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Research article

Stability and analyticity for a coupled plate-membrane system with Kelvin-Voigt damping

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Abstract: We present an operator-theoretic analysis of a coupled plate-membrane system with Kelvin-Voigt damping. The model consists of an Euler-Bernoulli plate coupled to a membrane through transmission conditions, with Kelvin-Voigt damping mechanisms acting on different components of the system. By formulating the problem within an abstract Hilbert space framework, we prove that the associated operator generates a strongly continuous and analytic semigroup. As a consequence, we derive regularity properties and describe the long-time behavior of solutions, including asymptotic stability results. The analysis relies on semigroup theory and establishes a rigorous functional-analytic framework for coupled elastic systems with Kelvin-Voigt damping, yielding analyticity and long-time stability properties.

Keywords: plate-membrane system; coupled PDE systems; Kelvin-Voigt damping; exponential stability; analytic semigroup; polynomial stability
Mathematics Subject Classification: 35M33, 35B35, 35B40, 47D06, 74K15, 74K20

1. Introduction

In recent years, the study of coupled systems with transmission conditions has attracted considerable attention in the analysis of partial differential equations, particularly in models involving interactions between different physical media through an interface. These problems naturally arise in biology, composite materials, hybrid structures, fluid-structure interactions, and vibration monitoring in civil infrastructure (see [2, 3, 18] and the references therein). In these scenarios, coupled transmission systems governed by plate and wave equations appear; they have become fundamental models for the study of several physical and engineering phenomena. Several authors have studied such structures under various damping mechanisms. Thermoelastic effects are present in the systems considered in [4, 7]. Frictional damping is considered in [4, 5, 11], and Kelvin-Voigt (K-V) damping appears in [7, 15, 17]. A description of the latter two damping mechanisms can be found in Subsections 1.1 and 1.2 of [25]. See also [1, Subsection 1.2] for additional comments on K-V damping. Here, we focus on K-V damping in the coupled structure under consideration.

In this paper, we consider a transmission problem involving an Euler-Bernoulli plate with K-V damping, coupled to a membrane with or without K-V damping. To cover both configurations, the equations below include K-V damping terms for both the plate and the membrane. Within the geometric framework of the problem, assume that Ω and Ω_2 are nonempty open, connected, and bounded sets in $\mathbb{R}^2$ with boundary of class C^4 such that $\overline{\Omega_2} \subset \Omega$, $\Omega_1 := \Omega \setminus \overline{\Omega_2}$, $\Gamma := \partial\Omega$, and $I := \partial\Omega_2$. Then $\partial\Omega_1 = \Gamma \cup I$. We assume that Ω_1 is occupied by the middle surface of the plate and that Ω_2 is occupied by the membrane. We consider the following equations:

$$u_{tt} + \Delta^2 u + \kappa_1 \Delta^2 u_t = 0 \quad \text{in} \quad (0,\infty)\times\Omega_1, \tag{1.1}$$
$$v_{tt} - \Delta v - \kappa_2 \Delta v_t = 0 \quad \text{in} \quad (0,\infty)\times\Omega_2, \tag{1.2}$$

where $u(t,x)$ denotes the transverse displacement at time t of the material point located at x on the middle surface of the plate and that $v(t,x)$ represents the corresponding displacement of the membrane. The constants are such that κ_1 is positive, and κ_2 is non-negative. When $\kappa_1, \kappa_2 > 0$, the term $\kappa_1\Delta^2 u_t$ in (1.1) represents K-V damping on the plate, whereas $\kappa_2\Delta v_t$ in (1.2) represents K-V damping acting on the membrane. We set the following boundary conditions:

$$u = \partial_\nu u = 0 \quad \text{on} \quad (0,\infty)\times\Gamma, \tag{1.3}$$
$$\mathscr{B}_1(u+\kappa_1 u_t) = 0 \quad \text{on} \quad (0,\infty)\times I, \tag{1.4}$$

together with the transmission conditions

$$u = v \quad \text{on} \quad (0,\infty)\times I, \tag{1.5}$$
$$\mathscr{B}_2(u+\kappa_1 u_t) = -\partial_\nu v - \kappa_2 \partial_\nu v_t \quad \text{on} \quad (0,\infty)\times I. \tag{1.6}$$

The conditions (1.3) are the so-called clamped boundary conditions. The system is completed by the initial conditions

$$\begin{aligned} u(0,\cdot) &= u^0, \quad u_t(0,\cdot) = u^1 \quad \text{in} \quad \Omega_1,\\ v(0,\cdot) &= v^0, \quad v_t(0,\cdot) = v^1 \quad \text{in} \quad \Omega_2. \end{aligned} \tag{1.7}$$

The operators $\mathscr{B}_1$ and $\mathscr{B}_2$ in the boundary and transmission conditions, acting on sufficiently regular functions w, are defined by

$$\mathscr{B}_1 w := \Delta w + (1-\mu)B_1 w \quad \text{and} \quad \mathscr{B}_2 w := \partial_\nu \Delta w + (1-\mu)\partial_\tau B_2 w,$$

where

$$B_1 w := -\langle \tau, (\nabla^2 w)\tau\rangle \quad \text{and} \quad B_2 w := \langle \tau, (\nabla^2 w)\nu\rangle,$$

where $\mu \in (0, 1/2)$ is the Poisson ratio of the plate, and $\nabla^2 w$ denotes the Hessian matrix of w. Here, $\nu = (\nu_1, \nu_2)^\top$ denotes the outward unit normal vector to the boundary of Ω_1 and $\tau := (-\nu_2, \nu_1)^\top$. On $I = \partial\Omega_2$, the outward unit normal vector to Ω_2 is therefore $-\nu|_I$. The energy of the system (1.1)–(1.6) is given by

$$E(t) := \frac{1}{2}\int_{\Omega_1}\left[\mu|\Delta u|^2 + (1-\mu)|\nabla^2 u|^2 + |u_t|^2\right]dx + \frac{1}{2}\int_{\Omega_2}\left(|\nabla v|^2 + |v_t|^2\right)dx. \tag{1.8}$$

A direct computation shows that for sufficiently smooth u and v, the following holds:

$$\frac{d}{dt}E(t) = -\kappa_1\mu \int_{\Omega_1} |\Delta u_t|^2 dx - \kappa_1(1-\mu) \int_{\Omega_1} |\nabla^2 u_t|^2 dx - \kappa_2 \int_{\Omega_2} |\nabla v_t|^2 dx. \tag{1.9}$$

The problem (1.1)–(1.7) is formulated as an abstract evolution equation in an appropriate Hilbert space, where the main analytical difficulty lies in the interaction between the boundary operators $\mathscr{B}_1$ and $\mathscr{B}_2$, which commonly appear in the so-called free-type boundary conditions in plate theory, and the transmission conditions, together with the different order of the partial differential equations (PDEs) involved in the system. This approach allows us to employ semigroup theory as a central tool for the analysis of the system. Stability and analyticity results are obtained for the semigroup associated with the abstract problem. Although one might expect such results from the behavior of the individual components, each of which generates an analytic semigroup under K-V damping, no proof seems to be available in the literature for a transmission problem of the type considered here. One of the main objectives of this work is to fill this gap. Despite the analyticity properties known for each uncoupled subsystem, the interaction through transmission conditions introduces nontrivial coupling effects that prevent a direct inheritance of regularity and stability properties.

Let us now review some studies related to our problem. In the one-dimensional case, the literature contains beam-string, string-beam-string, and beam-string-beam transmission problems. For instance, beam-string transmission problems were studied in [16,19,24], string-beam-string systems in [14], and beam-string-beam transmission problems in [6,8].

We next review some existing works concerning the n-dimensional case with $n \geq 2$. In [4], a transmission problem involving a thermoelastic plate, with or without rotational inertia, coupled to a membrane with and without frictional damping was analyzed. The authors examined the long-time dynamics of the solution based on damping and rotation effects. In [5], the authors studied a transmission problem coupling a structurally damped plate with a wave equation under frictional or no damping. They proved exponential stability in the fully damped case and only polynomial stability when damping was applied to the plate alone. In [7], the authors studied a transmission problem for a thermoelastic plate, with or without rotational inertia, coupled to a membrane. Structural and K-V damping were applied to the plate and membrane, respectively. They proved well-posedness, higher regularity, and exponential and polynomial stability as well as analyticity of the associated semigroup, depending on damping and rotational effects. In [11], the existence and uniqueness of solutions were proved for a coupled wave-plate system with variable coefficients and dynamic boundary conditions. Under geometric assumptions, exponential energy decay and strong stability were demonstrated using Riemannian geometry and Lyapunov methods. In [12], the authors studied a two-dimensional wave-plate system with localized frictional damping. They proved polynomial stability when the damping was applied only to the plate and exponential stability when it acted on the wave component. The proof was based on frequency domain techniques and multiplier methods to demonstrate exponential stability. In [13], the authors presented an analysis of energy decay in a coupled wave-plate system on concentric circular domains with localized frictional damping. Polynomial decay occurs with damping only on the plate, whereas exponential stability arises when damping is on the wave region. Using frequency domain techniques and multiplier methods, it was shown there that exponential stabilization is guaranteed only when the wave equation is damped. In [15], the authors investigated a transmission system where a membrane surrounds an Euler-Bernoulli plate, which has localized K-V damping. They showed that the system is dissipative, with energy decaying only at a logarithmic rate as time approached infinity. Their proof employed frequency domain techniques, combining Carleman estimates with contradiction arguments. In [17], the authors studied a Kirchhoff plate-wave system with localized K-V damping on the plate. They proved that the system is dissipative and that its energy decays at a logarithmic rate. The analysis used a new resolvent estimate via Carleman techniques and reduction to second-order elliptic systems. To the best of our knowledge, none of the papers mentioned above addressed a plate-membrane transmission problem involving an Euler-Bernoulli plate with K-V damping coupled with a membrane that may or may not be subject to K-V damping.

The main results of this article include the existence, uniqueness, and regularity of solutions to the system (1.1)–(1.7). When the plate and the membrane are endowed with K-V damping, the associated semigroup on the energy space is analytic. This property plays a central role in the analysis and provides a functional-analytic framework for the study of stability mechanisms as well as for the potential optimization and control of coupled elastic structures arising in engineering applications. On the other hand, if K-V damping is applied only to the plate while the membrane is undamped, the solution exhibits polynomial decay with rate $t^{-1/10}$, provided a suitable geometric condition holds on the interface. We emphasize that this polynomial decay rate is better than the rates obtained in [5], where structural damping acts on the plate, and in [7], where the plate is thermoelastic. However, the present analysis does not establish whether the decay rate $t^{-1/10}$ is optimal.

We recall that transmission structures closely related to the one considered here can be found in [4, 5, 7]. In [4] and [7], the plate is subject to thermal effects, in contrast to our model, in which on the plate acts a K-V damping. In [5], as in the present setting, the plate is assumed to be isothermal; however, the damping mechanisms differ because there, the plate experiences structural damping, whereas the membrane is either undamped or subject to frictional damping. In this context, it is worth emphasizing that the particular main result of the present work, namely the generation of an analytic semigroup associated with the coupled system, is not established in any of the aforementioned references. The analyticity result, together with the polynomial decay rate obtained in the partially damped configuration (which is faster than the corresponding rates established in [5] and [7]), further differentiates the present work from the related literature.

The paper is organized as follows: In Section 2, we prove the well-posedness of the problem (1.1)–(1.7) using standard methods. In Section 3, we prove the analyticity of the semigroup associated to the problem when $\kappa_1 > 0$, and $\kappa_2 > 0$ using the frequency-domain method of semigroup theory and an argument by contradiction. Finally, in Section 4, we prove the polynomial stability when $\kappa_1 > 0$, and $\kappa_2 = 0$ using a result by Borichev and Tomilov.

Throughout the paper, as usual, the symbol $\mathscr{L}(X)$ denotes the space of bounded linear operators from X to X, where X is a normed vector space. The letter C represents a constant which may change from line to line. For $\mathcal{O} \subset \mathbb{R}^2$ open with sufficiently smooth boundary $\partial\mathcal{O}$, $S \subset \partial\mathcal{O}$, $s \in \mathbb{R}$, and $G \in \{\mathcal{O}, S\}$, $H^s(G)$ denotes the standard Sobolev space (or fractional Sobolev space when $s \notin \mathbb{N}$) of order s on G.

2. The first-order system associated with the problem

Throughout the paper, all vector spaces are taken over the field of complex numbers. Setting $w := (w_1, w_2, w_3, w_4)^\top := (u, u_t, v, v_t)^\top$, we rewrite (1.1), (1.2), and (1.7) in the form of the first-order Cauchy problem

$$\partial_t w(t) - \mathcal{A}w(t) = 0 \quad (t > 0), \qquad w(0) = w_0,$$

where $w_0 := (u^0, u^1, v^0, v^1)^\top$, and $\mathcal{A}$ is the matrix operator

$$\mathcal{A} := \begin{pmatrix} 0 & \mathcal{I} & 0 & 0 \\ -\Delta^2 & -\kappa_1\Delta^2 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{I} \\ 0 & 0 & \Delta & \kappa_2\Delta \end{pmatrix}. \tag{2.1}$$

Here, $\mathcal{I}$ denotes the identity operator. Let $H^2_\Gamma(\Omega_1) := \{u \in H^2(\Omega_1) : u = \partial_\nu u = 0 \text{ on } \Gamma\}$. On $H^2_\Gamma(\Omega_1)$, we define the inner product

$$\langle u, v\rangle_{H^2_\Gamma(\Omega_1)} := \int_{\Omega_1} (\nabla^2 u : \nabla^2\overline{v} + \mu[u, \overline{v}])\, dx$$

for $u, v \in H^2_\Gamma(\Omega_1)$, where

$$[u, v] := u_{x_1x_1}v_{x_2x_2} + u_{x_2x_2}v_{x_1x_1} - 2u_{x_1x_2}v_{x_1x_2}$$

denotes the von Karman bracket (see [10, Section 1.4]), and

$$\nabla^2 u : \nabla^2\overline{v} := u_{x_1x_1}\overline{v}_{x_1x_1} + u_{x_2x_2}\overline{v}_{x_2x_2} + 2u_{x_1x_2}\overline{v}_{x_1x_2}.$$

Note that

$$\langle u, v\rangle_{H^2_\Gamma(\Omega_1)} = \mu\langle\Delta u, \Delta v\rangle_{L^2(\Omega_1)} + (1-\mu)\langle\nabla^2 u, \nabla^2 v\rangle_{L^2(\Omega_1)^{2\times2}}$$

for all $u, v \in H^2_\Gamma(\Omega_1)$, where

$$\langle\nabla^2 u, \nabla^2 v\rangle_{L^2(\Omega_1)^{2\times2}} := \int_{\Omega_1} \nabla^2 u : \nabla^2\overline{v}\, dx.$$

By Friedrichs' inequality (see [22, Theorem 1.9]), the corresponding norm $\|\cdot\|_{H^2_\Gamma(\Omega_1)}$ is equivalent to the standard $H^2(\Omega_1)$-norm on $H^2_\Gamma(\Omega_1)$. In particular, $(H^2_\Gamma(\Omega_1), \langle\cdot,\cdot\rangle_{H^2_\Gamma(\Omega_1)})$ is a Hilbert space. Because of Lemma 2.1 in [5], we have for all $u \in H^4(\Omega_1) \cap H^2_\Gamma(\Omega_1)$ and $v \in H^2_\Gamma(\Omega_1)$ that

$$\langle\Delta^2 u, v\rangle_{L^2(\Omega_1)} = \langle u, v\rangle_{H^2_\Gamma(\Omega_1)} - \langle\mathscr{B}_1 u, \partial_\nu v\rangle_{L^2(I)} + \langle\mathscr{B}_2 u, v\rangle_{L^2(I)}. \tag{2.2}$$

Now, we define

$$\mathscr{H} := \{w \in H^2_\Gamma(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2) : w_1 = w_3 \text{ on } I\}$$

endowed with the inner product

$$\langle w, \widetilde{w}\rangle_{\mathscr{H}} := \langle w_1, \widetilde{w}_1\rangle_{H^2_\Gamma(\Omega_1)} + \langle w_2, \widetilde{w}_2\rangle_{L^2(\Omega_1)} + \langle\nabla w_3, \nabla\widetilde{w}_3\rangle_{L^2(\Omega_2)^2} + \langle w_4, \widetilde{w}_4\rangle_{L^2(\Omega_2)},$$

and

$$\begin{aligned} D(\mathcal{A}) := \{w \in [H^2_\Gamma(\Omega_1)]^2 \times [H^1(\Omega_2)]^2 :\ & \Delta^2 w_1 + \kappa_1\Delta^2 w_2 \in L^2(\Omega_1), \\ & \Delta w_3 + \kappa_2\Delta w_4 \in L^2(\Omega_2) \text{ and } w_j = w_{j+2} \text{ on } I \text{ for } j = 1, 2\}. \end{aligned}$$

One can show that $(\mathscr{H}, \langle\cdot,\cdot\rangle_{\mathscr{H}})$ is a Hilbert space. Note that $D(\mathcal{A}) \subset \mathscr{H}$, and if $w \in D(\mathcal{A})$, then $\mathcal{A}w \in \mathscr{H}$. For sufficiently smooth $w \in D(\mathcal{A})$ and $\widetilde{w} \in [H^2_\Gamma(\Omega_1)]^2 \times [H^1(\Omega_2)]^2$ satisfying $\widetilde{w}_j = \widetilde{w}_{j+2}$ on I for $j = 1, 2$, we obtain by (2.2) and Green's formulas that

$$\begin{aligned}
&\langle \mathcal{A}w, \widetilde{w}\rangle_{\mathscr{H}} \\
=&\langle w_2, \widetilde{w}_1\rangle_{H^2_\Gamma(\Omega_1)} - \langle \Delta^2(w_1 + \kappa_1 w_2), \widetilde{w}_2\rangle_{L^2(\Omega_1)} + \langle \nabla w_4, \nabla\widetilde{w}_3\rangle_{L^2(\Omega_2)^2} + \langle \Delta(w_3 + \kappa_2 w_4), \widetilde{w}_4\rangle_{L^2(\Omega_2)} \\
=&\langle w_2, \widetilde{w}_1\rangle_{H^2_\Gamma(\Omega_1)} - \langle w_1 + \kappa_1 w_2, \widetilde{w}_2\rangle_{H^2_\Gamma(\Omega_1)} + \langle \mathscr{B}_1(w_1 + \kappa_1 w_2), \partial_\nu \widetilde{w}_2\rangle_{L^2(I)} - \langle \mathscr{B}_2(w_1 + \kappa_1 w_2), \widetilde{w}_2\rangle_{L^2(I)} \\
&+ \langle \nabla w_4, \nabla\widetilde{w}_3\rangle_{L^2(\Omega_2)^2} - \langle \nabla(w_3 + \kappa_2 w_4), \nabla\widetilde{w}_4\rangle_{L^2(\Omega_2)^2} - \langle \partial_\nu(w_3 + \kappa_2 w_4), \widetilde{w}_4\rangle_{L^2(I)} \\
=&\langle w_2, \widetilde{w}_1\rangle_{H^2_\Gamma(\Omega_1)} - \langle w_1, \widetilde{w}_2\rangle_{H^2_\Gamma(\Omega_1)} - \kappa_1\langle w_2, \widetilde{w}_2\rangle_{H^2_\Gamma(\Omega_1)} + \langle \nabla w_4, \nabla\widetilde{w}_3\rangle_{L^2(\Omega_2)^2} - \langle \nabla w_3, \nabla\widetilde{w}_4\rangle_{L^2(\Omega_2)^2} \\
&- \kappa_2\langle \nabla w_4, \nabla\widetilde{w}_4\rangle_{L^2(\Omega_2)^2} + \langle \mathscr{B}_1(w_1 + \kappa_1 w_2), \partial_\nu \widetilde{w}_2\rangle_{L^2(I)} - \langle \mathscr{B}_2(w_1 + \kappa_1 w_2) + \partial_\nu(w_3 + \kappa_2 w_4), \widetilde{w}_2\rangle_{L^2(I)}.
\end{aligned} \tag{2.3}$$

Now, note that the boundary and transmission conditions (1.4) and (1.6) read in terms of the components of the unknown w in the first-order problem associated with (1.1) and (1.2) as

$$\mathscr{B}_1(w_1 + \kappa_1 w_2) = 0 \quad \text{on} \quad I, \tag{2.4}$$

and

$$\mathscr{B}_2(w_1 + \kappa_1 w_2) = -\partial_\nu(w_3 + \kappa_2 w_4) \quad \text{on} \quad I. \tag{2.5}$$

Equation (2.3) motivates the following weak interpretation of the boundary and transmission conditions (2.4) and (2.5), which are associated with the conditions (1.4) and (1.6).

Definition 2.1. *We say that $w \in D(\mathcal{A})$ satisfies the boundary and transmission conditions* (2.4) *and* (2.5) *weakly if the equality*

$$\langle \mathcal{A}w, \widetilde{w}\rangle_{\mathscr{H}} = \langle w_2, \widetilde{w}_1\rangle_{H^2_\Gamma(\Omega_1)} - \langle w_1 + \kappa_1 w_2, \widetilde{w}_2\rangle_{H^2_\Gamma(\Omega_1)} + \langle \nabla w_4, \nabla\widetilde{w}_3\rangle_{L^2(\Omega_2)^2} - \langle \nabla(w_3 + \kappa_2 w_4), \nabla\widetilde{w}_4\rangle_{L^2(\Omega_2)^2}$$

is satisfied for all $\widetilde{w} \in [H^2_\Gamma(\Omega_1)]^2 \times [H^1(\Omega_2)]^2$ satisfying $\widetilde{w}_j = \widetilde{w}_{j+2}$ on I for $j = 1, 2$.

Now, we consider the unbounded linear operator $\mathscr{A} : D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ given by $\mathscr{A}w := \mathcal{A}w$ with the domain

$$D(\mathscr{A}) := \{w \in D(\mathcal{A}) : \text{(2.4) and (2.5) are weakly satisfied}\}.$$

Note that

$$\mathrm{Re}\langle \mathscr{A}w, w\rangle_{\mathscr{H}} = -\kappa_1\|w_2\|^2_{H^2_\Gamma(\Omega_1)} - \kappa_2\|\nabla w_4\|^2_{L^2(\Omega_2)^2} \tag{2.6}$$

for all $w \in D(\mathscr{A})$. Thus, the operator $\mathscr{A}$ is dissipative. Observe that the previous equality is related to the derivative of energy (1.9). This computation shows that for any smooth solution (u, v) of (1.1)–(1.6), the energy (1.8) is decreasing.

Theorem 2.1. *Suppose $\kappa_1 > 0$, and $\kappa_2 \geq 0$. The operator $\mathscr{A}$ generates a C_0-semigroup $(\mathscr{T}(t))_{t\geq 0}$ of contractions on $\mathscr{H}$. Therefore, for all $w_0 \in D(\mathscr{A})$, the Cauchy problem*

$$\partial_t w(t) = \mathscr{A}w(t) \quad (t > 0), \quad w(0) = w_0 \tag{2.7}$$

has a unique classical solution $w \in C^1([0, \infty), \mathscr{H})$ such that $w(t) \in D(\mathscr{A})$ for all $t \geq 0$, given by $w(t) = \mathscr{T}(t)w_0$.

Proof. We already know that $\mathscr{A}$ is dissipative. We now prove that the operator $\mathcal{I} - \mathscr{A} : D(\mathscr{A}) \to \mathscr{H}$ is surjective. Let $f \in \mathscr{H}$. We must find $w \in D(\mathscr{A})$ such that

$$w - \mathscr{A} w = f, \tag{2.8}$$

or, equivalently,

$$\begin{aligned} w_1 - w_2 &= f_1 \ \text{ in } \ H^2_\Gamma(\Omega_1), \\ w_2 + \Delta^2 w_1 + \kappa_1 \Delta^2 w_2 &= f_2 \ \text{ in } \ L^2(\Omega_1), \\ w_3 - w_4 &= f_3 \ \text{ in } \ H^1(\Omega_2), \\ w_4 - \Delta w_3 - \kappa_2 \Delta w_4 &= f_4 \ \text{ in } \ L^2(\Omega_2). \end{aligned} \tag{2.9}$$

To this end, we consider $\widetilde{w} \in [H^2_\Gamma(\Omega_1)]^2 \times [H^1(\Omega_2)]^2$ with $\widetilde{w}_j = \widetilde{w}_{j+2}$ on I for $j \in \{1, 2\}$, as in Definition 2.1. If we had a solution $w \in D(\mathscr{A})$ for (2.8) (equivalently (2.9)), we would have that $w_1 = w_2 + f_1$, $w_3 = w_4 + f_3$, and from Definition 2.1 that

$$\begin{aligned} \langle f, \widetilde{w}\rangle_{\mathscr{H}} &= \langle w, \widetilde{w}\rangle_{\mathscr{H}} - \langle \mathscr{A} w, \widetilde{w}\rangle_{\mathscr{H}} \\ &= \langle f_1, \widetilde{w}_1\rangle_{H^2_\Gamma(\Omega_1)} + \langle w_2, \widetilde{w}_2\rangle_{L^2(\Omega_1)} + \langle \nabla f_3, \nabla \widetilde{w}_3\rangle_{L^2(\Omega_2)^2} + \langle w_4, \widetilde{w}_4\rangle_{L^2(\Omega_2)} + \langle w_2, \widetilde{w}_2\rangle_{H^2_\Gamma(\Omega_1)} \\ &\quad + \kappa_1 \langle w_2, \widetilde{w}_2\rangle_{H^2_\Gamma(\Omega_1)} + \langle f_1, \widetilde{w}_2\rangle_{H^2_\Gamma(\Omega_1)} + (1 + \kappa_2)\langle \nabla w_4, \nabla \widetilde{w}_4\rangle_{L^2(\Omega_2)^2} + \langle \nabla f_3, \nabla \widetilde{w}_4\rangle_{L^2(\Omega_2)^2}. \end{aligned} \tag{2.10}$$

Furthermore, because

$$\langle f, \widetilde{w}\rangle_{\mathscr{H}} = \langle f_1, \widetilde{w}_1\rangle_{H^2_\Gamma(\Omega_1)} + \langle f_2, \widetilde{w}_2\rangle_{L^2(\Omega_1)} + \langle \nabla f_3, \nabla \widetilde{w}_3\rangle_{L^2(\Omega_2)^2} + \langle f_4, \widetilde{w}_4\rangle_{L^2(\Omega_2)},$$

we would obtain that

$$\begin{aligned} &\langle w_2, \widetilde{w}_2\rangle_{L^2(\Omega_1)} + (1 + \kappa_1)\langle w_2, \widetilde{w}_2\rangle_{H^2_\Gamma(\Omega_1)} + \langle w_4, \widetilde{w}_4\rangle_{L^2(\Omega_2)} + (1 + \kappa_2)\langle \nabla w_4, \nabla \widetilde{w}_4\rangle_{L^2(\Omega_2)^2} \\ &= -\langle f_1, \widetilde{w}_2\rangle_{H^2_\Gamma(\Omega_1)} + \langle f_2, \widetilde{w}_2\rangle_{L^2(\Omega_1)} - \langle \nabla f_3, \nabla \widetilde{w}_4\rangle_{L^2(\Omega_2)^2} + \langle f_4, \widetilde{w}_4\rangle_{L^2(\Omega_2)}. \end{aligned} \tag{2.11}$$

The equality (2.11) motivates the introduction of the following space. Let

$$\mathcal{V} := \{V = (V_1, V_2) \in H^2_\Gamma(\Omega_1) \times H^1(\Omega_2) : V_1|_I = V_2|_I\}$$

be endowed with the inner product

$$\langle V, \widetilde{V}\rangle_{\mathcal{V}} := \langle V_1, \widetilde{V}_1\rangle_{L^2(\Omega_1)} + (1 + \kappa_1)\langle V_1, \widetilde{V}_1\rangle_{H^2_\Gamma(\Omega_1)} + \langle V_2, \widetilde{V}_2\rangle_{L^2(\Omega_2)} + (1 + \kappa_2)\langle \nabla V_2, \nabla \widetilde{V}_2\rangle_{L^2(\Omega_2)^2}.$$

Note that the mapping $F : \mathcal{V} \to \mathbb{C}$ given by

$$F(V) := -\langle f_1, V_1\rangle_{H^2_\Gamma(\Omega_1)} + \langle f_2, V_1\rangle_{L^2(\Omega_1)} - \langle \nabla f_3, \nabla V_2\rangle_{L^2(\Omega_2)^2} + \langle f_4, V_2\rangle_{L^2(\Omega_2)}$$

is antilinear and continuous. Observe that (2.11) can be rewritten as

$$\langle (w_2, w_4), (\widetilde{w}_2, \widetilde{w}_4)\rangle_{\mathcal{V}} = F(\widetilde{w}_2, \widetilde{w}_4). \tag{2.12}$$

Because $(\mathcal{V}, \langle\cdot,\cdot\rangle_{\mathcal{V}})$ is a Hilbert space, the Riesz representation theorem implies that there exists a unique $W := (W_1, W_2) \in \mathcal{V}$ such that

$$\langle W, \widetilde{W}\rangle_{\mathcal{V}} = F(\widetilde{W}) \quad \text{for all} \quad \widetilde{W} \in \mathcal{V}. \tag{2.13}$$

Let $w := (w_1, w_2, w_3, w_4)^\top := (W_1 + f_1, W_1, W_2 + f_3, W_2)^\top$. We show that $w \in D(\mathscr{A})$ and that w satisfies (2.9). It is clear that $w \in H^2_\Gamma(\Omega_1) \times H^2_\Gamma(\Omega_1) \times H^1(\Omega_2) \times H^1(\Omega_2)$ and that the first and third equations in (2.9) are satisfied by w. Moreover,

$$w_1\big|_I = W_1 + f_1\big|_I = W_2 + f_3\big|_I = w_3\big|_I \quad \text{and} \quad w_2\big|_I = W_1\big|_I = W_2\big|_I = w_4\big|_I.$$

Let $\varphi \in C_c^\infty(\Omega_1)$, and $\widetilde{W} := (\varphi, 0)$. Because $\widetilde{W} \in \mathcal{V}$, it follows from (2.13) that

$$\langle W_1, \varphi\rangle_{L^2(\Omega_1)} + (1+\kappa_1)\langle W_1, \varphi\rangle_{H^2_\Gamma(\Omega_1)} = -\langle f_1, \varphi\rangle_{H^2_\Gamma(\Omega_1)} + \langle f_2, \varphi\rangle_{L^2(\Omega_1)}. \tag{2.14}$$

From (2.2), we obtain

$$\begin{aligned}\langle W_1, \varphi\rangle_{H^2_\Gamma(\Omega_1)} &= \overline{\langle \varphi, W_1\rangle}_{H^2_\Gamma(\Omega_1)} = \overline{\langle \Delta^2\varphi, W_1\rangle}_{L^2(\Omega_1)} = \langle W_1, \Delta^2\varphi\rangle_{L^2(\Omega_1)}\\ &= \langle W_1, \Delta^2\varphi\rangle_{\mathscr{D}'(\Omega_1)\times\mathscr{D}(\Omega_1)} = \langle \Delta^2 W_1, \varphi\rangle_{\mathscr{D}'(\Omega_1)\times\mathscr{D}(\Omega_1)}.\end{aligned} \tag{2.15}$$

Analogously,

$$\langle f_1, \varphi\rangle_{H^2_\Gamma(\Omega_1)} = \langle \Delta^2 f_1, \varphi\rangle_{\mathscr{D}'(\Omega_1)\times\mathscr{D}(\Omega_1)}. \tag{2.16}$$

It follows from (2.14)–(2.16) that

$$\langle W_1, \varphi\rangle_{\mathscr{D}'(\Omega_1)\times\mathscr{D}(\Omega_1)} + (1+\kappa_1)\langle \Delta^2 W_1, \varphi\rangle_{\mathscr{D}'(\Omega_1)\times\mathscr{D}(\Omega_1)} = -\langle \Delta^2 f_1, \varphi\rangle_{\mathscr{D}'(\Omega_1)\times\mathscr{D}(\Omega_1)} + \langle f_2, \varphi\rangle_{\mathscr{D}'(\Omega_1)\times\mathscr{D}(\Omega_1)}.$$

Then, we have

$$\langle \Delta^2[(W_1 + f_1) + \kappa_1 W_1], \varphi\rangle_{\mathscr{D}'(\Omega_1)\times\mathscr{D}(\Omega_1)} = \langle f_2 - W_1, \varphi\rangle_{\mathscr{D}'(\Omega_1)\times\mathscr{D}(\Omega_1)}.$$

Therefore,

$$\Delta^2 w_1 + \kappa_1 \Delta^2 w_2 = \Delta^2(w_1 + \kappa_1 w_2) = f_2 - W_1 = f_2 - w_2 \quad \text{in } \mathscr{D}'(\Omega_1).$$

As $f_2 - W_1 \in L^2(\Omega_1)$, we obtain that $\Delta^2 w_1 + \kappa_1\Delta^2 w_2 = \Delta^2(w_1 + \kappa_1 w_2) \in L^2(\Omega_1)$ and that w satisfies the second equation in (2.9). In a similar way, we have

$$\Delta w_3 + \kappa_2 \Delta w_4 = \Delta(w_3 + \kappa_2 w_4) = W_2 - f_4 = w_4 - f_4 \quad \text{in } \mathscr{D}'(\Omega_2).$$

As $W_2 - f_4 \in L^2(\Omega_2)$, we obtain that $\Delta w_3 + \kappa_2\Delta w_4 = \Delta(w_3 + \kappa_2 w_4) \in L^2(\Omega_2)$ and that w also satisfies the fourth equation in (2.9). Then, we have that $w \in D(\mathcal{A})$, and $w - \mathcal{A}w = f$. Therefore, for the surjectivity of $\mathcal{I} - \mathscr{A}$, it remains to show that w satisfies the boundary and transmission conditions (2.4) and (2.5) in the weak sense, which will imply that $w \in D(\mathscr{A})$, and $w - \mathscr{A}w = f$. Indeed, if we take again $\widetilde{w} \in [H^2_\Gamma(\Omega_1)]^2 \times [H^1(\Omega_2)]^2$ with $\widetilde{w}_j = \widetilde{w}_{j+2}$ on I for $j \in \{1, 2\}$, we have $(\widetilde{w}_2, \widetilde{w}_4) \in \mathcal{V}$. Then, (2.13) implies that (2.12) holds for our w and $\widetilde{w}$. In turn, (2.12) implies (2.11), and from this, (2.10) holds. From (2.10), we can see that w satisfies the boundary and transmission conditions (2.4) and (2.5) in the weak sense.

By [23, Theorem 4.6] and the Lumer-Phillips theorem, $\mathscr{A}$ generates a C_0-semigroup of contractions on $\mathscr{H}$. □

Similarly to Section 4 in [5], we can show that if $w = (w_1, w_2, w_3, w_4)^\top \in D(\mathscr{A})$, then $w_1 + \kappa_1 w_2$ and $w_3 + \kappa_2 w_4$ have enough regularity in order to satisfy the boundary and transmission conditions (2.4) and (2.5) in the strong sense of traces on I. We will use such a result in Section 3. Exactly, we have the following proposition.

Proposition 2.1. *If* $w = (w_1, w_2, w_3, w_4)^\top \in D(\mathscr{A})$*, then* $w_1 + \kappa_1 w_2 \in H^4(\Omega_1)$*, and* $w_3 + \kappa_2 w_4 \in H^2(\Omega_2)$*.*

Proof. Let $w = (w_1, w_2, w_3, w_4)^\top \in D(\mathscr{A})$, and $F = (f_1, f_2, f_3, f_4)^\top := \mathscr{A}w$. Then, $f_1 = w_2 \in H^2_\Gamma(\Omega_1)$, $f_3 = w_4 \in H^1(\Omega_2)$, $\Delta(w_3 + \kappa_2 w_4) = f_4$, and $\Delta^2(w_1 + \kappa_1 w_2) = -f_2$.

Because $f_4 \in L^2(\Omega_2)$, and $(w_1 + \kappa_2 w_2)\big|_I \in H^{3/2}(I)$, Remark 4.4(b) in [5] implies that there exists a unique $v \in H^2(\Omega_2)$ such that

$$\begin{aligned} \Delta v &= f_4 \quad \text{in} \quad \Omega_2, \\ v &= (w_1 + \kappa_2 w_2)\big|_I \quad \text{on} \quad I. \end{aligned}$$

Because $w_3 + \kappa_2 w_4 = w_1 + \kappa_2 w_2$ on I, we have $w_3 + \kappa_2 w_4 - v \in H^1_0(\Omega_2)$. Because $\Delta(w_3 + \kappa_2 w_4 - v) = 0$ in the weak sense in $H^1_0(\Omega_2)$, we have $w_3 + \kappa_2 w_4 - v = 0$. Therefore, $w_3 + \kappa_2 w_4 = v \in H^2(\Omega_2)$.

Now, for sufficiently large $\lambda_0 > 0$, we consider the following boundary value problem:

$$\begin{aligned} (\lambda_0 + \Delta^2)u &= \lambda_0(w_1 + \kappa_1 w_2) - f_2 \quad \text{in} \quad \Omega_1, \\ u = \partial_\nu u &= 0 \quad \text{on} \quad \Gamma, \\ \mathscr{B}_1 u &= 0 \quad \text{on} \quad I, \\ \mathscr{B}_2 u &= -\partial_\nu(w_3 + \kappa_2 w_4) \quad \text{on} \quad I. \end{aligned} \tag{2.17}$$

Because $\lambda_0(w_1+\kappa_1 w_2)-f_2 \in L^2(\Omega_1)$, and $-\partial_\nu(w_3+\kappa_2 w_4) \in H^{1/2}(I)$, we have by Corollary 4.3 in [5] that the problem (2.17) has a unique solution $u \in H^4(\Omega_1)$. Let $\varphi \in H^2_\Gamma(\Omega_1)$. Because $u \in H^4(\Omega_1) \cap H^2_\Gamma(\Omega_1)$, from (2.2) and the boundary conditions in (2.17), we obtain

$$\begin{aligned} \langle(\lambda_0 + \Delta^2)u, \varphi\rangle_{L^2(\Omega_1)} &= \lambda_0\langle u, \varphi\rangle_{L^2(\Omega_1)} + \langle u, \varphi\rangle_{H^2_\Gamma(\Omega_1)} + \langle \mathscr{B}_2 u, \varphi\rangle_{L^2(I)} \\ &= \lambda_0\langle u, \varphi\rangle_{L^2(\Omega_1)} + \langle u, \varphi\rangle_{H^2_\Gamma(\Omega_1)} - \langle \partial_\nu(w_3 + \kappa_2 w_4), \varphi\rangle_{L^2(I)}. \end{aligned} \tag{2.18}$$

Now, let $\varphi \in H^2_\Gamma(\Omega_1)$, and $\psi \in H^1(\Omega_2)$ with $\varphi\big|_I = \psi\big|_I$, and let $\widetilde{w} := (0, \varphi, 0, \psi)^\top$. By the definition of $D(\mathscr{A})$, (2.4) and (2.5) are weakly satisfied by w (see Definition 2.1). Then, it follows that

$$\begin{aligned} \langle \mathscr{A}w, \widetilde{w}\rangle_{\mathscr{H}} &= \langle -\Delta^2(w_1 + \kappa_1 w_2), \varphi\rangle_{L^2(\Omega_1)} + \langle \Delta(w_3 + \kappa_2 w_4), \psi\rangle_{L^2(\Omega_2)} \\ &= -\langle w_1 + \kappa_1 w_2, \varphi\rangle_{H^2_\Gamma(\Omega_1)} - \langle \nabla(w_3 + \kappa_2 w_4), \nabla\psi\rangle_{L^2(\Omega_2)^2}. \end{aligned}$$

Because $w_3 + \kappa_2 w_4 \in H^2(\Omega_2)$, integration by parts applied to the term $\langle \Delta(w_3 + \kappa_2 w_4), \psi\rangle_{L^2(\Omega_2)}$ in the last equality yields

$$\langle \Delta^2(w_1 + \kappa_1 w_2), \varphi\rangle_{L^2(\Omega_1)} = -\langle \partial_\nu(w_3 + \kappa_2 w_4), \psi\rangle_{L^2(I)} + \langle w_1 + \kappa_1 w_2, \varphi\rangle_{H^2_\Gamma(\Omega_1)}.$$

Then,

$$\begin{aligned} &\langle(\lambda_0 + \Delta^2)(w_1 + \kappa_1 w_2), \varphi\rangle_{L^2(\Omega_1)} \\ &= \lambda_0\langle w_1 + \kappa_1 w_2, \varphi\rangle_{L^2(\Omega_1)} + \langle w_1 + \kappa_1 w_2, \varphi\rangle_{H^2_\Gamma(\Omega_1)} - \langle \partial_\nu(w_3 + \kappa_2 w_4), \psi\rangle_{L^2(I)}. \end{aligned} \tag{2.19}$$

Note that $(\lambda_0 + \Delta^2)u = \lambda_0(w_1 + \kappa_1 w_2) - f_2 = (\lambda_0 + \Delta^2)(w_1 + \kappa_1 w_2)$ in Ω_1. Then, subtracting (2.19) from (2.18) we have

$$0 = \lambda_0\langle u - (w_1 + \kappa_1 w_2), \varphi\rangle_{L^2(\Omega_1)} + \langle u - (w_1 + \kappa_1 w_2), \varphi\rangle_{H^2_\Gamma(\Omega_1)}$$

for all $\varphi \in H^2_\Gamma(\Omega_1)$. Because $u - (w_1 + \kappa_1 w_2) \in H^2_\Gamma(\Omega_1)$, we obtain $u - (w_1 + \kappa_1 w_2) = 0$; that is, $w_1 + \kappa_1 w_2 = u \in H^4(\Omega_1)$. □

The following remark will be very useful in the next sections.

Remark 2.1. *If $u \in H^2_\Gamma(\Omega_1)$, and $v \in H^1(\Omega_2)$ with $u = v$ on I, then it holds that $u\chi_{\Omega_1} + v\chi_{\Omega_2} \in H^1_\Gamma(\Omega)$, where χ_{Ω_j} stand for the characteristic function of set Ω_j. From this and Poincaré's inequality, it follows that*

$$\|u\|^2_{L^2(\Omega_1)} + \|v\|^2_{L^2(\Omega_2)} \leq C_\Omega\Big(\|\nabla u\|^2_{L^2(\Omega_1)^2} + \|\nabla v\|^2_{L^2(\Omega_2)^2}\Big).$$

3. Analyticity

In this section, we will establish the analyticity of the solutions to the problem (1.1)–(1.7) when both the plate and the membrane have K-V damping. Our proof of analyticity, which will be by contradiction, is supported by the following well-known result.

Theorem 3.1 ([20, Theorem 1.3.3]). *Let $(T_A(t))_{t\geq 0}$ be a C_0-semigroup of contractions in a Hilbert space H generated by A. Suppose that $i\mathbb{R} \subset \rho(A)$. Then, $T_A(t)$ is analytic if and only if*

$$\limsup_{|\lambda|\to\infty} \|\lambda(i\lambda\mathcal{I} - A)^{-1}\|_{\mathscr{L}(H)} < \infty. \tag{3.1}$$

Proposition 3.1. *If $\kappa_1 > 0$, and $\kappa_2 \geq 0$, then $0 \in \rho(\mathscr{A})$.*

Proof. Let us see that operator $\mathscr{A} : D(\mathscr{A}) \to \mathscr{H}$ is bijective. Let $f \in \mathscr{H}$. Next, we will guarantee the existence of a unique $w \in D(\mathscr{A})$ such that $\mathscr{A}w = f$. For this reason, we put $w_2 := f_1$, $w_4 := f_3$ and consider the system

$$\begin{aligned} -\Delta^2 w_1 - \kappa_1\Delta^2 f_1 &= f_2 \ \text{ in } \ L^2(\Omega_1),\\ \Delta w_3 + \kappa_2\Delta f_3 &= f_4 \ \text{ in } \ L^2(\Omega_2). \end{aligned} \tag{3.2}$$

The space $\mathcal{W} := \{(\widetilde{w}_1, \widetilde{w}_3) \in H^2_\Gamma(\Omega_1) \times H^1(\Omega_2) : \widetilde{w}_1 = \widetilde{w}_3 \text{ on } I\}$ endowed with the inner product

$$\langle(\widetilde{w}_1, \widetilde{w}_3), (\widehat{w}_1, \widehat{w}_3)\rangle_{\mathcal{W}} := \langle\widetilde{w}_1, \widehat{w}_1\rangle_{H^2_\Gamma(\Omega_1)} + \langle\nabla\widetilde{w}_3, \nabla\widehat{w}_3\rangle_{L^2(\Omega_2)^2}$$

is a Hilbert space. Note that the application $\mathbb{A} : \mathcal{W} \to \mathbb{C}$ given by

$$\mathbb{A}(\phi_1, \phi_3) := -\kappa_1\langle f_1, \phi_1\rangle_{H^2_\Gamma(\Omega_1)} - \langle f_2, \phi_1\rangle_{L^2(\Omega_1)} - \kappa_2\langle\nabla f_3, \nabla\phi_3\rangle_{L^2(\Omega_2)^2} - \langle f_4, \phi_3\rangle_{L^2(\Omega_2)}$$

belongs to the antidual space $\mathcal{W}'$. By the Riesz representation theorem, there exists a unique $(w_1, w_3) \in \mathcal{W}$ such that

$$\langle(w_1, w_3), (\phi_1, \phi_3)\rangle_{\mathcal{W}} = \mathbb{A}(\phi_1, \phi_3) \quad \text{for all} \quad (\phi_1, \phi_3) \in \mathcal{W}. \tag{3.3}$$

Let $(\psi_1, \psi_3) \in C^\infty_c(\Omega_1) \times C^\infty_c(\Omega_2)$. Note that (ψ_1, ψ_3) is also an element of $\mathcal{W}$. Taking into account (2.2) and (3.3), and using an argument as in (2.15), we obtain

$$\langle\Delta^2 w_1 + \kappa_1\Delta^2 f_1 + f_2, \psi_1\rangle_{\Xi(\Omega_1)} + \langle-\Delta w_3 - \kappa_2\Delta f_3 + f_4, \psi_3\rangle_{\Xi(\Omega_2)} = 0, \tag{3.4}$$

where $\Xi(\Omega_j) := \mathscr{D}'(\Omega_j) \times \mathscr{D}(\Omega_j)$ for $j = 1, 2$. Because $f_2 \in L^2(\Omega_1)$, and $f_3, f_4 \in L^2(\Omega_2)$, Eq (3.4) implies that $\Delta^2 w_1 + \kappa_1\Delta^2 w_2 \in L^2(\Omega_1)$, and $\Delta w_3 + \kappa_2\Delta w_4 \in L^2(\Omega_2)$. Therefore, $w \in D(\mathcal{A})$. Observe that we have also established the validity of the equations in (3.2), and thus,

$$\mathcal{A}w = f.$$

Let $\widetilde{w} \in [H^2_\Gamma(\Omega_1)]^2 \times [H^1(\Omega_2)]^2$ with $\widetilde{w}_j = \widetilde{w}_{j+2}$ for $j = 1, 2$. By (2.1), (3.2), and (3.3), we compute

$$\begin{aligned}
&\langle \mathcal{A} w, \widetilde{w} \rangle_{\mathcal{H}} \\
&= \langle w_2, \widetilde{w}_1 \rangle_{H^2_\Gamma(\Omega_1)} + \langle -\Delta^2 w_1 - \kappa_1 \Delta^2 w_2, \widetilde{w}_2 \rangle_{L^2(\Omega_1)} + \langle \nabla w_4, \nabla \widetilde{w}_3 \rangle_{L^2(\Omega_2)^2} + \langle \Delta w_3 + \kappa_2 \Delta w_4, \widetilde{w}_4 \rangle_{L^2(\Omega_2)} \\
&= \langle w_2, \widetilde{w}_1 \rangle_{H^2_\Gamma(\Omega_1)} + \langle f_2, \widetilde{w}_2 \rangle_{L^2(\Omega_1)} + \langle \nabla w_4, \nabla \widetilde{w}_3 \rangle_{L^2(\Omega_2)^2} + \langle f_4, \widetilde{w}_4 \rangle_{L^2(\Omega_2)} \\
&= \langle w_2, \widetilde{w}_1 \rangle_{H^2_\Gamma(\Omega_1)} + \langle \nabla w_4, \nabla \widetilde{w}_3 \rangle_{L^2(\Omega_2)^2} - \langle w_1, \widetilde{w}_2 \rangle_{H^2_\Gamma(\Omega_1)} - \langle \nabla w_3, \nabla \widetilde{w}_4 \rangle_{L^2(\Omega_2)^2} - \kappa_1 \langle w_2, \widetilde{w}_2 \rangle_{H^2_\Gamma(\Omega_1)} \\
&\quad - \kappa_2 \langle \nabla w_4, \nabla \widetilde{w}_4 \rangle_{L^2(\Omega_2)^2}.
\end{aligned}$$

This shows that w weakly satisfies the boundary and transmission conditions (2.4) and (2.5). We have shown that there exists a unique $w \in D(\mathscr{A})$ satisfying $\mathscr{A} w = f$. Thus, $\mathscr{A} : D(\mathscr{A}) \to \mathscr{H}$ is bijective, and because it is also a closed operator (see Theorem 2.1 and [23, Corollary 2.5]), then $\mathscr{A}^{-1} : \mathscr{H} \to \mathscr{H}$ is a continuous operator. Hence, $0 \in \rho(\mathscr{A})$. □

The following result will be very useful for some estimates in the proof of the main result of this section and also in the last section.

Lemma 3.1. *Let O be a bounded domain in $\mathbb{R}^n$ with C^1 boundary ∂O. Then, for any function $u \in H^1(O)$, the following estimate holds:*

$$\|u\|_{L^2(\partial O)} \leq C \|u\|^{1/2}_{H^1(O)} \|u\|^{1/2}_{L^2(O)},$$

with C being a positive constant independent of u.

Proof. See Theorem 1.4.4 in [20]. □

Proposition 3.2. *If $\kappa_1 > 0$, and $\kappa_2 \geq 0$, then the imaginary axis is contained in the resolvent set of $\mathscr{A}$, that is, $i\mathbb{R} \subset \rho(\mathscr{A})$.*

Proof. Let $\kappa_1 > 0$, and $\kappa_2 \geq 0$. Because $\rho(\mathscr{A})$ is open in $\mathbb{C}$, and $0 \in \rho(\mathscr{A})$, there exists a ball $B_R(0)$ centered at the origin with radius $R > 0$ such that $B_R(0) \subset \rho(\mathscr{A})$. Let $\delta \in \mathbb{R}$ with $0 < \delta < R$. Observe that $i[-\delta, \delta] \subset \rho(\mathscr{A})$. Therefore, the set $\mathcal{R} := \{\lambda > 0 : i[-\lambda, \lambda] \subset \rho(\mathscr{A})\}$ is nonempty. Let $\lambda^* := \sup \mathcal{R}$. If $\lambda^* = \infty$, we have nothing to prove. Now, suppose $\lambda^* < \infty$. Note that $\lambda^* > 0$. Clearly, $\lambda^* \notin \mathcal{R}$. Then, there is $(\lambda_n)_{n\in\mathbb{N}} \subset \mathcal{R}$ such that $\lambda_n \to \lambda^*$, and $\|(i\lambda_n \mathcal{I} - \mathscr{A})^{-1}\|_{\mathcal{L}(\mathscr{H})} \to \infty$ as $n \to \infty$. Thus, there exists $(\widetilde{f}_n)_{n\in\mathbb{N}} \subset \mathscr{H}$ with

$$\|\widetilde{f}_n\|_{\mathscr{H}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \|(i\lambda_n \mathcal{I} - \mathscr{A})^{-1} \widetilde{f}_n\|_{\mathscr{H}} = \infty.$$

Now, we will only consider the nonzero terms of $((i\lambda_n \mathcal{I} - \mathscr{A})^{-1} \widetilde{f}_n)_{n\in\mathbb{N}}$. Putting $\widetilde{w}_n := (i\lambda_n \mathcal{I} - \mathscr{A})^{-1} \widetilde{f}_n$, $w_n := \widetilde{w}_n / \|\widetilde{w}_n\|_{\mathscr{H}}$, and $f_n := \widetilde{f}_n / \|\widetilde{w}_n\|_{\mathscr{H}}$, we obtain that

$$(i\lambda_n \mathcal{I} - \mathscr{A}) w_n = f_n \tag{3.5}$$

with

$$\|w_n\|_{\mathscr{H}} = 1 \quad \text{and} \quad \|(i\lambda_n \mathcal{I} - \mathscr{A}) w_n\|_{\mathscr{H}} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.6}$$

Note that $(w_n)_{n\in\mathbb{N}} \subset D(\mathscr{A})$. By (2.1) and Eq (3.5), we obtain

$$i\lambda_n w_1^n - w_2^n = f_1^n, \tag{3.7}$$

$$i\lambda_n w_2^n + \Delta^2 w_1^n + \kappa_1 \Delta^2 w_2^n = f_2^n, \tag{3.8}$$

$$i\lambda_n w_3^n - w_4^n = f_3^n, \tag{3.9}$$
$$i\lambda_n w_4^n - \Delta w_3^n - \kappa_2 \Delta w_4^n = f_4^n, \tag{3.10}$$

and

$$\|(i\lambda_n \mathcal{I} - \mathscr{A})w_n\|_{\mathscr{H}}^2 = \|f_1^n\|_{H_\Gamma^2(\Omega_1)}^2 + \|f_2^n\|_{L^2(\Omega_1)}^2 + \|\nabla f_3^n\|_{L^2(\Omega_2)^2}^2 + \|f_4^n\|_{L^2(\Omega_2)}^2.$$

Using the limit in (3.6) together with (3.7)–(3.10), we obtain that

$$i\lambda_n w_1^n - w_2^n \to 0 \ \text{ in } \ H_\Gamma^2(\Omega_1), \tag{3.11}$$
$$i\lambda_n w_2^n + \Delta^2 w_1^n + \kappa_1 \Delta^2 w_2^n \to 0 \ \text{ in } \ L^2(\Omega_1), \tag{3.12}$$
$$i\lambda_n \nabla w_3^n - \nabla w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2), \tag{3.13}$$
$$i\lambda_n w_4^n - \Delta w_3^n - \kappa_2 \Delta w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{3.14}$$

By the dissipativity of $\mathscr{A}$ (see (2.6)), we have

$$\mathrm{Re}\langle (i\lambda_n \mathcal{I} - \mathscr{A})w_n, w_n\rangle_{\mathscr{H}} = \kappa_1 \|w_2^n\|_{H_\Gamma^2(\Omega_1)}^2 + \kappa_2 \|\nabla w_4^n\|_{L^2(\Omega_2)^2}^2. \tag{3.15}$$

Case 1. We first consider the case $\kappa_2 > 0$. Using the Cauchy-Schwarz inequality, (3.6), and (3.15), it follows that

$$w_2^n \to 0 \ \text{ in } \ H_\Gamma^2(\Omega_1) \quad \text{and} \quad \nabla w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{3.16}$$

The limits $\lambda_n \to \lambda^*$, (3.11), (3.13), and (3.16) imply

$$w_1^n \to 0 \ \text{ in } \ H_\Gamma^2(\Omega_1) \quad \text{and} \quad \nabla w_3^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{3.17}$$

From the first limit in (3.16), it is immediate that

$$w_2^n \to 0 \ \text{ in } \ L^2(\Omega_1). \tag{3.18}$$

On the other hand,

$$\langle i\lambda_n w_4^n - \Delta w_3^n - \kappa_2 \Delta w_4^n, w_4^n\rangle_{L^2(\Omega_2)} = i\lambda_n \|w_4^n\|_{L^2(\Omega_2)}^2 - \langle \Delta w_3^n + \kappa_2 \Delta w_4^n, w_4^n\rangle_{L^2(\Omega_2)}. \tag{3.19}$$

By Proposition 2.1, we have that $w_3^n + \kappa_2 w_4^n \in H^2(\Omega_2)$. Then, using integration by parts and the fact that $w_2^n = w_4^n$ on I, we obtain

$$\langle \Delta w_3^n + \kappa_2 \Delta w_4^n, w_4^n\rangle_{L^2(\Omega_2)} = -\langle \nabla w_3^n, \nabla w_4^n\rangle_{L^2(\Omega_2)^2} - \kappa_2 \|\nabla w_4^n\|_{L^2(\Omega_2)^2}^2 - \langle \partial_\nu (w_3^n + \kappa_2 w_4^n), w_2^n\rangle_{L^2(I)}. \tag{3.20}$$

Thanks to the Cauchy-Schwarz inequality and the trace theorem,

$$\left|\langle \partial_\nu (w_3^n + \kappa_2 w_4^n), w_2^n\rangle_{L^2(I)}\right| \le C\|w_3^n + \kappa_2 w_4^n\|_{H^2(\Omega_2)} \|w_2^n\|_{H^1(\Omega_1)}. \tag{3.21}$$

Considering again the first limit in (3.16), we have

$$w_2^n \to 0 \ \text{ in } \ H^1(\Omega_1). \tag{3.22}$$

As $\Delta(w_3^n + \kappa_2 w_4^n) = i\lambda_n w_4^n - f_4^n$ (see (3.10)), and $w_3^n + \kappa_2 w_4^n = w_1^n + \kappa_2 w_2^n$ on I, Remark 3.1 in [7] implies

$$\begin{aligned}&\|w_3^n + \kappa_2 w_4^n\|_{H^2(\Omega_2)}\\ \leq & C(\|i\lambda_n w_4^n - f_4^n\|_{L^2(\Omega_2)} + \|w_1^n + \kappa_2 w_2^n\|_{H^{3/2}(I)})\\ \leq & C(\lambda_n\|w_4^n\|_{L^2(\Omega_2)} + \|f_4^n\|_{L^2(\Omega_2)} + \|w_1^n\|_{H^2(\Omega_1)} + \|w_2^n\|_{H^2(\Omega_1)}),\end{aligned}$$

and therefore,

$$\|w_3^n + \kappa_2 w_4^n\|_{H^2(\Omega_2)} \leq C. \tag{3.23}$$

From (3.6), (3.14), (3.16), (3.17), and (3.19)–(3.23), it follows that

$$w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{3.24}$$

The limits (3.17), (3.18), and (3.24) imply $w_n \to 0$ in $\mathscr{H}$, which contradicts the equality in (3.6). Hence, $\lambda^* = \infty$.

Case 2. We next consider the case $\kappa_2 = 0$. By Proposition 2.1, $w_1^n + \kappa_1 w_2^n \in H^4(\Omega_1)$, and $w_3^n \in H^2(\Omega_2)$. By (3.6), (3.11), and (3.15), we have

$$w_1^n \to 0 \ \text{ in } \ H^2(\Omega_1) \qquad \text{and} \qquad w_2^n \to 0 \ \text{ in } \ H^2(\Omega_1). \tag{3.25}$$

Inserting (3.9) into (3.10), we obtain

$$\lambda_n^2 w_3^n + \Delta w_3^n = -i\lambda_n f_3^n - f_4^n \ \text{ in } \ L^2(\Omega_2). \tag{3.26}$$

Because $f_3^n = f_1^n$ on I, Theorem 1.9 in [22] and the trace theorem imply

$$\|f_3^n\|_{H^1(\Omega_2)} \leq C\Big(\|f_1^n\|_{H^2(\Omega_1)} + \|\nabla f_3^n\|_{L^2(\Omega_2)^2}\Big). \tag{3.27}$$

Because $f_1^n \to 0$ in $H^2(\Omega_1)$, $\nabla f_3^n \to 0$ in $L^2(\Omega_2)$, and $f_4^n \to 0$ in $L^2(\Omega_2)$, then from (3.26) and (3.27), it follows that

$$\lambda_n^2 w_3^n + \Delta w_3^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{3.28}$$

Using integration by parts,

$$\langle \lambda_n^2 w_3^n + \Delta w_3^n, w_3^n\rangle_{L^2(\Omega_2)} = \lambda_n^2\|w_3^n\|^2_{L^2(\Omega_2)} - \|\nabla w_3^n\|^2_{L^2(\Omega_2)^2} - \langle \partial_\nu w_3^n, w_3^n\rangle_{L^2(I)}. \tag{3.29}$$

Arguing as in (3.23), we also obtain in this case the boundedness of $(w_3^n)_{n\in\mathbb{N}}$ in $H^2(\Omega_2)$. Therefore,

$$\Big|\langle \partial_\nu w_3^n, w_3^n\rangle_{L^2(I)}\Big| \leq C\|w_3^n\|_{H^2(\Omega_2)}\|w_1^n\|_{L^2(I)} \leq C\|w_1^n\|_{H^2(\Omega_1)}. \tag{3.30}$$

The first limit in (3.25) together with (3.28)–(3.30) allows us to write

$$\lambda_n^2\|w_3^n\|^2_{L^2(\Omega_2)} - \|\nabla w_3^n\|^2_{L^2(\Omega_2)^2} \to 0. \tag{3.31}$$

Let $x_0 \in \mathbb{R}^2$. Consider the vector field $h : \overline{\Omega_2} \to \mathbb{R}^2$ given by $h(x) := x - x_0$. By the Rellich identity (compare with equality (94) in [4]), we have

$$\operatorname{Re}\int_{\Omega_2} \Delta w_3^n (h\cdot \overline{\nabla w_3^n})dx = -\operatorname{Re}\int_I \partial_\nu w_3^n (h\cdot\overline{\nabla w_3^n})dS + \frac{1}{2}\int_I (h\cdot\nu)|\nabla w_3^n|^2 dS. \tag{3.32}$$

By performing calculations similar to those on page 12,904 of [4], we obtain

$$\lambda_n^2\|w_3^n\|^2_{L^2(\Omega_2)} = -\tfrac{1}{2}\lambda_n^2\int_I(h\cdot\nu)|w_3^n|^2dS - \mathrm{Re}\int_{\Omega_2}\lambda_n^2 w_3^n(h\cdot\overline{\nabla w_3^n})dx. \tag{3.33}$$

Substituting $\lambda_n^2 w_3^n$ for $-i\lambda_n f_3^n - f_4^n - \Delta w_3^n$ in (3.33) and using (3.32), we obtain

$$\begin{aligned}\lambda_n^2\|w_3^n\|^2_{L^2(\Omega_2)} &= -\tfrac{1}{2}\lambda_n^2\int_I(h\cdot\nu)|w_3^n|^2dS - \mathrm{Re}\int_I\partial_\nu w_3^n(h\cdot\overline{\nabla w_3^n})dS\\ &\quad + \tfrac{1}{2}\int_I(h\cdot\nu)|\nabla w_3^n|^2dS + \mathrm{Re}\int_{\Omega_2}(i\lambda_n f_3^n + f_4^n)(h\cdot\overline{\nabla w_3^n})dx.\end{aligned}$$

In consequence, due to $w_3^n = w_1^n$ on I, the trace theorem, and the boundedness of $(w_3^n)_{n\in\mathbb{N}}$ in $H^2(\Omega_2)$, we have

$$\lambda_n^2\|w_3^n\|^2_{L^2(\Omega_2)} \le C\Big(\|w_1^n\|^2_{H^2(\Omega_1)} + \|\nabla w_3^n\|^2_{L^2(I)^2} + \|f_3^n\|^2_{L^2(\Omega_2)} + \|f_4^n\|^2_{L^2(\Omega_2)}\Big). \tag{3.34}$$

As we have seen before, $\lim_{n\to\infty}\|w_1^n\|_{H^2(\Omega_1)} = \lim_{n\to\infty}\|f_3^n\|_{L^2(\Omega_2)} = \lim_{n\to\infty}\|f_4^n\|_{L^2(\Omega_2)} = 0$, and now we will establish that $\lim_{n\to\infty}\|\nabla w_3^n\|_{L^2(I)^2} = 0$. We have

$$\|\nabla w_3^n\|_{L^2(I)^2} = \|(\partial_\nu w_3^n)\nu + (\partial_\tau w_3^n)\tau\|_{L^2(I)^2} \le \|\partial_\nu w_3^n\|_{L^2(I)} + \|\partial_\tau w_3^n\|_{L^2(I)}. \tag{3.35}$$

As $w_1^n = w_3^n$ on I,

$$\|\partial_\tau w_3^n\|_{L^2(I)} = \|\partial_\tau w_1^n\|_{L^2(I)} \le C\|w_1^n\|_{H^2(\Omega_1)}. \tag{3.36}$$

Taking into account the transmission condition $\mathscr{B}_2(w_1^n + \kappa_1 w_2^n) = -\partial_\nu w_3^n$ on I, using Lemma 3.1 and the interpolation inequality with the spaces $H^2(\Omega_1)$ and $H^4(\Omega_1)$, we obtain

$$\begin{aligned}\|\partial_\nu w_3^n\|_{L^2(I)} &= \|\mathscr{B}_2(w_1^n + \kappa_1 w_2^n)\|_{L^2(I)}\\ &\le C\|w_1^n + \kappa_1 w_2^n\|^{1/2}_{H^4(\Omega_1)}\|w_1^n + \kappa_1 w_2^n\|^{1/2}_{H^3(\Omega_1)}\\ &\le C\|w_1^n + \kappa_1 w_2^n\|^{3/4}_{H^4(\Omega_1)}\|w_1^n + \kappa_1 w_2^n\|^{1/4}_{H^2(\Omega_1)}.\end{aligned} \tag{3.37}$$

Note that the components of w_n satisfy the system

$$\begin{cases}\Delta^2(w_1^n + \kappa_1 w_2^n) & = f_2^n - i\lambda_n w_2^n \ \text{ in } \ L^2(\Omega_1),\\ w_1^n + \kappa_1 w_2^n & = 0 \ \text{ on } \ \Gamma,\\ \partial_\nu(w_1^n + \kappa_1 w_2^n) & = 0 \ \text{ on } \ \Gamma,\\ \mathscr{B}_1(w_1^n + \kappa_1 w_2^n) & = 0 \ \text{ on } \ I,\\ \mathscr{B}_2(w_1^n + \kappa_1 w_2^n) & = -\partial_\nu w_3^n \ \text{ on } \ I.\end{cases}$$

By Corollary 4.3 in [5], there exists $\lambda_0 > 0$ such that

$$\|w_1^n + \kappa_1 w_2^n\|_{H^4(\Omega_1)} \le C\Big(\|\lambda_0(w_1^n + \kappa_1 w_2^n) + f_2^n - i\lambda_n w_2^n\|_{L^2(\Omega_1)} + \|\partial_\nu w_3^n\|_{H^{1/2}(I)}\Big),$$

and so

$$\|w_1^n + \kappa_1 w_2^n\|_{H^4(\Omega_1)} \le C. \tag{3.38}$$

By (3.35)–(3.38), we obtain $\nabla w_3^n \to 0$ in $L^2(I)$. Now, from (3.34), $w_3^n \to 0$ in $L^2(\Omega_2)$. A combination of the previous limit together with (3.14), (3.28), and (3.31) allows us to conclude that

$$\nabla w_3^n \to 0 \ \text{ in } \ L^2(\Omega_2), \qquad \text{and} \qquad w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{3.39}$$

Limits (3.25) and (3.39) lead to the same contradiction as in the previous case. □

The following is the main result of this section.

Theorem 3.2. *If $\kappa_1 > 0$, and $\kappa_2 > 0$, then the semigroup $(\mathscr{T}(t))_{t\geq 0}$ generated by $\mathscr{A}$ is analytic.*

Proof. We already have that $i\mathbb{R} \subset \rho(\mathscr{A})$; see Proposition 3.2. Let us suppose (3.1) is not true. Then, there are sequences $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ with $|\lambda_n| \to \infty$ and $(\widehat{f_n})_{n\in\mathbb{N}} \subset \mathscr{H}$ with $\|\widehat{f_n}\|_{\mathscr{H}} = 1$ such that

$$\lim_{n\to\infty} \|\lambda_n(i\lambda_n\mathcal{I} - \mathscr{A})^{-1}\widehat{f_n}\|_{\mathscr{H}} = \infty.$$

Henceforth, we only consider the terms of $(\lambda_n)_{n\in\mathbb{N}}$ and $(|\lambda_n|(i\lambda_n\mathcal{I} - \mathscr{A})^{-1}\widehat{f_n})_{n\in\mathbb{N}}$ that are nonzero. Doing $\widehat{w_n} := |\lambda_n|(i\lambda_n\mathcal{I} - \mathscr{A})^{-1}\widehat{f_n}$, $w_n := \widehat{w_n}/\|\widehat{w_n}\|_{\mathscr{H}}$ and $f_n := \widehat{f_n}/\|\widehat{w_n}\|_{\mathscr{H}}$, we obtain

$$\|w_n\|_{\mathscr{H}} = 1 \quad \text{and} \quad (\pm i\mathcal{I} - |\lambda_n|^{-1}\mathscr{A})w_n = f_n \tag{3.40}$$

with

$$\|(\pm i\mathcal{I} - |\lambda_n|^{-1}\mathscr{A})w_n\|_{\mathscr{H}} = \|f_n\|_{\mathscr{H}} = \frac{\|\widehat{f_n}\|_{\mathscr{H}}}{\|\widehat{w_n}\|_{\mathscr{H}}} = \frac{1}{\|\widehat{w_n}\|_{\mathscr{H}}} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.41}$$

Here, $\pm i = \operatorname{sign}(\lambda_n)i$, where $\operatorname{sign}(\lambda_n) := \lambda_n/|\lambda_n|$. Note that $(w_n)_{n\in\mathbb{N}} \subset D(\mathscr{A})$. Using (2.1) and the functional equality of (3.40), we obtain

$$\pm i w_1^n - |\lambda_n|^{-1} w_2^n = f_1^n, \tag{3.42}$$

$$\pm i w_2^n + |\lambda_n|^{-1}(\Delta^2 w_1^n + \kappa_1 \Delta^2 w_2^n) = f_2^n, \tag{3.43}$$

$$\pm i w_3^n - |\lambda_n|^{-1} w_4^n = f_3^n, \tag{3.44}$$

$$\pm i w_4^n - |\lambda_n|^{-1}(\Delta w_3^n + \kappa_2 \Delta w_4^n) = f_4^n. \tag{3.45}$$

By (3.41)–(3.45), we can write

$$\pm i w_1^n - |\lambda_n|^{-1} w_2^n \to 0 \ \text{ in } \ H^2_\Gamma(\Omega_1), \tag{3.46}$$

$$\pm i w_2^n + |\lambda_n|^{-1}(\Delta^2 w_1^n + \kappa_1 \Delta^2 w_2^n) \to 0 \ \text{ in } \ L^2(\Omega_1), \tag{3.47}$$

$$\pm i \nabla w_3^n - |\lambda_n|^{-1} \nabla w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2), \tag{3.48}$$

$$\pm i w_4^n - |\lambda_n|^{-1}(\Delta w_3^n + \kappa_2 \Delta w_4^n) \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{3.49}$$

The dissipation of $\mathscr{A}$ (see (2.6)) implies that

$$\operatorname{Re}\langle(\pm i\mathcal{I} - |\lambda_n|^{-1}\mathscr{A})w_n, w_n\rangle_{\mathscr{H}} = \frac{\kappa_1}{|\lambda_n|}\|w_2^n\|^2_{H^2_\Gamma(\Omega_1)} + \frac{\kappa_2}{|\lambda_n|}\|\nabla w_4^n\|^2_{L^2(\Omega_2)^2}.$$

This last equality together with the first statement of (3.40) and (3.41) allow us to obtain

$$|\lambda_n|^{-1/2} w_2^n \to 0 \ \text{ in } \ H^2_\Gamma(\Omega_1) \quad \text{and} \quad |\lambda_n|^{-1/2} \nabla w_4^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{3.50}$$

From (3.46) and the first limit in (3.50), it follows that

$$w_1^n \to 0 \ \text{ in } \ H^2_\Gamma(\Omega_1). \tag{3.51}$$

By (3.48) and the second limit in (3.50),

$$\nabla w_3^n \to 0 \ \text{ in } \ L^2(\Omega_2). \tag{3.52}$$

As $(w_1^n)_{n\in\mathbb{N}}$ is bounded in $H^2(\Omega_1)$, the limit (3.46) implies that $(|\lambda_n|^{-1}w_2^n)_{n\in\mathbb{N}}$ is also bounded in $H^2(\Omega_1)$. By the interpolation inequality and $\|w_2^n\|_{L^2(\Omega_1)} \leq \|w_n\|_{\mathscr{H}} = 1$, we obtain

$$\frac{1}{|\lambda_n|^{1/2}}\|w_2^n\|_{H^1(\Omega_1)} \leq C\frac{\|w_2^n\|^{1/2}_{H^2(\Omega_1)}}{|\lambda_n|^{1/2}}\|w_2^n\|^{1/2}_{L^2(\Omega_1)} \leq C. \tag{3.53}$$

Let us consider the following equality:

$$\langle \pm iw_4^n - |\lambda_n|^{-1}(\Delta w_3^n + \kappa_2\Delta w_4^n), w_4^n\rangle_{L^2(\Omega_2)} = \pm i\|w_4^n\|^2_{L^2(\Omega_2)} - |\lambda_n|^{-1}\langle\Delta(w_3^n + \kappa_2 w_4^n), w_4^n\rangle_{L^2(\Omega_2)}. \tag{3.54}$$

Using integration by parts and the equality $w_2^n = w_4^n$ on I, it follows that

$$\begin{aligned} |\lambda_n|^{-1}\langle\Delta(w_3^n + \kappa_2 w_4^n), w_4^n\rangle_{L^2(\Omega_2)} &= -|\lambda_n|^{-1}\langle\nabla w_3^n, \nabla w_4^n\rangle_{L^2(\Omega_2)^2} \\ - \kappa_2|\lambda_n|^{-1}\|\nabla w_4^n\|^2_{L^2(\Omega_2)^2} &- |\lambda_n|^{-1}\langle\partial_\nu(w_3^n + \kappa_2 w_4^n), w_2^n\rangle_{L^2(I)}. \end{aligned} \tag{3.55}$$

By the Cauchy-Schwarz inequality and the boundedness of $(|\lambda_n|^{-1}\nabla w_4^n)_{n\in\mathbb{N}}$ in $L^2(\Omega_2)$, we have

$$\left|\lambda_n^{-1}\langle\nabla w_3^n, \nabla w_4^n\rangle_{L^2(\Omega_2)^2}\right| \leq |\lambda_n|^{-1}\|\nabla w_3^n\|_{L^2(\Omega_2)^2}\|\nabla w_4^n\|_{L^2(\Omega_2)^2} \leq C\|\nabla w_3^n\|_{L^2(\Omega_2)^2}. \tag{3.56}$$

From (3.45), we have $\Delta(w_3^n + \kappa_2 w_4^n) = i\lambda_n w_4^n - |\lambda_n| f_4^n$, and as $w_3^n + \kappa_2 w_4^n = w_1^n + \kappa_2 w_2^n$ on I, then Proposition 2.1 and Remark 3.1 in [7] lead to

$$\begin{aligned} &\|w_3^n + \kappa_2 w_4^n\|_{H^2(\Omega_2)} \\ \leq& C\big(\|i\lambda_n w_4^n - |\lambda_n| f_4^n\|_{L^2(\Omega_2)} + \|w_1^n + \kappa_2 w_2^n\|_{H^{3/2}(I)}\big) \\ \leq& C\big(|\lambda_n|\|w_4^n\|_{L^2(\Omega_2)} + |\lambda_n|\|f_4^n\|_{L^2(\Omega_2)} + \|w_1^n\|_{H^2(\Omega_1)} + \|w_2^n\|_{H^2(\Omega_1)}\big), \end{aligned}$$

and therefore,

$$|\lambda_n|^{-1}\|w_3^n + \kappa_2 w_4^n\|_{H^2(\Omega_2)} \leq C\big(\|w_n\|_{\mathscr{H}} + \|f_n\|_{\mathscr{H}} + |\lambda_n|^{-1}\|w_n\|_{\mathscr{H}} + |\lambda_n|^{-1}\|w_2^n\|_{H^2(\Omega_1)}\big) \leq C. \tag{3.57}$$

Using the Cauchy-Schwarz inequality, Lemma 3.1, $\|w_2^n\|_{L^2(\Omega_1)} \leq 1$, and (3.53), we obtain

$$\begin{aligned} &\left|\lambda_n^{-1}\langle\partial_\nu(w_3^n + \kappa_2 w_4^n), w_2^n\rangle_{L^2(I)}\right| \\ \leq& |\lambda_n|^{-1}\|\partial_\nu(w_3^n + \kappa_2 w_4^n)\|_{L^2(I)}\|w_2^n\|_{L^2(I)} \\ \leq& C|\lambda_n|^{-3/4}\|\partial_\nu(w_3^n + \kappa_2 w_4^n)\|_{L^2(I)}|\lambda_n|^{-1/4}\|w_2^n\|^{1/2}_{H^1(\Omega_1)}\|w_2^n\|^{1/2}_{L^2(\Omega_1)} \\ \leq& C|\lambda_n|^{-3/4}\|\partial_\nu(w_3^n + \kappa_2 w_4^n)\|_{L^2(I)}. \end{aligned} \tag{3.58}$$

Using the estimate (3.57) and Corollary 4.3 in [7], we obtain

$$\begin{aligned} &|\lambda_n|^{-3/4}\|\partial_\nu(w_3^n + \kappa_2 w_4^n)\|_{L^2(I)} \\ \leq& C\frac{1}{|\lambda_n|^{1/2}}\|w_3^n + \kappa_2 w_4^n\|^{1/2}_{H^2(\Omega_2)}\frac{1}{|\lambda_n|^{1/4}}\|\nabla(w_3^n + \kappa_2 w_4^n)\|^{1/2}_{L^2(\Omega_2)^2} \\ \leq& C\big(|\lambda_n|^{-1/4}\|\nabla w_3^n\|^{1/2}_{L^2(\Omega_2)^2} + |\lambda_n|^{-1/4}\|\nabla w_4^n\|^{1/2}_{L^2(\Omega_2)^2}\big). \end{aligned} \tag{3.59}$$

Now, taking the limit as $n \to \infty$ in (3.54) and using (3.49), the boundedness of $(w_4^n)_{n\in\mathbb{N}}$ in $L^2(\Omega_2)$, (3.50), (3.52), (3.55), (3.56), (3.58), and (3.59), we obtain

$$w_4^n \to 0 \text{ in } L^2(\Omega_2). \tag{3.60}$$

On the other hand,

$$\langle \pm i w_2^n + |\lambda_n|^{-1}(\Delta^2 w_1^n + \kappa_1 \Delta^2 w_2^n), w_2^n \rangle_{L^2(\Omega_1)} = \pm i \|w_2^n\|^2_{L^2(\Omega_1)} + |\lambda_n|^{-1} \langle \Delta^2 (w_1^n + \kappa_1 w_2^n), w_2^n \rangle_{L^2(\Omega_1)}. \tag{3.61}$$

Because $\mathscr{B}_1(w_1^n + \kappa_1 w_2^n) = 0$, and $\mathscr{B}_2(w_1^n + \kappa_1 w_2^n) = -\partial_\nu w_3^n - \kappa_2 \partial_\nu w_4^n$ on I, (2.2) and Proposition 2.1 imply that

$$\begin{aligned} & |\lambda_n|^{-1} \langle \Delta^2 (w_1^n + \kappa_1 w_2^n), w_2^n \rangle_{L^2(\Omega_1)} \\ =& |\lambda_n|^{-1} \left[\langle w_1^n + \kappa_1 w_2^n, w_2^n \rangle_{H^2_\Gamma(\Omega_1)} + \langle -\partial_\nu (w_3^n + \kappa_2 w_4^n), w_2^n \rangle_{L^2(I)} \right] \\ =& |\lambda_n|^{-1} \langle w_1^n, w_2^n \rangle_{H^2_\Gamma(\Omega_1)} + \kappa_1 |\lambda_n|^{-1} \|w_2^n\|^2_{H^2_\Gamma(\Omega_1)} - |\lambda_n|^{-1} \langle \partial_\nu (w_3^n + \kappa_2 w_4^n), w_2^n \rangle_{L^2(I)}. \end{aligned} \tag{3.62}$$

Taking the limit as $n \to \infty$ in (3.61) and using (3.47), the boundedness of $(w_2^n)_{n\in\mathbb{N}}$ in $L^2(\Omega_1)$, (3.50), (3.52), (3.58), (3.59), and (3.62), we have

$$w_2^n \to 0 \text{ in } L^2(\Omega_1). \tag{3.63}$$

The limits (3.51), (3.52), (3.60), and (3.63) lead to $w_n \to 0$ in $\mathscr{H}$, which contradicts $\|w_n\|_{\mathscr{H}} = 1$. □

Remark 3.1. *Because the operator $\mathscr{A}$ generates an analytic semigroup $(\mathscr{T}(t))_{t\geq 0}$ on the phase space $\mathscr{H}$, the following classical consequences hold:*

i) Analyticity in time.
For every initial datum $w_0 \in \mathscr{H}$, the map $t \mapsto \mathscr{T}(t)w_0$ is analytic for all $t > 0$. In particular, the corresponding solutions are infinitely differentiable in time on $(0, \infty)$, even if $w_0 \notin D(\mathscr{A})$.

ii) Exponential stability.
Because $i\mathbb{R} \subset \rho(\mathscr{A})$, the analyticity of the semigroup generated by $\mathscr{A}$ implies the exponential stability of the semigroup.

iii) Robustness under bounded perturbations.
If $\mathcal{B} \in \mathscr{L}(\mathscr{H})$ is a bounded linear operator, then $\mathscr{A} + \mathcal{B}$ still generates an analytic semigroup on $\mathscr{H}$. In particular, analyticity is stable under bounded linear perturbations of the model (e.g., lower-order coupling terms or bounded feedback operators).

These properties are well-known in the theory of analytic semigroups; see, for example [23].

4. Polynomial stability in the case of the undamped membrane

In this section, we analyze the case of the undamped membrane for our problem (1.1)–(1.7), that is, $\kappa_2 = 0$. In this case, we prove that the solution of the first-order problem (2.7) associated with the abstract formulation of (1.1)–(1.7) exhibits polynomial decay under a geometric condition imposed on Ω_2, which frequently appears in the stability analysis of wave equations. Specifically, we assume that there exists some $x_0 \in \mathbb{R}^2$ such that

$$q \cdot \nu \leq 0 \quad \text{on } I, \tag{4.1}$$

where $q(x) := x - x_0$ with $x \in \overline{\Omega_2}$. Note that Condition (4.1) holds if, for example, Ω_2 is convex, or, more generally, if it is star-shaped with respect to a point x_0 located inside it, taking into account that the normal vector ν on I points towards the interior of Ω_2. Recall that a domain Ω_2 is said to be star-shaped with respect to a point $x_0 \in \Omega_2$ if, for every $x \in \Omega_2$ and every $t \in [0, 1]$,

$$x_0 + t(x - x_0) \in \Omega_2.$$

Indeed, if Ω_2 is star-shaped with respect to x_0, then

$$(x - x_0) \cdot \nu_{\Omega_2}(x) \geq 0 \quad \text{on } I,$$

where ν_{Ω_2} denotes the outward unit normal vector to Ω_2. Because the normal vector ν used here points toward the interior of Ω_2, we have $\nu = -\nu_{\Omega_2}$ on I, and hence,

$$q(x) \cdot \nu(x) = (x - x_0) \cdot \nu(x) \leq 0 \quad (x \in I).$$

For the proof, we use the following theorem, which provides a characterization of the polynomial stability of a C_0-semigroup of contractions due to Muñoz Rivera and Racke (see [21, Lemma 5.2]) and extends a well-known result of Borichev and Tomilov (see [9, Theorem 2.4]).

Theorem 4.1. *Let $(T(t))_{t\geq 0}$ be a contraction semigroup on a Hilbert space H with generator A such that $i\mathbb{R} \cap \sigma(A) = \emptyset$. Then, for $\alpha \in \mathbb{N}_0$ and $\beta > 0$ fixed, the following assertions are equivalent:*

i) There exist $C > 0$ and $\lambda_0 > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| > \lambda_0$ and all $f \in D(A^\alpha)$, it holds that

$$\left\|(i\lambda\mathcal{I} - A)^{-1} f\right\|_H \leq C|\lambda|^\beta \|A^\alpha f\|_H.$$

ii) There exists some $C > 0$ such that for all $t > 0$, it holds that

$$\left\|T(t)A^{-1}\right\|_{\mathscr{L}(H)} \leq Ct^{-\frac{1}{\alpha+\beta}}.$$

Note that Assertion *ii*) implies that $\|T(t)u_0\|_H \leq Ct^{-1/(\alpha+\beta)}\|Au_0\|_H$ for all $u_0 \in D(A)$. The following is the main result of this section.

Theorem 4.2. *Let $\kappa_1 > 0$, $\kappa_2 = 0$, and assume that the condition* (4.1) *holds. Then, for every $w_0 \in D(\mathscr{A})$, the solution of* (2.7) *decays polynomially at least at rate $t^{-1/10}$; that is, there exists a constant $C > 0$ such that*

$$\|\mathscr{T}(t)w_0\|_{\mathscr{H}} \leq Ct^{-\frac{1}{10}}\|\mathscr{A}w_0\|_{\mathscr{H}}$$

for all $t > 0$ and $w_0 \in D(\mathscr{A})$.

Proof. By Proposition 3.2, we have $i\mathbb{R} \cap \sigma(\mathscr{A}) = \emptyset$. Let $\lambda_0 > 0$, and $\lambda \in \mathbb{R}$ with $|\lambda| > \lambda_0$, $F = (f_1, f_2, f_3, f_4)^\top \in \mathscr{H}$, and $w = (w_1, w_2, w_3, w_4)^\top \in D(\mathscr{A})$ such that

$$(i\lambda\mathcal{I} - \mathscr{A})w = F, \tag{4.2}$$

which is equivalent to the following system of equations:

$$\begin{aligned} i\lambda w_1 - w_2 &= f_1 \in H^2_\Gamma(\Omega_1), && (4.3)\\ i\lambda w_2 + \Delta^2(w_1 + \kappa_1 w_2) &= f_2 \in L^2(\Omega_1), && (4.4)\\ i\lambda w_3 - w_4 &= f_3 \in H^1(\Omega_2), && (4.5) \end{aligned}$$

$$i\lambda w_4 - \Delta w_3 = f_4 \in L^2(\Omega_2). \tag{4.6}$$

Multiplying (4.4) by $-\overline{w_1}$ and (4.6) by $-\overline{w_3}$, integrating over the respective domains, and adding the resulting identities, we obtain

$$-i\lambda\langle w_2, w_1\rangle_{L^2(\Omega_1)} - \langle\Delta^2(w_1+\kappa_1 w_2), w_1\rangle_{L^2(\Omega_1)} - i\lambda\langle w_4, w_3\rangle_{L^2(\Omega_2)} + \langle\Delta w_3, w_3\rangle_{L^2(\Omega_2)} = -\langle f_2, w_1\rangle_{L^2(\Omega_1)} - \langle f_4, w_3\rangle_{L^2(\Omega_2)}.$$

Because $w_1 + \kappa_1 w_2 \in H^4(\Omega_1) \cap H^2_\Gamma(\Omega_1)$, $w_1 \in H^2_\Gamma(\Omega_1)$, and $w_3 \in H^2(\Omega_2)$, we may use (2.2), integrate by parts, and use the fact that w satisfies (2.4) and (2.5) in the strong sense of the traces to obtain

$$-i\lambda\langle w_2, w_1\rangle_{L^2(\Omega_1)} - \langle w_1+\kappa_1 w_2, w_1\rangle_{H^2_\Gamma(\Omega_1)} - i\lambda\langle w_4, w_3\rangle_{L^2(\Omega_2)} - \|\nabla w_3\|^2_{L^2(\Omega_2)^2} = -\langle f_2, w_1\rangle_{L^2(\Omega_1)} - \langle f_4, w_3\rangle_{L^2(\Omega_2)}.$$

Then, using (4.3) and (4.5), it follows that

$$\begin{aligned} &\lambda^2\|w_1\|^2_{L^2(\Omega_1)} + i\lambda\langle f_1, w_1\rangle_{L^2(\Omega_1)} - \|w_1\|^2_{H^2_\Gamma(\Omega_1)} - i\kappa_1\lambda\|w_1\|^2_{H^2_\Gamma(\Omega_1)} + \kappa_1\langle f_1, w_1\rangle_{H^2_\Gamma(\Omega_1)} \\ &+ \lambda^2\|w_3\|^2_{L^2(\Omega_2)} + i\lambda\langle f_3, w_3\rangle_{L^2(\Omega_2)} - \|\nabla w_3\|^2_{L^2(\Omega_2)^2} = -\langle f_2, w_1\rangle_{L^2(\Omega_1)} - \langle f_4, w_3\rangle_{L^2(\Omega_2)}. \end{aligned} \tag{4.7}$$

Using the Cauchy-Schwarz inequality after taking the real part of (4.7) and taking into account the equivalence of the norms $\|\cdot\|_{H^2(\Omega_1)}$ and $\|\cdot\|_{H^2_\Gamma(\Omega_1)}$, we have

$$\begin{aligned} &\|w_1\|^2_{H^2_\Gamma(\Omega_1)} + \|\nabla w_3\|^2_{L^2(\Omega_2)^2} \\ \leq &\lambda^2\|w_1\|^2_{L^2(\Omega_1)} + \lambda^2\|w_3\|^2_{L^2(\Omega_2)} + C(|\lambda|\|f_1\|_{H^2_\Gamma(\Omega_1)} + \kappa_1\|f_1\|_{H^2_\Gamma(\Omega_1)} + \|f_2\|_{L^2(\Omega_1)})\|w_1\|_{H^2_\Gamma(\Omega_1)} \\ &+ (|\lambda|\|f_3\|_{L^2(\Omega_2)} + \|f_4\|_{L^2(\Omega_2)})\|w_3\|_{L^2(\Omega_2)}. \end{aligned}$$

In view of Remark 2.1, the equivalence of the norms $\|\cdot\|_{H^2(\Omega_1)}$ and $\|\cdot\|_{H^2_\Gamma(\Omega_1)}$ again, and because $|\lambda| > \lambda_0$, we obtain

$$\|w_1\|^2_{H^2_\Gamma(\Omega_1)} + \|\nabla w_3\|^2_{L^2(\Omega_2)^2} \leq \lambda^2\|w_1\|^2_{L^2(\Omega_1)} + \lambda^2\|w_3\|^2_{L^2(\Omega_2)} + C|\lambda|\|w\|_{\mathscr{H}}\|F\|_{\mathscr{H}}. \tag{4.8}$$

Now, note that from (2.6), it follows that

$$\|w_2\|^2_{L^2(\Omega_1)} \leq C\|w_2\|^2_{H^2_\Gamma(\Omega_1)} \leq C\|w\|_{\mathscr{H}}\|F\|_{\mathscr{H}}, \tag{4.9}$$

and then, using (4.3), we obtain

$$\lambda^2\|w_1\|^2_{L^2(\Omega_1)} \leq C\lambda^2\|w_1\|^2_{H^2_\Gamma(\Omega_1)} = C\|f_1 + w_2\|^2_{H^2_\Gamma(\Omega_1)} \leq C\Big(\|F\|^2_{\mathscr{H}} + \|w\|_{\mathscr{H}}\|F\|_{\mathscr{H}}\Big). \tag{4.10}$$

Using (4.5) and Remark 2.1, we also obtain

$$\begin{aligned} \|w_4\|^2_{L^2(\Omega_2)} &\leq 2\lambda^2\|w_3\|^2_{L^2(\Omega_2)} + 2\|f_3\|^2_{L^2(\Omega_2)} \\ &\leq 2\lambda^2\|w_3\|^2_{L^2(\Omega_2)} + 2\Big(\|\nabla f_1\|^2_{L^2(\Omega_1)^2} + \|\nabla f_3\|^2_{L^2(\Omega_2)^2}\Big) \\ &\leq 2\lambda^2\|w_3\|^2_{L^2(\Omega_2)} + 2\Big(C\|f_1\|^2_{H^2_\Gamma(\Omega_1)} + \|\nabla f_3\|^2_{L^2(\Omega_2)^2}\Big) \\ &\leq C\Big(\lambda^2\|w_3\|^2_{L^2(\Omega_2)} + \|F\|^2_{\mathscr{H}}\Big). \end{aligned} \tag{4.11}$$

Using again the fact that $|\lambda| > \lambda_0$, from (4.8)–(4.11), it follows that

$$\|w\|^2_{\mathscr{H}} \le C\Big(\lambda^2\|w_3\|^2_{L^2(\Omega_2)} + \|F\|^2_{\mathscr{H}} + |\lambda|\|w\|_{\mathscr{H}}\|F\|_{\mathscr{H}}\Big). \tag{4.12}$$

Now, we must estimate $\lambda^2\|w_3\|^2_{L^2(\Omega_2)}$. Arguing similarly to the calculations on page 1100 in [5], we have

$$\lambda^2\|w_3\|^2_{L^2(\Omega_2)} \le C\Big(\|F\|^2_{\mathscr{H}} + |\lambda|\|w\|_{\mathscr{H}}\|F\|_{\mathscr{H}}\Big) + \int_I |\partial_\nu w_3(q\cdot\nabla\overline{w_3})|\,dS,$$

where q is the vector field given in (4.1). By (2.5), $\partial_\nu w_3 = -\mathscr{B}_2(w_1+\kappa_1 w_2)$ on I. Then,

$$\int_I |\partial_\nu w_3(q\cdot\nabla\overline{w_3})|\,dS = \int_I |\mathscr{B}_2(w_1+\kappa_1 w_2)||q\cdot\nabla\overline{w_3}|\,dS \le C\|\mathscr{B}_2(w_1+\kappa_1 w_2)\|_{L^2(I)}\|\nabla w_3\|_{L^2(I)^2}. \tag{4.13}$$

In order to estimate $\|\nabla w_3\|_{L^2(I)^2}$, we estimate the norm of w_3 in $H^2(\Omega_2)$. From (4.6), we have

$$\Delta w_3 = i\lambda w_4 - f_4.$$

Because $w_3 = w_1$ on I, by Remark 4.4(b) in [5], it follows that

$$\begin{aligned}\|w_3\|_{H^2(\Omega_2)} &\le C\Big(\|i\lambda w_4 - f_4\|_{L^2(\Omega_2)} + \|w_1\|_{H^{3/2}(I)}\Big)\\ &\le C\Big(|\lambda|\|w_4\|_{L^2(\Omega_2)} + \|f_4\|_{L^2(\Omega_2)} + \|w_1\|_{H^{3/2}(I)}\Big)\\ &\le C\Big(|\lambda|\|w\|_{\mathscr{H}} + \|F\|_{\mathscr{H}}\Big).\end{aligned} \tag{4.14}$$

Using Lemma 3.1, Remark 2.1, and the equivalence of the p-norms on $\mathbb{R}^2$, we obtain

$$\begin{aligned}\|\nabla w_3\|_{L^2(I)^2} &\le C\|w_3\|^{1/2}_{H^2(\Omega_2)}\|w_3\|^{1/2}_{H^1(\Omega_2)}\\ &\le C\Big(|\lambda|^{1/2}\|w\|^{1/2}_{\mathscr{H}} + \|F\|^{1/2}_{\mathscr{H}}\Big)\|w\|^{1/2}_{\mathscr{H}}\\ &= C\Big(|\lambda|^{1/2}\|w\|_{\mathscr{H}} + \|w\|^{1/2}_{\mathscr{H}}\|F\|^{1/2}_{\mathscr{H}}\Big).\end{aligned} \tag{4.15}$$

We now estimate the term $\|\mathscr{B}_2(w_1+\kappa_1 w_2)\|_{L^2(I)}$. We first fix a sufficiently large $\widetilde{\lambda}_0 > 0$. By (4.4), $w_1+\kappa_1 w_2$ satisfies

$$(\widetilde{\lambda}_0 + \Delta^2)(w_1+\kappa_1 w_2) = \widetilde{\lambda}_0[w_1 + (\kappa_1 - i)w_2] + f_2 - i\lambda w_2$$

in Ω_1. Furthermore, $w_1+\kappa_1 w_2$ satisfies the boundary conditions

$$\begin{aligned} w_1+\kappa_1 w_2 &= 0 \quad \text{on } \Gamma,\\ \partial_\nu(w_1+\kappa_1 w_2) &= 0 \quad \text{on } \Gamma,\\ \mathscr{B}_1(w_1+\kappa_1 w_2) &= 0 \quad \text{on } I,\\ \mathscr{B}_2(w_1+\kappa_1 w_2) &= -\partial_\nu w_3 \quad \text{on } I.\end{aligned}$$

Because $\widetilde{\lambda}_0[w_1+(\kappa_1-i)w_2] + f_2 - i\lambda w_2 \in L^2(\Omega_1)$, and $\partial_\nu w_3|_I \in H^{1/2}(I)$, we have the following a priori estimate (see Corollary 4.3 in [5]):

$$\|w_1+\kappa_1 w_2\|_{H^4(\Omega_1)} \le C_{\widetilde{\lambda}_0}\Big(\|\widetilde{\lambda}_0[w_1+(\kappa_1-i)w_2] + f_2 - i\lambda w_2\|_{L^2(\Omega_1)} + \|\partial_\nu w_3\|_{H^{1/2}(I)}\Big),$$

where $C_{\widetilde{\lambda}_0} > 0$ is a constant depending on $\widetilde{\lambda}_0$ but not on $w_1 + \kappa_1 w_2$ or the data. Then, using the trace theorem for w_3, it follows that

$$\begin{aligned}\|w_1 + \kappa_1 w_2\|_{H^4(\Omega_1)} &\le C\Big(\widetilde{\lambda}_0[\|w_1\|_{H^2_\Gamma(\Omega_1)} + \|w_2\|_{L^2(\Omega_1)}] + \|f_2\|_{L^2(\Omega_1)} + |\lambda|\|w_2\|_{L^2(\Omega_1)} + \|w_3\|_{H^2(\Omega_2)}\Big)\\ &\le C\Big(|\lambda|[\|w_1\|_{H^2_\Gamma(\Omega_1)} + \|w_2\|_{L^2(\Omega_1)}] + \|f_2\|_{L^2(\Omega_1)} + \|w_3\|_{H^2(\Omega_2)}\Big)\end{aligned}$$

for $|\lambda| > \lambda_0$, where the final constant $C > 0$ depends on λ_0 and $\widetilde{\lambda}_0$ but not on w, F, or λ. From the last inequality and (4.14), we obtain for $|\lambda| > \lambda_0$ that

$$\|w_1 + \kappa_1 w_2\|_{H^4(\Omega_1)} \le C\Big(|\lambda|\|w\|_{\mathscr{H}} + \|F\|_{\mathscr{H}}\Big). \tag{4.16}$$

From (4.9) and (4.10), it follows that

$$\begin{aligned}\|w_1 + \kappa_1 w_2\|_{H^2_\Gamma(\Omega_1)} &\le \|w_1\|_{H^2_\Gamma(\Omega_1)} + \kappa_1\|w_2\|_{H^2_\Gamma(\Omega_1)}\\ &\le \frac{C}{|\lambda|}\Big(\|w\|_{\mathscr{H}}\|F\|_{\mathscr{H}} + \|F\|^2_{\mathscr{H}}\Big)^{1/2} + C\|w\|^{1/2}_{\mathscr{H}}\|F\|^{1/2}_{\mathscr{H}}\\ &\le C\Big(\|w\|^{1/2}_{\mathscr{H}}\|F\|^{1/2}_{\mathscr{H}} + \|F\|_{\mathscr{H}}\Big).\end{aligned} \tag{4.17}$$

In the last inequality, we have taken into account that $\dfrac{1}{|\lambda|} < \dfrac{1}{\lambda_0}$, and $\dfrac{1}{\lambda_0}$ has been absorbed into the constant C.

Using Lemma 3.1, the interpolation inequality, (4.16), (4.17), and the equivalence of p-norms in $\mathbb{R}^2$, we obtain

$$\begin{aligned}\|\mathscr{B}_2(w_1 + \kappa_1 w_2)\|_{L^2(I)} &\le C\|w_1 + \kappa_1 w_2\|^{1/2}_{H^4(\Omega_1)}\|w_1 + \kappa_1 w_2\|^{1/2}_{H^3(\Omega_1)}\\ &\le C\|w_1 + \kappa_1 w_2\|^{1/2}_{H^4(\Omega_1)}\Big(\|w_1 + \kappa_1 w_2\|^{1/2}_{H^4(\Omega_1)}\|w_1 + \kappa_1 w_2\|^{1/2}_{H^2(\Omega_1)}\Big)^{1/2}\\ &\le C\|w_1 + \kappa_1 w_2\|^{3/4}_{H^4(\Omega_1)}\|w_1 + \kappa_1 w_2\|^{1/4}_{H^2(\Omega_1)}\\ &\le C\Big(|\lambda|\|w\|_{\mathscr{H}} + \|F\|_{\mathscr{H}}\Big)^{3/4}\Big(\|w\|^{1/2}_{\mathscr{H}}\|F\|^{1/2}_{\mathscr{H}} + \|F\|_{\mathscr{H}}\Big)^{1/4}\\ &\le C|\lambda|^{3/4}\Big(\|w\|^{3/4}_{\mathscr{H}} + \|F\|^{3/4}_{\mathscr{H}}\Big)\Big(\|w\|^{1/8}_{\mathscr{H}}\|F\|^{1/8}_{\mathscr{H}} + \|F\|^{1/4}_{\mathscr{H}}\Big)\\ &\le C|\lambda|^{3/4}\Big(\|w\|^{7/8}_{\mathscr{H}}\|F\|^{1/8}_{\mathscr{H}} + \|w\|^{3/4}_{\mathscr{H}}\|F\|^{1/4}_{\mathscr{H}} + \|w\|^{1/8}_{\mathscr{H}}\|F\|^{7/8}_{\mathscr{H}} + \|F\|_{\mathscr{H}}\Big).\end{aligned} \tag{4.18}$$

From (4.13), (4.15), and (4.18), it follows that

$$\begin{aligned}&\int_I |\partial_\nu w_3(q\cdot\nabla\overline{w_3})|\,dS\\ \le\, &C|\lambda|^{3/4}\Big(\|w\|^{7/8}_{\mathscr{H}}\|F\|^{1/8}_{\mathscr{H}} + \|w\|^{3/4}_{\mathscr{H}}\|F\|^{1/4}_{\mathscr{H}} + \|w\|^{1/8}_{\mathscr{H}}\|F\|^{7/8}_{\mathscr{H}} + \|F\|_{\mathscr{H}}\Big)\cdot\Big(|\lambda|^{1/2}\|w\|_{\mathscr{H}} + \|w\|^{1/2}_{\mathscr{H}}\|F\|^{1/2}_{\mathscr{H}}\Big)\\ \le\, &C|\lambda|^{5/4}\Big(\|w\|^{15/8}_{\mathscr{H}}\|F\|^{1/8}_{\mathscr{H}} + \|w\|^{11/8}_{\mathscr{H}}\|F\|^{5/8}_{\mathscr{H}} + \|w\|^{7/4}_{\mathscr{H}}\|F\|^{1/4}_{\mathscr{H}} + \|w\|^{5/4}_{\mathscr{H}}\|F\|^{3/4}_{\mathscr{H}}\\ &\quad + \|w\|^{9/8}_{\mathscr{H}}\|F\|^{7/8}_{\mathscr{H}} + \|w\|^{5/8}_{\mathscr{H}}\|F\|^{11/8}_{\mathscr{H}} + \|w\|_{\mathscr{H}}\|F\|_{\mathscr{H}} + \|w\|^{1/2}_{\mathscr{H}}\|F\|^{3/2}_{\mathscr{H}}\Big).\end{aligned}$$

Now, we can apply Young's inequality $a^{2-\theta}b^\theta \le \varepsilon a^2 + C_\varepsilon b^2$, which holds for fixed $\theta \in (0,2)$ and arbitrary $\varepsilon > 0$, to each summand on the right side of the last inequality. The estimate yielding the highest power of $|\lambda|$ is

$$|\lambda|^{5/4}\|w\|^{15/8}_{\mathscr{H}}\|F\|^{1/8}_{\mathscr{H}} = \|w\|^{15/8}_{\mathscr{H}}\Big(|\lambda|^{10}\|F\|_{\mathscr{H}}\Big)^{1/8} \le \varepsilon\|w\|^2_{\mathscr{H}} + C_\varepsilon|\lambda|^{20}\|F\|^2_{\mathscr{H}}.$$

Therefore, for $|\lambda| > \lambda_0$, we conclude that

$$\int_I |\partial_\nu w_3 (q \cdot \nabla \overline{w_3})| \, dS \le \varepsilon \|w\|_{\mathscr{H}}^2 + C_\varepsilon |\lambda|^{20} \|F\|_{\mathscr{H}}^2, \tag{4.19}$$

where $\varepsilon > 0$ is arbitrary, and C_ε is a positive constant depending on ε.

Using (4.12), (4.13) and (4.19) and applying Young's inequality to the term $|\lambda| \, \|w\|_{\mathscr{H}} \|F\|_{\mathscr{H}}$, we obtain

$$\|w\|_{\mathscr{H}}^2 \le \varepsilon \|w\|_{\mathscr{H}}^2 + C_\varepsilon |\lambda|^{20} \|F\|_{\mathscr{H}}^2 \quad (|\lambda| > \lambda_0)$$

for any $\varepsilon > 0$, where $C_\varepsilon > 0$ is a constant, which depends only on ε, λ_0, and $\widetilde{\lambda}_0$. Then, it follows that

$$\|w\|_{\mathscr{H}} \le C|\lambda|^{10} \|F\|_{\mathscr{H}} \quad (|\lambda| > \lambda_0), \tag{4.20}$$

where C is a positive constant depending on λ_0 and $\widetilde{\lambda}_0$ but not on w or F. The theorem now follows from Theorem 4.1. □

5. Conclusions

In this paper, we investigated a coupled plate-membrane transmission system with K-V damping acting on the plate and optionally on the membrane. The problem was formulated within an operator-theoretic framework, thereby allowing the system to be studied as an abstract evolution equation in an appropriate Hilbert space.

The main contribution of this work is the proof that the operator associated with the coupled system generates an analytic semigroup. This result provides a rigorous functional-analytic characterization of the dynamics and yields enhanced regularity properties for the corresponding solutions. In addition, the asymptotic behavior of the system was analyzed under different damping configurations. In particular, exponential stability follows when K-V damping acts on both components, whereas polynomial stability with an explicit decay rate is established when damping is applied only to the plate. Moreover, in the partially damped configuration, the polynomial stability obtained here yields a better decay rate than those previously reported for related transmission systems involving structural damping or thermoelastic effects. This observation highlights the influence of K-V damping on the long-time behavior of the coupled dynamics and further illustrates the impact of the transmission coupling on stability properties. An important consequence of the analyticity result is that solutions exhibit enhanced regularity in time, even for initial data belonging merely to the energy space. Furthermore, the generated semigroup inherits exponential stability properties and preserves analyticity under bounded perturbations. These features underscore the robustness of the proposed framework and demonstrate the effectiveness of semigroup methods in the analysis of coupled elastic transmission systems.

From an applied perspective, the analyticity and stability properties established in this work may be relevant for the mathematical analysis of vibration dissipation mechanisms in coupled elastic structures involving interacting plates and membranes. In particular, stability properties provide information on the long-time behavior of vibrations and their attenuation under damping effects. These results may also provide insight into the study of elastic structures composed of different materials coupled through transmission interfaces.

The results obtained in this work contribute to the mathematical understanding of transmission problems for coupled elastic structures with K-V damping and show that the interaction through interface conditions produces a nontrivial coupling between the subsystems. Future work may address localized damping effects, nonlinear interface interactions, or more general coupled elastic configurations involving heterogeneous materials.

Author contributions

Bienvenido Barraza Martínez, Jonathan González Ospino and Jairo Hernández Monzón: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. All authors of this article have been contributed equally.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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