}\phi_s(0)=\phi_s(I)=0,\\ \phi_s \text{ is additive on } \mathcal{N},\\ \phi_s(N)\in\mathcal{N} \quad \text{for all } N\in\mathcal{N},\end{cases}$$

$$\mathfrak{C}_s^{G}:\begin{cases}G_s(0)=0,\\ G_s \text{ is additive on } \mathcal{N},\\ G_s(N)\in\mathcal{N} \quad \text{for all } N\in\mathcal{N}.\end{cases}$$

For convenience, based on Properties 2.6 and 2.7, we set

$$\mathfrak{C}_1^{\phi}:\begin{cases}\phi_1(0)=\phi_1(I)=0,\\ \phi_1 \text{ is additive on } \mathcal{N},\\ \phi_1(N)\in\mathcal{N} \quad \text{for all } N\in\mathcal{N},\end{cases}$$

$$\mathfrak{C}_1^{G}:\begin{cases}G_1(0)=0,\\ G_1 \text{ is additive on } \mathcal{N},\\ G_1(N)\in\mathcal{N} \quad \text{for all } N\in\mathcal{N}.\end{cases}$$

Now, we will prove that ϕ_m and G_m are also additive on $\mathcal{N}$, aided by some lemmas.

Lemma 3.1. $G_m(0) = 0$, $\phi_m(0) = 0$, *and* $G_m(\mathcal{N}) \subseteq \mathcal{N}$, $\phi_m(\mathcal{N}) \subseteq \mathcal{N}$.

Proof. In view of the induction hypotheses $\mathfrak{C}_s^{\phi}$, $\mathfrak{C}_s^{G}$ $(1 \le s < m)$, we have $G_s(0) = \phi_s(0) = 0$ for all $1 \le s < m$. Hence,

$$\begin{aligned}G_m(0) =&G_m(q_n(0,0,\ldots,0))\\ =&q_n(G_m(0),0,\ldots,0)+q_n(0,\phi_m(0),\ldots,0)+\ldots+q_n(0,0,\ldots,\phi_m(0))\\ &+\sum_{\substack{i_1+i_2+\ldots+i_n=m\\ i_1,i_2,\ldots,i_n<m}} q_n(G_{i_1}(0),\phi_{i_2}(0),\ldots,\phi_{i_n}(0))\\ =&0.\end{aligned}$$

Similarly, we obtain $\phi_m(0) = 0$.

Furthermore, using the same hypotheses, we have

$$\begin{aligned}G_m(N) =&G_m(q_n(N,\frac{I}{2},\ldots,\frac{I}{2}))\\ =&q_n(G_m(N),\frac{I}{2},\ldots,\frac{I}{2})+q_n(N,\phi_m(\frac{I}{2}),\ldots,\frac{I}{2})+\ldots+q_n(N,\frac{I}{2},\ldots,\phi_m(\frac{I}{2}))\\ &+\sum_{\substack{i_1+i_2+\ldots+i_n=m\\ i_1,i_2,\ldots,i_n<m}} q_n(G_{i_1}(N),\phi_{i_2}(\frac{I}{2}),\ldots,\phi_{i_n}(\frac{I}{2}))\\ =&\frac{I}{2}(G_m(N)-G_m(N)^*)+(n-1)(N\phi_m(\frac{I}{2})^*+\phi_m(\frac{I}{2})N)\end{aligned}$$

for all $N \in \mathcal{N}$. Thus,

$$G_m(N) = \frac{I}{2}(G_m(N) - G_m(N)^*) + (n-1)(N\phi_m(\frac{I}{2})^* + \phi_m(\frac{I}{2})N).$$

Taking the involution on both sides, we obtain

$$G_m(N)^* = \frac{I}{2}(G_m(N)^* - G_m(N)) - (n-1)(N\phi_m(\frac{I}{2})^* + \phi_m(\frac{I}{2})N).$$

Comparing the last two equations, we obtain $G_m(N) + G_m(N)^* = 0$, which implies that $G_m(N)^* = -G_m(N)$. Consequently, we have $G_m(N) \in \mathcal{N}$, which means $G_m(\mathcal{N}) \subseteq \mathcal{N}$. In the same manner, we can prove $\phi_m(\mathcal{N}) \subseteq \mathcal{N}$. □

Lemma 3.2. *With the notations as above, we have*

$$G_m(B_{ii} + C_{12}) = G_m(B_{ii}) + G_m(C_{12})$$

and

$$\phi_m(B_{ii} + C_{12}) = \phi_m(B_{ii}) + \phi_m(C_{12})$$

for all $B_{ii} \in \mathcal{N}_{ii}$, $C_{12} \in \mathcal{N}_{12}$, $i \in \{1, 2\}$.

Proof. Let $T = G_m(B_{11} + C_{12}) - G_m(B_{11}) - G_m(C_{12})$. Based on Lemma 3.1, we have $T^* = -T$. In view of the induction hypotheses $\mathfrak{C}_s^{\phi}$ and $\mathfrak{C}_s^{G}$ $(1 \leq s < m)$ and the fact that $[B_{11}, P_2]_\diamond = 0$, it follows that

$$\begin{aligned}
&q_n(G_m(B_{11} + C_{12}), P_2, I, \ldots, I) + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11} + C_{12}), \phi_{i_2}(P_2), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
=&G_m(q_n(B_{11} + C_{12}, P_2, I, \ldots, I)) \\
=&G_m(q_n(B_{11}, P_2, I, \ldots, I)) + G_m(q_n(C_{12}, P_2, I, \ldots, I)) \\
=&q_n(G_m(B_{11}), P_2, I, \ldots, I) + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11}), \phi_{i_2}(P_2), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
&+ q_n(G_m(C_{12}), P_2, I, \ldots, I) + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(C_{12}), \phi_{i_2}(P_2), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
=&q_n(G_m(B_{11}) + G_m(C_{12}), P_2, I, \ldots, I) + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11} + C_{12}), \phi_{i_2}(P_2), \phi_{i_3}(I), \ldots, \phi_{i_n}(I))
\end{aligned}$$

for all $B_{11} \in \mathcal{N}_{11}$ and $C_{12} \in \mathcal{N}_{12}$. Hence, $q_n(T, P_2, I, \ldots, I) = 0$. It follows that $T_{22} = T_{12} = 0$, where $T_{12} \in \mathcal{N}_{12}$, $T_{22} \in \mathcal{N}_{22}$. Since $q_n(C_{12}, P_1 - P_2, I, \ldots, I) = 0$, we have

$$\begin{aligned}
&q_n(G_m(B_{11} + C_{12}), P_1 - P_2, I, \ldots, I) \\
&\quad + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11} + C_{12}), \phi_{i_2}(P_1 - P_2), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
=&G_m(q_n(B_{11} + C_{12}, P_1 - P_2, I, \ldots, I))
\end{aligned}$$

$$
\begin{aligned}
&= G_m(q_n(B_{11}, P_1 - P_2, I, \ldots, I)) + G_m(q_n(C_{12}, P_1 - P_2, I, \ldots, I)) \\
&= q_n(G_m(B_{11}), P_1 - P_2, I, \ldots, I) + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11}), \phi_{i_2}(P_1 - P_2), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
&\quad + q_n(G_m(C_{12}), P_1 - P_2, I, \ldots, I) + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(C_{12}), \phi_{i_2}(P_1 - P_2), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
&= q_n(G_m(B_{11}) + G_m(C_{12}), P_1 - P_2, I, \ldots, I) \\
&\quad + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11} + C_{12}), \phi_{i_2}(P_1 - P_2), \phi_{i_3}(I), \ldots, \phi_{i_n}(I))
\end{aligned}
$$

for all $B_{11} \in \mathcal{N}_{11}$ and $C_{12} \in \mathcal{N}_{12}$. This yields $T_{11} = 0$, and hence $T = 0$, that is, $G_m(B_{11} + C_{12}) = G_m(B_{11}) + G_m(C_{12})$. Similarly, we have $G_m(B_{22} + C_{12}) = G_m(B_{22}) + G_m(C_{12})$. By the same argument, we have $\phi_m(B_{ii} + C_{12}) = \phi_m(B_{ii}) + \phi_m(C_{12})$ for $i \in \{1, 2\}$. □

Lemma 3.3. *With notations as above, we have*

$$G_m(B_{11} + C_{12} + D_{22}) = G_m(B_{11}) + G_m(C_{12}) + G_m(D_{22})$$

and

$$\phi_m(B_{11} + C_{12} + D_{22}) = \phi_m(B_{11}) + \phi_m(C_{12}) + \phi_m(D_{22})$$

for all $B_{11} \in \mathcal{N}_{11}$, $C_{12} \in \mathcal{N}_{12}$, and $D_{22} \in \mathcal{N}_{22}$.

Proof. Let $T = G_m(B_{11} + C_{12} + D_{22}) - G_m(B_{11}) - G_m(C_{12}) - G_m(D_{22})$. By Lemma 3.1, we have $T^* = -T$. Next, we will show that $T = 0$. In view of the induction hypotheses $\mathfrak{C}_s^{\phi}$ and $\mathfrak{C}_s^{G}$ for $1 \leq s < m$, together with Lemma 3.2, it follows from $q_n(D_{22}, P_1, I, \ldots, I) = 0$ that

$$
\begin{aligned}
&q_n(G_m(B_{11} + C_{12} + D_{22}), P_1, I, \ldots, I) \\
&\quad + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11}) + G_{i_1}(C_{12}) + G_{i_1}(D_{22}), \phi_{i_2}(P_1), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
&= q_n(G_m(B_{11} + C_{12} + D_{22}), P_1, I, \ldots, I) \\
&\quad + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11} + C_{12} + D_{22}), \phi_{i_2}(P_1), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
&= G_m(q_n(B_{11} + C_{12} + D_{22}, P_1, I, \ldots, I)) \\
&= G_m(q_n(B_{11} + C_{12}, P_1, I, \ldots, I)) + G_m(q_n(D_{22}, P_1, I, \ldots, I)) \\
&= q_n(G_m(B_{11} + C_{12}), P_1, I, \ldots, I) \\
&\quad + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11} + C_{12}), \phi_{i_2}(P_1), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
&\quad + q_n(G_m(D_{22}), P_1, I, \ldots, I) \\
&\quad + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(D_{22}), \phi_{i_2}(P_1), \phi_{i_3}(I), \ldots, \phi_{i_n}(I))
\end{aligned}
$$

$$
\begin{aligned}
=&q_n(G_m(B_{11})+G_m(C_{12})+G_m(D_{22}),P_1,I,\ldots,I)\\
&+\sum_{\substack{i_1+i_2+\ldots+i_n=m\\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11})+G_{i_1}(C_{12})+G_{i_1}(D_{22}),\phi_{i_2}(P_1),\phi_{i_3}(I),\ldots,\phi_{i_n}(I))
\end{aligned}
$$

for all $B_{11}\in\mathcal{N}_{11}$, $C_{12}\in\mathcal{N}_{12}$, $D_{22}\in\mathcal{N}_{22}$. Hence,

$$
q_n(T,P_1,I,\ldots,I)=0,
$$

which means $T_{11}=T_{12}=0$, where $T_{11}\in\mathcal{N}_{11}$, $T_{12}\in\mathcal{N}_{12}$.

Since $q_n(B_{11},P_2,I,\ldots,I)=0$, we conclude

$$
\begin{aligned}
&q_n(G_m(B_{11}+C_{12}+D_{22}),P_2,I,\ldots,I)\\
&\quad+\sum_{\substack{i_1+i_2+\ldots+i_n=m\\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11})+G_{i_1}(C_{12})+G_{i_1}(D_{22}),\phi_{i_2}(P_2),\phi_{i_3}(I),\ldots,\phi_{i_n}(I))\\
=&q_n(G_m(B_{11}+C_{12}+D_{22}),P_2,I,\ldots,I)\\
&\quad+\sum_{\substack{i_1+i_2+\ldots+i_n=m\\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11}+C_{12}+D_{22}),\phi_{i_2}(P_2),\phi_{i_3}(I),\ldots,\phi_{i_n}(I))\\
=&G_m(q_n(B_{11}+C_{12}+D_{22},P_2,I,\ldots,I))\\
=&G_m(q_n(B_{11},P_2,I,\ldots,I))+G_m(q_n(C_{12}+D_{22},P_2,I,\ldots,I))\\
=&q_n(G_m(B_{11}),P_2,I,\ldots,I)\\
&\quad+\sum_{\substack{i_1+i_2+\ldots+i_n=m\\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11}),\phi_{i_2}(P_2),\phi_{i_3}(I),\ldots,\phi_{i_n}(I))\\
&\quad+q_n(G_m(C_{12}+D_{22}),P_2,I,\ldots,I)\\
&\quad+\sum_{\substack{i_1+i_2+\ldots+i_n=m\\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(C_{12}+D_{22}),\phi_{i_2}(P_2),\phi_{i_3}(I),\ldots,\phi_{i_n}(I))\\
=&q_n(G_m(B_{11})+G_m(C_{12})+G_m(D_{22}),P_2,I,\ldots,I)\\
&\quad+\sum_{\substack{i_1+i_2+\ldots+i_n=m\\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(B_{11})+G_{i_1}(C_{12})+G_{i_1}(D_{22}),\phi_{i_2}(P_2),\phi_{i_3}(I),\ldots,\phi_{i_n}(I)).
\end{aligned}
$$

This yields $q_n(T,P_2,I,\ldots,I)=0$, which implies that $T_{22}=0$. Hence, $T=0$. Consequently,

$$
G_m(B_{11}+C_{12}+D_{22})=G_m(B_{11})+G_m(C_{12})+G_m(D_{22}).
$$

Similarly,

$$
\phi_m(B_{11}+C_{12}+D_{22})=\phi_m(B_{11})+\phi_m(C_{12})+\phi_m(D_{22}).
$$

□

Lemma 3.4. *As previously defined, we have*

$$
G_m(B_{12}+C_{12})=G_m(B_{12})+G_m(C_{12})
$$

and

$$\phi_m(B_{12} + C_{12}) = \phi_m(B_{12}) + \phi_m(C_{12})$$

for all $B_{12}, C_{12} \in \mathcal{N}_{12}$.

Proof. Let $B_{12} = U_{12} - U_{12}^*$ and $C_{12} = V_{12} - V_{12}^*$, where $U_{12}, V_{12} \in \mathcal{A}_{12}$. It follows that

$$q_n(iP_1 + iU_{12} + iU_{12}^*, iP_2 + iV_{12} + iV_{12}^*, \frac{I}{2}, \ldots, \frac{I}{2}) = B_{12} + C_{12} + B_{12}C_{12}^* - C_{12}B_{12}^*.$$

According to the definition of involution, we have $B_{12}^* = -B_{12}$ and $C_{12}^* = -C_{12}$. Clearly, $B_{12}+C_{12} \in \mathcal{N}_{12}$ and

$$B_{12}C_{12}^* - C_{12}B_{12}^* = P_1(U_{12}V_{12}^* - V_{12}U_{12}^*)P_1 + P_2(U_{12}^*V_{12} - V_{12}^*U_{12})P_2 \in \mathcal{N}_{11} + \mathcal{N}_{22}.$$

Therefore, by the induction hypotheses $\mathfrak{C}_s^{\phi}$ and $\mathfrak{C}_s^{G}$ for $1 \leq s < m$, and Lemmas 3.1–3.3, we have

$$\begin{aligned}
&G_m(B_{12} + C_{12}) + G_m(B_{12}C_{12}^* - C_{12}B_{12}^*)\\
&=G_m(B_{12} + C_{12} + B_{12}C_{12}^* - C_{12}B_{12}^*)\\
&=G_m(q_n(iP_1 + iU_{12} + iU_{12}^*, iP_2 + iV_{12} + iV_{12}^*, \frac{I}{2}, \ldots, \frac{I}{2}))\\
&= \sum_{i_1+i_2+\ldots+i_n=m} q_n(G_{i_1}(iP_1 + iU_{12} + iU_{12}^*), \phi_{i_2}(iP_2 + iV_{12} + iV_{12}^*), \phi_{i_3}(\frac{I}{2}), \ldots, \phi_{i_n}(\frac{I}{2}))\\
&= \sum_{i_1+i_2+\ldots+i_n=m} q_n(G_{i_1}(iP_1), \phi_{i_2}(iP_2), \phi_{i_3}(\frac{I}{2}), \ldots, \phi_{i_n}(\frac{I}{2}))\\
&\quad + \sum_{i_1+i_2+\ldots+i_n=m} q_n(G_{i_1}(iP_1), \phi_{i_2}(iV_{12} + iV_{12}^*), \phi_{i_3}(\frac{I}{2}), \ \ldots, \ \phi_{i_n}(\frac{I}{2}))\\
&\quad + \sum_{i_1+i_2+\ldots+i_n=m} q_n(G_{i_1}(iU_{12} + iU_{12}^*), \phi_{i_2}(iV_{12} + iV_{12}^*), \phi_{i_3}(\frac{I}{2}), \ldots, \phi_{i_n}(\frac{I}{2}))\\
&\quad + \sum_{i_1+i_2+\ldots+i_n=m} q_n(G_{i_1}(iU_{12} + iU_{12}^*), \phi_{i_2}(iP_2), \phi_{i_3}(\frac{I}{2}), \ldots, \phi_{i_n}(\frac{I}{2}))\\
&=G_m(q_n(iP_1, iP_2, \frac{I}{2}, \ldots, \frac{I}{2})) + G_m(q_n(iP_1, iV_{12} + iV_{12}^*, \ \frac{I}{2}, \ldots, \frac{I}{2}))\\
&\quad + G_m(q_n(iU_{12} + iU_{12}^*, iV_{12} + iV_{12}^*, \frac{I}{2}, \ldots, \frac{I}{2})) + G_m(q_n(iU_{12} + iU_{12}^*, iP_2, \frac{I}{2}, \ldots, \frac{I}{2}))\\
&=G_m(B_{12}) + G_m(C_{12}) + G_m(B_{12}C_{12}^* - C_{12}B_{12}^*)
\end{aligned}$$

for all $B_{12}, C_{12} \in \mathcal{N}_{12}$. Hence, we obtain $G_m(B_{12} + C_{12}) = G_m(B_{12}) + G_m(C_{12})$. Similarly, we have $\phi_m(B_{12} + C_{12}) = \phi_m(B_{12}) + \phi_m(C_{12})$. □

Lemma 3.5. *As previously defined, we have*

$$G_m(C_{ii} + D_{ii}) = G_m(C_{ii}) + G_m(D_{ii})$$

and

$$\phi_m(C_{ii} + D_{ii}) = \phi_m(C_{ii}) + \phi_m(D_{ii})$$

for all $C_{ii}, D_{ii} \in \mathcal{N}_{ii}$, $i \in \{1, 2\}$.

Proof. Set $T = G_m(C_{11} + D_{11}) - G_m(C_{11}) - G_m(D_{11})$. Applying Lemma 3.1, we get $T^* = -T$. By the induction hypotheses $\mathfrak{C}_s^G$ $(1 \le s < m)$, and the fact that $[C_{11} + D_{11}, P_2]_\diamond = [C_{11}, P_2]_\diamond = [D_{11}, P_2]_\diamond = 0$, it follows that

$$\begin{aligned}
q_n(T, P_2, I, \ldots, I) = & q_n(G_m(C_{11} + D_{11}), P_2, I, \ldots, I) - q_n(G_m(C_{11}), P_2, I, \ldots, I) \\
& - q_n(G_m(D_{11}), P_2, I, \ldots, I) \\
=& q_n(G_m(C_{11} + D_{11}), P_2, I, \ldots, I) \\
& + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(C_{11} + D_{11}), \phi_{i_2}(P_2), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
& - q_n(G_m(C_{11}), P_2, I, \ldots, I) - \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(C_{11}), \phi_{i_2}(P_2), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
& - q_n(G_m(D_{11}), P_2, I, \ldots, I) - \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(D_{11}), \phi_{i_2}(P_2), \phi_{i_3}(I), \ldots, \phi_{i_n}(I)) \\
=& G_m(q_n(C_{11} + D_{11}, P_2, I, \ldots, I)) - G_m(q_n(C_{11}, P_2, I, \ldots, I)) - G_m(q_n(D_{11}, P_2, I, \ldots, I)) \\
=& 0.
\end{aligned}$$

Therefore, $q_n(T, P_2, I, \ldots, I) = 0$, which implies that $T_{12} = T_{22} = 0$, where $T_{12} \in \mathcal{N}_{12}, T_{22} \in \mathcal{N}_{22}$. Consequently, $T = P_1 T P_1$. Let $N = A_{12} - A_{12}^*$ with $A_{12} \in \mathcal{A}_{12}$. Then,

$$q_n(C_{11}, N, \frac{I}{2}, \ldots, \frac{I}{2}) \in \mathcal{N}_{12}, \qquad q_n(D_{11}, N, \frac{I}{2}, \ldots, \frac{I}{2}) \in \mathcal{N}_{12}.$$

Using the induction hypotheses $\mathfrak{C}_s^G$ $(1 \le s < m)$, Lemma 3.4, and the above relations, we obtain

$$\begin{aligned}
q_n(T, N, \frac{I}{2}, \ldots, \frac{I}{2}) =& q_n(G_m(C_{11} + D_{11}), N, \frac{I}{2}, \ldots, \frac{I}{2}) - q_n(G_m(C_{11}), N, \frac{I}{2}, \ldots, \frac{I}{2}) \\
& - q_n(G_m(D_{11}), N, \frac{I}{2}, \ldots, \frac{I}{2}) \\
=& q_n(G_m(C_{11} + D_{11}), N, \frac{I}{2}, \ldots, \frac{I}{2}) \\
& + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(C_{11} + D_{11}), \phi_{i_2}(N), \phi_{i_3}(\frac{I}{2}), \ldots, \phi_{i_n}(\frac{I}{2})) \\
& - q_n(G_m(C_{11}), N, \frac{I}{2}, \ldots, \frac{I}{2}) \\
& - \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(C_{11}), \phi_{i_2}(N), \phi_{i_3}(\frac{I}{2}), \ldots, \phi_{i_n}(\frac{I}{2})) \\
& - q_n(G_m(D_{11}), N, \frac{I}{2}, \ldots, \frac{I}{2}) \\
& - \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1\in\{0,\ldots,m-1\}}} q_n(G_{i_1}(D_{11}), \phi_{i_2}(N), \phi_{i_3}(\frac{I}{2}), \ldots, \phi_{i_n}(\frac{I}{2}))
\end{aligned}$$

$$
\begin{aligned}
&=G_m(q_n(C_{11}+D_{11},N,\frac{I}{2},\ldots,\frac{I}{2}))-G_m(q_n(C_{11},N,\frac{I}{2},\ldots,\frac{I}{2}))\\
&\quad -G_m(q_n(D_{11},N,\frac{I}{2},\ldots,\frac{I}{2}))\\
&=G_m(q_n(C_{11},N,\frac{I}{2},\ldots,\frac{I}{2}))+G_m(q_n(D_{11},N,\frac{I}{2},\ldots,\frac{I}{2}))\\
&\quad -G_m(q_n(C_{11},N,\frac{I}{2},\ldots,\frac{I}{2}))-G_m(q_n(D_{11},N,\frac{I}{2},\ldots,\frac{I}{2}))\\
&=0.
\end{aligned}
$$

Hence, we have $q_n(T,N,\frac{I}{2},\ldots,\frac{I}{2})=0$, along with $N=A_{12}-A_{12}^*$ and $T=P_1TP_1$. It follows that

$$A_{12}^*T+TA_{12}=0.$$

Multiplying the above relation on the left by P_1 yields $P_1TA_{12}=0$. In view of (2.1), we conclude $T=P_1TP_1=0$, that is,

$$G_m(C_{11}+D_{11})=G_m(C_{11})+G_m(D_{11}).$$

Similarly, $G_m(C_{22}+D_{22})=G_m(C_{22})+G_m(D_{22})$. By the same reasoning, it follows that

$$\phi_m(C_{ii}+D_{ii})=\phi_m(C_{ii})+\phi_m(D_{ii}).$$

□

By Lemmas 3.2–3.5, we obtain the following result.

Lemma 3.6. *Under the induction hypotheses $\mathfrak{C}_s^{\phi}$ and $\mathfrak{C}_s^{G}$, G_m and ϕ_m are additive on $\mathcal{N}$.*

Next, we study the associated nonlinear bi-skew Lie n-higher derivation $\Phi=\{\phi_m\}_{m\in\mathbb{N}}$ of the nonlinear generalized bi-skew Lie n-higher derivation $G=\{G_m\}_{m\in\mathbb{N}}$. We first assume that ϕ_m ($m\in\mathbb{N}$) satisfies

$$P_1\phi_m(P_1)P_2=0. \tag{3.1}$$

It follows from [3, Lemma 2.9] that $\phi_1(P_1)=W_1$ with $W_1\in\mathcal{M}$ and $P_1W_1P_1=P_2W_1P_2=0$. Thus,

$$\phi_1(P_1)=P_1\phi_1(P_1)P_2+P_2\phi_1(P_1)P_1.$$

Applying (3.1), we obtain $\phi_1(P_1)=P_2\phi_1(P_1)P_1$. Meanwhile, [3, Lemma 2.11] shows that $\phi_1(P_1)^*=\phi_1(P_1)$, which forces

$$\phi_1(P_1)=0. \tag{3.2}$$

Now, using Property 2.7 and (3.2), we assume inductively that for each s with $1<s<m$, ϕ_s satisfies the following condition

$$\mathfrak{C}_s:\begin{cases}\phi_s(iA)=i\phi_s(A),\quad \phi_s(A^*)=\phi_s(A)^* \quad \text{for all } A\in\mathcal{A},\\ \phi_s(P_1)=0,\quad \phi_s(iI)=0,\\ \phi_s(M)\in\mathcal{M}\quad \text{for all } M\in\mathcal{M},\\ \phi_s \text{ is additive on } \mathcal{A}.\end{cases}$$

Specifically, we can define

$$\mathfrak{C}_1 : \begin{cases} \phi_1(iA) = i\phi_1(A), \quad \phi_1(A^*) = \phi_1(A)^* \quad \text{for all } A \in \mathcal{A}, \\ \phi_1(P_1) = 0, \quad \phi_1(iI) = 0, \\ \phi_1(M) \in \mathcal{M} \quad \text{for all } M \in \mathcal{M}, \\ \phi_1 \text{ is additive on } \mathcal{A}. \end{cases}$$

We will now prove several lemmas showing that ϕ_m also satisfies all the conditions in $\mathfrak{C}_s$, along with the additional relation $\phi_m(I) = 0$.

Lemma 3.7. *Based on the aforementioned registered symbols, we have*

(i) $\phi_m(I) = 0$*,*

(ii) $\phi_m(M) \in \mathcal{M}$ *for all* $M \in \mathcal{M}$*,*

(iii) $\phi_m(P_1) = 0$*.*

Proof. (i) Applying the induction hypotheses $\mathfrak{C}_s^\phi$ $(1 \le s < m)$, and Lemma 3.6, we have

$$\begin{aligned} 2^{n-1}\phi_m(N) =& \phi_m(2^{n-1}N) \\ =& \phi_m(q_n(N, I, \dots, I)) \\ =& q_n(\phi_m(N), I, \dots, I) + q_n(N, \phi_m(I), \dots, I) + \ldots + q_n(N, I, \dots, \phi_m(I)) \\ &+ \sum_{\substack{i_1+\ldots+i_n=m \\ i_1,\dots,i_n \in \{0,\dots,m-1\}}} q_n(\phi_{i_1}(N), \phi_{i_2}(I), \dots, \phi_{i_n}(I)) \\ =& 2^{n-1}\phi_m(N) + (n-1)2^{n-2}(N\phi_m(I)^* + \phi_m(I)N) \end{aligned}$$

for all $N \in \mathcal{N}$. Thus, $N\phi_m(I)^* + \phi_m(I)N = 0$. Choosing $N = iI$, we obtain $\phi_m(I)^* = -\phi_m(I)$, that is, $\phi_m(I) \in \mathcal{N}$. It follows that $N\phi_m(I) = \phi_m(I)N$ for all $N \in \mathcal{N}$. For any $B \in \mathcal{A}$, let $B = N_1 + iN_2$, where $N_1 = \frac{1}{2}(B - B^*) \in \mathcal{N}$, $N_2 = \frac{1}{2i}(B + B^*) \in \mathcal{N}$. Then, $B\phi_m(I) = \phi_m(I)B$ for all $B \in \mathcal{A}$. This implies $\phi_m(I) \in Z(\mathcal{A}) \cap \mathcal{N}$.

According to the induction hypotheses $\mathfrak{C}_s$ $(1 \le s < m)$, we have

$$\begin{aligned} 0 =& \phi_m(q_n(N, iI, I, \dots, I)) \\ =& q_n(\phi_m(N), iI, I, \dots, I) + q_n(N, \phi_m(iI), I, \dots, I) + q_n(N, iI, \phi_m(I), \dots, I) \\ &+ \ldots + q_n(N, iI, I, \dots, \phi_m(I)) + \sum_{\substack{i_1+i_2+\ldots+i_n=m \\ i_1,\dots,i_n \in \{0,\dots,m-1\}}} q_n(\phi_{i_1}(N), \phi_{i_2}(iI), \phi_{i_3}(I), \dots, \phi_{i_n}(I)) \\ =& 2^{n-2}(\phi_m(iI)N - N\phi_m(iI)) \end{aligned}$$

for all $N \in \mathcal{N}$. It follows that $\phi_m(iI) \in Z(\mathcal{A}) \cap \mathcal{N}$.

Let $\phi_m(P_1) = W_1 + iW_2$, where $W_1, W_2 \in \mathcal{M}$. Then,

$$\begin{aligned} 0 =& \phi_m(q_n(I, P_1, \frac{I}{2}, \dots, \frac{I}{2})) \\ =& q_n(\phi_m(I), P_1, \frac{I}{2}, \dots, \frac{I}{2}) + q_n(I, \phi_m(P_1), \frac{I}{2}, \dots, \frac{I}{2}) + q_n(I, P_1, \phi_m(\frac{I}{2}), \dots, \frac{I}{2}) \end{aligned}$$

$$+\ldots+q_n(I,P_1,\frac{I}{2},\ldots,\phi_m(\frac{I}{2}))+\sum_{\substack{i_1+i_2+\ldots+i_n=m\\ i_1,\ldots,i_n\in\{0,\ldots,m-1\}}} q_n(\phi_{i_1}(I),\phi_{i_2}(P_1),\phi_{i_3}(\frac{I}{2}),\ldots,\phi_{i_n}(\frac{I}{2}))$$
$$=2\phi_m(I)P_1-2iW_2.$$

This implies $iW_2=\phi_m(I)P_1$. Consequently,

$$\phi_m(P_1)=W_1+\phi_m(I)P_1.$$

In view of the induction hypotheses $\mathfrak{C}_s^{\phi}$ and $\mathfrak{C}_s$ $(1\le s<m)$ and Lemma 3.6, we conclude

$$\begin{aligned}2^{n-1}\phi_m(iP_1)=&\phi_m(q_n(iI,P_1,I,\ldots,I))\\
=&q_n(\phi_m(iI),P_1,I,\ldots,I)+q_n(iI,\phi_m(P_1),I,\ldots,I)\\
&+q_n(iI,P_1,\phi_m(I),\ldots,I)+\ldots+q_n(iI,P_1,I,\ldots,\phi_m(I))\\
&+\sum_{\substack{i_1+i_2+\ldots+i_n=m\\ i_1,\ldots,i_n\in\{0,\ldots,m-1\}}} q_n(\phi_{i_1}(iI),\phi_{i_2}(P_1),\phi_{i_3}(I),\ldots,\phi_{i_n}(I))\\
=&2^{n-1}(\phi_m(iI)P_1+iW_1).\end{aligned}$$

Therefore,

$$\phi_m(iP_1)=\phi_m(iI)P_1+iW_1. \tag{3.3}$$

Using the induction hypotheses $\mathfrak{C}_s^{\phi}$ and $\mathfrak{C}_s$ $(1\le s<m)$ and Lemma 3.6, we have

$$\begin{aligned}2^{n-1}\phi_m(iP_1)=&\phi_m(q_n(iP_1,P_1,I,\ldots,I))\\
=&q_n(\phi_m(iP_1),P_1,I,\ldots,I)+q_n(iP_1,\phi_m(P_1),I,\ldots,I)\\
&+q_n(iP_1,P_1,\phi_m(I),\ldots,I)+\ldots+q_n(iP_1,P_1,I,\ldots,\phi_m(I))\\
&+\sum_{\substack{i_1+i_2+\ldots+i_n=m\\ i_1,\ldots,i_n\in\{0,\ldots,m-1\}}} q_n(\phi_{i_1}(iP_1),\phi_{i_2}(P_1),\phi_{i_3}(I),\ldots,\phi_{i_n}(I))\\
=&2^{n-1}\phi_m(iI)P_1+2^{n-1}i(W_1P_1+P_1W_1).\end{aligned}$$

Thus,

$$\phi_m(iP_1)=\phi_m(iI)P_1+i(W_1P_1+P_1W_1). \tag{3.4}$$

Comparing (3.3) and (3.4), we obtain $W_1P_1+P_1W_1=W_1$, which implies $P_1W_1P_1=P_2W_1P_2=0$. Hence,

$$\phi_m(P_1)=\phi_m(I)P_1+W_1=\phi_m(I)P_1+P_1W_1P_2+P_2W_1P_1. \tag{3.5}$$

For any $A_{12}\in\mathcal{A}_{12}$, let $N=A_{12}-A_{12}^*$. Then, $N\in\mathcal{N}$. By the induction hypotheses $\mathfrak{C}_s^{\phi}$ and $\mathfrak{C}_s$ for

$1 \leq s < m$ and Lemma 3.6, we have

$$\begin{aligned}
-2^{n-2}\phi_m(N) =&\phi_m(q_n(P_1, N, I, \dots, I)) \\
=&q_n(\phi_m(P_1), N, I, \dots, I) + q_n(P_1, \phi_m(N), I, \dots, I) \\
&+ q_n(P_1, N, \phi_m(I), \dots, I) + \dots + q_n(P_1, N, I, \dots, \phi_m(I)) \\
&+ \sum_{\substack{i_1+i_2+\dots+i_n=m \\ i_1,\dots,i_n\in\{0,\dots,m-1\}}} q_n(\phi_{i_1}(P_1), \phi_{i_2}(N), \phi_{i_3}(I), \dots, \phi_{i_n}(I)) \\
=&-2^{n-2}(\phi_m(P_1)N + N\phi_m(P_1)^* + P_1\phi_m(N) + \phi_m(N)P_1).
\end{aligned} \tag{3.6}$$

Multiplying (3.6) on the left by P_1 and on the right by P_2 yields $\phi_m(I)A_{12} = 0$. By (2.1), we obtain $\phi_m(I)P_1 = 0$. Since $\phi_m(I) \in Z(\mathcal{A}) \cap \mathcal{N}$, it follows that $\phi_m(I)A_{12}^* = 0$, and applying (2.1) again gives $\phi_m(I)P_2 = 0$. Consequently,

$$\phi_m(I) = \phi_m(I)P_1 + \phi_m(I)P_2 = 0.$$

(ii) Since $\phi_m(I) = 0$, we have

$$\begin{aligned}
0 =&\phi_m(q_n(I, M, I, \dots, I)) \\
=&q_n(I, \phi_m(M), I, \dots, I) + \sum_{\substack{i_1+i_2+\dots+i_n=m \\ i_2\in\{0,\dots,m-1\}}} q_n(\phi_{i_1}(I), \phi_{i_2}(M), \phi_{i_3}(I), \dots, \phi_{i_n}(I)) \\
=&2^{n-2}(\phi_m(M)^* - \phi_m(M))
\end{aligned}$$

for all $M \in \mathcal{M}$. This yields that $\phi_m(M)^* = \phi_m(M)$ for all $M \in \mathcal{M}$.

(iii) Using $\phi_m(I) = 0$ together with (3.5), we obtain $\phi_m(P_1) = P_1W_1P_2 + P_2W_1P_1$. By (3.1), we have $P_1W_1P_2 = 0$, which implies $\phi_m(P_1) = P_2W_1P_1$. Since $\phi_m(M) \in \mathcal{M}$ and $P_1 \in \mathcal{M}$, it follows that $\phi_m(P_1) = \phi_m(P_1)^* = P_1W_1P_2 = 0$. □

Lemma 3.8. *Based on the aforementioned registered symbols, we have*

(i) ϕ_m *is additive on* $\mathcal{A}$,

(ii) $\phi_m(A^*) = \phi_m(A)^*$ *for all* $A \in \mathcal{A}$.

Proof. (i) By the induction hypotheses $\mathfrak{C}_s^\phi$ and $\mathfrak{C}_s$ for $1 \leq s < m$ together with Lemmas 3.6 and 3.7, we have

$$\begin{aligned}
2^{n-1}\phi_m(iM) =&\phi_m(q_n(iI, M, I, \dots, I)) \\
=&q_n(\phi_m(iI), M, I, \dots, I) + q_n(iI, \phi_m(M), I, \dots, I) + q_n(iI, M, \phi_m(I), \dots, I) \\
&+ \dots + q_n(iI, M, I, \dots, \phi_m(I)) + \sum_{\substack{i_1+i_2+\dots+i_n=m \\ i_1,\dots,i_n\in\{0,\dots,m-1\}}} q_n(\phi_{i_1}(iI), \phi_{i_2}(M), \phi_{i_3}(I), \dots, \phi_{i_n}(I)) \\
=&2^{n-1}(\phi_m(iI)M + i\phi_m(M))
\end{aligned}$$

for all $M \in \mathcal{M}$. Thus,

$$\phi_m(iM) = i\phi_m(M) + \phi_m(iI)M. \tag{3.7}$$

On the one hand, applying the induction hypotheses $\mathfrak{C}_s^{\phi}$ for $1 \leq s < m$, Lemmas 3.6 and 3.7 and Eq (3.7) lead to

$$\begin{aligned} 2^{n-1}(i\phi_m(K) + \phi_m(iI)K) &= \phi_m(2^{n-1}iK) \\ &= \phi_m(q_n(H + iK, I, \ldots, I)) \\ &= q_n(\phi_m(H + iK), I, \ldots, I) \\ &= 2^{n-2}(\phi_m(H + iK) - \phi_m(H + iK)^*) \end{aligned} \tag{3.8}$$

for all $H, K \in \mathcal{M}$. On the other hand,

$$\begin{aligned} 2^{n-1}(i\phi_m(H) + \phi_m(iI)H) &= \phi_m(2^{n-1}iH) \\ &= \phi_m(q_n(iI, H + iK, I, \ldots, I)) \\ &= q_n(\phi_m(iI), H + iK, I, \ldots, I) + q_n(iI, \phi_m(H + iK), I, \ldots, I) \\ &= 2^{n-1}\phi_m(iI)H + 2^{n-2}i(\phi_m(H + iK) + \phi_m(H + iK)^*) \end{aligned} \tag{3.9}$$

for all $H, K \in \mathcal{M}$. Comparing (3.8) and (3.9), we obtain

$$\phi_m(H + iK) = \phi_m(H) + i\phi_m(K) + \phi_m(iI)K. \tag{3.10}$$

According to Lemma 3.6 and (3.7), we have

$$\begin{aligned} i\phi_m(H + K) + \phi_m(iI)(H + K) &= \phi_m(i(H + K)) \\ &= \phi_m(iH) + \phi_m(iK) \\ &= i(\phi_m(H) + \phi_m(K)) + \phi_m(iI)(H + K). \end{aligned}$$

Therefore, $i\phi_m(H + K) = i(\phi_m(H) + \phi_m(K))$, which implies

$$\phi_m(H + K) = \phi_m(H) + \phi_m(K)$$

for all $H, K \in \mathcal{M}$. This means ϕ_m is additive on $\mathcal{M}$.

For any $A, B \in \mathcal{A}$, write $A = A_1 + iA_2$ and $B = B_1 + iB_2$, where $A_k, B_k \in \mathcal{M}$ $(k = 1, 2)$. Applying (3.10) we obtain

$$\begin{aligned} \phi_m(A + B) &= \phi_m((A_1 + B_1) + i(A_2 + B_2)) \\ &= \phi_m(A_1 + B_1) + i\phi_m(A_2 + B_2) + \phi_m(iI)(A_2 + B_2) \\ &= (\phi_m(A_1) + i\phi_m(A_2) + \phi_m(iI)A_2) + (\phi_m(B_1) + i\phi_m(B_2) + \phi_m(iI)B_2) \\ &= \phi_m(A) + \phi_m(B). \end{aligned}$$

Therefore, ϕ_m is additive on $\mathcal{A}$.

(ii) For any $A = A_1 + iA_2 \in \mathcal{A}$ with $A_1, A_2 \in \mathcal{M}$, Lemma 3.1, Lemma 3.7, and (3.10) imply

$$\begin{aligned} \phi_m(A)^* &= \phi_m(A_1 + iA_2)^* \\ &= (\phi_m(A_1) + i\phi_m(A_2) + \phi_m(iI)A_2)^* \\ &= \phi_m(A_1) - i\phi_m(A_2) - \phi_m(iI)A_2 \\ &= \phi_m(A_1 - iA_2) \\ &= \phi_m(A^*) \end{aligned}$$

for all $A \in \mathcal{A}$. $\square$

Lemma 3.9. *In accordance with the symbols defined above, we have*

(i) $\phi_m(iI) = 0$,

(ii) $\phi_m(iA) = i\phi_m(A)$ *for all* $A \in \mathcal{A}$.

Proof. (i) It follows from Lemma 3.7 that

$$\phi_m(iP_1) = \phi_m(iI)P_1.$$

The induction hypotheses $\mathfrak{C}_s$ for $1 \leq s < m$ imply $\phi_s(iP_1) = i\phi_s(P_1) = 0$. Now, apply the induction hypotheses $\mathfrak{C}_s$ for $1 \leq s < m$, Lemma 3.7, Lemma 3.8, and (3.7) to get

$$\begin{aligned}
-2^{n-2}(i(\phi_m(A_{12})&+\phi_m(A_{12}^*)) + \phi_m(iI)(A_{12} + A_{12}^*)) \\
&= -2^{n-2}\phi_m(i(A_{12} + A_{12}^*)) \\
&=\phi_m(q_n(iP_1, A_{12} - A_{12}^*, I, \dots, I)) \\
&=q_n(\phi_m(iP_1), A_{12} - A_{12}^*, I, \dots, I) + q_n(iP_1, \phi_m(A_{12} - A_{12}^*), I, \dots, I) \\
&=2^{n-2}((\phi_m(iI)P_1)(A_{12}^* - A_{12}) + (A_{12} - A_{12}^*)(\phi_m(iI)P_1) \\
&\quad + iP_1(\phi_m(A_{12})^* - \phi_m(A_{12})) + i(\phi_m(A_{12}) - \phi_m(A_{12})^*)P_1)
\end{aligned}$$

for all $A_{12} \in \mathcal{A}_{12}$. Multiplying the left-hand side of the above relation by P_1 and the right-hand side by P_2, we arrive at $P_1\phi_m(A_{12})^*P_2 = 0$.

Meanwhile,

$$\begin{aligned}
2^{n-2}(\phi_m(A_{12}) - \phi_m(A_{12})^*) &=\phi_m(q_n(iP_1, i(A_{12} + A_{12}^*), I, \dots, I)) \\
&=q_n(\phi_m(iP_1), i(A_{12} + A_{12}^*), I, \dots, I) + q_n(iP_1, \phi_m(i(A_{12} + A_{12}^*)), I, \dots, I) \\
&=2^{n-2}((-i\phi_m(iI)P_1)(A_{12}^* + A_{12}) - (A_{12} + A_{12}^*)(-i\phi_m(iI)P_1) \\
&\quad + P_1(\phi_m(A_{12})^* + \phi_m(A_{12}) - i\phi_m(iI)(A_{12} + A_{12}^*)) \\
&\quad - (\phi_m(A_{12}) + \phi_m(A_{12})^* - i\phi_m(iI)(A_{12} + A_{12}^*))P_1)
\end{aligned}$$

for all $A_{12} \in \mathcal{A}_{12}$. Multiplying the left-hand side of the above relation by P_1 and the right-hand side by P_2, and using $P_1\phi_m(A_{12})^*P_2 = 0$, we get $\phi_m(iI)A_{12} = 0$. By (2.1), this implies $\phi_m(iI)P_1 = 0$. Since $\phi_m(iI) \in Z(\mathcal{A}) \cap \mathcal{N}$, it follows that $\phi_m(iI)A_{12}^* = 0$. Applying (2.1) again gives $\phi_m(iI)P_2 = 0$. Consequently,

$$\phi_m(iI) = \phi_m(iI)P_1 + \phi_m(iI)P_2 = 0.$$

(ii) Using (3.7) together with $\phi_m(iI) = 0$ gives $\phi_m(iM) = i\phi_m(M)$ for all $M \in \mathcal{M}$. For any $A \in \mathcal{A}$, write $A = A_1 + iA_2$ with $A_1, A_2 \in \mathcal{M}$. Then,

$$\phi_m(iA) = \phi_m(i(A_1 + iA_2)) = \phi_m(iA_1 - A_2) = i(\phi_m(A_1) + i\phi_m(A_2)) = i\phi_m(A).$$

□

In what follows, we shall apply the preceding lemmas to derive several properties of nonlinear generalized bi-skew Lie n-higher derivations $G = \{G_m\}_{m\in\mathbb{N}}$, and then establish the main result of this paper.

Lemma 3.10. *Based on the aforementioned registered symbols, we have*

(i) G_m *is additive on* $\mathcal{A}$,

(ii) $G_m(iA) = iG_m(A)$ *for all* $A \in \mathcal{A}$,

(iii) $G_m(A^*) = G_m(A)^*$ *for all* $A \in \mathcal{A}$.

Proof. (i) By the induction hypotheses $\mathfrak{C}_s^{\phi}$ with $1 \leq s < m$ and Lemma 3.7, we get

$$0 = G_m(q_n(M, I, \ldots, I)) = q_n(G_m(M), I, \ldots, I) = 2^{n-2}(G_m(M) - G_m(M)^*)$$

for all $M \in \mathcal{M}$. Hence,

$$G_m(M)^* = G_m(M). \tag{3.11}$$

Applying the induction hypotheses $\mathfrak{C}_s^{\phi}$ and $\mathfrak{C}_s$ ($1 \leq s < m$) together with Lemma 3.6, Lemma 3.7, Lemma 3.9, and (3.11) yields

$$2^{n-1}G_m(iM) = -G_m(q_n(M, iI, I, \ldots, I)) = -q_n(G_m(M), iI, I, \ldots, I) = 2^{n-1}iG_m(M)$$

for all $M \in \mathcal{M}$. Consequently,

$$G_m(iM) = iG_m(M). \tag{3.12}$$

Consider the induction hypotheses $\mathfrak{C}_s^{\phi}$ ($1 \leq s < m$) along with Lemmas 3.6, 3.7, and Eq (3.12) to obtain

$$\begin{aligned} 2^{n-1}iG_m(K) =& G_m(2^{n-1}iK) \\ =& G_m(q_n(H + iK, I, \ldots, I)) \\ =& q_n(G_m(H + iK), I, \ldots, I) \\ =& 2^{n-2}(G_m(H + iK) - G_m(H + iK)^*) \end{aligned}$$

for all $H, K \in \mathcal{M}$. A similar argument, using the induction hypotheses $\mathfrak{C}_s^{\phi}$ and $\mathfrak{C}_s$ ($1 \leq s < m$) together with Lemma 3.6, 3.7, 3.9, and Eq (3.12), gives

$$\begin{aligned} 2^{n-1}iG_m(H) =& G_m(2^{n-1}iH) \\ =& -G_m(q_n(H + iK, iI, I, \ldots, I)) \\ =& -q_n(G_m(H + iK), iI, I, \ldots, I) \\ =& 2^{n-2}i(G_m(H + iK) + G_m(H + iK)^*) \end{aligned}$$

for all $H, K \in \mathcal{M}$. Comparing the two above relations, we conclude

$$G_m(H + iK) = G_m(H) + iG_m(K) \tag{3.13}$$

for all $H, K \in \mathcal{M}$.

Using Lemma 3.6 and (3.12), we have

$$iG_m(H+K) = G_m(i(H+K)) = G_m(iH) + G_m(iK) = i(G_m(H) + G_m(K))$$

for all $H, K \in \mathcal{M}$. That is, $G_m(H+K) = G_m(H) + G_m(K)$. So, G_m is additive on $\mathcal{M}$.

Take any $A, B \in \mathcal{A}$ and write $A = A_1 + iA_2$, $B = B_1 + iB_2$ with $A_k, B_k \in \mathcal{M}$ $(k = 1, 2)$. By (3.13) and the additivity of G_m on $\mathcal{M}$, we obtain

$$\begin{aligned} G_m(A+B) &= G_m((A_1+B_1) + i(A_2+B_2)) \\ &= G_m(A_1+B_1) + iG_m(A_2+B_2) \\ &= (G_m(A_1) + iG_m(A_2)) + (G_m(B_1) + iG_m(B_2)) \\ &= G_m(A) + G_m(B). \end{aligned}$$

Therefore, G_m is additive on $\mathcal{A}$.

(ii) Let $A = A_1 + iA_2$ with $A \in \mathcal{A}$, $A_k \in \mathcal{M}$ $(k = 1, 2)$. Applying (3.12) and the additivity of G_m on $\mathcal{A}$, we obtain

$$\begin{aligned} G_m(iA) &= G_m(i(A_1 + iA_2)) = G_m(iA_1 - A_2) \\ &= G_m(iA_1) - G_m(A_2) = i(G_m(A_1) + iG_m(A_2)) \\ &= iG_m(A). \end{aligned}$$

Thus, $G_m(iA) = iG_m(A)$ for all $A \in \mathcal{A}$.

(iii) For any $A \in \mathcal{A}$, write $A = A_1 + iA_2$ with $A_1, A_2 \in \mathcal{M}$. Using (i), (ii), and (3.11), we obtain

$$\begin{aligned} G_m(A)^* &= G_m(A_1 + iA_2)^* = (G_m(A_1) + iG_m(A_2))^* \\ &= G_m(A_1) - iG_m(A_2) = G_m(A_1 - iA_2) \\ &= G_m(A^*). \end{aligned}$$

Hence, $G_m(A^*) = G_m(A)^*$ for all $A \in \mathcal{A}$. □

Lemma 3.11. *Using the symbols introduced above, G_m satisfies*

$$G_m(AB) = \sum_{j_1+j_2=m} G_{j_1}(A)\phi_{j_2}(B),$$

and ϕ_m satisfies

$$\phi_m(AB) = \sum_{i_1+i_2=m} \phi_{i_1}(A)\phi_{i_2}(B)$$

for all $A, B \in \mathcal{A}$.

Proof. By Lemma 3.7, Lemma 3.10, and the induction hypotheses $\mathfrak{C}_s^\phi$ $(1 \le s < m)$, on the one hand, we obtain

$$\begin{aligned} 2^{n-2} iG_m(AB^* + BA^*) &= G_m(q_n(iA, B, I, \ldots, I)) \\ &= \sum_{j_1+\ldots+j_n=m} q_n(G_{j_1}(iA), \phi_{j_2}(B), \phi_{j_3}(I), \ldots, \phi_{j_n}(I)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j_1+j_2=m} q_n(iG_{j_1}(A), \phi_{j_2}(B), I, \ldots, I) \\
&= 2^{n-2} \sum_{j_1+j_2=m} i(G_{j_1}(A)\phi_{j_2}(B)^* + \phi_{j_2}(B)G_{j_1}(A)^*)
\end{aligned}
$$

for all $A, B \in \mathcal{A}$. Hence,

$$
G_m(AB^* + BA^*) = \sum_{j_1+j_2=m} (G_{j_1}(A)\phi_{j_2}(B)^* + \phi_{j_2}(B)G_{j_1}(A)^*).
$$

On the other hand, we have

$$
\begin{aligned}
2^{n-2}G_m(AB^* - BA^*) &= G_m(q_n(A, B, I, \ldots, I)) \\
&= \sum_{j_1+\ldots+j_n=m} q_n(G_{j_1}(A), \phi_{j_2}(B), \phi_{j_3}(I), \ldots, \phi_{j_n}(I)) \\
&= \sum_{j_1+j_2=m} q_n(G_{j_1}(A), \phi_{j_2}(B), I, \ldots, I) \\
&= 2^{n-2} \sum_{j_1+j_2=m} (G_{j_1}(A)\phi_{j_2}(B)^* - \phi_{j_2}(B)G_{j_1}(A)^*),
\end{aligned}
$$

it follows that

$$
G_m(AB^* - BA^*) = \sum_{j_1+j_2=m} (G_{j_1}(A)\phi_{j_2}(B)^* - \phi_{j_2}(B)G_{j_1}(A)^*).
$$

Comparing the two above relations, we obtain

$$
G_m(AB^*) = \sum_{j_1+j_2=m} G_{j_1}(A)\phi_{j_2}(B)^*.
$$

Moreover, Lemma 3.8 yields

$$
G_m(AB) = G_m(A(B^*)^*) = \sum_{j_1+j_2=m} G_{j_1}(A)\phi_{j_2}(B^*)^* = \sum_{j_1+j_2=m} G_{j_1}(A)\phi_{j_2}(B)
$$

for all $A, B \in \mathcal{A}$. Similarly,

$$
\phi_m(AB) = \sum_{i_1+i_2=m} \phi_{i_1}(A)\phi_{i_2}(B).
$$

□

The preceding lemmas and Definitions 2.2 and 2.4 now enable us to establish the main result of this paper.

Theorem 3.12. *Let $\mathcal{A}$ be a $*$-algebra with identity element I and a nontrivial projection P_1 satisfying condition (2.1). Suppose $G = \{G_m\}_{m\in\mathbb{N}}$ is a nonlinear generalized bi-skew Lie n-higher derivation on $\mathcal{A}$, and $\Phi = \{\phi_m\}_{m\in\mathbb{N}}$ is the associated nonlinear bi-skew Lie n-higher derivation of G that satisfies (3.1). Then, G is an additive generalized higher $*$-derivation.*

4. Applications

Assume that for each nonlinear generalized bi-skew Lie n-higher derivation listed below, the associated bi-skew Lie n-higher derivation satisfies condition (3.1). We obtain the following corollaries.

The algebra $\mathcal{A}$ is said to be prime if for all $A, B \in \mathcal{A}$, $A\mathcal{A}B$ implies $A = 0$ or $B = 0$. It is easy to see that prime $*$-algebras satisfy condition (2.1). Applying Theorem 3.12 to prime $*$-algebras, we have the following corollary.

Corollary 4.1. *Let $\mathcal{A}$ be a prime $*$-algebra with identity element I, containing a nontrivial projection P_1. Then, every nonlinear generalized bi-skew Lie n-higher derivation on $\mathcal{A}$ is an additive generalized higher $*$-derivation.*

A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathcal{H}$ containing the identity element I. A von Neumann algebra $\mathcal{A}$ is a factor if its center is $\mathbb{C}I$. It is well known that a factor von Neumann algebra is prime. Hence, as an immediate consequence of Corollary 4.1, we get the following corollary.

Corollary 4.2. *Let $\mathcal{A}$ be a factor von Neumann algebra with $\dim \mathcal{A} \geq 2$. Then, every nonlinear generalized bi-skew Lie n-higher derivation on $\mathcal{A}$ is an additive generalized higher $*$-derivation.*

As demonstrated in [14, 15], if a von Neumann algebra has no central summands of type I_1, then it satisfies condition (2.1). Therefore, we can obtain the following statement.

Corollary 4.3. *Let $\mathcal{B}$ be a von Neumann algebra with no central summands of type I_1. Then, every nonlinear generalized bi-skew Lie n-higher derivation on $\mathcal{B}$ is an additive generalized higher $*$-derivation.*

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. Let $\mathcal{F}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ be the subalgebra of all bounded finite rank operators. A subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called a standard operator algebra if it contains $\mathcal{F}(\mathcal{H})$. Since the standard operator algebra is prime, by Corollary 4.1, we arrive at the following corollary.

Corollary 4.4. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space, and $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$ containing the identity element I. Suppose that $\mathcal{A}$ is closed under the adjoint operation. Then, every nonlinear generalized bi-skew Lie n-higher derivation $G = \{G_m\}_{(m\in\mathbb{N})}$ from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$ is an additive generalized higher $*$-derivation.*

5. Conclusions

Motivated by existing results on nonlinear bi-skew Lie-type derivations on $*$-algebras, we systematically analyze nonlinear generalized bi-skew Lie n-higher derivations on $*$-algebras using mathematical induction. It is shown that, under the given conditions, every such derivation on the $*$-algebra $\mathcal{A}$ is an additive generalized higher $*$-derivation. This result enriches the theory of nonlinear Lie-type derivations on $*$-algebras and provides a theoretical reference for the structural study of von Neumann algebras, $\mathbb{C}^*$-algebras, and other operator algebras. Moreover, this paper offers a feasible analytical method for the study of other generalized higher derivations. Our results also lay a theoretical foundation for applying the properties of nonlinear bi-skew Lie n-higher derivations on $*$-algebras in related fields.

Finally, we outline several promising directions for future research. First, we extend our investigation to nonlinear generalized bi-skew Jordan n-higher derivations, and explore whether analogous results can be obtained under similar assumptions. Second, we study nonlinear generalized bi-skew Lie n-higher derivations on a broader class of algebraic structures, such as non-unital $*$-algebras. Third, our main theorem can be applied to more operator algebras such as Banach algebras and C*-algebras, so that we can derive more practically valuable corollaries. Fourth, we weaken the hypotheses of the main theorem, aiming to find the minimal sufficient conditions that guarantee the additivity of such nonlinear mappings.

Author contributions

Chuqi Jia: Formal analysis, Writing–original draft, Writing–review and editing; He Yuan: Conceptualization, Funding acquisition, Writing–review and editing; Ke Shi: Formal analysis, Writing–original draft. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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