AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 23792–23816.
DOI: 10.3934/math.2026958
Received: 09 May 2026
Revised: 08 July 2026
Accepted: 28 July 2026
Published: 04 August 2026

Research article

Continuous warped-shift wavelets on nonlinear spectral groups

Fethi Bouzeffour and Mhamed Eddahbi*

Department of Mathematics, College of Science, King Saud University, P. O. Box 2455, Z. C. 11451, Riyadh, Saudi Arabia

* **Correspondence:** Email: meddahbi@ksu.edu.sa.

Abstract: We developed a continuous wavelet theory on nonlinear spectral groups induced by frequency warpings. Let $\varphi\colon I \to \mathbb{R}$ be a real C^1-diffeomorphism and define

$$(\mathcal{F}_\varphi f)(t) = \widehat{f}(\varphi(t)), \quad d\mu_\varphi(t) = \frac{|\varphi'(t)|}{2\pi}\, dt.$$

The transform $\mathcal{F}_\varphi$ is unitary from $L^2(\mathbb{R})$ onto $L^2(I, d\mu_\varphi)$, and it diagonalizes the warped spectral operator

$$T_\varphi = i\varphi^{-1}(D), \quad D = -i\partial_x.$$

The change of variables $\xi = \varphi(t)$ transports the additive Fourier group to the nonlinear spectral group

$$t \oplus_\varphi s = \varphi^{-1}(\varphi(t) + \varphi(s)).$$

Using the corresponding warped shifts and automorphic dilations, we constructed continuous warped-shift wavelets and proved their Calderón admissibility, Plancherel, and reconstruction formulas. The theory becomes the classical continuous wavelet transform after linearization by φ, while in the variable t, it yields a nonlinear spectral wavelet analysis. We also separated this construction from Fourier-adapted and operator-adapted warped wavelets, and explained its connection with finite operator calculus, umbral generating functions, and the Meixner–Pollaczek central-difference operator. Examples include logarithmic, exponential, sine, tangent, hyperbolic tangent, and hyperbolic-sine warpings.

Keywords: continuous wavelet transform; warped Fourier transform; nonlinear spectral group; Calderón formula; finite operator calculus; umbral calculus; Meixner–Pollaczek polynomials
Mathematics Subject Classification: 60H15, 60H30, 60H35, 76V05

1. Introduction

Wavelet analysis arose from the need for representations that are localized simultaneously in space and frequency. Classical Fourier analysis decomposes a signal into global oscillations, whereas wavelet analysis decomposes it into translated and dilated copies of a single analyzing function. This makes wavelets effective in multiscale analysis, singularity detection, time-frequency localization, numerical approximation, and signal processing.

The continuous wavelet transform has its roots in Calderón's reproducing formula [1], which already contains the resolution-of-the-identity structure behind wavelet reconstruction. The modern continuous wavelet transform was introduced by Morlet and collaborators in connection with seismic signal analysis [2]. Its mathematical foundation was developed by Grossmann and Morlet through square-integrable representations of the affine group [3], and further by Grossmann et al. [4]. In this setting, a mother wavelet $\psi \in L^2(\mathbb{R})$ generates

$$\psi_{a,b}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right), \qquad a \in \mathbb{R}^*,\ b \in \mathbb{R},$$

and reconstruction is governed by the Calderón admissibility condition

$$\int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}\, d\xi < \infty.$$

The later development of compactly supported wavelets and multiresolution analysis by Daubechies and Mallat connected wavelets with filter banks, approximation theory, and fast algorithms [5–8]. This connection was extended in the theory of subband coding and multirate systems [9, 10].

Classical continuous wavelets are based on the affine geometry of the real line: Translations are additive and dilations are multiplicative. In the Fourier domain, this means

$$\widehat{\psi}_{a,b}(\xi) = |a|^{1/2} e^{-ib\xi}\widehat{\psi}(a\xi).$$

Thus the classical transform is naturally adapted to the Fourier operator

$$D = -i\partial_x, \quad De^{ix\xi} = \xi e^{ix\xi}.$$

In many problems, however, the relevant spectral coordinate is not the ordinary Fourier variable ξ, but a nonlinear variable t related to it by

$$\xi = \varphi(t).$$

Such nonlinear coordinates occur in perceptual frequency scales, nonuniform filter-bank design, finite-difference operators, delta operators, and spectral transforms associated with special functions and orthogonal polynomials.

Frequency warping has a long history in signal processing. Unequal-bandwidth spectral analysis and digital frequency warping were studied by Oppenheim et al. [11, 12]. Baraniuk and Jones emphasized the role of unitary equivalence in frequency-warped signal processing [13]. In the wavelet setting, Evangelista and Cavaliere constructed frequency-warped filter banks and wavelet transforms using Laguerre expansions and allpass frequency transformations [14]. Evangelista's dyadic warped wavelet theory showed that a full dyadic warped construction is not merely a global change of frequency variable; it also involves generalized shifts, warped multiresolution spaces, warped Riesz bases, warped quadrature mirror filters, and iterated warping [15].

The present paper develops a continuous warped-shift wavelet theory in a related but different direction. We begin with a real C^1-diffeomorphism

$$\varphi : I \to \mathbb{R},$$

and define the warped Fourier transform

$$(\mathcal{F}_\varphi f)(t) = \widehat{f}(\varphi(t)).$$

The natural spectral measure is

$$d\mu_\varphi(t) = \frac{|\varphi'(t)|}{2\pi}\, dt.$$

With this measure, $\mathcal{F}_\varphi$ is unitary from $L^2(\mathbb{R})$ onto $L^2(I, d\mu_\varphi)$. It diagonalizes

$$T_\varphi = i\varphi^{-1}(D)$$

in the sense that

$$\mathcal{F}_\varphi(T_\varphi f)(t) = it(\mathcal{F}_\varphi f)(t).$$

Hence t is not only a change of coordinate; it is the spectral eigenvalue of the warped operator T_φ.

The diffeomorphism φ transports the additive group law of the Fourier line to the interval I. This gives the nonlinear spectral group

$$t \oplus_\varphi s = \varphi^{-1}(\varphi(t) + \varphi(s)).$$

The identity and inverse are

$$e_\varphi = \varphi^{-1}(0), \quad \ominus_\varphi t = \varphi^{-1}(-\varphi(t)),$$

and $d\mu_\varphi$ is the Haar measure for this transported group up to the normalization factor $1/(2\pi)$. Warped shifts and dilations are defined by

$$(\mathsf{T}_s^\varphi F)(t) = F(t \ominus_\varphi s), \quad \delta_a^\varphi(t) = \varphi^{-1}(a\varphi(t)).$$

They generate the warped-shift wavelets

$$\Phi_{a,s}^{\oplus_\varphi}(t) = |a|^{-1/2}\Phi\left(\varphi^{-1}\left(\frac{\varphi(t) - \varphi(s)}{a}\right)\right), \quad a \in \mathbb{R}^*, \; s \in I.$$

Here s is a translation parameter in the nonlinear spectral group, not a physical translation in x-space. The scale parameter a dilates the linearized coordinate $\varphi(t)$. The associated transform is

$$\mathcal{W}_\Phi^{\oplus_\varphi} F(a, s) = \langle F, \Phi_{a,s}^{\oplus_\varphi} \rangle_{L^2(I, d\mu_\varphi)}.$$

We prove its Calderón admissibility condition, Plancherel identity, and reconstruction formula. The proof is based on the observation that the change of variables $\xi = \varphi(t)$ and $\eta = \varphi(s)$ converts the warped-shift system into the ordinary affine wavelet system in the linear spectral variable ξ.

It is important to distinguish this construction from two related ones. A Fourier-adapted warped wavelet transform dilates the ordinary Fourier frequency $\xi = \varphi(t)$; it is just the classical continuous wavelet transform transported through the unitary change of variable. An operator-adapted warped wavelet transform dilates the eigenvalue t of T_φ directly. The present construction is instead the affine wavelet theory of the nonlinear spectral group $(I, \oplus_\varphi)$.

There is also an algebraic interpretation. In finite operator calculus, Rota et al. study shift-invariant operators $Q = q(\partial)$ with $q(0) = 0$ and $q'(0) \neq 0$ [16]. The inverse $\varphi = q^{-1}$ appears in the umbral generating function $e^{x\varphi(z)}$. Zeilberger emphasized that this formal theory can be viewed through Fourier analysis because shift-invariant operators become multipliers after Fourier transform [17]. Replacing ∂ by $D = -i\partial_x$ gives

$$q(D)e^{ix\varphi(t)} = te^{ix\varphi(t)}.$$

Thus $e^{ix\varphi(t)}$ is the oscillatory analogue of the umbral generating function. The detailed connection with Shen's umbral refinement equation for wavelets [18] is developed later in the paper.

A guiding example is the hyperbolic-sine symbol

$$q(\xi) = 2\sinh\frac{\xi}{2}, \quad \varphi(t) = q^{-1}(t) = 2\,\mathrm{arcsinh}\,\frac{t}{2}.$$

The warped Fourier kernel is

$$E_\varphi(x,t) = \exp\left(2ix\,\mathrm{arcsinh}\,\frac{t}{2}\right).$$

This is the T-exponential for the central-difference operator studied by Ismail and Saad in connection with the Meixner–Pollaczek polynomials [19]. In this case,

$$(Tf)(x) = \frac{f(x+i/2) - f(x-i/2)}{i}, \quad TE_\varphi(x,t) = itE_\varphi(x,t),$$

and the group law is

$$t \oplus_\varphi s = t\sqrt{1+\frac{s^2}{4}} + s\sqrt{1+\frac{t^2}{4}}.$$

Thus the hyperbolic-sine warping links the central-difference operator, the Meixner–Pollaczek polynomial calculus, the arcsinh warped Fourier kernel, and the nonlinear spectral group used in this paper. $J \subsetneq \mathbb{R}$ are treated on the spectral subspace $\mathbb{H}_J$ or locally, operator domains are defined by spectral calculus, and all constants are tied to the Fourier normalization in (1.1).

The main contributions are as follows:

(i) We develop the warped Fourier transform and the nonlinear spectral group $(I, \oplus_\varphi)$, including its Haar measure, warped shifts, and automorphic dilations.

(ii) We construct the continuous warped-shift wavelet transform and prove its Calderón admissibility condition, Plancherel formula, and reconstruction formula.

(iii) We transfer the construction to physical space through $\mathcal{F}_\varphi^{-1}$, obtaining physical warped-shift wavelets.

(iv) We distinguish warped-shift wavelets from Fourier-adapted and operator-adapted warped wavelets.

(v) We explain the connection with shift-invariant operators, finite operator calculus, and umbral generating functions.

(vi) We provide explicit examples, including linear, logarithmic, exponential, sine, tangent, hyperbolic tangent, and hyperbolic-sine warpings.

The paper is organized as follows. Section 2 develops the warped Fourier transform and the nonlinear spectral group. Section 3 constructs the warped-shift wavelet transform and proves the Calderón theorem and reconstruction formula. Section 4 discusses the warped spectral operator, the connection with finite operator calculus and umbral methods, operator-adapted wavelets, and concrete examples.

Throughout the paper, we use the Fourier transform convention

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}\,dx, \quad f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)e^{ix\xi}\,d\xi. \tag{1.1}$$

With this normalization,

$$\|f\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f}(\xi)|^2\,d\xi.$$

For a Borel function m, the spectral calculus for $D = -i\partial_x$ is

$$\widehat{m(D)f}(\xi) = m(\xi)\widehat{f}(\xi).$$

We write $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. If a warping maps $\varphi\colon I \to J \subseteq \mathbb{R}$ onto a proper interval J, the natural physical space is the Fourier spectral subspace

$$\mathcal{H}_J = \left\{ f \in L^2(\mathbb{R}) : \widehat{f}(\xi) = 0 \text{ for a.e. } \xi \notin J \right\}. \tag{1.2}$$

For the global nonlinear group construction, we assume $J = \mathbb{R}$, so that $\mathcal{H}_J = L^2(\mathbb{R})$.

If $J \subsetneq \mathbb{R}$, all warped Fourier identities are understood on the closed spectral subspace $\mathcal{H}_J$, and the transported law is used only where $\varphi(t) + \varphi(s) \in J$. Thus bounded or one-sided spectral intervals give band-limited or branch-local versions of the same warped Fourier principle, not a global group on all of I.

2. The warped Fourier transform and nonlinear spectral groups

This section develops the analytic and algebraic foundations of the theory. We begin by defining and proving unitarity of the warped Fourier transform, then constructing the nonlinear spectral group induced by the diffeomorphism φ, and, finally proving its key properties.

2.1. The warped Fourier transform

Let $I, J \subseteq \mathbb{R}$ be non-empty open intervals and let $\varphi\colon I \to J$ be a real C^1-diffeomorphism with inverse $\psi = \varphi^{-1}\colon J \to I$. When $J \subsetneq \mathbb{R}$, the natural function space is the spectral subspace $\mathcal{H}_J$ defined in (1.2); when $J = \mathbb{R}$, we have $\mathcal{H}_{\mathbb{R}} = L^2(\mathbb{R})$.

Definition 2.1. *The* warped exponential kernel *associated with φ is*

$$E_\varphi(x,t) := e^{ix\varphi(t)}, \quad x \in \mathbb{R},\ t \in I.$$

The warped Fourier transform *is*

$$(\mathcal{F}_\varphi f)(t) := \int_{\mathbb{R}} f(x)\,\overline{E_\varphi(x,t)}\,dx = \widehat{f}(\varphi(t)), \quad t \in I, \tag{2.1}$$

initially for $f \in L^1(\mathbb{R}) \cap \mathcal{H}_J$, and extended below. The warped spectral measure *is the pullback of $d\xi/(2\pi)$:*

$$d\mu_\varphi(t) := \frac{|\varphi'(t)|}{2\pi}\,dt. \tag{2.2}$$

The absolute value $|\varphi'(t)|$ ensures that $d\mu_\varphi$ is a positive measure regardless of whether φ is increasing or decreasing.

Theorem 2.1. *The warped Fourier transform extends uniquely to a unitary operator*

$$\mathcal{F}_\varphi : \mathcal{H}_J \longrightarrow L^2(I, d\mu_\varphi). \tag{2.3}$$

The inverse is given by the formula

$$(\mathcal{F}_\varphi^{-1}F)(x) = \int_I F(t)\,E_\varphi(x,t)\,d\mu_\varphi(t), \quad F \in L^2(I, d\mu_\varphi), \tag{2.4}$$

where the integral is understood as an $L^2(\mathbb{R})$ limit over an exhaustion of I by compact intervals. More precisely, if $K_n \Subset I$ and $K_n \uparrow I$, then

$$\mathcal{F}_\varphi^{-1}F = \lim_{n\to\infty} \int_{K_n} F(t)E_\varphi(\cdot,t)\,d\mu_\varphi(t) \quad \text{in } L^2(\mathbb{R}).$$

Consequently, for every $f \in \mathcal{H}_J$,

$$f(x) = \int_I (\mathcal{F}_\varphi f)(t)\,E_\varphi(x,t)\,d\mu_\varphi(t). \tag{2.5}$$

Proof. First, let f belong to the dense subclass $L^1(\mathbb{R}) \cap \mathcal{H}_J$. Since $(\mathcal{F}_\varphi f)(t) = \widehat{f}(\varphi(t))$ and

$$d\mu_\varphi(t) = \frac{|\varphi'(t)|}{2\pi}\,dt,$$

the change of variables $\xi = \varphi(t)$ gives

$$\begin{aligned}\int_I |(\mathcal{F}_\varphi f)(t)|^2\,d\mu_\varphi(t) &= \int_I |\widehat{f}(\varphi(t))|^2 \frac{|\varphi'(t)|}{2\pi}\,dt \\ &= \frac{1}{2\pi}\int_J |\widehat{f}(\xi)|^2\,d\xi.\end{aligned}$$

Here the absolute value in $|\varphi'(t)|$ accounts for the possibility that φ is decreasing. Since $f \in \mathcal{H}_J$, its Fourier transform is supported in J, and Plancherel's theorem therefore yields

$$\frac{1}{2\pi}\int_J |\widehat{f}(\xi)|^2\, d\xi = \|f\|^2_{L^2(\mathbb{R})}.$$

Hence

$$\left\|\mathcal{F}_\varphi f\right\|_{L^2(I,d\mu_\varphi)} = \|f\|_{L^2(\mathbb{R})}$$

on $L^1(\mathbb{R}) \cap \mathcal{H}_J$. By density and continuity, $\mathcal{F}_\varphi$ extends uniquely to an isometry

$$\mathcal{F}_\varphi : \mathcal{H}_J \longrightarrow L^2(I, d\mu_\varphi).$$

It remains to prove that this isometry is surjective. Let $F \in L^2(I, d\mu_\varphi)$ be arbitrary, and define a function G on J by

$$G(\xi) = F(\psi(\xi)), \quad \psi = \varphi^{-1},$$

extending G by zero outside J. Using the substitution $\xi = \varphi(t)$, we obtain

$$\frac{1}{2\pi}\int_J |G(\xi)|^2\, d\xi = \int_I |F(t)|^2\, d\mu_\varphi(t) < \infty.$$

Thus, $G \in L^2(\mathbb{R}, d\xi/(2\pi))$. Let $f = \mathcal{F}^{-1}G$. Since G is supported in J, we have $f \in \mathcal{H}_J$. Moreover, for almost every $t \in I$,

$$(\mathcal{F}_\varphi f)(t) = \widehat{f}(\varphi(t)) = G(\varphi(t)) = F(t).$$

Therefore, every element of $L^2(I, d\mu_\varphi)$ has a preimage under $\mathcal{F}_\varphi$, so the isometric extension is surjective and hence unitary. Finally, for $F \in L^2(I, d\mu_\varphi)$, we set

$$G(\xi) = F(\psi(\xi)), \quad \xi \in J,$$

extended by zero outside J. Since $\mathcal{F}_\varphi^{-1}F = \mathcal{F}^{-1}G$, the inverse Fourier formula gives the following L^2 limit: If $J_n \Subset J$ and $J_n \uparrow J$, then

$$\mathcal{F}_\varphi^{-1}F = \lim_{n\to\infty} \frac{1}{2\pi}\int_{J_n} G(\xi)e^{i(\cdot)\xi}\, d\xi \quad \text{in } L^2(\mathbb{R}).$$

Substituting $\xi = \varphi(t)$ and using $E_\varphi(x,t) = e^{ix\varphi(t)}$ yields, in the same L^2 sense,

$$(\mathcal{F}_\varphi^{-1}F)(x) = \int_I F(t)E_\varphi(x,t)\, d\mu_\varphi(t),$$

which is (2.4). Applying this formula to $F = \mathcal{F}_\varphi f$ gives the reconstruction identity (2.5). □

2.2. The nonlinear spectral group

Throughout the remainder of this section and whenever the warped group law is used globally, we assume $J = \mathbb{R}$, i.e., φ: $I \to \mathbb{R}$ is a global diffeomorphism.

If instead $J \subsetneq \mathbb{R}$, the same formulas are used only on the branch for which the sums in the φ-coordinate remain in J; this is the situation in the band-limited sine and arctangent examples below. The diffeomorphism φ transports the additive group law of $(\mathbb{R}, +)$ to the interval I.

Definition 2.2. *For $t, s \in I$, define*

$$t \oplus_\varphi s := \varphi^{-1}(\varphi(t) + \varphi(s)), \tag{2.6}$$
$$t \ominus_\varphi s := \varphi^{-1}(\varphi(t) - \varphi(s)). \tag{2.7}$$

The identity element is $e_\varphi := \varphi^{-1}(0)$, and the group inverse of t is $\ominus_\varphi t := \varphi^{-1}(-\varphi(t))$.

Proposition 2.1. *$(I, \oplus_\varphi)$ is an abelian locally compact group. The map φ: $(I, \oplus_\varphi) \to (\mathbb{R}, +)$ is a topological group isomorphism. The measure $d\mu_\varphi$ is a (left and right) Haar measure on $(I, \oplus_\varphi)$, up to the normalization factor $1/(2\pi)$. Moreover, for every $x \in \mathbb{R}$, the warped exponential $t \mapsto E_\varphi(x, t) = e^{ix\varphi(t)}$ is a unitary character of $(I, \oplus_\varphi)$:*

$$E_\varphi(x, t \oplus_\varphi s) = E_\varphi(x, t)\, E_\varphi(x, s). \tag{2.8}$$

Proof. All group axioms follow by transporting the corresponding properties of $(\mathbb{R}, +)$ through φ. For instance, associativity holds because

$$\varphi\big((t \oplus_\varphi s) \oplus_\varphi r\big) = \varphi(t) + \varphi(s) + \varphi(r) = \varphi\big(t \oplus_\varphi (s \oplus_\varphi r)\big),$$

and injectivity of φ yields the identity of group elements. Commutativity follows from commutativity of addition. The identity and inverse are clearly $e_\varphi = \varphi^{-1}(0)$ and $\ominus_\varphi t = \varphi^{-1}(-\varphi(t))$. Since φ is a homeomorphism, $(I, \oplus_\varphi)$ is locally compact.

The Haar measure property follows by the standard change of variables. Let $f \in C_c(I)$ and $r \in I$. Set $\eta = \varphi(t)$ and $\rho = \varphi(r)$. Since $d\mu_\varphi(t) = d\eta/(2\pi)$, we have

$$\begin{aligned}
\int_I f(t \oplus_\varphi r)\, d\mu_\varphi(t) &= \frac{1}{2\pi} \int_{\mathbb{R}} f(\varphi^{-1}(\eta + \rho))\, d\eta \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} f(\varphi^{-1}(u))\, du \\
&= \int_I f(t)\, d\mu_\varphi(t).
\end{aligned}$$

The same computation with $r \oplus_\varphi t$ gives right invariance. Finally, $\varphi(t \oplus_\varphi s) = \varphi(t) + \varphi(s)$ gives (2.8) immediately. □

2.3. Warped shifts, dilations, and the affine representation

Definition 2.3. *For $s \in I$ and $a \in \mathbb{R}^*$, define the* warped translation operator *and the* warped dilation operator *on $L^2(I, d\mu_\varphi)$ by*

$$(\mathsf{T}_s^\varphi F)(t) := F(t \ominus_\varphi s) = F\Big(\varphi^{-1}(\varphi(t) - \varphi(s))\Big), \tag{2.9}$$
$$(\mathsf{D}_a^\varphi F)(t) := |a|^{-1/2}\, F\left(\varphi^{-1}\left(\frac{\varphi(t)}{a}\right)\right). \tag{2.10}$$

The automorphic dilation *of $(I, \oplus_\varphi)$ by scale a is $\delta_a^\varphi(t) := \varphi^{-1}(a\varphi(t))$, which is related to D_a^φ by $(\mathsf{D}_a^\varphi F)(t) = |a|^{-1/2} F(\delta_{1/a}^\varphi(t))$.*

Proposition 2.2. *The following hold:*

(1) T_s^φ *is unitary on* $L^2(I, d\mu_\varphi)$*, with* $(\mathsf{T}_s^\varphi)^{-1} = \mathsf{T}_{(\ominus_\varphi s)}^\varphi$ *and* $\mathsf{T}_s^\varphi \mathsf{T}_r^\varphi = \mathsf{T}_{s \oplus_\varphi r}^\varphi$*.*

(2) D_a^φ *is unitary on* $L^2(I, d\mu_\varphi)$*, with* $(\mathsf{D}_a^\varphi)^{-1} = \mathsf{D}_{1/a}^\varphi$ *and* $\mathsf{D}_a^\varphi \mathsf{D}_b^\varphi = \mathsf{D}_{ab}^\varphi$*.*

(3) The automorphism property holds: $\delta_a^\varphi(t \oplus_\varphi s) = \delta_a^\varphi(t) \oplus_\varphi \delta_a^\varphi(s)$*.*

(4) The affine commutation relation holds: $\mathsf{D}_a^\varphi \mathsf{T}_s^\varphi = \mathsf{T}_{\delta_a^\varphi(s)}^\varphi \mathsf{D}_a^\varphi$*.*

(5) The operators $\mathsf{U}^\varphi(a, s) := \mathsf{T}_s^\varphi \mathsf{D}_a^\varphi$ *form a strongly continuous unitary representation of the warped affine group with the product law* $(a, s) \cdot (b, r) = (ab,\ s \oplus_\varphi \delta_a^\varphi(r))$*.*

Proof. The unitary isomorphism U_φ: $L^2(I, d\mu_\varphi) \to L^2(\mathbb{R}, d\xi/(2\pi))$ defined by $(U_\varphi F)(\xi) = F(\psi(\xi))$ intertwines the warped operators with their classical counterparts:

$$U_\varphi\, \mathsf{T}_s^\varphi\, U_\varphi^{-1} = \tau_{\varphi(s)}, \quad U_\varphi\, \mathsf{D}_a^\varphi\, U_\varphi^{-1} = D_a,$$

where $(\tau_\eta g)(\xi) = g(\xi - \eta)$ and $(D_a g)(\xi) = |a|^{-1/2} g(\xi/a)$ are ordinary translation and dilation on $L^2(\mathbb{R}, d\xi/(2\pi))$. All claimed identities follow from the corresponding classical identities for τ_η and D_a, pulled back through U_φ. For instance, unitarity of T_s^φ follows from unitarity of $\tau_{\varphi(s)}$ and that of U_φ. The affine commutation $\mathsf{D}_a^\varphi \mathsf{T}_s^\varphi = \mathsf{T}_{\delta_a^\varphi(s)}^\varphi \mathsf{D}_a^\varphi$ is the pullback of $D_a \tau_\eta = \tau_{a\eta} D_a$. □

2.4. Examples of the nonlinear spectral group

We record the group law, measure, and group difference for the most important warpings.

Example 2.1. *(Linear warping) Set* $\varphi(t) = t$*,* $I = \mathbb{R}$*:* $t \oplus_\varphi s = t + s$*,* $e_\varphi = 0$*,* $d\mu_\varphi(t) = dt/(2\pi)$*. This is the ordinary additive group.*

Example 2.2. *(The multiplicative group) For* $\varphi(t) = \log t$*,* $I = (0, \infty)$*:*

$$t \oplus_\varphi s = ts, \quad e_\varphi = 1, \quad \ominus_\varphi t = \frac{1}{t}, \quad t \ominus_\varphi s = \frac{t}{s}, \quad d\mu_\varphi(t) = \frac{dt}{2\pi t}.$$

The warped group is the multiplicative group of positive reals, i.e., $((0, \infty), \times)$*. The warped Fourier transform is the Fourier form of Mellin analysis, with the kernel* $E_\varphi(x, t) = t^{ix}$*.*

Example 2.3. *(Hyperbolic-sine warping) Set* $\varphi(t) = 2 \operatorname{arcsinh}(t/2)$*,* $I = \mathbb{R}$*:*

$$t \oplus_\varphi s = t \sqrt{1 + \tfrac{s^2}{4}} + s \sqrt{1 + \tfrac{t^2}{4}}, \tag{2.11}$$

$$t \ominus_\varphi s = t \sqrt{1 + \tfrac{s^2}{4}} - s \sqrt{1 + \tfrac{t^2}{4}}, \tag{2.12}$$

$$d\mu_\varphi(t) = \frac{dt}{\pi \sqrt{t^2 + 4}}.$$

This group law is related to the addition formula for hyperbolic sine: $\sinh(\alpha + \beta) = \sinh \alpha \cosh \beta + \cosh \alpha \sinh \beta$*.*

Example 2.4. *(Sine warping: band-limited analysis) Set* $\varphi(t) = 2 \arcsin(t/2)$*,* $I = (-2, 2)$*,* $J = (-\pi, \pi)$*:*

$$t \oplus_\varphi s = t \sqrt{1 - \tfrac{s^2}{4}} + s \sqrt{1 - \tfrac{t^2}{4}}, \quad d\mu_\varphi(t) = \frac{dt}{\pi \sqrt{4 - t^2}}.$$

Since $J = (-\pi, \pi) \subsetneq \mathbb{R}$, the warped Fourier transform acts on the Nyquist subspace $\mathcal{H}_{(-\pi,\pi)}$, i.e., on functions band-limited to $(-\pi, \pi)$. The group law is local: it is well-defined for $|t|, |s| < 2$ as long as the resulting phases remain in $(-\pi, \pi)$. At the boundary, one must either restrict to such admissible pairs or pass to a periodic/circle interpretation of the phase variable; the global group theorem above is not being invoked on all of $(-2, 2)$.

Example 2.5. *(Arctangent warping) Set $\varphi(t) = 2\arctan(t/2)$, $I = \mathbb{R}$, $J = (-\pi, \pi)$:*

$$t \oplus_\varphi s = \frac{t+s}{1 - ts/4}, \quad t \ominus_\varphi s = \frac{t-s}{1+ts/4}, \quad d\mu_\varphi(t) = \frac{2\,dt}{\pi(t^2+4)}.$$

This is a scaled version of Möbius addition on the real line, and the group is isomorphic to the circle group.

Because $J = (-\pi, \pi)$ is a proper Fourier band, this example is used as a band-limited or local model, not as an application of the global group theorem on all of I.

Example 2.6. *(Hyperbolic-tangent warping: relativistic velocity addition) Set $\varphi(t) = \operatorname{artanh}(t)$, $I = (-1, 1)$:*

$$t \oplus_\varphi s = \frac{t+s}{1+ts}, \quad t \ominus_\varphi s = \frac{t-s}{1-ts}, \quad d\mu_\varphi(t) = \frac{dt}{2\pi(1-t^2)}.$$

The group law is precisely the one-dimensional relativistic velocity addition formula, and $(I, \oplus_\varphi)$ is isomorphic to $(\mathbb{R}, +)$ through artanh.

Example 2.7. *(Logit warping) Set $\varphi(t) = \log(t/(1-t))$, $I = (0, 1)$:*

$$t \oplus_\varphi s = \frac{ts}{ts + (1-t)(1-s)}, \quad e_\varphi = \tfrac{1}{2}, \quad \ominus_\varphi t = 1 - t, \quad d\mu_\varphi(t) = \frac{dt}{2\pi t(1-t)}.$$

Table 1 provides a comparative summary of the principal features of the standard warping functions. The entries in the table have been recalculated from the formula $d\mu_\varphi(t) = |\varphi'(t)|dt/(2\pi)$. In particular, the hyperbolic-tangent row gives $dt/(2\pi(1-t^2))$, while the logit row gives $dt/(2\pi t(1-t))$.

Table 1. The entries have been recalculated from the formula $d\mu_\varphi(t) = |\varphi'(t)|dt/(2\pi)$.

$\varphi(t)$	I	Group law $t \oplus_\varphi s$	$d\mu_\varphi(t)$
t	$\mathbb{R}$	$t + s$	$dt/(2\pi)$
$\log t$	$(0, \infty)$	ts	$dt/(2\pi t)$
$2\operatorname{arcsinh}(t/2)$	$\mathbb{R}$	$t\sqrt{1+s^2/4} + s\sqrt{1+t^2/4}$	$dt/(\pi\sqrt{t^2+4})$
$2\arcsin(t/2)$	$(-2, 2)$	$t\sqrt{1-s^2/4} + s\sqrt{1-t^2/4}$	$dt/(\pi\sqrt{4-t^2})$
$2\arctan(t/2)$	$\mathbb{R}$	$(t+s)/(1-ts/4)$	$2\,dt/(\pi(t^2+4))$
$\operatorname{artanh}(t)$	$(-1, 1)$	$(t+s)/(1+ts)$	$dt/(2\pi(1-t^2))$
$\log(t/(1-t))$	$(0, 1)$	$ts/[ts + (1-t)(1-s)]$	$dt/(2\pi t(1-t))$

3. Warped-shift wavelets and the Calderón theorem

We now construct the continuous wavelet theory associated with the nonlinear spectral group $(I, \oplus_\varphi)$. The essential point is that translations are no longer ordinary translations in the variable t, but translations with respect to the transported group law induced by the diffeomorphism $\varphi\colon I \to \mathbb{R}$. Likewise, dilations act automorphically through the linearized coordinate $\xi = \varphi(t)$. Thus the natural wavelet atoms are obtained by first dilating the spectral profile in the linearized variable and then shifting it in the nonlinear spectral group.

Definition 3.1. *Let $\Phi \in L^2(I, d\mu_\varphi)$. For $a \in \mathbb{R}^*$ and $s \in I$, the* warped-shift wavelet *with mother wavelet Φ, scale a, and spectral center s is defined by*

$$\Phi^{\oplus_\varphi}_{a,s}(t) := (\mathsf{T}^\varphi_s \, \mathsf{D}^\varphi_a \, \Phi)(t) = |a|^{-1/2} \, \Phi\left(\varphi^{-1}\left(\frac{\varphi(t) - \varphi(s)}{a}\right)\right). \tag{3.1}$$

This formula becomes transparent after passing to the linearized coordinate. If

$$\widetilde{\Phi}(\xi) := \Phi(\psi(\xi)), \quad \psi = \varphi^{-1},$$

then the unitary isomorphism U_φ of Proposition 2.2 gives

$$(U_\varphi \, \Phi^{\oplus_\varphi}_{a,s})(\xi) = |a|^{-1/2} \, \widetilde{\Phi}\left(\frac{\xi - \varphi(s)}{a}\right).$$

Thus warped-shift wavelets are precisely ordinary affine wavelet atoms in the linear spectral variable ξ, centered at $\eta = \varphi(s)$ and pulled back to the nonlinear spectral variable t through $\psi = \varphi^{-1}$. This linearization principle is the mechanism behind the Calderón theorem below.

Once the atoms have been defined, the associated transform is obtained by taking their Hilbert-space coefficients in $L^2(I, d\mu_\varphi)$. The parameter a measures scale, while s records the spectral center in the nonlinear group.

Definition 3.2. *For $F \in L^2(I, d\mu_\varphi)$, the* warped-shift wavelet transform *is*

$$\mathcal{W}^{\oplus_\varphi}_\Phi F(a, s) := \left\langle F, \, \Phi^{\oplus_\varphi}_{a,s} \right\rangle_{L^2(I, d\mu_\varphi)} = \int_I F(t) \, \overline{\Phi^{\oplus_\varphi}_{a,s}(t)} \, d\mu_\varphi(t). \tag{3.2}$$

The admissibility condition is inherited from the ordinary Calderón condition after linearization. In other words, admissibility is not imposed directly on the nonlinear variable t, but on the corresponding profile $\widetilde{\Phi}$ in the variable $\xi = \varphi(t)$. This reflects the fact that the warped-shift system is an affine wavelet system after conjugation by U_φ.

Definition 3.3. *Let $\widetilde{\Phi}(\xi) = \Phi(\psi(\xi))$ be the linearized mother wavelet, and let $\widehat{\xi \, \widetilde{\Phi}}$ denote its Fourier transform in the ξ variable, namely,*

$$(\widehat{\xi \, \widetilde{\Phi}})(\omega) = \int_{\mathbb{R}} \widetilde{\Phi}(\xi) \, e^{-i\omega\xi} \, d\xi.$$

The mother wavelet $\Phi \in L^2(I, d\mu_\varphi)$ is called warped-shift admissible *if*

$$C^{\oplus_\varphi}_\Phi := \int_{\mathbb{R}} \frac{|(\widehat{\xi \, \widetilde{\Phi}})(\omega)|^2}{|\omega|} \, d\omega, \quad 0 < C^{\oplus_\varphi}_\Phi < \infty. \tag{3.3}$$

Equivalently, returning to the warped variable through $\xi = \varphi(t)$,

$$(\widehat{\xi\widetilde{\Phi}})(\omega) = \int_I \Phi(t)\, e^{-i\omega\varphi(t)}\, |\varphi'(t)|\, dt.$$

A standard sufficient condition for admissibility, together with the usual local regularity and decay assumptions near the origin and infinity in the linearized variable, is the vanishing moment condition

$$(\widehat{\xi\widetilde{\Phi}})(0) = 0,$$

that is,

$$\int_I \Phi(t)\, |\varphi'(t)|\, dt = 0,$$

or equivalently,

$$\int_I \Phi(t)\, d\mu_\varphi(t) = 0.$$

This moment condition is only a convenient sufficient criterion; the definition of admissibility is the non-zero finiteness of $C_\Phi^{\oplus_\varphi}$. Thus the vanishing integral is not used as an equivalent characterization. The actual hypothesis in Theorem 3.1 is exactly $0 < C_\Phi^{\oplus_\varphi} < \infty$.

When $\varphi(t) = t$, the linearized function is simply $\widetilde{\Phi} = \Phi$, and condition (3.3) reduces to the classical Calderón condition

$$\int_{\mathbb{R}} \frac{|\widehat{\Phi}(\omega)|^2}{|\omega|}\, d\omega < \infty.$$

More generally, the constant $C_\Phi^{\oplus_\varphi}$ depends only on $\widetilde{\Phi} = \Phi \circ \psi$. Therefore admissibility of Φ is exactly the classical admissibility of its linearized profile.

The following theorem is the central result of the section. It states that the warped-shift wavelet transform satisfies the same structural identities as the ordinary continuous wavelet transform: a Plancherel identity and a weak reconstruction formula. The only change is that the translation variable is measured using the density $|\varphi'(s)|\, ds$, which is the pullback of Lebesgue measure under $\eta = \varphi(s)$.

Theorem 3.1. *(Calderón theorem for warped-shift wavelets) Let φ: $I \to \mathbb{R}$ be a C^1-diffeomorphism, and let $\Phi \in L^2(I, d\mu_\varphi)$ be warped-shift admissible. Then, for every $F \in L^2(I, d\mu_\varphi)$,*

$$\int_{\mathbb{R}^*} \int_I \left| \mathcal{W}_\Phi^{\oplus_\varphi} F(a,s) \right|^2 |\varphi'(s)|\, ds\, \frac{da}{a^2} = \frac{C_\Phi^{\oplus_\varphi}}{2\pi}\, \|F\|^2_{L^2(I, d\mu_\varphi)}. \tag{3.4}$$

Moreover, F is recovered in the weak topology of $L^2(I, d\mu_\varphi)$ by

$$F(t) = \frac{2\pi}{C_\Phi^{\oplus_\varphi}} \int_{\mathbb{R}^*} \int_I \mathcal{W}_\Phi^{\oplus_\varphi} F(a,s)\, \Phi_{a,s}^{\oplus_\varphi}(t)\, |\varphi'(s)|\, ds\, \frac{da}{a^2}. \tag{3.5}$$

More precisely, for

$$E_N = \{(a,s) : N^{-1} \le |a| \le N,\ |\varphi(s)| \le N\},$$

the truncated integrals over E_N converge to F weakly in $L^2(I, d\mu_\varphi)$; equivalently, their inner products against every $G \in L^2(I, d\mu_\varphi)$ converge to $\langle F, G\rangle$.

Proof. Let U_φ: $L^2(I, d\mu_\varphi) \longrightarrow L^2(\mathbb{R}, d\xi/(2\pi))$ be the unitary isomorphism defined by

$$(U_\varphi F)(\xi) = F(\psi(\xi)), \quad \psi = \varphi^{-1},$$

and set $\widetilde{F} = U_\varphi F$. By the linearization of the warped-shift atoms,

$$(U_\varphi\, \Phi^{\oplus_\varphi}_{a,s})(\xi) = |a|^{-1/2}\widetilde{\Phi}\Big(\frac{\xi-\eta}{a}\Big), \quad \eta = \varphi(s).$$

Therefore, using the inner product on $L^2(\mathbb{R}, d\xi/(2\pi))$, we obtain

$$\begin{aligned}\mathcal{W}^{\oplus_\varphi}_\Phi F(a,s) &= \left\langle F,\, \Phi^{\oplus_\varphi}_{a,s}\right\rangle_{L^2(I,d\mu_\varphi)}\\ &= \left\langle U_\varphi F,\, U_\varphi \Phi^{\oplus_\varphi}_{a,s}\right\rangle_{L^2(d\xi/(2\pi))}\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \widetilde{F}(\xi)\, \overline{|a|^{-1/2}\widetilde{\Phi}\Big(\frac{\xi-\eta}{a}\Big)}\, d\xi.\end{aligned}$$

This is exactly the ordinary continuous wavelet transform of $\widetilde{F}$ in the variable ξ, with mother wavelet $\widetilde{\Phi}$, scale a, and translation parameter η.

The classical Calderón's Plancherel identity in the ξ variable, with spectral measure $d\xi/(2\pi)$, gives

$$\int_{\mathbb{R}^*}\int_{\mathbb{R}} \left|\mathcal{W}^{\oplus_\varphi}_\Phi F(a,\psi(\eta))\right|^2 d\eta\, \frac{da}{a^2} = \frac{C^{\oplus_\varphi}_\Phi}{2\pi}\left\|\widetilde{F}\right\|^2_{L^2(d\xi/(2\pi))}.$$

Substituting $\eta = \varphi(s)$, $d\eta = |\varphi'(s)|\, ds$, and using the unitarity of U_φ, namely,

$$\left\|\widetilde{F}\right\|_{L^2(d\xi/(2\pi))} = \|F\|_{L^2(I,d\mu_\varphi)},$$

yields the Plancherel identity (3.4).

For reconstruction, the ordinary Calderón formula in the ξ variable states that

$$\widetilde{F}(\xi) = \frac{2\pi}{C^{\oplus_\varphi}_\Phi}\int_{\mathbb{R}^*}\int_{\mathbb{R}} \mathcal{W}^{\oplus_\varphi}_\Phi F(a,\psi(\eta))\, |a|^{-1/2}\widetilde{\Phi}\Big(\frac{\xi-\eta}{a}\Big)\, d\eta\, \frac{da}{a^2}$$

weakly in $L^2(\mathbb{R}, d\xi/(2\pi))$. Applying U_φ^{-1} and substituting

$$\xi = \varphi(t), \quad \eta = \varphi(s), \quad d\eta = |\varphi'(s)|\, ds$$

gives exactly (3.5). □

The factor $2\pi/C^{\oplus_\varphi}_\Phi$ in (3.5) is a consequence of the Fourier convention (1.1), for which the spectral measure is $d\xi/(2\pi)$. There is no discrepancy with the usual constant $1/C^{\oplus_\varphi}_\Phi$: That constant is obtained when the linearized spectral measure is taken to be $d\xi$ instead of $d\xi/(2\pi)$. Equivalently, if one writes the warped measure as $d\mu_\varphi(s) = |\varphi'(s)|ds/(2\pi)$ rather than using the translation measure $|\varphi'(s)|ds$, the Plancherel identity becomes

This paragraph fixes the normalization ambiguity: the translation variable in Theorem 3.1 is integrated with $d\eta = |\varphi'(s)|ds$, whereas the Hilbert space norm uses $d\mu_\varphi(t) = |\varphi'(t)|dt/(2\pi)$.

$$\int_{\mathbb{R}^*}\int_I |\mathcal{W}^{\oplus_\varphi}_\Phi F(a,s)|^2\, d\mu_\varphi(s)\, \frac{da}{a^2} = \frac{C^{\oplus_\varphi}_\Phi}{(2\pi)^2}\|F\|^2_{L^2(I,d\mu_\varphi)}.$$

The preceding theorem lives entirely on the spectral side. Since the warped Fourier transform $\mathcal{F}_\varphi$ is unitary, the same wavelet system can be transported back to physical space. The resulting physical wavelets are not ordinary translations and dilations in the variable x; rather, they are inverse warped Fourier transforms of spectral wavelets adapted to the nonlinear group $(I, \oplus_\varphi)$.

Definition 3.4. *Let $f \in \mathcal{H}_J$ and set $F = \mathcal{F}_\varphi f$. For $a \in \mathbb{R}^*$ and $s \in I$, define*

$$\Psi_{a,s}^{\oplus_\varphi} := \mathcal{F}_\varphi^{-1} \Phi_{a,s}^{\oplus_\varphi}.$$

By the inversion formula (2.4)*, this physical wavelet is given explicitly by*

$$\Psi_{a,s}^{\oplus_\varphi}(x) = \int_I |a|^{-1/2}\, \Phi\left(\varphi^{-1}\left(\frac{\varphi(t) - \varphi(s)}{a}\right)\right) e^{ix\varphi(t)}\, d\mu_\varphi(t). \tag{3.6}$$

The physical warped-shift wavelet transform is

$$\mathcal{W}_\Phi^{\oplus_\varphi} f(a, s) := \left\langle f,\, \Psi_{a,s}^{\oplus_\varphi} \right\rangle_{L^2(\mathbb{R})}.$$

The next result is the physical-space version of the Calderón reconstruction formula. It follows immediately from the spectral formula because $\mathcal{F}_\varphi$ is unitary.

Corollary 3.1. *Under the hypotheses of Theorem 3.1, for every $f \in \mathcal{H}_J$,*

$$f(x) = \frac{2\pi}{C_\Phi^{\oplus_\varphi}} \int_{\mathbb{R}^*} \int_I \mathcal{W}_\Phi^{\oplus_\varphi} f(a, s)\, \Psi_{a,s}^{\oplus_\varphi}(x)\, |\varphi'(s)|\, ds\, \frac{da}{a^2} \tag{3.7}$$

weakly in $L^2(\mathbb{R})$.

Proof. Since $\mathcal{F}_\varphi$ is unitary and $\Psi_{a,s}^{\oplus_\varphi} = \mathcal{F}_\varphi^{-1} \Phi_{a,s}^{\oplus_\varphi}$, we have

$$\mathcal{W}_\Phi^{\oplus_\varphi} f(a, s) = \mathcal{W}_\Phi^{\oplus_\varphi} (\mathcal{F}_\varphi f)(a, s).$$

The result follows by applying $\mathcal{F}_\varphi^{-1}$ to the spectral reconstruction formula (3.5). □

The parameter s shifts the spectral profile in the nonlinear group $(I, \oplus_\varphi)$. It does not translate the physical variable x. A physical translation of f corresponds on the spectral side to multiplication of $\mathcal{F}_\varphi f$ by $e^{ib\varphi(t)}$, whereas the present construction uses the warped translation $t \mapsto t \ominus_\varphi s$. Thus warped-shift wavelets are spectral wavelets adapted to $(I, \oplus_\varphi)$ rather than physical-space wavelets adapted to $(\mathbb{R}, +)$.

We now identify the spectral operator naturally diagonalized by the warped Fourier transform. This operator explains why the variable t has intrinsic meaning: it is not merely a coordinate introduced by the change of variables $\xi = \varphi(t)$, but the eigenvalue of the operator obtained by applying φ^{-1} to the Fourier multiplier D. Here and below, $\varphi^{-1}(D)$ is not meant as an algebraic substitution. It is defined by the Borel functional calculus for the self-adjoint Fourier multiplier $D = -i\partial_x$, with the maximal L^2 domain written explicitly in the next statement.

Proposition 3.1. *Let $m(\xi) = \varphi^{-1}(\xi)$, $\xi \in \mathbb{R}$. Define the Fourier multiplier*

$$A_\varphi := m(D) = \varphi^{-1}(D), \quad \mathrm{Dom}(A_\varphi) = \left\{ f \in L^2(\mathbb{R}) : m(\xi)\widehat{f}(\xi) \in L^2\left(\mathbb{R}, \frac{d\xi}{2\pi}\right) \right\},$$

and set $T_\varphi := iA_\varphi$ on the same domain. Then:

(1) $T_\varphi E_\varphi(x,t) = it\, E_\varphi(x,t)$, so t is the eigenvalue of T_φ associated with the warped exponential.
(2) $\mathcal{F}_\varphi(T_\varphi f)(t) = it\,(\mathcal{F}_\varphi f)(t)$ for all $f \in \mathrm{Dom}(T_\varphi)$.
(3) A_φ is self-adjoint on its natural domain, and T_φ is skew-adjoint.

Proof. Since

$$D\, e^{ix\varphi(t)} = \varphi(t)\, e^{ix\varphi(t)},$$

the spectral calculus gives

$$A_\varphi E_\varphi(x,t) = \varphi^{-1}(\varphi(t))E_\varphi(x,t) = t\, E_\varphi(x,t).$$

Multiplying by i yields

$$T_\varphi E_\varphi(x,t) = it\, E_\varphi(x,t),$$

which proves the first claim. The second claim follows from the same diagonalization identity and the unitarity of $\mathcal{F}_\varphi$. Finally, since φ^{-1} is real-valued, $A_\varphi = \varphi^{-1}(D)$ is a real Fourier multiplier and is therefore self-adjoint on its natural domain. Hence $T_\varphi = iA_\varphi$ is skew-adjoint. □

For comparison, it is useful to record a different construction associated with the same operator. In the warped-shift transform, dilation acts through the automorphisms of the group $(I, \oplus_\varphi)$, namely, through $\delta_a^\varphi(t) = \varphi^{-1}(a\varphi(t))$. In the operator-adapted construction below, dilation acts directly on the eigenvalue t of T_φ. This produces a genuinely different transform unless $\varphi(t) = t$.

Definition 3.5. *Assume $J = \mathbb{R}$. For $a \in \mathbb{R}^*$, $b \in \mathbb{R}$, and $\Phi \in L^2(\mathbb{R}, d\mu_\varphi)$, set*

$$\Phi_{a,b}^{T_\varphi}(t) := |a|^{1/2} e^{-ib\varphi(t)}\Phi(at),$$

and define

$$\psi_{a,b}^{T_\varphi} := \mathcal{F}_\varphi^{-1}\Phi_{a,b}^{T_\varphi}.$$

The operator-adapted warped wavelet transform is

$$W_\Phi^{T_\varphi} f(a,b) := \left\langle f,\, \psi_{a,b}^{T_\varphi}\right\rangle_{L^2(\mathbb{R})}.$$

The following Calderón identity for the operator-adapted transform is stated to clarify the contrast with the warped-shift theorem. Its admissibility condition is expressed directly in the eigenvalue variable t, rather than in the linearized group coordinate $\varphi(t)$.

Proposition 3.2. *If*

$$C_\Phi^{T_\varphi} := \int_{\mathbb{R}} \frac{|\Phi(u)|^2}{|u|}\, du < \infty,$$

then

$$\int_{\mathbb{R}^*}\int_{\mathbb{R}} |W_\Phi^{T_\varphi} f(a,b)|^2 \, \frac{db\, da}{a^2} = C_\Phi^{T_\varphi}\, \|f\|_{L^2(\mathbb{R})}^2.$$

Moreover, f is recovered by the corresponding weak L^2 reconstruction formula with normalization constant $1/C_\Phi^{T_\varphi}$.

Proof. Let $F = \mathcal{F}_\varphi f$. By the unitarity of $\mathcal{F}_\varphi$,

$$W_\Phi^{T_\varphi} f(a,b) = \left\langle F,\, \Phi_{a,b}^{T_\varphi} \right\rangle_{L^2(\mathbb{R},d\mu_\varphi)}.$$

Using

$$\Phi_{a,b}^{T_\varphi}(t) = |a|^{1/2} e^{-ib\varphi(t)} \Phi(at),$$

integration over the b variable gives

$$\int_{\mathbb{R}} |W_\Phi^{T_\varphi} f(a,b)|^2\, db = \int_{\mathbb{R}} |F(t)|^2\, |a|\, |\Phi(at)|^2\, d\mu_\varphi(t).$$

Integrating this identity with respect to da/a^2 and using the substitution $u = at$ yields

$$\int_{\mathbb{R}^*} |a|\, |\Phi(at)|^2 \frac{da}{a^2} = \int_{\mathbb{R}} \frac{|\Phi(u)|^2}{|u|}\, du = C_\Phi^{T_\varphi}.$$

Therefore,

$$\int_{\mathbb{R}^*} \int_{\mathbb{R}} |W_\Phi^{T_\varphi} f(a,b)|^2 \frac{db\, da}{a^2} = C_\Phi^{T_\varphi} \int_{\mathbb{R}} |F(t)|^2\, d\mu_\varphi(t).$$

Since $\mathcal{F}_\varphi$ is unitary, the last integral equals $\|f\|^2_{L^2(\mathbb{R})}$, proving the stated identity. The reconstruction formula follows from the same frame identity by the standard weak L^2 argument. □

The key distinction is therefore structural. The operator-adapted transform dilates the eigenvalue t of T_φ by the ordinary rule $t \mapsto at$, while the warped-shift transform dilates by the group automorphism

$$\delta_a^\varphi(t) = \varphi^{-1}(a\varphi(t)).$$

These two mechanisms agree only in the linear case $\varphi(t) = t$.

From a practical point of view, the warped-shift construction is useful when the design grid is naturally uniform in the linearized coordinate $\xi = \varphi(t)$ but must be represented in the nonlinear spectral variable t. Its atoms therefore move and scale according to the geometry of $(I, \oplus_\varphi)$. The operator-adapted construction is useful when the eigenvalue t itself is the quantity to be scaled. Thus the latter is better viewed as an operator-adapted time-frequency transform, whereas the former is the genuine affine wavelet transform on the nonlinear spectral group.

4. Umbral calculus and nonlinear warpings

This section gives conceptual background for the warped spectral groups used in the paper. The aim is not to reprove the standard results of finite operator calculus, but to explain how the present warped-wavelet construction fits between several related traditions: Rota's finite operator calculus [16], Zeilberger's Fourier-analytic interpretation of umbral calculus [17], Shen's umbral approach to wavelet refinement [18], and the Meixner–Pollaczek central-difference calculus studied by Ismail and Saad [19].

In Rota's finite operator calculus, one studies shift-invariant operators on the polynomial algebra $\mathcal{P} = \mathbb{C}[x]$. If $\partial = \frac{d}{dx}$, then every shift-invariant operator is formally a power series in ∂: $Q = q(\partial)$.

When $q(0) = 0$, $q'(0) \neq 0$, the operator Q is called a delta operator, and the symbol q has a formal compositional inverse $\varphi = q^{-1}$. The corresponding basic polynomial sequence is generated by

$$e^{x\varphi(z)} = \sum_{n=0}^{\infty} p_n(x)\frac{z^n}{n!},$$

and satisfies the lowering relation

$$Qp_n = np_{n-1}.$$

Thus, on the umbral side, the inverse function $\varphi = q^{-1}$ is the coordinate that turns the delta operator $Q = q(\partial)$ into multiplication by the formal variable z.

Zeilberger emphasized that this formal picture can be interpreted through Fourier analysis: Shift-invariant operators correspond to convolution operators, and after Fourier transform, they become multiplication by symbols [17]. This observation is the bridge to warped Fourier analysis. Replacing the formal differentiation operator ∂ by the Fourier multiplier $D = -i\partial_x$ turns the formal symbol q into the spectral operator $q(D)$. The same inverse function $\varphi = q^{-1}$ now defines the warped Fourier kernel

$$E_\varphi(x,t) = e^{ix\varphi(t)}.$$

Since

$$q(D)E_\varphi(x,t) = q(\varphi(t))E_\varphi(x,t) = tE_\varphi(x,t),$$

the variable t is the spectral eigenvalue of $q(D)$. Equivalently, the warped Fourier transform

$$(\mathcal{F}_\varphi f)(t) = \widehat{f}(\varphi(t))$$

diagonalizes $q(D)$:

$$\mathcal{F}_\varphi(q(D)f)(t) = t(\mathcal{F}_\varphi f)(t).$$

Therefore the same function $\varphi = q^{-1}$ appears in two parallel forms:

$$e^{x\varphi(z)} \quad \text{in umbral calculus}$$

and

$$e^{ix\varphi(t)} \quad \text{in warped Fourier analysis.}$$

The first generates polynomial sequences lowered by $q(\partial)$; the second diagonalizes the Fourier multiplier $q(D)$. The correspondence is,

Umbral calculus	Warped spectral analysis
$Q = q(\partial)$	$q(D)$
$\varphi = q^{-1}$	$\varphi = q^{-1}$
$e^{x\varphi(z)}$	$e^{ix\varphi(t)}$
$Qp_n = np_{n-1}$	$q(D)E_\varphi(x,t) = tE_\varphi(x,t)$
z formal variable	t spectral eigenvalue
e^{yQ} formal Taylor shift	$\mathsf{T}^{\varphi}_{\varphi^{-1}(y)}$ warped group shift

On the umbral side, this dictionary is, in general, a formal power-series dictionary; analytic convergence is asserted only on domains where the corresponding series converge. On the warped Fourier side, the identities are interpreted through the spectral calculus of the Fourier multiplier $q(D)$.

This dictionary also explains the nonlinear spectral group law. Since $t = q(\xi)$ and $\xi = \varphi(t)$, ordinary addition in the Fourier variable ξ is transported to the warped variable t by

$$t \oplus_\varphi s = \varphi^{-1}(\varphi(t) + \varphi(s)).$$

Thus the group law used in the warped-shift wavelet transform is not imposed externally. It is the additive Fourier group rewritten in the spectral coordinate of the operator $q(D)$. This is the main structural point: the symbol q determines an operator, the inverse coordinate $\varphi = q^{-1}$, a warped Fourier kernel, and a nonlinear spectral group.

This viewpoint also clarifies the relation with Shen's work on wavelets and umbral calculus. Shen showed that the refinement equation of wavelet theory can be transformed into an umbral refinement equation by encoding the moments of the scaling function and the filter through umbrae [18]. In the classical refinement equation

$$\phi(x) = 2 \sum_{n=0}^{L} h_n \phi(2x - n),$$

the affine maps $x \mapsto 2x - n$ describe the dyadic self-similarity of the scaling function. Shen passes to the moment sequence of ϕ and encodes it by a scaling umbra α. The filter coefficients $h_0, h_1, \ldots, h_L$ are encoded by a filter umbra γ, whose moments are

$$\gamma^k \sim \sum_{n=0}^{L} h_n n^k.$$

Using the Cauchy generating function, Shen obtains the umbral refinement equation

$$\alpha \equiv \frac{\alpha + \gamma}{2}.$$

Thus the refinement equation becomes a symbolic fixed-point equation for moments.

When the filter coefficients satisfy

$$h_n \geq 0, \quad \sum_{n=0}^{L} h_n = 1,$$

the filter also has a probabilistic interpretation. Let Γ be the discrete random variable defined by

$$\mathbb{P}(\Gamma = n) = h_n, \quad n = 0, \ldots, L.$$

Then the filter umbra represents the moments of Γ:

$$\gamma^k \sim \mathbb{E}[\Gamma^k].$$

In this case, Shen's umbral refinement equation is the moment form of the distributional fixed-point equation

$$X \stackrel{d}{=} \frac{X' + \Gamma}{2},$$

where X' is an independent copy of X, independent of Γ. Thus the scaling object is decomposable into a contracted copy of itself plus a new random noise term determined by the filter. Iterating gives

$$X \overset{d}{=} \frac{\Gamma_1}{2} + \frac{\Gamma_2}{4} + \frac{\Gamma_3}{8} + \cdots ,$$

where the Γ_j's are independent copies of Γ. This is the probabilistic analogue of Shen's infinite umbral sum for the scaling umbra.

The same random refinement process can be transported to the warped spectral setting.

Let $T = \varphi^{-1}(X)$ and $X = \varphi(T)$, then,

$$X \overset{d}{=} \frac{X' + \Gamma}{2}$$

becomes

$$\varphi(T) \overset{d}{=} \frac{\varphi(T') + \Gamma}{2},$$

or equivalently,

$$T \overset{d}{=} \varphi^{-1}\left(\frac{\varphi(T') + \Gamma}{2}\right).$$

If $S = \varphi^{-1}(\Gamma)$, this can be written intrinsically on the nonlinear spectral group as

$$T \overset{d}{=} \delta^{\varphi}_{1/2}\left(T' \oplus_{\varphi} S\right), \quad \delta^{\varphi}_{a}(t) = \varphi^{-1}(a\varphi(t)).$$

Thus the ordinary random refinement process becomes a random affine process on the warped spectral group $(I, \oplus_{\varphi})$. Equivalently,

$$T \overset{d}{=} \varphi^{-1}\left(\sum_{j=1}^{\infty} \frac{\Gamma_j}{2^j}\right),$$

or, if $S_j = \varphi^{-1}(\Gamma_j)$,

$$T \overset{d}{=} \bigoplus_{j=1}^{\infty} \delta^{\varphi}_{2^{-j}}(S_j).$$

The infinite warped sum is defined as the limit in the group topology transported by φ; equivalently,

$$\varphi\left(\bigoplus_{j=1}^{N} \delta^{\varphi}_{2^{-j}}(S_j)\right) = \sum_{j=1}^{N} 2^{-j}\Gamma_j.$$

Since the refinement filter is finite, the real series on the right converges absolutely almost surely, and hence the warped sum is well-defined.

The infinite symbol $\oplus_{j=1}^{\infty}$ is therefore a shorthand for this limit and is not an additional operation assumed without proof. Therefore Shen's umbral refinement process has a warped counterpart: The scaling variable is obtained by adding infinitely many randomly shifted spectral digits, but the addition and dilation are those of the nonlinear spectral group.

For example, for the Haar filter

$$h_0 = h_1 = \frac{1}{2},$$

the noise variable Γ is Bernoulli:

$$\mathbb{P}(\Gamma = 0) = \mathbb{P}(\Gamma = 1) = \frac{1}{2}.$$

Then

$$X = \sum_{j=1}^{\infty} \frac{\Gamma_j}{2^j}$$

is uniformly distributed on $[0, 1]$. The corresponding warped random variable is $T = \varphi^{-1}(X)$.

If $\varphi(t) = \log t$, for $t > 0$, then $T = e^X$, and T has density $\frac{1}{t}\mathbf{1}_{[1,e]}(t)$. Here the support is $[1, e]$ because X is uniform on $[0, 1]$ and $T = e^X$.

The warped random refinement equation becomes

$$T \stackrel{d}{=} \exp\left(\frac{\log T' + \Gamma}{2}\right) = \sqrt{T' e^{\Gamma}}.$$

Thus the Haar binary refinement process becomes a multiplicative random process on the logarithmic warped group.

A particularly important example for the present paper is the hyperbolic-sine symbol

$$q(\xi) = 2 \sinh \frac{\xi}{2}.$$

Its inverse is

$$\varphi(t) = q^{-1}(t) = 2 \operatorname{arcsinh} \frac{t}{2}.$$

The corresponding warped Fourier kernel is

$$E_\varphi(x, t) = \exp\left(2ix \operatorname{arcsinh} \frac{t}{2}\right),$$

and it satisfies

$$q(D)E_\varphi(x, t) = tE_\varphi(x, t).$$

Equivalently, if

$$T_\varphi = iq(D) = i\, 2 \sinh \frac{D}{2},$$

then

$$T_\varphi E_\varphi(x, t) = itE_\varphi(x, t).$$

Since

$$2 \sinh \frac{D}{2} = e^{D/2} - e^{-D/2}$$

and $D = -i\partial_x$, this operator is formally the imaginary-step central difference

$$(T_\varphi f)(x) = \frac{f(x + i/2) - f(x - i/2)}{i}.$$

On the Paley–Wiener core

$$\mathcal{PW} = \bigcup_{R>0} \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [-R, R] \right\},$$

the expression is interpreted through the entire extension supplied by the compact Fourier support. This core is invariant under the multiplier and is dense in $L^2(\mathbb{R})$. On $L^2(\mathbb{R})$, the closed realization is not defined by pointwise analytic continuation but by the Fourier multiplier

$$\widehat{T_\varphi f}(\xi) = 2i\sinh\Big(\frac{\xi}{2}\Big)\widehat{f}(\xi).$$

This is exactly the central-difference structure appearing in the Meixner–Pollaczek calculus. For $0 < \theta < \pi$ and $\lambda > 0$, the Meixner–Pollaczek polynomials may be defined by

$$P_n^{(\lambda)}(x;\theta) = \frac{(2\lambda)_n}{n!} e^{in\theta} {}_2F_1\left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \,\middle|\, 1 - e^{-2i\theta}\right).$$

Equivalently, with the normalization used here, they have the generating function

$$\sum_{n=0}^{\infty} P_n^{(\lambda)}(x;\theta)t^n = (1 - te^{i\theta})^{-(\lambda - ix)}(1 - te^{-i\theta})^{-(\lambda + ix)}, \qquad |t| < 1.$$

The factor $(2\lambda)_n$ appears in the hypergeometric definition above. Therefore it is already contained in $P_n^{(\lambda)}(x;\theta)$ and should not be repeated in this generating-function normalization. If another normalization is used, the generating function must be modified accordingly.

Ismail and Saad studied the central-difference operator

$$(Tf)(x) = \frac{f(x + i/2) - f(x - i/2)}{i}$$

in connection with these polynomials [19]. They also introduced the polynomial basis

$$\phi_n^{(\lambda)}(x) = \left(\lambda + \frac{1-n}{2} + ix\right)_n, \quad n \geq 0,$$

which serves for T-calculus the role played by monomials x^n in ordinary differential calculus. In particular,

$$T\phi_n^{(\lambda)} = in\,\phi_{n-1}^{(\lambda)},$$

and more generally

$$T^k\phi_n^{(\lambda)} = (-i)^k(-n)_k\phi_{n-k}^{(\lambda)}.$$

For the Meixner–Pollaczek polynomials themselves, the same operator gives the lowering relation

$$TP_n^{(\lambda)}(x;\theta) = 2\sin\theta\, P_{n-1}^{(\lambda+1/2)}(x;\theta),$$

and, more generally,

$$T^k P_n^{(\lambda)}(x;\theta) = (2\sin\theta)^k P_{n-k}^{(\lambda+k/2)}(x;\theta).$$

Finally, Ismail and Saad identified the corresponding T-exponential as

$$E(x,t) = \left(\frac{t}{2} + \sqrt{1 + \frac{t^2}{4}}\right)^{2ix} = \exp\Big(2ix \operatorname{arcsinh}\frac{t}{2}\Big).$$

This is precisely the arcsinh-warped Fourier kernel. Therefore the Meixner–Pollaczek central-difference calculus, the arcsinh warped Fourier transform, and the nonlinear spectral group are different manifestations of the same structure.

For the arcsinh warping, the induced nonlinear spectral group is

$$t \oplus_\varphi s = t\sqrt{1+\frac{s^2}{4}} + s\sqrt{1+\frac{t^2}{4}}.$$

Thus the arcsinh example connects the central-difference operator, Meixner–Pollaczek polynomial calculus, the warped Fourier kernel

$$\exp\Big(2ix \operatorname{arcsinh}\frac{t}{2}\Big)$$

and the nonlinear spectral group $(I, \oplus_\varphi)$. The warped-shift wavelets associated with this φ are therefore wavelets on the nonlinear spectral group naturally determined by the central-difference operator.

The arcsinh example is only one instance of a general principle. Any symbol q with a local inverse $\varphi = q^{-1}$ gives both an umbral calculus and a warped spectral geometry.

If $q(\xi) = \xi$, for $\varphi(t) = t$, then $q(D) = D$, the kernel is e^{ixt}, and the group law is ordinary addition. This is the classical wavelet case.

If $q(\xi) = e^{\xi}$, $\varphi(t) = \log t$, for $t > 0$, then $E_\varphi(x, t) = t^{ix}$ and the group law becomes multiplicative:

$$t \oplus_\varphi s = ts.$$

This gives a Mellin-type spectral geometry.

If instead $q(\xi) = e^{\xi} - 1$, $\varphi(t) = \log(1 + t)$, then $q(D) = e^D - I$, and the induced group law is

$$t \oplus_\varphi s = t + s + ts.$$

On the umbral side, this is the forward-difference symbol, whose basic polynomials are the falling factorials.

If $q(\xi) = 2\sin\frac{\xi}{2}$, then, on the principal interval $\xi \in (-\pi, \pi)$, $\varphi(t) = 2\arcsin\frac{t}{2}$. This gives the real-centered finite-difference symbol on a bounded Fourier band. The corresponding group law is

$$t \oplus_\varphi s = t\sqrt{1-\frac{s^2}{4}} + s\sqrt{1-\frac{t^2}{4}}$$

on the branch where the expression is defined. Thus the sine warping is the bounded analogue of the hyperbolic-sine warping.

If $q(\xi) = 2\tan\frac{\xi}{2}$, for $\xi \in (-\pi, \pi)$, then $\varphi(t) = 2\arctan\frac{t}{2}$. This warping maps an unbounded spectral variable t to a bounded Fourier band.

Finally, if $q(\xi) = \tanh\xi$, then

$$\varphi(t) = \operatorname{artanh} t = \frac{1}{2}\log\frac{1+t}{1-t}, \quad |t| < 1,$$

and the group law becomes

$$t \oplus_\varphi s = \frac{t+s}{1+ts}.$$

This is the familiar relativistic addition law, now interpreted as ordinary addition in ξ-space transported through the bounded symbol $q(\xi) = \tanh \xi$.

These examples show that warped spectral groups are not arbitrary nonlinear groups. They are the spectral geometries naturally attached to symbols of shift-invariant operators. In the umbral setting, the same symbols generate polynomial sequences. In the warped Fourier setting, they generate kernels, spectral measures, nonlinear group laws, and wavelet geometries.

The connection with Shen's refinement equation also suggests a possible discrete multi-resolution theory. Such a multi-resolution analysis would require, in addition to the continuous Calderón theory developed here, a compatible lattice or discrete subgroup in the linearized coordinate, stable refinement masks, and orthogonality or frame conditions after transport by φ. These issues are not automatic consequences of the present continuous theory and will be treated separately.

5. Conclusions

In this paper, we developed a continuous wavelet theory adapted to nonlinear spectral coordinates. Starting from a real C^1-diffeomorphism $\varphi\colon I \to \mathbb{R}$, we introduced the warped Fourier transform

$$(\mathcal{F}_\varphi f)(t) = \widehat{f}(\varphi(t))$$

and identified its natural Plancherel measure as

$$d\mu_\varphi(t) = \frac{|\varphi'(t)|}{2\pi}\, dt.$$

This choice makes $\mathcal{F}_\varphi$ unitary and realizes the variable t as the spectral coordinate of the warped operator $T_\varphi = i\varphi^{-1}(D)$. The warping map transports ordinary addition on the Fourier line to the nonlinear group law

$$t \oplus_\varphi s = \varphi^{-1}(\varphi(t) + \varphi(s)).$$

Within this group structure, we defined warped translations and automorphic dilations and showed that they form a unitary affine-type representation. The associated continuous warped-shift wavelet transform satisfies a Calderón admissibility condition, an exact Plancherel identity, and a reconstruction formula. Although the construction is unitarily equivalent to the classical continuous wavelet transform in the linearized variable $\xi = \varphi(t)$, its formulation in the original variable t reveals the nonlinear spectral geometry determined by the operator under consideration. A further contribution of the paper is the connection between this wavelet framework and finite operator calculus. When $\varphi = q^{-1}$ for the symbol q of a shift-invariant operator, the same function q simultaneously determines the delta operator, its umbral polynomial sequence, the warped Fourier kernel, the spectral measure, and the nonlinear group law. The hyperbolic-sine example illustrates this correspondence particularly clearly: the arcsinh-warped Fourier kernel, the central-difference operator, and the Meixner–Pollaczek polynomial calculus arise from a common spectral structure. The logarithmic, exponential, sine, tangent, hyperbolic-tangent, and logit examples demonstrate that the construction also encompasses multiplicative, bounded-band, and relativistic-type spectral geometries. The present results provide the continuous harmonic-analytic foundation for wavelet analysis on warped spectral groups. Several natural extensions remain open. These include the construction of discrete warped wavelet frames and multiresolution analyses, the identification of compatible lattices and refinement masks, the study of stability under perturbations of the warping map, and the development of numerical algorithms for nonlinear spectral decomposition. Another promising direction is to apply the theory to difference and differential-difference operators whose symbols generate classical or nonclassical families of special functions. Such developments would further clarify how operator symbols, nonlinear spectral geometry, umbral calculus, and multiscale analysis interact within a unified framework.

Author contributions

Fethi Bouzeffour: wrote original versions of Sections 1 and 2, contributed to the conceptualization, methodology, formal analysis, writing, editing, and critical revision of the manuscript; Mhamed Eddahbi: wrote original versions of Sections 3 and 4, contributed to the conceptualization, methodology, formal analysis, writing, editing, and critical revision of the manuscript. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The second-named author gratefully acknowledges the support provided by the Ongoing Research Funding Program (ORF-2026-1075), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare no conflicts of interest.

References

1. A. P. Calderón, Intermediate spaces and interpolation, the complex method, *Stud. Math.*, **24** (1964), 113–190. https://doi.org/10.4064/SM-24-2-113-190
2. J. Morlet, G. Arens, E. Fourgeau, D. Giard, Wave propagation and sampling theory; Part I, complex signal and scattering in multilayered media, *Geophysics*, **47** (1982), 203–221. https://doi.org/10.1190/1.1441328
3. A. Grossmann, J. Morlet, Decomposition of Hardy functions into square integrable wavelets of constant shape, *SIAM J. Math. Anal.*, **15** (1984), 723–736. https://doi.org/10.1137/0515056
4. A. Grossmann, J. Morlet, T. Paul, Transforms associated to square integrable group representations II: examples, *Ann. Inst. Henri Poincaré*, **45** (1986), 293–309.
5. I. Daubechies, Orthonormal bases of compactly supported wavelets, *Commun. Pure Appl. Math.*, **41** (1988), 909–996. https://doi.org/10.1002/cpa.3160410705

6. I. Daubechies, *Ten lectures on wavelets*, Philadelphia, 1992. https://doi.org/10.1137/1.9781611970104
7. S. Mallat, A theory for multiresolution signal decomposition: the wavelet representation, *IEEE Trans. Pattern Anal. Mach. Intell.*, **11** (1989), 674–693. https://doi.org/10.1109/34.192463
8. S. Mallat, *A wavelet tour of signal processing*, 2 Eds., Academic Press, 1999.
9. P. P. Vaidyanathan, *Multirate systems and Filter banks*, Prentice Hall, 1993.
10. M. Vetterli, J. Kovačević, *Wavelets and subband coding*, Prentice Hall, 1995.
11. A. V. Oppenheim, D. H. Johnson, K. Steiglitz, Computation of spectra with unequal resolution using the fast Fourier transform, *Proc. IEEE*, **59** (1971), 299–301. https://doi.org/10.1109/PROC.1971.8146
12. C. Braccini, A. V. Oppenheim, Unequal bandwidth spectral analysis using digital frequency warping, *IEEE Trans. Acoust. Speech Signal Process.*, **22** (1974), 236–244. https://doi.org/10.1109/TASSP.1974.1162582
13. R. G. Baraniuk, D. L. Jones, Unitary equivalence: a new twist on signal processing, *IEEE Trans. Signal Process.*, **43** (1995), 2269–2282. https://doi.org/10.1109/78.469861
14. G. Evangelista, S. Cavaliere, Frequency-warped filter banks and wavelet transforms: a discrete-time approach via Laguerre expansion, *IEEE Trans. Signal Process.*, **46** (1998), 2638–2650. https://doi.org/10.1109/78.720367
15. G. Evangelista, Dyadic warped wavelets, *Adv. Imaging Electron. Phys.*, **117** (2001), 73–170. https://doi.org/10.1016/S1076-5670(01)80112-9
16. G. C. Rota, D. Kahaner, A. Odlyzko, Finite operator calculus, *J. Math. Anal. Appl.*, **42** (1973), 684–760.
17. D. Zeilberger, Some comments on Rota's umbral calculus, *J. Math. Anal. Appl.*, **74** (1980), 456–463. https://doi.org/10.1016/0022-247X(80)90141-9
18. J. Shen, Combinatorics for wavelets: the umbral refinement equation, *Stud. Appl. Math.*, **103** (1999), 121–147. https://doi.org/10.1111/1467-9590.00123
19. M. E. H. Ismail, N. Saad, On the Meixner–Pollaczek polynomials and the Sturm–Liouville problems, *J. Math. Anal. Appl.*, **552** (2025), 129794. https://doi.org/10.1016/j.jmaa.2025.129794