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Research article

Nonlinear $*$-commuting maps on rings with involution

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Abstract: Let $\mathcal{R}$ be a $*$-ring with the unit I. Assume that $\mathcal{R}$ contains a symmetric idempotent P satisfying $A\mathcal{R}P = 0$ implies $A = 0$ and $A\mathcal{R}(I - P) = 0$ implies $A = 0$. We prove that if $\varphi : \mathcal{R} \to \mathcal{R}$ is a nonlinear $*$-commuting map, then there exist $Z \in \mathcal{Z}(\mathcal{R})$ and a map $h : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ such that $\varphi(A) = ZA^* + h(A)$ for all $A \in \mathcal{R}$.

Keywords: $*$-commuting map; $*$-ring; $*$-Lie product; prime $*-$ring; factor von Neumann algebra
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1. Introduction

Let $\mathcal{R}$ be a ring. A map φ: $\mathcal{R} \to \mathcal{R}$ is called commuting if $\varphi(A)A = A\varphi(A)$ for all $A \in \mathcal{R}$. For $A, B \in \mathcal{R}$, denote by $[A, B] = AB - BA$ the Lie product. Evidently, the commuting maps can also be written as $[\varphi(A), A] = 0$ for all $A \in \mathcal{R}$ by Lie product. If φ is additive, then for any $A, B \in \mathcal{R}$, replacing A by $A + B$ in the above equation yields that $[\varphi(A), B] = [A, \varphi(B)]$ for all $A, B \in \mathcal{R}$. The commuting maps are closely related to the theory of functional identities. When treating commuting maps, the main goal is to characterize its concrete form. The first important result on commuting maps is Posner's theorem [1]. Brešar [2] proved that the form of additive commuting maps φ on a simple unital ring $\mathcal{R}$ is $\varphi(A) = ZA + f(A)$ for some $Z \in \mathcal{Z}(\mathcal{R})$ and additive map $f : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$, where $\mathcal{Z}(\mathcal{R})$ is the center of $\mathcal{R}$. We also refer the reader to the well-written paper [3] and the book [4]. Recently, Bounds [5] characterized commuting maps over the ring of strictly upper triangular matrices. Zhang and Mosić [6] investigated the representations of a special commuting map of matrix rings. Jia and Xiao [7] studied commuting maps on certain incidence algebras. Ko and Liu [8] studied commuting maps on strictly upper triangular matrix rings. Furthermore, without the additivity assumption, a map $\varphi : \mathcal{R} \to \mathcal{R}$ is called nonlinear commuting if it satisfies only $[\varphi(A), B] = [A, \varphi(B)]$ for all $A, B \in \mathcal{R}$. Liu and Chen [9] gave the form of nonlinear commuting maps on triangular n-matrix rings.

Recall that a ring $\mathcal{R}$ is called a ring with involution or a $*$-ring if there exists an additive map $*:\mathcal{R}\to\mathcal{R}$ such that $(AB)^*=B^*A^*$ and $(A^*)^*=A$ for all $A,B\in\mathcal{R}$. For $A,B\in\mathcal{R}$, the $*$-Lie product of A and B is defined by $[A,B]_\bullet=A^*B-BA^*$. For literature on $*$-Lie product, one can refer to [10,11]. Inspired by the work on commuting maps, it is natural to consider $*$-commuting maps. An additive map $\varphi:\mathcal{R}\to\mathcal{R}$ is called $*$-commuting if $[\varphi(A),A]_\bullet=0$ for all $A\in\mathcal{R}$. Replacing A with $A+B$ in the above equation implies $[\varphi(A),B]_\bullet=-[\varphi(B),A]_\bullet$ for all $A,B\in\mathcal{R}$. Furthermore, without the additivity assumption, a map $\varphi:\mathcal{R}\to\mathcal{R}$ is called nonlinear $*$-commuting if it satisfies only

$$[\varphi(A),B]_\bullet=-[\varphi(B),A]_\bullet$$

for all $A,B\in\mathcal{R}$.

Let $\mathcal{Z}(\mathcal{R})$ be the center of $\mathcal{R}$. In this paper, we will characterize the concrete form of nonlinear $*$-commuting maps on rings with involution. Moreover, we apply the above result to prime $*$-rings, factor von Neumann algebras and standard operator algebras.

2. Main result

In this section, we will prove the following theorem.

Theorem 2.1. *Let $\mathcal{R}$ be a $*$-ring with unit I. Assume that $\mathcal{R}$ contains a symmetric idempotent P satisfying: (Q_1) $A\mathcal{R}P=0$ implies $A=0$, (Q_2) $A\mathcal{R}(I-P)=0$ implies $A=0$. If a map $\varphi:\mathcal{R}\to\mathcal{R}$ satisfies*

$$[\varphi(A),B]_\bullet=-[\varphi(B),A]_\bullet$$

for all $A,B\in\mathcal{R}$, then there exist $Z\in\mathcal{Z}(\mathcal{R})$ and a map $h:\mathcal{R}\to\mathcal{Z}(\mathcal{R})$ such that

$$\varphi(A)=ZA^*+h(A)$$

for all $A\in\mathcal{R}$.

It is clear that $P\neq 0$ and $P\neq I$. Write $P_1=P$ and $P_2=I-P_1$, $\mathcal{R}_{ij}=P_i\mathcal{R}P_j$, $i,j=1,2$. Then $\mathcal{R}=\mathcal{R}_{11}+\mathcal{R}_{12}+\mathcal{R}_{21}+\mathcal{R}_{22}$. For any $A\in\mathcal{R}$, we have $A=A_{11}+A_{12}+A_{21}+A_{22}$, where $A_{ij}\in\mathcal{R}_{ij}$ $(i,j=1,2)$. To complete the proof of Theorem 2.1, the following lemmas are required.

Lemma 2.1. *Let $A\in\mathcal{R}$. If $[A,T]_\bullet=0$ for all $T\in\mathcal{R}$, then $A\in\mathcal{Z}(\mathcal{R})$.*

Proof. Since $[A,T]_\bullet=0$ for all $T\in\mathcal{R}$, we have

$$A^*T=TA^*$$

for all $T\in\mathcal{R}$, which implies $T^*A=AT^*$. Replacing T^* by T in $T^*A=AT^*$, it follows that $TA=AT$ for all $T\in\mathcal{R}$. Therefore, $A\in\mathcal{Z}(\mathcal{R})$.

Lemma 2.2. *For every $A,B\in\mathcal{R}$, we have*

$$\varphi(A+B)-\varphi(A)-\varphi(B)\in\mathcal{Z}(\mathcal{R}).$$

Proof. Let $A,B\in\mathcal{R}$. For every $T\in\mathcal{R}$, it follows that

$$\begin{aligned}[\varphi(A+B)-\varphi(A)-\varphi(B),T]_\bullet &= [\varphi(A+B),T]_\bullet-[\varphi(A),T]_\bullet-[\varphi(B),T]_\bullet\\ &= -[\varphi(T),A+B]_\bullet+[\varphi(T),A]_\bullet+[\varphi(T),B]_\bullet\\ &= 0.\end{aligned}$$

Hence $\varphi(A+B)-\varphi(A)-\varphi(B)\in\mathcal{Z}(\mathcal{R})$ by Lemma 2.1.
Lemma 2.3. *There exist elements* $Z_1, Z_2 \in \mathcal{Z}(\mathcal{R})$ *such that*

$$\varphi(P_1) = Z_1P_1 + Z_2I.$$

Proof. For every $A\in\mathcal{R}$, it is easy to check that

$$[[[\varphi(A), P_1]_\bullet, P_1]_\bullet, P_1]_\bullet = -[\varphi(A), P_1]_\bullet.$$

It follows that

$$[[[-\varphi(P_1), A]_\bullet, P_1]_\bullet, P_1]_\bullet = [\varphi(P_1), A]_\bullet.$$

Since

$$\begin{aligned}[-[\varphi(P_1), A]_\bullet, P_1]_\bullet &= [-\varphi(P_1)^*A + A\varphi(P_1)^*, P_1]_\bullet\\ &= -A^*\varphi(P_1)P_1 + \varphi(P_1)A^*P_1 + P_1A^*\varphi(P_1) - P_1\varphi(P_1)A^*.\end{aligned}$$

Hence,

$$\begin{aligned}[[-[\varphi(P_1), A]_\bullet, P_1]_\bullet, P_1]_\bullet &= -P_1\varphi(P_1)^*AP_1 + P_1A\varphi(P_1)^*P_1 + \varphi(P_1)^*AP_1 - A\varphi(P_1)^*P_1\\ &\quad + P_1\varphi(P_1)^*A - P_1A\varphi(P_1)^* - P_1\varphi(P_1)^*AP_1 + P_1A\varphi(P_1)^*P_1\\ &= -2P_1\varphi(P_1)^*AP_1 + 2P_1A\varphi(P_1)^*P_1 + \varphi(P_1)^*AP_1 - A\varphi(P_1)^*P_1\\ &\quad + P_1\varphi(P_1)^*A - P_1A\varphi(P_1)^*.\end{aligned}$$

For every $A\in\mathcal{R}$, we have

$$\begin{aligned}&-2P_1\varphi(P_1)^*AP_1 + 2P_1A\varphi(P_1)^*P_1 + \varphi(P_1)^*AP_1 - A\varphi(P_1)^*P_1 + P_1\varphi(P_1)^*A - P_1A\varphi(P_1)^*\\ =&\varphi(P_1)^*A - A\varphi(P_1)^*. \end{aligned}\tag{2.1}$$

Taking $A = A_{12}$ in Eq (2.1), we get

$$2A_{12}\varphi(P_1)^*P_1 - A_{12}\varphi(P_1)^*P_1 + P_1\varphi(P_1)^*A_{12} - A_{12}\varphi(P_1)^* = \varphi(P_1)^*A_{12} - A_{12}\varphi(P_1)^*. \tag{2.2}$$

Multiplying Eq (2.2) by P_2 from the left, we obtain that $P_2\varphi(P_1)^*A_{12} = 0$ for all $A_{12}\in\mathcal{R}_{12}$. It follows from the condition (Q_2) of Theorem 2.1 that $P_2\varphi(P_1)^*P_1 = 0$, then

$$P_1\varphi(P_1)P_2 = 0. \tag{2.3}$$

Taking $A = A_{21}$ in Eq (2.1), we have

$$-P_1\varphi(P_1)^*A_{21} - A_{21}\varphi(P_1)^*P_1 = -A_{21}\varphi(P_1)^*. \tag{2.4}$$

Multiplying Eq (2.2) by P_1 from both sides, we have $P_1\varphi(P_1)^*A_{21} = 0$. It follows from the condition (Q_1) of Theorem 2.1 that

$$P_2\varphi(P_1)P_1 = 0. \tag{2.5}$$

Taking $A = A_{11}$ in Eq (2.1), we have

$$P_1\varphi(P_1)^*A_{11} = A_{11}\varphi(P_1)^*P_1. \tag{2.6}$$

It follows from Lemma 4, [12] that there exists $C_1^* \in \mathcal{Z}(\mathcal{R})$ such that $P_1\varphi(P_1)^*P_1 = C_1^*P_1$. This implies that

$$P_1\varphi(P_1)P_1 = C_1P_1, \tag{2.7}$$

where $C_1 \in \mathcal{Z}(\mathcal{R})$.

Taking $A = A_{22}$ in Eq (2.1), we have

$$-A_{22}\varphi(P_1)^*P_1 + P_1\varphi(P_1)^*A_{22} = \varphi(P_1)^*A_{22} - A_{22}\varphi(P_1)^*. \tag{2.8}$$

Multiplying Eq (2.8) by P_2 from both sides, we have

$$P_2\varphi(P_1)^*A_{22} = A_{22}\varphi(P_1)^*P_2$$

for all $A_{22} \in \mathcal{R}_{22}$. It follows from Lemma 4, [12] that there exists $C_2 \in \mathcal{Z}(\mathcal{R})$, such that

$$P_2\varphi(P_1)P_2 = C_2P_2. \tag{2.9}$$

Hence,

$$\begin{aligned}\varphi(P_1) &= P_1\varphi(P_1)P_1 + P_1\varphi(P_1)P_2 + P_2\varphi(P_1)P_1 + P_2\varphi(P_1)P_2\\ &= C_1P_1 + C_2P_2\\ &= Z_1P_1 + Z_2I,\end{aligned}$$

where $Z_1 = C_1 - C_2$, $Z_2 = C_2$ and $Z_1, Z_2 \in \mathcal{Z}(\mathcal{R})$.

Lemma 2.4. *For every $A_{ij} \in \mathcal{R}_{ij}$ $(1 \le i \ne j \le 2)$, there exists $Z_{A_{ij}} \in \mathcal{Z}(\mathcal{R})$ such that*

$$\varphi(A_{ij}) = Z_1A_{ij}^* + Z_{A_{ij}}.$$

Proof. Let $A_{12} \in \mathcal{R}_{12}$. Write $\varphi(A_{12}) = U_{11}+U_{12}+U_{21}+U_{22}$. It follows from the nonlinear $*-$commuting condition and Lemma 2.3 that

$$\begin{aligned}\varphi(A_{12})^*P_1 - P_1\varphi(A_{12})^* &= [\varphi(A_{12}), P_1]_\bullet\\ &= -[\varphi(P_1), A_{12}]_\bullet\\ &= -[Z_1P_1 + Z_2I, A_{12}]_\bullet\\ &= -Z_1^*A_{12}.\end{aligned} \tag{2.10}$$

Multiplying Eq (2.10) by P_2 from the left and by P_1 from the right, we have $P_2\varphi(A_{12})^*P_1 = 0$, which implies that $U_{12} = P_1\varphi(A_{12})P_2 = 0$. Therefore,

$$\varphi(A_{12}) = U_{11} + U_{21} + U_{22}. \tag{2.11}$$

For any $B \in \mathcal{R}$, let $\varphi(B) = W_{11} + W_{12} + W_{21} + W_{22}$. Since $[\varphi(B), A_{12}]_\bullet = -[\varphi(A_{12}), B]_\bullet$ and by Eq (2.11), we can obtain

$$\varphi(B)^* A_{12} - A_{12}\varphi(B)^* = -[U_{11} + U_{21} + U_{22}, B]_\bullet.$$

$$W_{11}^* A_{12} + W_{12}^* A_{12} - A_{12} W_{12}^* - A_{12} W_{22}^* = -U_{11}^* B - U_{21}^* B - U_{22}^* B + BU_{11}^* + BU_{21}^* + BU_{22}^*. \quad (2.12)$$

Multiplying Eq (2.12) by P_1 from the right, we get

$$-A_{12} W_{12}^* = -U_{11}^* BP_1 - U_{21}^* BP_1 - U_{22}^* BP_1 + BU_{11}^*. \quad (2.13)$$

For any $B_{21} \in \mathcal{R}_{21}$, replacing B by B_{21} in Eq (2.13), we have

$$-A_{12} W_{12}^* = -U_{21}^* B_{21} - U_{22}^* B_{21} + B_{21} U_{11}^*. \quad (2.14)$$

Multiplying Eq (2.14) by P_2 from the left, we get $0 = -U_{22}^* B_{21} + B_{21} U_{11}^*$, so

$$U_{22}^* B_{21} = B_{21} U_{11}^*. \quad (2.15)$$

It follows from Lemma 1.6, [13] that there exists $Z_{A_{12}}^* \in \mathcal{Z}(\mathcal{R})$ such that $U_{22}^* + U_{11}^* = Z_{A_{12}}^*$, and so $U_{22} + U_{11} = Z_{A_{12}} \in \mathcal{Z}(\mathcal{R})$. Multiplying Eq (2.10) by P_2 from the right,

$$U_{21} = P_2 \varphi(A_{12}) P_1 = Z_1 A_{12}^*.$$

Therefore,

$$\varphi(A_{12}) = Z_1 A_{12}^* + Z_{A_{12}}. \quad (2.16)$$

Let $A_{21} \in \mathcal{R}_{21}$ and write $\varphi(A_{21}) = S_{11} + S_{12} + S_{21} + S_{22}$. It follows from Lemma 2.3 that

$$\varphi(A_{21})^* P_1 - P_1 \varphi(A_{21})^* = [\varphi(A_{21}), P_1]_\bullet = -[\varphi(P_1), A_{21}]_\bullet = Z_1^* A_{21}. \quad (2.17)$$

Multiplying Eq (2.17) by P_2 from the right, we get $P_1 \varphi(A_{21})^* P_2 = 0$, and so $S_{21} = P_2 \varphi(A_{21}) P_1 = 0$. Hence,

$$\varphi(A_{21}) = S_{11} + S_{12} + S_{22}. \quad (2.18)$$

Since $[\varphi(B), A_{21}]_\bullet = -[\varphi(A_{21}), B]_\bullet$ and by Eq (2.18), we have

$$[W_{11} + W_{12} + W_{21} + W_{22}, A_{21}]_\bullet = -[S_{11} + S_{12} + S_{22}, B]_\bullet. \quad (2.19)$$

Multiplying Eq (2.19) by P_2 from the right, we get

$$-A_{21} W_{21}^* = -S_{11}^* BP_2 - S_{12}^* BP_2 - S_{22}^* BP_2 + BS_{22}^* P_2. \quad (2.20)$$

For any $B_{12} \in \mathcal{R}_{12}$, replacing B by B_{12} in Eq (2.20), we have

$$-A_{21} W_{21}^* = -S_{11}^* B_{12} - S_{12}^* B_{12} + B_{12} S_{22}^* P_2. \quad (2.21)$$

Multiplying Eq (2.21) by P_1 from the left, we have $S_{11}^*B_{12} = B_{12}S_{22}^*$. It follows from Lemma 1.6, [13] that there exists $Z_{A_{21}}^* \in \mathcal{Z}(\mathcal{R})$ such that $S_{11}^* + S_{22}^* = Z_{A_{21}}^*$. Therefore, $S_{11} + S_{22} = Z_{A_{21}} \in \mathcal{Z}(\mathcal{R})$. Multiplying Eq (2.17) by P_2 from the left and by P_1 from the right, we have $P_2\varphi(A_{21})^*P_1 = Z_1^*A_{21}$. It follows that

$$S_{12} = P_1\varphi(A_{21})P_2 = Z_1A_{21}^*.$$

Hence,

$$\varphi(A_{21}) = Z_1A_{21}^* + Z_{A_{21}}.$$

Lemma 2.5. *For every $A_{ii} \in \mathcal{R}_{ii}$ $(i = 1, 2)$, there exists $Z_{A_{ii}} \in \mathcal{Z}(\mathcal{R})$ such that*

$$\varphi(A_{ii}) = Z_1A_{ii}^* + Z_{A_{ii}}.$$

Proof. Let $A_{11} \in \mathcal{R}_{11}$ and write $\varphi(A_{11}) = T_{11} + T_{12} + T_{21} + T_{22}$. It follows from Lemma 2.3 that

$$\varphi(A_{11})^*P_1 - P_1\varphi(A_{11})^* = [\varphi(A_{11}), P_1]_\bullet = -[\varphi(P_1), A_{11}]_\bullet = 0. \tag{2.22}$$

Multiplying Eq (2.22) by P_2 from the left, we get $P_2\varphi(A_{11})^*P_1 = 0$, and so

$$T_{12} = P_1\varphi(A_{11})P_2 = 0.$$

Multiplying Eq (2.22) by P_2 from the right, we get $P_1\varphi(A_{11})^*P_2 = 0$, and so

$$T_{21} = P_2\varphi(A_{11})P_1 = 0.$$

Hence, $\varphi(A_{11}) = T_{11} + T_{22}$.

For any $A_{21} \in \mathcal{R}_{21}$, it follows from Lemma 2.4 that

$$[Z_1A_{21}^* + Z_{A_{21}}, A_{11}]_\bullet = [\varphi(A_{21}), A_{11}]_\bullet = -[\varphi(A_{11}), A_{21}]_\bullet = -[T_{11} + T_{22}, A_{21}]_\bullet.$$

It follows that

$$A_{21}Z_1^*A_{11} = -T_{22}^*A_{21} + A_{21}T_{11}^*.$$

$$T_{22}^*A_{21} = -A_{21}Z_1^*A_{11} + A_{21}T_{11}^* = A_{21}(-Z_1^*A_{11} + T_{11}^*).$$

It follows from Lemma 1.6, [13] that there exists $Z_{A_{11}}^* \in \mathcal{Z}(\mathcal{R})$ such that

$$T_{11}^* + T_{22}^* = Z_1^*A_{11} + Z_{A_{11}}^*.$$

Therefore,

$$\varphi(A_{11}) = T_{11} + T_{22} = Z_1A_{11}^* + Z_{A_{11}}. \tag{2.23}$$

Let $A_{22} \in \mathcal{R}_{22}$ and write $\varphi(A_{22}) = V_{11} + V_{12} + V_{21} + V_{22}$. It follows from Lemma 2.3 that

$$\varphi(A_{22})^*P_1 - P_1\varphi(A_{22})^* = [\varphi(A_{22}), P_1]_\bullet = -[\varphi(P_1), A_{22}]_\bullet = 0. \tag{2.24}$$

Multiplying Eq (2.24) by P_2 from the left, we have

$$V_{12} = P_1\varphi(A_{22})P_2 = 0.$$

Multiplying Eq (2.24) by P_2 from the right, we get

$$V_{21} = P_2\varphi(A_{22})P_1 = 0.$$

Hence, $\varphi(A_{22}) = V_{11} + V_{22}$.

For any $A_{12} \in \mathcal{R}_{12}$, it follows from Lemma 2.4 that

$$[Z_1A_{12}^* + Z_{A_{12}}, A_{22}]_\bullet = [\varphi(A_{12}), A_{22}]_\bullet = -[\varphi(A_{22}), A_{12}]_\bullet = -[V_{11} + V_{22}, A_{12}]_\bullet,$$

which implies that

$$A_{12}Z_1^*A_{22} = -V_{11}^*A_{12} + A_{12}V_{22}^*.$$

Then we have

$$V_{11}^*A_{12} = A_{12}(-Z_1^*A_{22} + V_{22}^*).$$

It follows from Lemma 1.6, [13] that there exists $Z_{A_{22}}^* \in \mathcal{Z}(\mathcal{R})$ such that

$$V_{11}^* + V_{22}^* = Z_1^*A_{22} + Z_{A_{22}}^*.$$

$$\varphi(A_{22}) = V_{11} + V_{22} = Z_1A_{22}^* + Z_{A_{22}}.$$

Proof of Theorem 2.1. It follows from Lemmas 2.4 and 2.5 that

$$\varphi(A_{ij}) = Z_1A_{ij}^* + Z_{A_{ij}} \quad (i, j = 1, 2), \tag{2.25}$$

where $Z_1, Z_{A_{ij}} \in \mathcal{Z}(\mathcal{R})$. For any $A = A_{11} + A_{12} + A_{21} + A_{22}$, let

$$Z_A = Z_{A_{11}} + Z_{A_{12}} + Z_{A_{21}} + Z_{A_{22}}.$$

Then $Z_A \in \mathcal{Z}(\mathcal{R})$. It follows from Eq (2.25) and Lemma 2.2 that

$$\begin{aligned}\varphi(A) - Z_1A^* - Z_A &= \varphi(A) - Z_1(A_{11}^* + A_{12}^* + A_{21}^* + A_{22}^*) - (Z_{A_{11}} + Z_{A_{12}} + Z_{A_{21}} + Z_{A_{22}})\\ &= \varphi(A) - (\varphi(A_{11}) + \varphi(A_{12}) + \varphi(A_{21}) + \varphi(A_{22}))\\ &= \varphi(A) - \sum_{i,j=1}^{2}\varphi(A_{ij}) \in \mathcal{Z}(\mathcal{R}).\end{aligned}$$

Now we define a map $f : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ by

$$f(A) = \varphi(A) - Z_1A^* - Z_A$$

for all $A \in \mathcal{R}$. Then

$$\varphi(A) = Z_1A^* + Z_A + f(A)$$

for all $A \in \mathcal{R}$. Since $Z_A, f(A) \in \mathcal{Z}(\mathcal{R})$, we define a map $h : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ by

$$h(A) = Z_A + f(A)$$

for all $A \in \mathcal{R}$. Then we have

$$\varphi(A) = ZA^* + h(A)$$

for all $A \in \mathcal{R}$, where $Z = Z_1 \in \mathcal{Z}(\mathcal{R})$.

As applications of Theorem 2.1, the concrete form of $*-$commuting maps on prime $*-$rings, factor von Neumann algebras and standard operator algebras are obtained. We have the following corollaries.

Corollary 2.1. *Let $\mathcal{R}$ be a unital prime $*-$ring with the unit I containing a nontrivial symmetric idempotent P. If a map $\varphi : \mathcal{R} \to \mathcal{R}$ satisfies $[\varphi(A), B]_\bullet = -[\varphi(B), A]_\bullet$ for all $A, B \in \mathcal{R}$, then there exists a map $h : \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ such that*

$$\varphi(A) = ZA^* + h(A)$$

for all $A \in \mathcal{R}$, where $Z \in \mathcal{Z}(\mathcal{R})$.

Corollary 2.2. *Let $\mathcal{A}$ be a factor von Neumann algebra on a complex Hilbert apace H with $\dim\mathcal{A} > 1$. If a map $\varphi : \mathcal{A} \to \mathcal{A}$ satisfies $[\varphi(A), B]_\bullet = -[\varphi(B), A]_\bullet$ for all $A, B \in \mathcal{A}$, then there exists a map $h : \mathcal{A} \to \mathbb{C}I$ such that*

$$\varphi(A) = \alpha A^* + h(A)$$

for all $A \in \mathcal{A}$, where $\alpha \in \mathbb{C}I$.

Proof. Recall that $\mathcal{A}$ is called a factor von Neumann algebra if its center contains only scalar operators, that is $\mathcal{Z}(\mathcal{A}) = \mathbb{C}I$. Note that factor von Neumann algebras are prime, so Corollary 2.2 is trivial.

Corollary 2.3. *Let $\mathcal{A}$ be a standard operator algebra on a complex Hilbert apace H, which is closed under the adjoint operation and $\dim\mathcal{A} > 1$. If a map $\varphi : \mathcal{A} \to \mathcal{A}$ satisfies $[\varphi(A), B]_\bullet = -[\varphi(B), A]_\bullet$ for all $A, B \in \mathcal{A}$, then there exists a map $h : \mathcal{A} \to \mathbb{C}I$ such that*

$$\varphi(A) = \alpha A^* + h(A)$$

for all $A \in \mathcal{A}$, where $\alpha \in \mathbb{C}I$.

Proof. Since $\mathcal{A}$ is prime and $\mathcal{Z}(\mathcal{A}) = \mathbb{C}I$, Corollary 2.3 holds by Theorem 2.1.

Example 2.1. Let $\mathbb{C}$ be the complex number field. Consider

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{C} \right\}.$$

The map $* : \mathcal{A} \to \mathcal{A}$ defined by $*(A) = A^*$ is an involution, where A^* is the conjugate transpose of $A \in \mathcal{A}$.

Define $\varphi : \mathcal{A} \to \mathcal{A}$ by

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2\bar{a} + \mathrm{i} & 2\bar{c} \\ 2\bar{b} & 2\bar{d} + \mathrm{i} \end{pmatrix}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{A}$. Then $\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = Z \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* + h \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $Z = 2I$, $h \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{i}I \in \mathcal{Z}(\mathcal{A})$. It is easy to verify that $[\varphi(A), B]_\bullet = -[\varphi(B), A]_\bullet$ for all $A, B \in \mathcal{A}$.

3. Conclusions

In this paper, we characterize the structure of $*-$commuting maps on rings with involution. As applications, the main results are applied on prime $*-$algebras, factor von Neumann algebras and standard operator algebras, the concrete form of $*-$commuting maps on these algebras are obtained. This study greatly enriches the theory of commuting maps in operator algebras.

Use of Generative-AI tools declaration

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Conflict of interest

The author confirms that there is no conflict of interest.

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