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Research article

Characterizations of the Bertrand offsets of ruled surfaces with the Laplace operator in Euclidean 3-space $\mathbb{E}^3$

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Abstract: In the present paper, the Laplace operators with respect to the first and second fundamental forms of the Bertrand offset of a non-developable ruled surface were investigated, taking into account the structure functions of the non-developable ruled surface. For this purpose, the Bertrand offset of a non-developable ruled surface was considered, and its fundamental forms, mean curvature, and Gaussian curvature were calculated. Thus, finite (or infinite) type characterizations of the surface were obtained, and the relationships between Bertrand ruled surface pairs were established through the resulting Laplace operators.

Keywords: non-developable ruled surface; Bertrand offset; fundamental forms; Laplace operator; finite type surface
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1. Introduction

The concept of finite type was first defined by Chen in the late 1970s on submanifolds in Euclidean and semi-Euclidean spaces and has become an important area of research in differential geometry. A submanifold M^n of a Euclidean space $\mathbb{E}^{n+m}$ is said to be of finite type if each component of its position vector field X can be written as a finite sum of eigenfunctions of the Laplace operator Δ of M^n, that is,

$$X = x_0 + \sum_{i=1}^{k} x_i, \quad \Delta x_i = \lambda_i x_i, \quad i = 1, \ldots, k, \tag{1.1}$$

where x_0 is a constant vector and λ_i are real numbers. In particular, if all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are mutually distinct, then M^n is said to be of k-type. If $\lambda_i = 0$ for some $i = 1, \ldots, k$, then M^n is said to be of null k-type. Otherwise, M^n is said to be of infinite type. In general, when M^n is of finite type k, it follows from (1.1) that there exists a nonzero monic polynomial P of degree k such that $P(\Delta)(X - x_0) = 0$, [1]. In particular, Takahashi's theorem ($\Delta X = \lambda X$) states that a submanifold is of 1-type if and only if it is either a minimal submanifold of Euclidean space ($\lambda = 0$) or a minimal submanifold of a hypersphere ($\lambda \neq 0$) [2].

Subsequently, Chen proposed the classification problem for finite type surfaces in Euclidean spaces. This led to an investigation of surfaces from the viewpoint of finite type geometry [3]. In 1988, Garay examined surfaces of revolution [4], and in 1990, Chen et al. studied ruled surfaces [5] and quadrics [6], which inspired further research on various surfaces. In these initial studies, the surfaces were examined via their parametric expressions using the Laplace operator with respect to the first fundamental form, and their characterizations were provided. In Chen's original notion of finite type, the classical Laplace-Beltrami operator is obtained with the first fundamental form. Later, researchers extended these characterizations of surfaces via the Laplace operator with respect to the second and third fundamental forms, leading to a uniform notation covering all fundamental forms. Therefore, in this extended notation, the notations Δ and Δ^I, as well as the notions of k-type and finite I-type k, have the same meaning. In addition, Garay [7] established the foundations of the matrix relation approach in finite type theory by generalizing Takahashi's theorem. Coordinate finite type submanifolds were introduced through Garay's generalization of Takahashi's theorem. More precisely, Garay considered submanifolds of Euclidean space whose position vector field X satisfies $\Delta X = AX$ for a constant diagonal matrix A, and called them coordinate finite type submanifolds.

Thus, coordinate finite type surfaces have also been investigated, where the position vector field satisfies an equation of the form*

$$\Delta^{\mathcal{J}} X = \Lambda X, \quad \mathcal{J} = I, II, III, \tag{1.2}$$

where Λ is a constant matrix. Following the foundational studies on classical surfaces such as ruled surfaces, surfaces of revolution, and quadrics, the research was also extended with respect to the second and third fundamental forms, both Chen's finite type spectral framework defined by Eq (1.1) and the coordinate finite type matrix framework defined by Eq (1.2). For ruled surfaces with respect to the third fundamental form, ruled surfaces satisfying the condition $\Delta^{III} X = \Lambda X$ were investigated in [8], while finite III-type ruled surfaces were studied within Chen's approach in [9]. Similarly, ruled surfaces satisfying $\Delta^{II} X = \Lambda X$ in [10] and finite II-type ruled surfaces within Chen's approach in [11] were examined. In a similar manner, surfaces of revolution have been classified with respect to the third fundamental form within Chen's finite type approach [12], including coordinate type studies for both the third [13] and the second [14] fundamental forms. For quadrics, investigations on surfaces satisfying $\Delta^{III} X = \Lambda X$ [8], as well as studies in III-type [15] and II-type [16] within Chen's approach, have also been carried out. Also, the literature has been significantly expanded through characterizations via the Laplace operator with respect to the fundamental forms of various surfaces, such as translation [17], helicoidal [18], conchoidal [19], and Hasimoto [20]. Following a similar approach, pioneering studies [21, 22] investigated the concept of finite type for the Gauss map of surfaces. This subject remains a topic of significant interest in recent literature across various surface classes [20, 23, 24].

*For the notation $\mathcal{J}$, I: first fundamental form, II: second fundamental form, III: third fundamental form.

In differential geometry, ruled surfaces play a fundamental role and arise naturally in various applications such as computer graphics, architectural design, and engineering modeling. Ruled surfaces are defined as a one-parameter family of lines. In other words, they can also be defined as a surface obtained by moving a line around a curve. Ruled surfaces have been studied in the literature in various spaces, with the properties of the curves that form the surface. In 1991, the Bertrand offset of ruled surfaces was defined [25]. Then, the involute-evolute offset of ruled surfaces was studied in [26], and the Mannheim offset of ruled surfaces was examined in [27].

In 2014, Liu et al. defined invariants of non-developable ruled surfaces in Euclidean 3-space $\mathbb{E}^3$ using the classical methods of differential geometry [28]. They investigated invariants of the non-developable ruled surfaces in Euclidean 3-space, called structure functions, and showed the kinematics meaning of these invariants [29]. Then, in 2016, Yoon studied the involute-evolute offset of non-developable ruled surfaces in $\mathbb{E}^3$ [30]. Research on non-developable ruled surfaces and their invariants remains an active topic in the literature [31, 32].

In this paper, we investigate the Bertrand offset of a non-developable ruled surface via the Laplace operator with respect to the fundamental forms, under the coordinate finite type condition, i.e., their position vectors satisfy Eq (1.2), and we focus on the calculations for $\mathcal{J} = I$ and $\mathcal{J} = II$. For this purpose, we consider the Bertrand offset of a non-developable ruled surface given in [28] and obtain its fundamental forms, curvatures, and the Laplace operators. Then, based on these considerations, some characterizations are given via the Laplace operator with respect to the first and second fundamental forms. The novelty of this study lies in the fact that, motivated by the question of how a surface pair can be analyzed via the Laplace operator, it considers the Bertrand ruled surface pair under the coordinate finite type condition. The analysis is carried out through the structure functions of a non-developable ruled surface, and the Laplace operators of the Bertrand offset are expressed in terms of these functions. Moreover, the relations among the structure functions are used not only to compute the curvatures and Laplace operators, but also to characterize the developable, non-developable, and minimal cases of the Bertrand offset. Also, it provides a new perspective on the Laplace operator of a non-developable ruled surface via structure functions. Moreover, fundamental relations are established between the Bertrand ruled surface pair by the structure functions of the non-developable ruled surfaces.

The paper is organized as follows: In Section 2, we present the necessary preliminaries. In Section 3, the main results and characterizations are derived for non-developable ruled surfaces and their Bertrand offsets with respect to the Laplace operators of the first and second fundamental forms. In addition, the minimal developable and minimal non-developable cases of the Bertrand offsets are also discussed. Besides, two explicit examples of non-developable ruled surfaces are provided in Section 4 to illustrate the obtained results. These examples are selected to represent different cases of the derived characterizations. For each example, the structure functions, fundamental forms, curvatures, and Laplace operators are computed both for the non-developable ruled surface and for its Bertrand offset. Then, the corresponding surfaces are determined to be of finite or infinite type with respect to the Laplace operators of the first and second fundamental forms. Finally, the paper concludes with a section that includes discussions, conclusions, and suggestions for future studies.

2. Preliminaries

In this section, we present the fundamental concepts to establish the necessary framework for our main results. Initially, we present essential terminology from the theory of curves and surfaces [33–35]. In particular, we present the basic geometric properties of ruled surfaces [36, 37]. Then, our main focus is directed toward the Laplace operator and the geometric properties of non-developable ruled surfaces [28, 30] and Bertrand offsets [25].

Throughout the paper, the notation $\dot{(\cdot)}$ and $(\cdot)'$ represent differentiation with respect to s and u, respectively.

Definition 2.1. *Let $\alpha : I \to \mathbb{E}^3$ be a regular curve in Euclidean 3-space $\mathbb{E}^3$ parameterized by its arc-length s and $\{t(s), n(s), b(s)\}$ be the Frenet frame along $\alpha(s)$. Then, the Frenet formulas of $\alpha(s)$ are given by*

$$\begin{cases} \dot{t}(s) = \kappa(s)n(s), \\ \dot{n}(s) = -\kappa(s)t(s) + \tau(s)b(s), \\ \dot{b}(s) = -\tau(s)n(s). \end{cases}$$

Here, the functions $\kappa(s) = \|\ddot{\alpha}(s)\|$ and $\tau(s) = \frac{\langle \dot{\alpha}(s), \ddot{\alpha}(s) \times \dddot{\alpha}(s) \rangle}{\|\ddot{\alpha}(s)\|^2}$ are the curvature and torsion functions of $\alpha(s)$, respectively [34, 35].

For a spherical curve, the following definition can be given.

Definition 2.2. *Let $\alpha(s)$ be a spherical curve parameterized by its arc-length s in $\mathbb{E}^3$ with $\langle \alpha(s), \alpha(s) \rangle$=1. Let $t(s) = \dot{\alpha}(s)$ and $g(s) = \alpha(s) \times \dot{\alpha}(s)$. Then, $\{\alpha(s), t(s), g(s)\}$ is an orthonormal frame along $\alpha(s)$ in $\mathbb{E}^3$ and is called the spherical Frenet frame of the spherical curve $\alpha(s)$ in $\mathbb{E}^3$. Thus, spherical Frenet frame formulas of $\alpha(s)$ are*

$$\begin{cases} \dot{\alpha}(s) = t(s), \\ \dot{t}(s) = -\alpha(s) - \kappa_g(s)g(s), \\ \dot{g}(s) = \kappa_g(s)t(s). \end{cases} \tag{2.1}$$

Here, the function

$$\kappa_g(s) = \det(\ddot{\alpha}(s), \dot{\alpha}(s), \alpha(s)) \tag{2.2}$$

is called the geodesic curvature function of the spherical curve $\alpha(s)$ [34, 35].

Proposition 2.1. *Let $\alpha(s)$ be a spherical curve parameterized by its arc-length s in $\mathbb{E}^3$. The geodesic curvature $\kappa_g(s)$, the curvature $\kappa(s)$, and the torsion $\tau(s)$ of $\alpha(s)$ satisfy the following relations [34, 35]:*

$$\kappa(s) = \sqrt{1 + \kappa_g^2(s)}, \quad \tau(s) = \pm\frac{\dot{\kappa}_g(s)}{\kappa_g^2(s) + 1}.$$

Definition 2.3. *Let $\mathcal{M}$ be a smooth surface given with the patch $X : \Omega \subset \mathbb{E}^2 \to \mathbb{E}^3$ in $\mathbb{E}^3$. Then, the unit normal vector field $\mathcal{U}$ of $\mathcal{M}$ at points $(u, v) \in \Omega \subset \mathbb{E}^2$ is defined by [34, 35]:*

$$\mathcal{U} = \frac{X_u \times X_v}{\|X_u \times X_v\|}, \tag{2.3}$$

at which $X_u \times X_v$ does not vanish, i.e., X is regular. For a regular patch X, the Gaussian curvature and the mean curvature of the surface $\mathcal{M}$ are given by

$$\mathcal{K} = \frac{LN - M^2}{EG - F^2}, \qquad \mathcal{H} = \frac{LG - 2MF + NE}{2(EG - F^2)}, \tag{2.4}$$

where

$$E = \langle X_u, X_u \rangle\,,\; F = \langle X_u, X_v \rangle\,,\; G = \langle X_v, X_v \rangle\,, \tag{2.5}$$

and

$$L = \langle X_{uu}, \mathcal{U} \rangle\,,\; M = \langle X_{uv}, \mathcal{U} \rangle\,,\; N = \langle X_{vv}, \mathcal{U} \rangle\,, \tag{2.6}$$

are coefficients of the first fundamental form I of $\mathcal{M}$ and coefficients of the second fundamental form II of $\mathcal{M}$, respectively. Here, the notation $\langle\,,\rangle$ represents Euclidean inner product.

In $\mathbb{E}^3$, a surface is called flat if its Gaussian curvature vanishes identically, and it is called minimal if its mean curvature is zero.

Definition 2.4. *The Laplace operator Δ with respect to the induced metric on the surface $\mathcal{M}$ is defined as [1]:*

$$\Delta^{\mathcal{J}} = -\frac{1}{\sqrt{d}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u_i} \left(\sqrt{d} d^{ij} \frac{\partial}{\partial u_j} \right), \tag{2.7}$$

where u_1, u_2 are the local coordinates, that is, $u_1 = u, u_2 = v$ on $\mathcal{M}$ and $\mathcal{J} = I, II$ or III. The notation $\mathcal{J}$ indicates that the fundamental form is used in computing the Laplace operator. The matrix (d^{ij}) is the inverse of $(d_{ij}) = \left(\begin{smallmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{smallmatrix}\right)$, then we write $(d^{ij}) = \left(\begin{smallmatrix} d^{11} & d^{12} \\ d^{21} & d^{22} \end{smallmatrix}\right) = \frac{1}{d}\left(\begin{smallmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{smallmatrix}\right)$, where $\det(d_{ij}) = d$ is nonzero. Expanding the righthand side of Eq (2.7), we obtain

$$\begin{aligned}
\Delta^{\mathcal{J}} =& -\frac{d_{22}}{d}\frac{\partial^2}{\partial u^2} + \frac{2d_{12}}{d}\frac{\partial^2}{\partial u \partial v} - \frac{d_{11}}{d}\frac{\partial^2}{\partial v^2} \\
&+ \left(-\frac{1}{d}\frac{\partial d_{22}}{\partial u} + \frac{d_{22}\left(\frac{\partial d_{11}}{\partial u} d_{22} + d_{11}\frac{\partial d_{22}}{\partial u} - 2d_{12}\frac{\partial d_{12}}{\partial u}\right)}{2d^2} + \frac{1}{d}\frac{\partial d_{12}}{\partial v} - \frac{d_{12}\left(\frac{\partial d_{11}}{\partial v} d_{22} + d_{11}\frac{\partial d_{22}}{\partial v} - 2d_{12}\frac{\partial d_{12}}{\partial v}\right)}{2d^2} \right) \frac{\partial}{\partial u} \\
&+ \left(\frac{1}{d}\frac{\partial d_{12}}{\partial u} - \frac{d_{12}\left(\frac{\partial d_{11}}{\partial u} d_{22} + d_{11}\frac{\partial d_{22}}{\partial u} - 2d_{12}\frac{\partial d_{12}}{\partial u}\right)}{2d^2} - \frac{1}{d}\frac{\partial d_{11}}{\partial v} + \frac{d_{11}\left(\frac{\partial d_{11}}{\partial v} d_{22} + d_{11}\frac{\partial d_{22}}{\partial v} - 2d_{12}\frac{\partial d_{12}}{\partial v}\right)}{2d^2} \right) \frac{\partial}{\partial v}.
\end{aligned} \tag{2.8}$$

Note that $d_{12} = d_{21}$ here.

Having defined the Laplace operator $\Delta^{\mathcal{J}}$, we now recall the finite type notion for surfaces in Chen's spectral sense.

If the position vector X of $\mathcal{M}$ can be expressed as a finite sum of eigenvectors of the Laplace $\Delta^{\mathcal{J}}$ of $\mathcal{M}$, then $\mathcal{M}$ is said to be a finite type surface, that is,

$$X = x_0 + \sum_{i=1}^{k} x_i, \quad \Delta^{\mathcal{J}} x_i = \lambda_i x_i, \quad 1 \leq i \leq k,$$

where x_0 is a constant map and $x_1, \ldots, x_k$ are nonconstant maps. If the eigenvalues λ_i are all distinct from each other, then the surface is called a finite k-type surface with respect to the fundamental form $\mathcal{J}$, or simply a finite $\mathcal{J}$-type k surface. If any of the eigenvalues is zero, then the surface is called a finite null k-type surface with respect to the fundamental form $\mathcal{J}$, or a finite null $\mathcal{J}$-type k surface. Otherwise, the surface is an infinite type surface.

On the other hand, a surface $\mathcal{M}$ in $\mathbb{E}^3$ is said to be of coordinate finite type with respect to the fundamental form $\mathcal{J}$ if its position vector $X = (x_1, x_2, x_3)$ satisfies the following condition:

$$\Delta^{\mathcal{J}} X = \Lambda X, \tag{2.9}$$

where $\Delta^{\mathcal{J}}$ is the Laplace operator with respect to the fundamental form $\mathcal{J}$ and $\Lambda \in \mathbb{R}^{3\times 3}$ is a real matrix of order 3. If no such matrix Λ exists, then $\mathcal{M}$ is said to be of coordinate infinite type with respect to the fundamental form $\mathcal{J}$. In particular, in case $\Delta^{\mathcal{J}} X = 0$, $\mathcal{M}$ is called a $\mathcal{J}$-harmonic surface.

Definition 2.5. *In $\mathbb{E}^3$, a one-parameter family of straight lines is called a ruled surface. These lines are called rulings (or generators) of the surface. A ruled surface $\mathcal{M}$ in $\mathbb{E}^3$ can be parametrized as*

$$X(u, v) = \alpha(u) + ve(u), \ \langle e, e\rangle = 1, \tag{2.10}$$

where $\alpha(u)$ is the base curve and $e(u)$ is the unit vector in the direction of the rulings of $\mathcal{M}$. The curve drawn by $e(u)$ on the unit sphere S^2 is called the spherical indicatrix curve, denoted by (e), and e is also called the spherical indicatrix vector of $\mathcal{M}$. Here, u is the arc-length parameter of α [36, 37].

Definition 2.6. *Let $\mathcal{M}$ be a ruled surface with parametric representation $X(u, v)$ given in Eq* (2.10) *and $\mathcal{U}$ denotes its unit normal vector field. The unit normal along a generator $X(u_0, v)$ of the ruled surface approaches a limiting direction as v infinitely decreases. This direction is called the asymptotic normal direction and given by [25]*

$$g = \frac{e \times e'}{\|e'\|}.$$

The point at which $\mathcal{U}$ has rotated only $\pi/2$ and is perpendicular to g is called the striction point[†] *(or central point) on $X(u_0, v)$. The direction of $\mathcal{U}$ at this point, denoted by t, is called the central normal of the ruled surface and is given by*

$$t = \frac{e'}{\|e'\|}.$$

The set of striction points on a ruled surface defines the striction curve. The striction curve $\beta(u)$ can be written such that

$$\beta = \alpha - \frac{\langle \alpha', e'\rangle}{\langle e', e'\rangle} e. \tag{2.11}$$

Definition 2.7. *The ratio of the shortest distance between two adjacent generators of a ruled surface to the angle between them is called the distribution parameter of the ruled surface, such that [25, 36, 37]:*

$$\frac{\det(\alpha', e, e')}{\|e'\|^2}. \tag{2.12}$$

If this value is zero, then the ruled surface is developable[‡]*. Otherwise, the ruled surface is non-developable.*

[†]The striction point is also the foot of the common normal between two consecutive generators.
[‡]If consecutive generators of a ruled surface intersect, then the surface is said to be developable.

Let $\mathcal{M}$ be a non-developable ruled surface in $\mathbb{E}^3$ such that

$$X(u,v) = \beta(u) + ve(u),\ \langle e, e\rangle = 1, \tag{2.13}$$

where $\beta(u)$ is the striction curve of the surface $X(u,v)$ and (e) is the spherical indicatrix curve with $\langle e'(u), e'(u)\rangle = 1$. Since $\beta(u)$ is the striction curve of $X(u,v)$, via Eq (2.11), we have $\langle \beta'(u), e'(u)\rangle = 0$. Then, the set $\{e, t, g\}$ is the spherical Frenet frame of (e) and the vectors

$$t = e', \quad g = e \times e' \tag{2.14}$$

are the central normal and the asymptotic normal of $X(u,v)$, respectively [28].

Here, one can realize that using Definition 2.2, Eq (2.1) turns to the following:

$$\begin{cases} e' = t, \\ t' = -e - Jg, \\ g' = Jt, \end{cases} \tag{2.15}$$

where

$$J = \langle e'', e' \times e\rangle \tag{2.16}$$

denotes the geodesic curvature, previously denoted by κ_g (see Eq (2.2)), of the spherical indicatrix curve (e) [28, 30].

On the other hand, the derivative of the striction curve β is given by

$$\beta' = Fe + Qg, \tag{2.17}$$

where

$$F = \langle \beta', e\rangle, \tag{2.18}$$

$$Q = \langle \beta', e \times e'\rangle. \tag{2.19}$$

One can also see F and Q in Eqs (2.5) and (2.12), respectively. Here, the function Q, that is, $Q \neq 0$, is the parameter of the distribution of $X(u,v)$. The functions J, F, and Q of $X(u,v)$ are called structure functions of a non-developable ruled surface $X(u,v)$ given in Eq (2.13) [28, 30].

Proposition 2.2. *Let $X(u,v)$ be a non-developable ruled surface given in Eq (2.13) in $\mathbb{E}^3$. Then, $X(u,v)$ is uniquely determined by its structure functions J, F, and Q, up to orthonormal transformations in $\mathbb{E}^3$ [30].*

On the other hand, the parameter u is the arc-length parameter of (e), but usually u is not the arc-length parameter of the curve β. By Eq (2.17), we have the following.

Proposition 2.3. *Let $X(u,v)$ be a non-developable ruled surface given in Eq (2.13) in $\mathbb{E}^3$. If the parameter u is also the arc-length parameter of the striction curve β of $X(u,v)$, then the structure functions F and Q of $X(u,v)$ satisfy $F^2 + Q^2 = 1$ [30].*

Now, we examine the geometric properties of a non-developable ruled surface $X(u, v)$ given in Eq (2.13). Then, Eqs (2.15) and (2.17) enable to determine the following [30]:

Considering Eq (2.5), the coefficients of the first fundamental form of $X(u, v)$ are given by

$$E = F^2 + Q^2 + v^2, \quad F = \langle \beta', e \rangle, \quad G = 1. \tag{2.20}$$

Then, using Eq (2.3), the unit normal vector $\mathcal{U}$ of $X(u, v)$ is written as

$$\mathcal{U} = \frac{1}{\sqrt{\mathsf{d}}}(\beta' \times e + v\,(e' \times e)) = \frac{1}{\sqrt{\mathsf{d}}}(Qt - vg), \tag{2.21}$$

where

$$\mathsf{d} = EG - F^2 = Q^2 + v^2 > 0, \tag{2.22}$$

(which, in particular, implies $\mathsf{d} \neq 0$) with the help of Eq (2.20). This leads to the coefficients L, M, and N of the second fundamental form via Eq (2.6) as

$$L = \frac{1}{\sqrt{\mathsf{d}}}(Q(F + QJ) - Q'v + Jv^2), \quad M = \frac{Q}{\sqrt{\mathsf{d}}}, \quad N = 0. \tag{2.23}$$

Subsequently, via Eq (2.4), these coefficients allow us to determine the Gaussian curvature $\mathcal{K}$ and the mean curvature $\mathcal{H}$ of $X(u, v)$ as

$$\mathcal{K} = -\frac{Q^2}{\mathsf{d}^2}, \qquad \mathcal{H} = \frac{1}{2\mathsf{d}^{3/2}}(Jv^2 - Q'v + Q(QJ - F)), \tag{2.24}$$

respectively.

Definition 2.8. *Two ruled surfaces are said to be Bertrand offsets of one another if there exists a one to one correspondence between their rulings such that both surfaces have a common principal normal at the striction points of their corresponding rulings [25].*

In view of this definition, let $X^*(u, v)$ be the Bertrand offset of the non-developable ruled surface $X(u, v)$ given in Eq (2.13). Then, the parametrization of $X^*(u, v)$ is given as [25]:

$$X^*(u, v) = \beta^*(u) + ve^*(u) = \beta(u) + ct(u) + v(\cos\theta e(u) + \sin\theta g(u)), \tag{2.25}$$

via relationship

$$\begin{bmatrix} e^* \\ t^* \\ g^* \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} e \\ t \\ g \end{bmatrix},$$

between the spherical Frenet frame $\{e, t, g\}$ of $X(u, v)$ and spherical Frenet frame $\{e^*, t^*, g^*\}$ of $X^*(u, v)$. Here, c and θ are constants representing the linear offset and the angular offset between the two surfaces, respectively.

Remark 2.1. *If $\theta = 0$, then X and X^* are referred to as oriented offsets, where*

$$X^*(u, v) = \beta(u) + ct(u) + ve(u).$$

If $\theta = \pi/2$, then X and X^* are referred to as right offsets, where

$$X^*(u, v) = \beta(u) + ct(u) + vg(u).$$

3. Characterizations of the Bertrand offsets of non-developable ruled surfaces with the Laplace operator via the first and second fundamental forms in $\mathbb{E}^3$

In this section, we characterize the Bertrand offsets of a non-developable ruled surface in $\mathbb{E}^3$ given in Eq (2.13) by the Laplace operators via the first and second fundamental forms. Our analysis is based on spherical Frenet frames and structure functions of the non-developable ruled surface. To establish a foundation, we revisit the existing characterizations of ruled surfaces with the Laplace operator from a new perspective. Subsequently, we focus on our primary objective: characterizing the Bertrand offsets via the Laplace operator and investigating whether these surfaces are of finite type.

In this context, we establish the fundamental geometric structures required for the original results obtained in Sections 3.1 and 3.2 of this article.

The abbreviations frequently used in this section, together with the relevant notations introduced in Section 2, are summarized in Table 1.

Table 1. Frequently used symbols.

Symbol	Expression
J	$\langle e'', e' \times e\rangle$
F	$\langle \beta', e\rangle$
Q	$\langle \beta', e \times e'\rangle$
d	$Q^2 + v^2$
ξ_1	$F - c$
ξ_2	$Q - cJ$
A	$\cos\theta + J\sin\theta$
B	$\xi_2\cos\theta - \xi_1\sin\theta$
d^*	$v^2A^2 + B^2$

Let us examine the geometric properties of $X^*(u, v)$ given in Eq (2.25). From Eqs (2.15) and (2.17), we obtain the following:

Considering Eq (2.25), the partial derivatives of the Bertrand offset $X^*(u, v)$ with respect to u and v are obtained as follows:

$$\begin{cases} X^*_u = \beta' + ct' + v(\cos\theta e' + \sin\theta g') = (F - c)e + v(\cos\theta + J\sin\theta)t + (Q - cJ)g, \\ X^*_v = \cos\theta e + \sin\theta g. \end{cases} \tag{3.1}$$

Thus, taking into account Eqs (2.5) and (3.1), the coefficients of the first fundamental form of $X^*(u, v)$ are

$$E^* = \xi_1^2 + v^2A^2 + \xi_2^2, \quad F^* = \xi_1\cos\theta + \xi_2\sin\theta, \quad G^* = 1, \tag{3.2}$$

where

$$\xi_1 = F - c, \quad \xi_2 = Q - cJ, \quad A = \cos\theta + J\sin\theta. \tag{3.3}$$

Additionally, considering Eqs (2.3) and (3.1), the unit normal vector $\mathcal{U}^*$ of $X^*(u, v)$ is written as

$$\mathcal{U}^* = \frac{1}{\sqrt{\mathsf{d}^*}}\left(vA\sin\theta e + Bt - vA\cos\theta g\right), \tag{3.4}$$

where

$$\mathsf{d}^* = E^*G^* - F^{*2} = v^2A^2 + B^2 > 0, \tag{3.5}$$

(which, in particular, implies $\mathsf{d}^* \neq 0$) and

$$B = \xi_2 \cos\theta - \xi_1 \sin\theta. \tag{3.6}$$

To determine the coefficients of the second fundamental form, we first calculate the second order partial derivatives of the Bertrand offset $X^*(u, v)$ with respect to u and v as

$$X^*_{uu} = (F' - vA)e + (\xi_1 + vA' + \xi_2 J)t + (Q' - cJ' - vAJ)g, \quad X^*_{uv} = At, \quad X^*_{vv} = 0. \tag{3.7}$$

Considering these derivatives, Eqs (2.6) and (3.4), the coefficients of the second fundamental form of $X^*(u, v)$ are obtained as

$$\begin{cases} L^* = \dfrac{1}{\sqrt{\mathsf{d}^*}}\left((F' - vA)(vA\sin\theta) + (\xi_1 + vA' + \xi_2 J)B - (Q' - cJ' - vAJ)(vA\cos\theta)\right), \\ M^* = \dfrac{AB}{\sqrt{\mathsf{d}^*}}, \\ N^* = 0. \end{cases} \tag{3.8}$$

Thus, through the calculation of the coefficients of the first and second fundamental forms given in Eqs (3.2) and (3.8), the Gaussian curvature $\mathcal{K}^*$ and the mean curvature $\mathcal{H}^*$ of $X^*(u, v)$ are given, respectively, via Eq (2.4) as

$$\begin{aligned} \mathcal{K}^* &= -\frac{A^2 B^2}{\mathsf{d}^{*2}}, \\ \mathcal{H}^* &= \frac{vA\sin\theta(F' - vA) + B(\xi_1 + vA' + \xi_2 J) - vA\cos\theta(Q' - cJ' - vAJ) - 2ABF^*}{2\mathsf{d}^{*3/2}}. \end{aligned} \tag{3.9}$$

Additionally, from Eq (2.12), the parameter of the distribution for Bertrand offset X^* of the non-developable ruled surface X is calculated by

$$Q^* = \frac{\det(\beta^{*\prime}, e^*, e^{*\prime})}{\|e^{*\prime}\|^2} = \frac{\xi_2\cos\theta - \xi_1\sin\theta}{\cos\theta + J\sin\theta} = \frac{B}{A},$$

where ξ_1, ξ_2, A, and B are given in Eqs (3.3) and (3.6). In order for Q^* to be well-defined, e^* must be a regular curve. Therefore, it becomes apparent from this part of the paper that

$$A = \cos\theta + J\sin\theta \neq 0. \tag{3.10}$$

Theorem 3.1. *Let X^* be a Bertrand offset of the non-developable ruled surface X given in Eq* (2.13). *Let J, F, and Q be structure functions of X.*

(i) If X^* has a constant Gaussian curvature $\mathcal{K}^*_0$, then X^* has zero Gaussian curvature. In this case, the structure functions of X satisfy $(Q - cJ)\cos\theta - (F - c)\sin\theta = 0$.

(ii) If X^* has a constant mean curvature $\mathcal{H}^*_0$, then X^* has zero mean curvature. In this case, the structure functions of X satisfy $J = \tan\theta$ and $Q' = JF'$, together with one of the following relations:

- $Q = JF$.
- $Q \neq JF$ *and* $c = \dfrac{F + QJ}{1 + J^2}$.

Proof. (i) Let us suppose that X^* has a constant Gaussian curvature $\mathcal{K}_0^*$. From Eqs (3.5) and (3.9), we obtain the following polynomial of v as

$$\mathcal{K}_0^*(v^4A^4 + 2v^2A^2B^2 + B^4) + A^2B^2 = 0.$$

Thus, all coefficients of this polynomial with respect to v are equal to zero, then

$$A^4\mathcal{K}_0^* = 0, \tag{3.11}$$
$$2A^2B^2\mathcal{K}_0^* = 0, \tag{3.12}$$
$$B^4\mathcal{K}_0^* + A^2B^2 = 0, \tag{3.13}$$

are obtained. Since $A \neq 0$ (see Eq (3.10)), Eqs (3.11)–(3.13) yield $\mathcal{K}_0^* = 0$ and $B = 0$. Hence, it is observed that from Eqs (3.3) and (3.6), the equality $(Q - cJ)\cos\theta - (F - c)\sin\theta = 0$ includes structure functions of X, when X^* has zero Gaussian curvature.

(ii) Let us suppose that X^* has a constant mean curvature $\mathcal{H}_0^*$. From Eqs (3.2), (3.5), and (3.9), we obtain

$$2\mathcal{H}_0^*(v^2A^2+B^2)^{3/2} = vA\sin\theta(F'-vA)+B(\xi_1+vA'+\xi_2J)-vA\cos\theta(Q'-cJ'-vAJ)-2AB\left(\xi_1\cos\theta+\xi_2\sin\theta\right).$$

Squaring both sides of the equation, we obtain the following polynomial in v as

$$\begin{aligned}
&4v^6\mathcal{H}_0^{*2}A^6+A^4v^4\left(12\mathcal{H}_0^{*2}B^2-(\sin\theta-J\cos\theta)^2\right)-2v^3A^2(J\cos\theta-\sin\theta)\left(BA'+A\cos\theta\left(cJ'-Q'\right)+AF'\sin\theta\right)\\
&+v^2\Big(2A^2B\left(6\mathcal{H}_0^{*2}B^3-(J\cos\theta-\sin\theta)(\xi_1+J\xi_2-2A(\xi_1\cos\theta+\xi_2\sin\theta)\right)-\left(BA'+A\cos\theta\left(cJ'-Q'\right)+AF'\sin\theta\right)^2\Big)\\
&+2vB(-\xi_1-J\xi_2+2A(\xi_1\cos\theta+\xi_2\sin\theta))\left(BA'+A\cos\theta\left(cJ'-Q'\right)+AF'\sin\theta\right)\\
&+B^2\left(2\mathcal{H}_0^*B^2-\xi_1-J\xi_2+2A(\xi_1\cos\theta+\xi_2\sin\theta)\right)\left(2\mathcal{H}_0^*B^2+\xi_1+J\xi_2-2A(\xi_1\cos\theta+\xi_2\sin\theta)\right)=0.
\end{aligned}$$

Thus, all coefficients of this polynomial with respect to v are equal to zero, and we have

$$4\mathcal{H}_0^{*2}A^6 = 0, \tag{3.14}$$
$$A^4\left(12\mathcal{H}_0^{*2}B^2 - (\sin\theta - J\cos\theta)^2\right) = 0, \tag{3.15}$$
$$-2A^2(J\cos\theta - \sin\theta)\left(BA' + A\cos\theta\left(cJ' - Q'\right) + AF'\sin\theta\right) = 0, \tag{3.16}$$
$$\begin{aligned}&\Big(2A^2B\left(6\mathcal{H}_0^{*2}B^3 - (J\cos\theta - \sin\theta)(\xi_1 + J\xi_2 - 2A(\xi_1\cos\theta + \xi_2\sin\theta)\right)\\ &\quad - \left(BA' + A\cos\theta\left(cJ' - Q'\right) + AF'\sin\theta\right)^2\Big) = 0,\end{aligned} \tag{3.17}$$
$$2B(-\xi_1 - J\xi_2 + 2A(\xi_1\cos\theta + \xi_2\sin\theta))\left(BA' + A\cos\theta\left(cJ' - Q'\right) + AF'\sin\theta\right) = 0, \tag{3.18}$$
$$B^2\left(2\mathcal{H}_0^*B^2-\xi_1-J\xi_2+2A(\xi_1\cos\theta+\xi_2\sin\theta)\right)\left(2\mathcal{H}_0^*B^2+\xi_1+J\xi_2-2A(\xi_1\cos\theta+\xi_2\sin\theta)\right)=0. \tag{3.19}$$

Since $A \neq 0$ (see Eq (3.10)), $\mathcal{H}_0^* = 0$ via Eq (3.14). Substituting $\mathcal{H}_0^* = 0$ into the other equations yields: As $A \neq 0$, $J = \tan\theta$ (Since $\cos\theta = 0$ leads to a contradiction, it follows that $\cos\theta \neq 0$) is obtained from Eq (3.15). It is clear that Eq (3.16) is satisfied by $J = \tan\theta$. Besides, since $J = \tan\theta$ is constant, then $A = \cos\theta + J\sin\theta = \sec\theta$ is constant and $A' = 0$. Thus, Eq (3.17) yields $F'\sin\theta - Q'\cos\theta = 0$ using $J = \tan\theta$, $J' = 0$, and $A' = 0$. From the resulting equality, $Q' = JF'$ is obtained. It can be easily verified that Eq (3.18) is satisfied by the obtained conditions.

Once these findings are substituted into Eq (3.19) and the abbreviations are expanded (see Eqs (3.3) and (3.6)),

$$-B^2(\xi_1 + \xi_2\tan\theta)^2 = -((Q - cJ)\cos\theta - (F - c)\sin\theta)^2 \frac{((F - c)\cos\theta + (Q - cJ)\sin\theta))^2}{\cos^2\theta} = 0$$

is obtained. Therefore, either $B = (Q-cJ)\cos\theta-(F-c)\sin\theta = 0$ or $F^* = (F-c)\cos\theta+(Q-cJ)\sin\theta = 0$ (see Eq (3.2)).

If $B = (Q-cJ)\cos\theta-(F-c)\sin\theta = 0$, then $Q = JF$ is obtained. Thus, if X^* has zero mean curvature $\mathcal{H}_0^* = 0$, the structure functions of X satisfy $Q' = JF'$ under conditions $Q = JF$ and $J = \tan\theta$.

In the other case, if $F^* = (F - c)\cos\theta + (Q - cJ)\sin\theta = 0$, then $c = \frac{F+QJ}{1+J^2}$ is obtained. Thus, if X^* has zero mean curvature $\mathcal{H}_0^* = 0$, the structure functions of X satisfy $Q' = JF'$ and $c = \frac{F+QJ}{1+J^2}$ under conditions $Q \neq JF$ and $J = \tan\theta$. □

Corollary 3.1. *Let X^* be a Bertrand offset of the non-developable ruled surface X given in Eq* (2.13)*. Let J, F, and Q be structure functions of X. Then, the following hold:*

(i) X^* is a developable ruled surface (flat surface) when $(Q - cJ)\cos\theta - (F - c)\sin\theta = 0$.
(ii) X^* is a non-developable ruled surface when $(Q - cJ)\cos\theta - (F - c)\sin\theta \neq 0$.
(iii) X^* is a developable ruled surface when $Q = JF$ and $J = \tan\theta$.
(iv) X^* is a non-developable ruled surface when $Q \neq JF$ and $J = \tan\theta$.
(v) In the case $Q = JF$, X^* is a minimal developable ruled surface when $Q' = JF'$ where $J = \tan\theta$.
(vi) In the case $Q \neq JF$, X^* is a minimal non-developable ruled surface when $Q' = JF'$ and $c = \frac{F+QJ}{1+J^2}$ where $J = \tan\theta$.

Proof. The proofs are given as follows: Items (i) and (ii) can be seen from Theorem 3.1(i).

Besides, from Eq (3.6), $B = (Q - JF)\cos\theta$ is obtained under condition $J = \tan\theta$. Therefore, if $Q = JF$, then the Bertrand offset X^* is developable ($Q^* = 0$), whereas if $Q \neq JF$, then X^* is non-developable ($Q^* \neq 0$). Then, Items (iii) and (iv) are verified.

Items (v) and (vi) can be seen from Theorem 3.1(ii) and Corollary 3.1(iii) and (iv). □

3.1. Characterizations of non-developable ruled surfaces and their Bertrand offsets with the Laplace operator via the first fundamental form

In this subsection, we establish the Laplace operator of Bertrand offsets of non-developable ruled surfaces with respect to the first fundamental form. Besides, certain characterizations are obtained in accordance with the obtained results.

Considering Eq (2.8) with $d_{11} = E$, $d_{12} = d_{21} = F$, $d_{22} = G$, and $d = EG - F^2$ the Laplace operator with respect to the first fundamental form of any surface is rewritten as

$$\Delta^I = -\frac{G}{d}\frac{\partial^2}{\partial u^2} + \frac{2F}{d}\frac{\partial^2}{\partial u\partial v} - \frac{E}{d}\frac{\partial^2}{\partial v^2} + \left(-\frac{G_u}{d} + \frac{G(E_uG+EG_u-2FF_u)}{2d^2} + \frac{F_v}{d} - \frac{F(E_vG+EG_v-2FF_v)}{2d^2}\right)\frac{\partial}{\partial u}$$
$$+\left(\frac{F_u}{d} - \frac{F(E_uG + EG_u - 2FF_u)}{2d^2} - \frac{E_v}{d} + \frac{E(E_vG + EG_v - 2FF_v)}{2d^2}\right)\frac{\partial}{\partial v}.$$

Also, for a ruled surface, we can write

$$\Delta^I = -\frac{1}{E-F^2}\frac{\partial^2}{\partial u^2} + \frac{2F}{E-F^2}\frac{\partial^2}{\partial u \partial v} - \frac{E}{E-F^2}\frac{\partial^2}{\partial v^2} + \left(\frac{E_u - 2FF_u - FE_v}{2(E-F^2)^2}\right)\frac{\partial}{\partial u} + \left(\frac{F_u}{E-F^2} - \frac{F(E_u - 2FF_u)}{2(E-F^2)^2} - \frac{E_v}{E-F^2} + \frac{EE_v}{2(E-F^2)^2}\right)\frac{\partial}{\partial v}. \tag{3.20}$$

3.1.1. Characterizations of non-developable ruled surfaces with the Laplace operator via the first fundamental form

The Laplace operator of non-developable ruled surface X given in Eq (2.13) with respect to the first fundamental form can be obtained by putting Eq (2.20) and their partial derivatives in Eq (3.20) as

$$\Delta^I X = -\frac{Jv^2 - Q'v + Q(QJ - F)}{(Q^2+v^2)^2}(Qt - vg), \tag{3.21}$$

where $d = \mathsf{d} = Q^2 + v^2 \neq 0$. Since X is non-developable, it is clear that $Q \neq 0$. From Eqs (2.21) and (2.24), it is evident that $\Delta^I X = -2\mathcal{H}\mathcal{U}$. On the other hand, from Eq (2.9), we can write

$$\Delta^I X = \lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3 + v\left(\lambda_{i1}e_1 + \lambda_{i2}e_2 + \lambda_{i3}e_3\right), \tag{3.22}$$

for $i = 1, 2, 3$. Here, $X = (x_1, x_2, x_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$, and $e = (e_1, e_2, e_3)$ are the coordinate functions of X, β, and e in $\mathbb{E}^3$, respectively. Besides, we denote the entries of the 3×3 matrix Λ by λ_{ij} for $i, j = 1, 2, 3$. Considering Eqs (3.21) and (3.22) together, we have

$$\begin{aligned}
&- v^5(\lambda_{i1}e_1 + \lambda_{i2}e_2 + \lambda_{i3}e_3) - v^4(\lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3)\\
&+ v^3(Jg - 2Q^2(\lambda_{i1}e_1 + \lambda_{i2}e_2 + \lambda_{i3}e_3)) + v^2(-JQt - Q'g - 2Q^2(\lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3))\\
&+ v(-FQg + JQ^2g + QQ't - Q^4(\lambda_{i1}e_1 + \lambda_{i2}e_2 + \lambda_{i3}e_3)) + FQ^2t - JQ^3t - Q^4(\lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3) = 0.
\end{aligned}$$

This last equation implies that the coefficients of the powers of v must be zero, so we obtain the following system:

$$\lambda_{i1}e_1 + \lambda_{i2}e_2 + \lambda_{i3}e_3 = 0, \tag{3.23}$$

$$\lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3 = 0, \tag{3.24}$$

$$Jg - 2Q^2(\lambda_{i1}e_1 + \lambda_{i2}e_2 + \lambda_{i3}e_3) = 0, \tag{3.25}$$

$$-JQt - Q'g - 2Q^2(\lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3) = 0, \tag{3.26}$$

$$-FQg + JQ^2g + QQ't - Q^4(\lambda_{i1}e_1 + \lambda_{i2}e_2 + \lambda_{i3}e_3) = 0, \tag{3.27}$$

$$FQ^2t - JQ^3t - Q^4(\lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3) = 0. \tag{3.28}$$

Substituting Eqs (3.23) and (3.24) into Eq (3.22) implies that $\Delta^I X = 0$. Although the first two equations seem to provide the result, the surface is only concluded to be finite type surface after the system of equations is analyzed collectively.

Besides, if Eq (3.23) is substituted into Eq (3.25), the geodesic curvature J of the spherical indicatrix curve (e) is obtained zero. It means that $J = \langle e'', e' \times e \rangle = 0$ from Eq (2.16). It is observed that the vectors e'', e', and e are linearly dependent. Therefore, there are two functions $a(u)$ and $b(u)$ such that

$$e'' = a(u)e' + b(u)e.$$

It is known that spherical Frenet frame $\{e, t, g\}$ enables $\langle e, e\rangle = 1$, $\langle e', e'\rangle = 1$, $\langle e', e\rangle = 0$, then $e'' = -e$ ($a(u) = 0$, $b(u) = -1$) is obtained since $\langle e'', e'\rangle = 0$ and $\langle e'', e\rangle = -1$. One can also see $e'' = -e$ via $J = 0$ considering Eq (2.15).

Let us substitute the conditions obtained from Eq (3.24) and $J = 0$ into Eq (3.26). Then, $Q' = 0$ is obtained, implying that Q is constant.

Furthermore, substituting $Q' = 0$ into Eqs (3.27) and (3.28), together with the result obtained previously, yields $F = 0$ since the ruled surface X is non-developable, i.e., $Q \neq 0$.

Via these findings, it follows that the non-developable ruled surface X in $\mathbb{E}^3$ is a finite I-type surface and the structure functions of this ruled surface are $J = 0$, $F = 0$, and $Q \neq 0 =$ constant. Thus, as reported in prior work [5], the only finite I-type non-developable ruled surface in $\mathbb{E}^3$ is the helicoid[§]. Also, X is minimal since $\Delta^I X = -2\mathcal{H}\mathcal{U} = 0$, under these conditions.

3.1.2. Characterizations of Bertrand offsets of non-developable ruled surfaces with the Laplace operator via the first fundamental form

Building on the results obtained above, and in line with the main aim of this paper, we now extend this approach to the Bertrand offsets.

By analogous reasoning, via Eq (2.8) with $d_{11} = E^*$, $d_{12} = d_{21} = F^*$, and $d_{22} = G^*$, the Laplace operator with respect to the first fundamental form of the Bertrand offset of a non-developable ruled surface X (see Eq (2.13)) can be found. Therefore, by putting Eq (3.2) and its derivatives and Eqs (3.1), (3.7) in Eq (3.20), Laplace operator for X^* (see Eq (2.25)) is obtained as

$$\begin{aligned}\Delta^I X^* = &\left(\frac{vA - F'}{\mathsf{d}^*} + \eta_1\xi_1 + \eta_2\cos\theta\right)e + \left(\frac{2AF^*}{\mathsf{d}^*} - \frac{(\xi_1 + vA' + \xi_2 J)}{\mathsf{d}^*} + v\eta_1 A\right)t \\ &+ \left(\frac{vAJ - Q' + cJ'}{\mathsf{d}^*} + \eta_1\xi_2 + \eta_2\sin\theta\right)g,\end{aligned} \tag{3.29}$$

where

$$\begin{cases}\eta_1 = \dfrac{2v^2AA' + 2\xi_1F' + 2\xi_2(Q' - cJ') - 2F^*\left(F'\cos\theta + (Q' - cJ')\sin\theta\right) - 2vF^*A^2}{2{\mathsf{d}^*}^2}, \\ \eta_2 = \dfrac{F'\cos\theta + (Q' - cJ')\sin\theta}{\mathsf{d}^*} \\ \quad - \dfrac{F^*}{2{\mathsf{d}^*}^2}\left(2\xi_1F' + 2\xi_2(Q' - cJ') + 2v^2AA' - 2F^*\left(F'\cos\theta + (Q' - cJ')\sin\theta\right)\right) \\ \quad - \dfrac{2vA^2}{\mathsf{d}^*} + \dfrac{(\xi_1^2 + v^2A^2 + \xi_2^2)(2vA^2)}{2{\mathsf{d}^*}^2},\end{cases}$$

and $\mathsf{d}^* = v^2A^2 + B^2 \neq 0$ from Eq (3.5). Here, the equalities $E^*_u = 2F'\xi_1 + 2v^2AA' + 2(Q' - cJ')\xi_2$, $E^*_v = 2vA^2$, $F^*_u = F'\cos\theta + (Q' - cJ')\sin\theta$, $F^*_v = 0$, $G^*_u = 0$, and $G^*_v = 0$ are used considering Eq (3.2). From Eqs (3.4) and (3.9), it is clear that $\Delta^I X^* = -2\mathcal{H}^*\mathcal{U}^*$.

To avoid confusion, the notation d^* for the Bertrand offset X^* is preferred instead of d in Eq (3.20).

We now state the following corollary.

[§]The result obtained in [5] is recovered here through an alternative perspective.

Corollary 3.2. *Let X^* be the Bertrand offset of the non-developable ruled surface X given in Eq* (2.13). *Considering Eq* (3.29)*, the following statements can be given.*

(i) If X and X^* are oriented offsets, namely, $\theta = 0$, we have

$$\Delta^I X^* = \left(-\frac{F' - v}{v^2 + \xi_2^2} + \eta_1\xi_1 + \eta_2\right)e + \left(\frac{\xi_1 - \xi_2 J}{v^2 + \xi_2^2} + v\,\eta_1\right)t + \left(-\frac{Q' - cJ' - vJ}{v^2 + \xi_2^2} + \eta_1\xi_2\right)g,$$

where

$$\begin{cases} \eta_1 = \dfrac{2\xi_2(Q' - cJ') - 2v\xi_1}{2(v^2 + \xi_2^2)^2}, \\ \eta_2 = \dfrac{F'}{v^2 + \xi_2^2} - \dfrac{\xi_1\,(2\xi_2(Q' - cJ'))}{2(v^2 + \xi_2^2)^2} - \dfrac{2v}{v^2 + \xi_2^2} + \dfrac{2v(\xi_1^2 + v^2 + \xi_2^2)}{2(v^2 + \xi_2^2)^2}. \end{cases}$$

(ii) If X and X^* are right offsets, namely, $\theta = \pi/2$, we have

$$\Delta^I X^* = \left(-\frac{F' - vJ}{v^2J^2 + \xi_1^2} + \eta_1\xi_1\right)e + \left(\frac{-\xi_1 - vJ' + \xi_2 J}{v^2J^2 + \xi_1^2} + v\,\eta_1 J\right)t + \left(-\frac{Q' - cJ' - vJ^2}{v^2J^2 + \xi_1^2} + \eta_1\xi_2 + \eta_2\right)g,$$

where

$$\begin{cases} \eta_1 = \dfrac{2v^2JJ' + 2\xi_1F' - 2v\xi_2J^2}{2(v^2J^2 + \xi_1^2)^2}, \\ \eta_2 = \dfrac{Q' - cJ'}{v^2J^2 + \xi_1^2} - \dfrac{\xi_2\left(2\xi_1F' + 2v^2JJ'\right)}{2(v^2J^2 + \xi_1^2)^2} - \dfrac{2vJ^2}{v^2J^2 + \xi_1^2} + \dfrac{(\xi_1^2 + v^2J^2 + \xi_2^2)(2vJ^2)}{2(v^2J^2 + \xi_1^2)^2}. \end{cases}$$

Now, let us examine whether the Bertrand offset of the non-developable ruled surface (see Eq (2.13)) is finite via the Laplace operator with respect to the first fundamental form.

Considering Eq (2.9) for X^* (see Eq (2.25)) for $i = 1, 2, 3$, we write

$$\Delta^I X^* = \lambda_{i1}^*\beta_1^* + \lambda_{i2}^*\beta_2^* + \lambda_{i3}^*\beta_3^* + v\left(\lambda_{i1}^*e_1^* + \lambda_{i2}^*e_2^* + \lambda_{i3}^*e_3^*\right). \tag{3.30}$$

Here, $X^* = (x_1^*, x_2^*, x_3^*)$, $\beta^* = (\beta_1^*, \beta_2^*, \beta_3^*)$, and $e^* = (e_1^*, e_2^*, e_3^*)$ are the coordinate functions of X^*, β^*, and e^*, respectively. Furthermore, we denote the entries of the 3×3 matrix Λ^* by λ_{ij}^* for $i, j = 1, 2, 3$. Thus, with the help of Eqs (3.29) and (3.30), we have

$$
\begin{aligned}
&- v^5A^4(\lambda^*_{i1}e^*_1 + \lambda^*_{i2}e^*_2 + \lambda^*_{i3}e^*_3) - v^4A^4(\lambda^*_{i1}\beta^*_1 + \lambda^*_{i2}\beta^*_2 + \lambda^*_{i3}\beta^*_3)\\
&+ v^3\left(A^3(e + Jg) - A^4(\cos\theta e + \sin\theta g) - 2A^2B^2(\lambda^*_{i1}e^*_1 + \lambda^*_{i2}e^*_2 + \lambda^*_{i3}e^*_3)\right)\\
&+ v^2(- A^2(\xi_1 + J\xi_2)t + A^3(\xi_1\cos\theta + \xi_2\sin\theta)t - 2A^2B^2(\lambda^*_{i1}\beta^*_1 + \lambda^*_{i2}\beta^*_2 + \lambda^*_{i3}\beta^*_3)\\
&+ A\cos\theta(\xi_2\cos\theta - \xi_1\sin\theta)A'g + A\sin\theta(-\xi_2\cos\theta + \xi_1\sin\theta)A'e\\
&+ A^2\cos\theta(\sin\theta F' + \cos\theta(cJ' - Q'))g - A^2\sin\theta(\sin\theta F' + \cos\theta(cJ' - Q'))e)\\
&+ v(AB^2e + AB^2Jg - 2A^2B^2\cos\theta e - A^2\xi_1\xi_2\cos\theta g + A^2\xi_2^2\cos\theta e - 2A^2B^2\sin\theta g\\
&+ A^2\xi_1^2\sin\theta g - A^2\xi_1\xi_2\sin\theta e - B^4(\lambda^*_{i1}e^*_1 + \lambda^*_{i2}e^*_2 + \lambda^*_{i3}e^*_3)\\
&- B^2A't - A(\xi_2\cos\theta - \xi_1\sin\theta)(\sin\theta F' + \cos\theta(cJ' - Q'))t)\\
&+ \frac{1}{2}(- 2B^2(\xi_1 + J\xi_2)t + 4AB^2(\xi_1\cos\theta + \xi_2\sin\theta)t - 2B^4(\lambda^*_{i1}\beta^*_1 + \lambda^*_{i2}\beta^*_2 + \lambda^*_{i3}\beta^*_3)\\
&+ [2B^2\cos\theta g - (\xi_1^2 + \xi_2^2)\cos\theta g + (\xi_1 - \xi_2)(\xi_1 + \xi_2)\cos\theta\cos 2\theta g\\
&- 2B^2\sin\theta e + (\xi_1^2 + \xi_2^2)\sin\theta e + 4\xi_1\xi_2\cos^2\theta\sin\theta g - (\xi_1^2 - \xi_2^2)\cos 2\theta\sin\theta e\\
&- 4\xi_1\xi_2\cos\theta\sin^2\theta e](\sin\theta F' + \cos\theta(cJ' - Q'))) = 0.
\end{aligned}
$$

This polynomial implies that the coefficients of the powers of v must be zero, so we obtain the following system:

$$
A^4(\lambda^*_{i1}e^*_1 + \lambda^*_{i2}e^*_2 + \lambda^*_{i3}e^*_3) = 0, \tag{3.31}
$$

$$
A^4(\lambda^*_{i1}\beta^*_1 + \lambda^*_{i2}\beta^*_2 + \lambda^*_{i3}\beta^*_3) = 0, \tag{3.32}
$$

$$
A^3(e + Jg) - A^4(\cos\theta e + \sin\theta g) - 2A^2B^2(\lambda^*_{i1}e^*_1 + \lambda^*_{i2}e^*_2 + \lambda^*_{i3}e^*_3) = 0, \tag{3.33}
$$

$$
\begin{aligned}
&- A^2(\xi_1 + J\xi_2)t + A^3(\xi_1\cos\theta + \xi_2\sin\theta)t - 2A^2B^2(\lambda^*_{i1}\beta^*_1 + \lambda^*_{i2}\beta^*_2 + \lambda^*_{i3}\beta^*_3)\\
&\quad + A(\cos\theta g - \sin\theta e)(\xi_2\cos\theta - \xi_1\sin\theta)A' + A^2(\cos\theta g - \sin\theta e)(\sin\theta F' + \cos\theta(cJ' - Q')) = 0,
\end{aligned} \tag{3.34}
$$

$$
\begin{aligned}
&AB^2e + AB^2Jg - 2A^2B^2\cos\theta e - A^2\xi_1\xi_2\cos\theta g + A^2\xi_2^2\cos\theta e - 2A^2B^2\sin\theta g\\
&\quad + A^2\xi_1^2\sin\theta g - A^2\xi_1\xi_2\sin\theta e - B^4(\lambda^*_{i1}e^*_1 + \lambda^*_{i2}e^*_2 + \lambda^*_{i3}e^*_3)\\
&\quad - B^2A't - A(\xi_2\cos\theta - \xi_1\sin\theta)(\sin\theta F' + \cos\theta(cJ' - Q'))t = 0,
\end{aligned} \tag{3.35}
$$

$$
\begin{aligned}
&- 2B^2(\xi_1 + J\xi_2)t + 4AB^2(\xi_1\cos\theta + \xi_2\sin\theta)t - 2B^4(\lambda^*_{i1}\beta^*_1 + \lambda^*_{i2}\beta^*_2 + \lambda^*_{i3}\beta^*_3)\\
&\quad + (\cos\theta g - \sin\theta e)\left(2B^2 + 2\xi_1\xi_2\sin 2\theta + \cos 2\theta(\xi_1^2 - \xi_2^2) - \xi_1^2 - \xi_2^2\right)(\sin\theta F' + \cos\theta\,(cJ' - Q')) = 0.
\end{aligned} \tag{3.36}
$$

Since $A \neq 0$ (see Eq (3.10)), it follows from Eqs (3.30)–(3.32) that $\Delta^I X^* = 0$. Although the first two equations seem to provide the result, the surface is only concluded to be finite type surface after the system of equations is analyzed collectively. Therefore, we can state the following theorem:

Theorem 3.2. *Let X^* be a Bertrand offset of a non-developable ruled surface X given in Eq* (2.13)*. Let J, F, and Q be structure functions of X. In this case, X^* is a finite I-type surface under conditions $J = \tan\theta$ and $Q' = JF'$, together with one of the following relations:*

(i) $Q = JF$.

(ii) $Q \neq JF$ *and* $c = \dfrac{F + QJ}{1 + J^2}$.

Proof. Since $A \neq 0$ (see Eq (3.10)), from Eqs (3.31) and (3.32), it follows:

$$\lambda^*_{i1}e^*_1 + \lambda^*_{i2}e^*_2 + \lambda^*_{i3}e^*_3 = 0, \tag{3.37}$$

and

$$\lambda^*_{i1}\beta^*_1 + \lambda^*_{i2}\beta^*_2 + \lambda^*_{i3}\beta^*_3 = 0. \tag{3.38}$$

Therefore, substituting Eq (3.37) into Eq (3.33) and rearranging it, we have

$$(J\cos\theta - \sin\theta)(\cos\theta g - \sin\theta e) = 0.$$

Since $\cos\theta g - \sin\theta e \neq 0$, it follows that $J = \tan\theta$ (since $\cos\theta = 0$ leads to a contradiction, it follows that $\cos\theta \neq 0$).

If Eq (3.38) and $J = \tan\theta$ are substituted in Eq (3.34), it yields $\sec\theta(\cos\theta g - \sin\theta e)(F'\tan\theta - Q')$=0. Thus, $F'\tan\theta - Q' = 0$ since $\sec\theta \neq 0$ and $\cos\theta g - \sin\theta e \neq 0$. This shows that $Q' = JF'$ when $J = \tan\theta$.

Finally, substituting all these findings into Eqs (3.35) and (3.36), it is seen that either $c = \frac{1}{2}(F\cos 2\theta + F + Q\sin 2\theta)$ or $Q = FJ$. The first equation is transformed into $c = \frac{F+JQ}{1+J^2}$ using $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$ and $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$ obtained from $J = \tan\theta$.

Consequently, it is observed that whether the Bertrand offset X^* is a finite type surface depends on the obtained conditions between structure functions of the non-developable ruled surface X. Thus, the Bertrand offset of the non-developable ruled surface given in Eq (2.13) is a finite I-type surface by the conditions: If $Q = FJ$, then $Q' = JF'$ where $J = \tan\theta$, or if $Q \neq FJ$, then $Q' = JF'$ and $c = \frac{F+JQ}{1+J^2}$ where $J = \tan\theta$. Additionally, under these conditions, X^* is a minimal surface, since $\Delta^I X^* = -2\mathcal{H}^*\mathcal{U}^* = 0$. Indeed, the obtained conditions in items *i* and *ii* of Theorem 3.2 clearly satisfy items *v* and *vi* of Corollary 3.1, respectively, which implies that $\mathcal{H}^* = 0$. □

Remark 3.1. *Let X^* be the Bertrand offset of the non-developable ruled surface X given in Eq* (2.13). *Except for the conditions between the structure functions J, F, and Q of X discussed in Theorem 3.2, X^* is infinite I-type.*

According to Theorem 3.2 and Corollary 3.1, the following corollary can be stated.

Corollary 3.3. *Let X^* be the Bertrand offset of the non-developable ruled surface X given in Eq* (2.13). *Under condition $J = \tan\theta$,*

(i) X^* is I-type developable ruled surface when $Q = JF$ and $Q' = JF'$.
(ii) X^* is I-type non-developable ruled surface when $Q \neq JF$, $Q' = JF'$, and $c = \frac{F+JQ}{1+J^2}$.

Corollary 3.4. *The Bertrand offset of a finite I-type non-developable ruled surface is a finite I-type surface in the trivial case;*[¶] *otherwise, it is infinite I-type.*

Proof. It is known that the only non-developable finite I-type ruled surface is a helicoid. Since the structure functions of the helicoid are $J = 0$, $F = 0$, and $Q = 1$, it is clear that $Q \neq JF$. According to Theorem 3.2(*ii*) holds under condition $\tan\theta = 0$. Thus, Bertrand offset of the helicoid is a I-type surface with $c = \frac{F+JQ}{1+J^2} = 0$, $\theta = k\pi$ (trivial case). Otherwise, it is an infinite I-type surface (for details see Example 4.1). □

[¶]Namely, the Bertrand offset of the helicoid is a finite I-type surface under conditions $c = 0$, $\theta = k\pi$. For $k = 2n$, $n \in \mathbb{Z}$, the helicoid and its Bertrand offset are the same surfaces

3.2. *Characterizations of non-developable ruled surfaces and their Bertrand offsets with the Laplace operator via the second fundamental form*

In this subsection, we obtain the Laplace operator of Bertrand offsets of non-developable ruled surfaces with respect to the second fundamental form. Furthermore, certain characterizations are derived from the obtained results.

Considering Eq (2.8) with $d_{11} = L$, $d_{12} = d_{21} = M$, $d_{22} = N$, and $d = LN - M^2$, the Laplace operator with respect to the second fundamental form of any surface is rewritten as

$$\begin{aligned}\Delta^{II} = &-\frac{N}{d}\frac{\partial^2}{\partial u^2} + \frac{2M}{d}\frac{\partial^2}{\partial u\partial v} - \frac{L}{d}\frac{\partial^2}{\partial v^2}\\ &+\left(-\frac{N_u}{d} + \frac{N(L_uN + LN_u - 2MM_u)}{2d^2} + \frac{M_v}{d} - \frac{M(L_vN + LN_v - 2MM_v)}{2d^2}\right)\frac{\partial}{\partial u}\\ &+\left(\frac{M_u}{d} - \frac{M(L_uN + LN_u - 2MM_u)}{2d^2} - \frac{L_v}{d} + \frac{L(L_vN + LN_v - 2MM_v)}{2d^2}\right)\frac{\partial}{\partial v}.\end{aligned}$$

Also, for a ruled surface, we can write

$$\Delta^{II} = -\frac{2}{M}\frac{\partial^2}{\partial u\partial v} + \frac{L}{M^2}\frac{\partial^2}{\partial v^2} + \left(\frac{ML_v - LM_v}{M^3}\right)\frac{\partial}{\partial v}. \tag{3.39}$$

3.2.1. Characterizations of non-developable ruled surfaces with the Laplace operator via the second fundamental form

The Laplace operator of non-developable ruled surface X given in Eq (2.13) with respect to the second fundamental form can be obtained by putting Eq (2.23) and their partial derivatives in Eq (3.39) as

$$\Delta^{II}X = \frac{(2Jv - Q')\sqrt{\mathsf{d}}}{Q^2}e - \frac{2\sqrt{\mathsf{d}}}{Q}t, \tag{3.40}$$

where d is given in Eq (2.22). Since X is non-developable, we know that $Q \neq 0$. On the other hand, from Eq (2.9), we can write

$$\Delta^{II}X = \lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3 + v\left(\lambda_{i1}e_1 + \lambda_{i2}e_2 + \lambda_{i3}e_3\right). \tag{3.41}$$

for $i = 1, 2, 3$. Here, $X = (x_1, x_2, x_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$, and $e = (e_1, e_2, e_3)$ are the coordinate functions of X, β, and e, respectively. Besides, we denote the entries of the 3×3 matrix Λ by λ_{ij} for $i, j = 1, 2, 3$. Thus, from Eqs (3.40) and (3.41), we have

$$v\left(2J\sqrt{\mathsf{d}}e - Q^2\left(\lambda_{i1}e_1 + \lambda_{i2}e_2 + \lambda_{i3}e_3\right)\right) - 2Q\sqrt{\mathsf{d}}t - Q'\sqrt{\mathsf{d}}e - Q^2\left(\lambda_{i1}\beta_1 + \lambda_{i2}\beta_2 + \lambda_{i3}\beta_3\right) = 0. \tag{3.42}$$

Since $Q \neq 0$, and $\mathsf{d} = Q^2 + v^2$ depends on the parameter v, no v-independent real coefficients λ_{ij} can be obtained from the equation (Eq (3.42)). Hence, there exists no real constant matrix Λ satisfying $\Delta^{II}X = \Lambda X$. Consequently, X is an infinite II-type surface. All ruled surfaces are infinite II-type surfaces[‖], as documented in previous work [11].

[‖]The result obtained in [11] is recovered here through an alternative perspective.

3.2.2. Characterizations of Bertrand offsets of non-developable ruled surfaces with the Laplace operator via the second fundamental form

In light of the results obtained above, and aligned with the main purpose of this paper, we now extend this approach to the Bertrand offsets.

Analogously, via Eq (2.8) with $d_{11} = L^*$, $d_{12} = d_{21} = M^*$, and $d_{22} = N^*$, the Laplace operator with respect to the second fundamental form of X^* (see Eq (2.25)) can be computed. For this computation, we need to suppose that

$$B = (Q - cJ)\cos\theta - (F - c)\sin\theta \neq 0, \tag{3.43}$$

(see Eq (3.6)) which also means X^* is a non-developable ruled surface. Therefore, putting Eqs (3.1), (3.7), (3.8), and their derivatives in Eq (3.39), we have

$$\begin{aligned}\Delta^{II}X^* =& \frac{\cos\theta\sqrt{\mathsf{d}^*}\,(BA' + A\cos\theta\,(2AJv + cJ' - Q') + A\sin\theta\,(F' - 2Av))\,e}{A^2B^2} - \frac{2\sqrt{\mathsf{d}^*}t}{B}\\ &+ \frac{\sin\theta\sqrt{\mathsf{d}^*}\,(BA' + A\cos\theta\,(2AJv + cJ' - Q') + A\sin\theta\,(F' - 2Av))\,g}{A^2B^2},\end{aligned} \tag{3.44}$$

where d^* is given in Eq (3.5) with $A \neq 0$ and $B \neq 0$.

It clearly follows from this part of this section that $B \neq 0$.

We now present the following corollary.

Corollary 3.5. *Let the non-developable ruled surfaces X and X^* be Bertrand offsets. Considering Eq* (3.44)*, the following statement can be given:*

(i) If X and X^* are oriented offsets, we have

$$\Delta^{II}X^* = \frac{\sqrt{(Q - cJ)^2 + v^2}\,(cJ' + 2Jv - Q')\,e}{(Q - cJ)^2} - \frac{2\sqrt{(Q - cJ)^2 + v^2}t}{Q - cJ}.$$

Here, since $\theta = 0$, we have $A = 1 \neq 0$ and $B = Q - cJ \neq 0$ (see Eqs (3.10) *and* (3.43)*).*

(ii) If X and X^* are right offsets, we have

$$\Delta^{II}X^* = -\frac{2\sqrt{(c - F)^2 + J^2v^2}t}{c - F} + \frac{\sqrt{(c - F)^2 + J^2v^2}\,((c - F)J' + J\,(F' - 2Jv))\,g}{J^2(c - F)^2}.$$

Here, since $\theta = \frac{\pi}{2}$, we have $A = J \neq 0$ and $B = c - F \neq 0$ (see Eqs (3.10) *and* (3.43)*).*

Now, let us examine whether the Bertrand offset of a non-developable ruled surface (given in Eq (2.13)) is finite via the Laplace operator with respect to the second fundamental form.

Considering Eq (2.9) for X^* given in Eq (2.25) for $i = 1, 2, 3$, we write

$$\Delta^{II}X^* = \lambda^*_{i1}\beta^*_1 + \lambda^*_{i2}\beta^*_2 + \lambda^*_{i3}\beta^*_3 + v\,(\lambda^*_{i1}e^*_1 + \lambda^*_{i2}e^*_2 + \lambda^*_{i3}e^*_3)\,. \tag{3.45}$$

Here, $X^* = (x^*_1, x^*_2, x^*_3)$, $\beta^* = (\beta^*_1, \beta^*_2, \beta^*_3)$, and $e^* = (e^*_1, e^*_2, e^*_3)$ are the coordinate functions of X^*, β^*, and e^*, respectively. Furthermore, we denote the entries of the 3×3 matrix Λ^* by λ^*_{ij} for $i, j = 1, 2, 3$. Using

Eqs (3.44) and (3.45), we have

$$\begin{aligned}
&v\left(A^2\left(2\sqrt{\mathsf{d}^*}(J\cos\theta-\sin\theta)(\cos\theta e+\sin\theta g)-B^2\left(\lambda^*_{i1}e^*_1+\lambda^*_{i2}e^*_2+\lambda^*_{i3}e^*_3\right)\right)\right)\\
&+Ac\cos^2\theta J'\sqrt{\mathsf{d}^*}e+Ac\sin\theta\cos\theta J'\sqrt{\mathsf{d}^*}g+AF'\sin\theta\cos\theta\sqrt{\mathsf{d}^*}e\\
&-A\cos^2\theta Q'\sqrt{\mathsf{d}^*}e+AF'\sin^2\theta\sqrt{\mathsf{d}^*}g-A\sin\theta\cos\theta Q'\sqrt{\mathsf{d}^*}g\\
&-2A^2B\sqrt{\mathsf{d}^*}t+BA'\cos\theta\sqrt{\mathsf{d}^*}e+BA'\sin\theta\sqrt{\mathsf{d}^*}g\\
&-A^2B^2\left(\lambda^*_{i1}\beta^*_1+\lambda^*_{i2}\beta^*_2+\lambda^*_{i3}\beta^*_3\right)=0.
\end{aligned}\tag{3.46}$$

Therefore, Eq (3.46) shows that there exist no real constant coefficients λ^*_{ij} satisfying the equation. Thus, we are now able to state the following theorem.

Theorem 3.3. *The Bertrand offset of a non-developable ruled surface is an infinite II-type surface.*

Proof. Since $A \neq 0$ and $B \neq 0$, and $\mathsf{d}^* = A^2v^2 + B^2$ depends on parameter v, no v-independent real coefficients λ_{ij} can be obtained from the equation (Eq (3.46)). Hence, there is no Bertrand offset of a non-developable ruled surface that satisfies the condition $\Delta^{II}X^* = \Lambda^*X^*$ (see Eq (2.9)) for a real 3×3 matrix Λ^*. Consequently, X^* is an infinite II-type surface. □

Corollary 3.6. *All non-developable Bertrand offsets of the infinite II-type non-developable ruled surface are infinite II-type surface.*

4. Implementation

In this section, we present two examples and examine the proposed characterizations using computational procedures. The figures and numerical results presented here are generated using Wolfram Mathematica.

Example 4.1. *Let us consider a helicoid with parametric representation*

$$X(u,v)=(v\cos u, v\sin u, u).$$

Here, $\beta(u) = (0,0,u)$ *and* $e(u) = (\cos u, \sin u, 0)$ $(\langle e', e'\rangle = 1)$ *are the base curve and the spherical indicatrix curve of this ruled surface, respectively. Thus,* $\beta' = (0,0,1)$*,* $e' = (-\sin u, \cos u, 0)$*, and* $e'' = (-\cos u, -\sin u, 0)$*. It is clear that* β *is the striction curve since* $\langle\beta', e'\rangle = 0$ *(see Eq* (2.13)*).*

Initially, the structure functions of X *are calculated as*

$$\begin{aligned}
J&=\langle e'', e'\times e\rangle=\langle(-\cos u,-\sin u,0),(0,0,-1)\rangle=0,\\
F&=\langle\beta', e\rangle=\langle(0,0,1),(\cos u,\sin u,0)\rangle=0,\\
Q&=\langle\beta', e\times e'\rangle=\langle(0,0,1),(0,0,1)\rangle=1,
\end{aligned}$$

via Eqs (2.16)*,* (2.18)*, and* (2.19)*, respectively, where*

$$\begin{aligned}
e&=(\cos u,\sin u,0),\\
t&=e'=(-\sin u,\cos u,0),\\
g&=e\times e'=(0,0,1),
\end{aligned}\tag{4.1}$$

(see Eq (2.14)*). Since $Q = 1 \neq 0$, it is apparent that the helicoid is a non-developable ruled surface. Then, with the help of Eq* (2.20)*, the coefficients of the first fundamental form of X are $E = F^2 + Q^2 + v^2 = 1 + v^2$, $F = 0$, $G = 1$. Besides, via Eq* (2.21) *and $\mathsf{d} = Q^2 + v^2 = v^2 + 1$, the unit normal vector of the helicoid is*

$$\mathcal{U} = \frac{1}{\sqrt{\mathsf{d}}}(Qt - vg) = \frac{1}{\sqrt{v^2+1}}(1(-\sin u, \cos u, 0) - v(0,0,1)) = \frac{1}{\sqrt{v^2+1}}(-\sin u, \cos u, -v).$$

Thus, by Eq (2.23)*, the coefficients of the second fundamental form of X are $L = \frac{1}{\sqrt{\mathsf{d}}}(Q(F + QJ) - Q'v + Jv^2) = 0$, $M = \frac{Q}{\sqrt{\mathsf{d}}} = \frac{1}{\sqrt{1+v^2}}$, $N = 0$. Also, with Eq* (2.24)*, the Gauss and mean curvatures of the helicoid are $\mathcal{K} = -\frac{1}{(1+v^2)^2}$ and $\mathcal{H} = 0$, respectively. Besides, taking into account Eq* (3.21)*, the Laplace operator with respect to the first fundamental form of X is calculated as $\Delta^I X = (0,0,0)$. Hence, it is verified that the helicoid is of finite I-type. It can also be seen that $\Delta^I X = -2\mathcal{H}\mathcal{U} = 0$.*

Following this construction, taking into account Eq (3.40)*, the Laplace operator with respect to the second fundamental form of X is calculated as*

$$\Delta^{II} X = \left(2\sqrt{v^2+1}\sin u, -2\sqrt{v^2+1}\cos u, 0\right). \tag{4.2}$$

Hence, there exists no real constant matrix Λ satisfying $\Delta^{II}X = \Lambda X$. It means that the helicoid is an infinite II-type surface.

Now, let us examine the Bertrand offset of the given helicoid. With Eqs (2.25) *and* (4.1)*, it is written as*

$$\begin{aligned} X^*(u,v) &= \beta^*(u) + ve^*(u) \\ &= \beta(u) + ct(u) + v(\cos\theta e(u) + \sin\theta g(u)) \\ &= (-c\sin u, c\cos u, u) + v(\cos\theta\cos u, \cos\theta\sin u, \sin\theta). \end{aligned} \tag{4.3}$$

As shown in Figure 1, the solid black curve denotes the base curve $\beta(u)$ of helicoid X, while the dashed black curve denotes the base curve $\beta^*(u)$ of its Bertrand offset X^*. The red arrow indicates the ruling direction $e(u)$ of X, and the blue arrow indicates the ruling direction $e^*(u)$ of X^*.

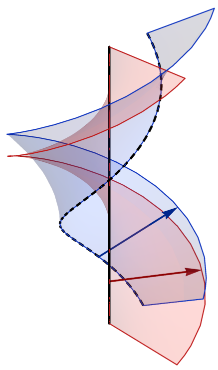


Figure 1. Helicoid (red) and its Bertrand offset (blue) for $c = 1$ and $\theta = \pi/6$.

Considering Eq (3.2)*, the coefficients of the first fundamental form of* X^* *are*

$$\begin{aligned} E^* &= \xi_1^2 + v^2 A^2 + \xi_2^2 = c^2 + v^2\cos^2\theta + 1, \\ F^* &= \xi_1\cos\theta + \xi_2\sin\theta = -c\cos\theta + \sin\theta, \\ G^* &= 1, \end{aligned}$$

where $\xi_1 = -c$*,* $\xi_2 = 1$*,* $A = \cos\theta$*, and* $B = \cos\theta + c\sin\theta$ *have been utilized via Eqs* (3.3) *and* (3.6)*. Also, from Eq* (3.4)*, the unit normal vector of* X^* *is*

$$\mathcal{U}^* = \left(\frac{-\sin u(\cos\theta + c\sin\theta) + v\cos\theta\sin\theta\cos u}{\sqrt{v^2\cos^2\theta + (\cos\theta + c\sin\theta)^2}}, \frac{\cos u(\cos\theta + c\sin\theta) + v\cos\theta\sin\theta\sin u}{\sqrt{v^2\cos^2\theta + (\cos\theta + c\sin\theta)^2}}, -\frac{v\cos^2\theta}{\sqrt{v^2\cos^2\theta + (\cos\theta + c\sin\theta)^2}} \right).$$

Here, the expression inside the square root in the denominator is strictly positive (see Eq (3.5)*). Then, from Eq* (3.8)*, the coefficients of the second fundamental form of* X^* *are*

$$L^* = -\frac{c\cos\theta + \left(c^2 + v^2\cos^2\theta\right)\sin\theta}{\sqrt{v^2\cos^2\theta + (\cos\theta + c\sin\theta)^2}}, \quad M^* = \frac{\cos\theta(\cos\theta + c\sin\theta)}{\sqrt{v^2\cos^2\theta + (\cos\theta + c\sin\theta)^2}}, \quad N^* = 0.$$

Besides, via Eq (3.9)*, the Gaussian curvature and the mean curvature of* X^* *are, respectively:*

$$\mathcal{K}^* = -\frac{\cos^2\theta(\cos\theta + c\sin\theta)^2}{\left(v^2\cos^2\theta + (\cos\theta + c\sin\theta)^2\right)^2},$$

$$\mathcal{H}^* = \frac{2c\cos 3\theta - \left(2 + v^2 + (2 - 2c^2 + v^2)\cos 2\theta\right)\sin\theta}{4\left(v^2\cos^2\theta + (\cos\theta + c\sin\theta)^2\right)^{3/2}}.$$

Using Eq (3.29)*, the Laplace operator of* X^* *with respect to the first fundamental form is*

$$\Delta^I X^* = \begin{bmatrix} \dfrac{\left(2c\cos 3\theta - \left(2 + v^2 + (2 - 2c^2 + v^2)\cos 2\theta\right)\sin\theta\right)(\sin u(\cos\theta + c\sin\theta) - v\cos\theta\sin\theta\cos u)}{2\left(v^2\cos^2\theta + (\cos\theta + c\sin\theta)^2\right)^2} \\ \dfrac{\left(-4c\cos 3\theta + 2\left(2 + v^2 + (2 - 2c^2 + v^2)\cos 2\theta\right)\sin\theta\right)(\cos u(\cos\theta + c\sin\theta) + v\cos\theta\sin\theta\sin u)}{4\left(v^2\cos^2\theta + (\cos\theta + c\sin\theta)^2\right)^2} \\ \dfrac{\left(2c\cos 3\theta - \left(2 + v^2 + (2 - 2c^2 + v^2)\cos 2\theta\right)\sin\theta\right)v\cos^2\theta}{2\left(v^2\cos^2\theta + (\cos\theta + c\sin\theta)^2\right)^2} \end{bmatrix}^t. \tag{4.4}$$

It can also be verified that $\Delta^I X^* = -2\mathcal{H}^*\mathcal{U}^*$*.*

Let us consider Theorem 3.2 by reviewing the conditions. Since $J = 0$*,* $F = 0$*, and* $Q = 1$*, it is evident that* $Q \neq JF$ *is satisfied. Then, by Theorem 3.2(ii), under condition* $J = \tan\theta = 0$*, the relation* $Q' = JF'$ *is satisfied and* $c = \frac{F+QJ}{1+J^2} = 0$ *is obtained. Hence,* X^* *is of finite I-type under conditions* $c = 0$ *and* $\theta = k\pi$ *with solving* $J = \tan\theta = 0$*. In fact, for* $\theta = k\pi$*, we calculate*

$$\Delta^I X^* = \left(\frac{c\sin u}{\left(1 + v^2\right)^2}, -\frac{c\cos u}{\left(1 + v^2\right)^2}, \pm\frac{cv}{\left(1 + v^2\right)^2} \right).$$

Thus, the Bertrand offset of the helicoid is of finite I-type with $c = 0$*. Also,* X^* *is minimal (*$\mathcal{H}^* = 0$*) under conditions* $\theta = k\pi$ *and* $c = 0$ *(see Corollary 3.1(vi)). It can be seen that the findings for this example verify Corollary 3.4.*

In line with this structure, taking into account Eq (3.44)*, the Laplace operator with respect to the second fundamental form of* X^* *is calculated as*

$$\Delta^{II}X^* = \begin{bmatrix} \dfrac{2\sqrt{(c\sin\theta+\cos\theta)^2+v^2\cos^2\theta}\,(c\sin\theta\sin u+\cos\theta(\sin u - v\sin\theta\cos u))}{(\cos\theta+c\sin\theta)^2} \\ -\dfrac{2\sqrt{(c\sin\theta+\cos\theta)^2+v^2\cos^2\theta}\,(\cos u(c\sin\theta+\cos\theta)+v\sin\theta\cos\theta\sin u)}{(\cos\theta+c\sin\theta)^2} \\ -\dfrac{2v\sin^2\theta\sqrt{(c\sin\theta+\cos\theta)^2+v^2\cos^2\theta}}{(\cos\theta+c\sin\theta)^2} \end{bmatrix}^t. \tag{4.5}$$

It should be emphasized that $B = \cos\theta + c\sin\theta \neq 0$ *(see Eq* (3.43)*). It is observed that* X^* *is an infinite II-type surface, and thus Theorem 3.3 is satisfied. Also, the findings illustrate the validity of Corollary 3.6.*

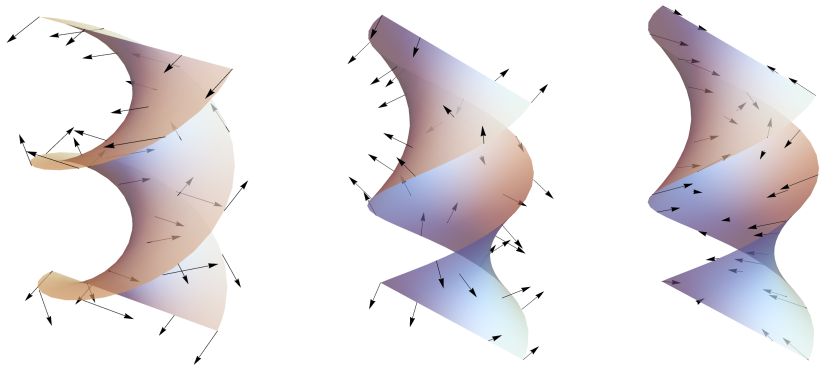
Additionally, one can see the visual vectorial representation of the Laplace operators with respect to the first and second fundamental forms for the helicoid ($\Delta^I X = (0,0,0)$ *and Eq* (4.2)*) and its Bertrand offset for* $u \in [0,2\pi]$*,* $v \in [-2,2]$*,* $c = 1$*,* $\theta = \pi/6$ *in Figure 2. Considering these values, from Eqs* (4.3)–(4.5)*, we obtain:*

$$X^*(u,v) = \left(-\sin u + \frac{\sqrt{3}\cos u}{2}v, \cos u + \frac{\sqrt{3}\sin u}{2}v, u + \frac{v}{2}\right),$$

$$\Delta^I X^* = \left(\frac{\left(3v^2+4\right)\left(\sqrt{3}v\cos u - 2\left(\sqrt{3}+1\right)\sin u\right)}{2\left(3v^2+2\sqrt{3}+4\right)^2}, \frac{\left(3v^2+4\right)\left(\sqrt{3}v\sin u + 2\left(\sqrt{3}+1\right)\cos u\right)}{2\left(3v^2+2\sqrt{3}+4\right)^2}, -\frac{\left(3v^2+4\right)3v}{2\left(3v^2+2\sqrt{3}+4\right)^2}\right),$$

$$\Delta^{II} X^* = \left(\frac{\sqrt{3v^2+2\sqrt{3}+4}\,(2(\sqrt{3}+1)\sin u - \sqrt{3}v\cos u)}{(\sqrt{3}+1)^2}, -\frac{\sqrt{3v^2+2\sqrt{3}+4}\,(\sqrt{3}v\sin u + 2(\sqrt{3}+1)\cos u)}{(\sqrt{3}+1)^2}, -\frac{\sqrt{3v^2+2\sqrt{3}+4}\,v}{(\sqrt{3}+1)^2}\right).$$

The arrows in Figure 2 represent the vectorial form of the Laplace operator and explicitly depict the direction and behavior of the operator over the surface. These arrows are evaluated at $X(u_0,v_0) + k\,\Delta^{II}X(u_0,v_0)$*, where* $u_0 \in [0,2\pi]$*,* $v_0 \in [-2,2]$*,* $k \in \mathbb{R}$ *for Figure 2(a) and the value* $k = 0.25$ *is chosen to represent the magnitude. Similarly, for Figures 2(b) and (c),* $X^*(u_0,v_0) + k\,\Delta^{\mathcal{J}}X^*(u_0,v_0)$*, where* $u_0 \in [0,2\pi]$*,* $v_0 \in [-2,2]$*,* $k \in \mathbb{R}$*,* $\mathcal{J} = I$*, and* $\mathcal{J} = II$ *are considered. The values* $k = 4$ *and* $k = 0.25$ *are chosen for Figures 2(b) and (c), respectively.*



(a) **(b)** **(c)**

Figure 2. (a) Helicoid and vectorial form of $\Delta^{II}X$ for $u \in [0, 2\pi]$, $v \in [-2, 2]$; (b) Bertrand offset of (a) and vectorial form of $\Delta^{I}X^{*}$ for $u \in [0, 2\pi]$, $v \in [-2, 2]$, $c = 1$, $\theta = \pi/6$; (c) Bertrand offset of (a) and vectorial form of $\Delta^{II}X^{*}$ for $u \in [0, 2\pi]$, $v \in [-2, 2]$, $c = 1$, $\theta = \pi/6$.

Example 4.2. *Let us consider a hyperboloid of one sheet with parametric representation*

$$X(u,v) = \left(\cos\sqrt{2}u - \frac{v\sin\sqrt{2}u}{\sqrt{2}}, \sin\sqrt{2}u + \frac{v\cos\sqrt{2}u}{\sqrt{2}}, \frac{v}{\sqrt{2}}\right).$$

Here, $\beta(u) = \left(\cos\sqrt{2}u, \sin\sqrt{2}u, 0\right)$ *and* $e(u) = \left(-\frac{\sin\sqrt{2}u}{\sqrt{2}}, \frac{\cos\sqrt{2}u}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $(\langle e', e'\rangle = 1)$ *are the base curve and the spherical indicatrix curve of this ruled surface, respectively. It is clear that* $\beta(u)$ *is the striction curve since* $\langle \beta', e'\rangle = 0$ *(see Eq* (2.13)*).*

The structure functions of X are $J = -1$*,* $F = 1$*, and* $Q = -1$ *via Eqs* (2.16), (2.18)*, and* (2.19)*, respectively, where*

$$e(u) = \left(-\frac{\sin\sqrt{2}u}{\sqrt{2}}, \frac{\cos\sqrt{2}u}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), t(u) = \left(-\cos\sqrt{2}u, -\sin\sqrt{2}u, 0\right), g(u) = \left(\frac{\sin\sqrt{2}u}{\sqrt{2}}, -\frac{\cos\sqrt{2}u}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad (4.6)$$

(see Eq (2.14)*). Since* $Q = -1 \neq 0$*, it is apparent that the hyperboloid of one sheet is a non-developable ruled surface. From Eq* (2.20)*, the coefficients of the first fundamental form of X are calculated as* $E = v^2 + 2$*,* $F = 1$*,* $G = 1$*. With the help of Eq* (2.21)*, the unit normal vector of X is*

$$\mathcal{U} = \frac{1}{\sqrt{v^2+1}}\left(\cos\sqrt{2}u - \frac{v\sin\sqrt{2}u}{\sqrt{2}}, \sin\sqrt{2}u + \frac{v\cos\sqrt{2}u}{\sqrt{2}}, -\frac{v}{\sqrt{2}}\right).$$

Thus, with Eq (2.23)*, the coefficients of the second fundamental form of X are* $L = -\frac{v^2+2}{\sqrt{v^2+1}}$*,* $M = -\frac{1}{\sqrt{v^2+1}}$*,* $N = 0$*. Also, with Eq* (2.24)*, the Gauss and the mean curvature of X are:* $\mathcal{K} = -\frac{1}{(v^2+1)^2}$*,* $\mathcal{H} = -\frac{v^2}{2(v^2+1)^{3/2}}$*. Considering Eq* (3.21)*, the Laplace operator with respect to the first fundamental form of X is calculated as*

$$\Delta^{I}X = \left(-\frac{v^2\left(\sqrt{2}v\sin\sqrt{2}u - 2\cos\sqrt{2}u\right)}{2\left(v^2+1\right)^2}, \frac{v^2\left(\sqrt{2}v\cos\sqrt{2}u + 2\sin\sqrt{2}u\right)}{2\left(v^2+1\right)^2}, -\frac{v^3}{\sqrt{2}\left(v^2+1\right)^2}\right) = -2\mathcal{H}\mathcal{U}. \quad (4.7)$$

Since no real values of λ_{ij} independent of v can be obtained such that $\Delta^I X = \Lambda X$, the surface is of infinite I-type. Thus, the hyperboloid of one sheet is infinite I-type.

Employing this construction, taking into account Eq (3.40)*, the Laplace operator with respect to the second fundamental form of X is calculated as*

$$\Delta^{II} X = \left(\sqrt{v^2+1}\left(\sqrt{2}v\sin\sqrt{2}u - 2\cos\sqrt{2}u\right), -\sqrt{v^2+1}\left(\sqrt{2}v\cos\sqrt{2}u + 2\sin\sqrt{2}u\right), -\sqrt{2}v\sqrt{v^2+1}\right). \tag{4.8}$$

Hence, there exists no real constant matrix Λ satisfying $\Delta^{II} X = \Lambda X$. It means that the hyperboloid of one sheet is an infinite II-type surface.

Let us examine the Bertrand offset of the given hyperboloid of one sheet. Then, it is written via Eqs (2.25) *and* (4.6) *as*

$$X^*(u,v) = \left(\frac{v\sin\sqrt{2}u(\sin\theta-\cos\theta)}{\sqrt{2}} - (c-1)\cos\sqrt{2}u, \frac{v\cos\sqrt{2}u(\cos\theta-\sin\theta)}{\sqrt{2}} - (c-1)\sin\sqrt{2}u, \frac{v(\sin\theta+\cos\theta)}{\sqrt{2}}\right). \tag{4.9}$$

With the help of Eq (3.2)*, the coefficients of the first fundamental form of X^* are*

$$E^* = 2(c-1)^2 - 2v^2\sin\theta\cos\theta + v^2, \quad F^* = (c-1)(\sin\theta-\cos\theta), \quad G^* = 1,$$

where $A = \cos\theta - \sin\theta$ and $B = (c-1)(\cos\theta+\sin\theta)$ have been utilized via Eqs (3.3) *and* (3.6)*. Also, with Eq* (3.4)*, the unit normal vector of X^* is*

$$\mathcal{U}^* = \begin{bmatrix} \dfrac{-(\cos\theta+\sin\theta)\left(\sqrt{2}\,v(\cos\theta-\sin\theta)\sin\sqrt{2}u + 2(c-1)\cos\sqrt{2}u\right)}{2\sqrt{v^2(\cos\theta-\sin\theta)^2+(c-1)^2(\cos\theta+\sin\theta)^2}} \\ \dfrac{(\cos\theta+\sin\theta)\left(\sqrt{2}\,v(\cos\theta-\sin\theta)\cos\sqrt{2}u - 2(c-1)\sin\sqrt{2}u\right)}{2\sqrt{v^2(\cos\theta-\sin\theta)^2+(c-1)^2(\cos\theta+\sin\theta)^2}} \\ -\dfrac{v(\cos\theta-\sin\theta)^2}{\sqrt{2}\sqrt{v^2(\cos\theta-\sin\theta)^2+(c-1)^2(\cos\theta+\sin\theta)^2}} \end{bmatrix}^t.$$

It is worth emphasizing that the expression inside the square root in the denominator is strictly positive (see Eq (3.5)*). Then, from Eq* (3.8)*, the coefficients of the second fundamental form of X^* are*

$$\begin{aligned} L^* &= -\frac{(\cos\theta+\sin\theta)\left(2(c-1)^2 + v^2(\cos\theta-\sin\theta)^2\right)}{\sqrt{v^2(\cos\theta-\sin\theta)^2+(c-1)^2(\cos\theta+\sin\theta)^2}}, \\ M^* &= \frac{(c-1)\cos 2\theta}{\sqrt{v^2(\cos\theta-\sin\theta)^2+(c-1)^2(\cos\theta+\sin\theta)^2}}, \\ N^* &= 0. \end{aligned}$$

Considering Eq (3.9)*, we have the Gaussian curvature and the mean curvature of X^*, respectively:*

$$\begin{aligned} \mathcal{K}^* &= -\frac{(c-1)^2\cos^2 2\theta}{\left(v^2(\cos\theta-\sin\theta)^2+(c-1)^2(\cos\theta+\sin\theta)^2\right)^2}, \\ \mathcal{H}^* &= \frac{(\cos\theta+\sin\theta)\left(\left(v^2-2(c-1)^2\right)\sin(2\theta) - v^2\right)}{2\left(v^2(\cos\theta-\sin\theta)^2+(c-1)^2(\cos\theta+\sin\theta)^2\right)^{3/2}}. \end{aligned}$$

Hence, using Eq (3.29)*, the Laplace operator with respect to the first fundamental form of* X^* *is*

$$\Delta^I X^* = \begin{bmatrix} \dfrac{(\cos\theta+\sin\theta)^2\left(\left(v^2-2(c-1)^2\right)\sin 2\theta - v^2\right)\left(\sqrt{2}v(\cos\theta-\sin\theta)\sin\sqrt{2}u + 2(c-1)\cos\sqrt{2}u\right)}{2\left(v^2(\cos\theta-\sin\theta)^2+(c-1)^2(\cos\theta+\sin\theta)^2\right)^2} \\ -\dfrac{(\cos\theta+\sin\theta)^2\left(\left(v^2-2(c-1)^2\right)\sin 2\theta - v^2\right)\left(\sqrt{2}v(\cos\theta-\sin\theta)\cos\sqrt{2}u - 2(c-1)\sin\sqrt{2}u\right)}{2\left(v^2(\cos\theta-\sin\theta)^2+(c-1)^2(\cos\theta+\sin\theta)^2\right)^2} \\ \dfrac{v(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)^2\left(\left(v^2-2(c-1)^2\right)\sin 2\theta - v^2\right)}{\sqrt{2}\left(v^2(\cos\theta-\sin\theta)^2+(c-1)^2(\cos\theta+\sin\theta)^2\right)^2} \end{bmatrix}^t. \tag{4.10}$$

It can be clearly seen that $\Delta^I X^* = -2\mathcal{H}^*\mathcal{U}^*$.

By examining the conditions, let us consider Theorem 3.2. Since $J = -1$, $F = 1$, *and* $Q = -1$, *it is clear that* $Q = JF$ *is satisfied. Then, by Theorem 3.2-i, under condition* $J = \tan\theta = -1$, *the relations* $Q = JF$ *and* $Q' = JF'$ *are satisfied. Thus,* X^* *is a finite I-type surface where* $J = \tan\theta$. *In fact, for* $\theta = -\frac{\pi}{4} + k\pi$ *with solving* $J = \tan\theta = -1$, *we calculate* $\Delta^I X^* = (0, 0, 0)$. *Thus, the Bertrand offset of the hyperboloid of one sheet is a finite I-type surface. In addition,* X^* *is minimal (*$\mathcal{H}^* = 0$*) under condition* $\theta = -\frac{\pi}{4} + k\pi$ *(see Corollary 3.1(v)).*

After building this, taking into account Eq (3.39)*, the Laplace operator with respect to the second fundamental form of* X^* *is calculated as*

$$\Delta^{II} X^* = \begin{bmatrix} \dfrac{\sqrt{(c-v-1)(c+v-1)\sin 2\theta + (c-1)^2 + v^2}\left(2(c-1)\cos\sqrt{2}u + \sqrt{2}v\sin\sqrt{2}u(\cos\theta-\sin\theta)\right)}{(c-1)^2(\cos\theta+\sin\theta)} \\ \dfrac{\sqrt{(c-v-1)(c+v-1)\sin 2\theta + (c-1)^2 + v^2}\left(2(c-1)\sin\sqrt{2}u + \sqrt{2}v\cos\sqrt{2}u(\sin\theta-\cos\theta)\right)}{(c-1)^2(\cos\theta+\sin\theta)} \\ -\dfrac{\sqrt{2}v\sqrt{(c-v-1)(c+v-1)\sin 2\theta + (c-1)^2 + v^2}}{(c-1)^2} \end{bmatrix}^t. \tag{4.11}$$

It is worth noting that $B = (c-1)(\cos\theta + \sin\theta) \neq 0$ *(see Eq* (3.43)*). The results show that* X^* *is an infinite II-type surface, and thus Theorem 3.3 is satisfied. Also, the results confirm the validity of Corollary 3.6.*

It can be observed that the visual vectorial representation of the Laplace operators with respect to the first and second fundamental forms for the hyperboloid of one sheet (see Eqs (4.7) *and* (4.8)*) and its Bertrand offset for* $u \in [0, 2\pi]$, $v \in [-2, 2]$, $c = 2$, $\theta = \pi/6$ *can be seen in Figure 3. Considering these values, from Eqs* (4.9)–(4.11)*, we have*

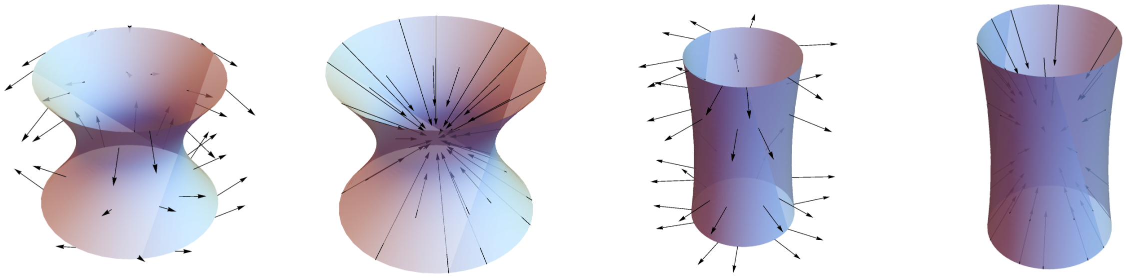
$$X^*(u,v) = \left(-\frac{1}{2}\sqrt{2-\sqrt{3}}\,v\sin\sqrt{2}u - \cos\sqrt{2}u, \frac{1}{2}\sqrt{2-\sqrt{3}}\,v\cos\sqrt{2}u - \sin\sqrt{2}u, \frac{1}{2}\sqrt{\sqrt{3}+2}\,v\right),$$

$$\Delta^I X^* = \begin{bmatrix} \dfrac{(\sqrt{3}+2)\left((\sqrt{3}-2)v^2-2\sqrt{3}\right)\left(\sqrt{2}(\sqrt{3}-1)v\sin(\sqrt{2}u)+4\cos(\sqrt{2}u)\right)}{4\left(-((\sqrt{3}-2)v^2)+\sqrt{3}+2\right)^2} \\ -\dfrac{(\sqrt{3}+2)\left((\sqrt{3}-2)v^2-2\sqrt{3}\right)\left(\sqrt{2}(\sqrt{3}-1)v\cos(\sqrt{2}u)-4\sin(\sqrt{2}u)\right)}{4\left(-((\sqrt{3}-2)v^2)+\sqrt{3}+2\right)^2} \\ \dfrac{(5-3\sqrt{3})v^3+2(\sqrt{3}-3)v}{2\sqrt{2}\left(-((\sqrt{3}-2)v^2)+\sqrt{3}+2\right)^2} \end{bmatrix}^t,$$

and

$$\Delta^{II}X^* = \begin{bmatrix} \frac{\sqrt{-((\sqrt{3}-2)v^2)+\sqrt{3}+2}(\sqrt{2}(\sqrt{3}-1)v\sin(\sqrt{2}u)+4\cos(\sqrt{2}u))}{\sqrt{2}+\sqrt{6}} \\ \frac{\sqrt{-((\sqrt{3}-2)v^2)+\sqrt{3}+2}(4\sin(\sqrt{2}u)-\sqrt{2}(\sqrt{3}-1)v\cos(\sqrt{2}u))}{\sqrt{2}+\sqrt{6}} \\ -v\sqrt{-\left(\left(\sqrt{3}-2\right)v^2\right)+\sqrt{3}+2} \end{bmatrix}^t.$$

Likewise, as shown in Figure 2, the arrows in Figure 3 represent the vectorial form of the Laplace operator and depict its direction and behavior over the surface. These arrows are evaluated at $X(u_0, v_0) + k\Delta^{\mathcal{J}}X(u_0, v_0)$, *where* $u_0 \in [0, 2\pi]$, $v_0 \in [-2, 2]$, $k \in \mathbb{R}$, $\mathcal{J} = I$, *and* $\mathcal{J} = II$ *for Figures 3(a) and (b). The values* $k = 2$ *and* $k = 0.2$ *are chosen for these figures, respectively. Analogously, for Figures 3(c) and (d),* $X^*(u_0, v_0) + k\Delta^{\mathcal{J}}X^*(u_0, v_0)$, *where* $u_0 \in [0, 2\pi]$, $v_0 \in [-2, 2]$, $k \in \mathbb{R}$, $\mathcal{J} = I$, *and* $\mathcal{J} = II$ *are considered. The values* $k = 1$ *and* $k = 0.2$ *are chosen for these figures, respectively.*



(a) **(b)** **(c)** **(d)**

Figure 3. (a) Hyperboloid of one sheet and vectorial form of $\Delta^I X$ for $u \in [0, 2\pi]$, $v \in [-2, 2]$; (b) Hyperboloid of one sheet and vectorial form of $\Delta^{II}X$ for $u \in [0, 2\pi]$, $v \in [-2, 2]$; (c) Bertrand offset of (a) and vectorial form of $\Delta^I X^*$ for $u \in [0, 2\pi]$, $v \in [-2, 2]$, $c = 2$, $\theta = \pi/6$; (d) Bertrand offset of (a) and vectorial form of $\Delta^{II}X^*$ for $u \in [0, 2\pi]$, $v \in [-2, 2]$, $c = 2$, $\theta = \pi/6$.

As a final remark of this section, it is worth noting that the vector fields visualized in the figures represent the curvature-related geometric effect of the Laplace operators on the surface. Besides, the vectors of the Laplace operator with respect to the first fundamental form at any point on the surface are directed along its unit normal vector field. It is clear that no vector field occurs for the helicoid with respect to the first fundamental form, since $\Delta^I X = 0$, indicating that $\Delta^I X$ is harmonic and therefore the surface is minimal.

5. Discussion and conclusions

In surface theory and differential geometry, ruled surfaces occupy a prominent place and play a key role in many application areas, including computer graphics, architectural design, and engineering modeling. The offsets of ruled surfaces provide deeper insight into the geometric behavior of surface pairs. Also, the Laplace operator holds an important position in differential geometry, providing a natural link between the analytical and geometric properties of a surface. In this context, the investigation of ruled surfaces and their Bertrand offsets via the Laplace operator provides a comprehensive framework for understanding the geometric structure of surface pairs.

In this study, we have examined the characterizations of Bertrand offset X^* of the non-developable ruled surface X with the Laplace operator via the first and second fundamental forms and established a novel perspective including the structure functions and provided characterizations. The findings can be concisely and effectively summarized; considering the first fundamental form, X being finite implies that X^* is infinite except for the trivial case. X being infinite implies that X^* is finite by the conditions given by Theorem 3.2. Otherwise, X^* is infinite. Besides, considering the second fundamental form, X and X^* are infinite.

Consequently, we believe that this study makes an important contribution to the theory of surfaces, and that new characterizations for surface pairs can be obtained in future studies by using a similar approach.

Author contributions

Duygu Çağlar Çay: Investigation, Methodology, Resources, Software, Validation, Visualization, Writing–original draft, Writing–review & editing; Nurten Gürses: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Resources, Supervision, Validation, Writing–original draft, Writing–review & editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that no artificial intelligence (AI) tools were used in the preparation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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