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Research article

The inverse Pareto–Chen distribution with properties and applications to real-world data across multiple fields

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Abstract: In this article, we propose a flexible extension of the Chen model, termed the odd inverse Pareto–Chen (OIPC) distribution. The proposed distribution accommodates a wide range of failure rate shapes, including decreasing, increasing, J-shaped, reversed J-shaped, bathtub, and upside-down bathtub forms. In addition, its density function is capable of capturing right-skewed, left-skewed, symmetric, J-shaped, reversed J-shaped, and concave-down shapes. Its fundamental mathematical properties are thoroughly investigated. The parameters of the OIPC distribution are estimated using eight well-established estimation methods. An extensive simulation study is conducted to assess the performance of these estimators under both small and large sample sizes. The practical relevance of the proposed model is demonstrated through the analysis of four real-world datasets drawn from engineering and medical applications. The empirical findings highlight the remarkable flexibility of the OIPC distribution, which consistently outperforms several competing models, suggesting it as a promising alternative for modeling data in insurance and finance sectors.

Keywords: Chen distribution; actuarial measures; percentile estimators; insurance data; simulations
Mathematics Subject Classification: 60E05, 62E10, 62F10, 62N05

1. Introduction

Lifetime and reliability data arise in engineering, medicine, and economics, motivating flexible parametric models for inference and prediction. As a result, researchers have developed different methods to create new families of distributions. They have worked to define new types of probability distributions that extend well-known ones, allowing for better data modeling in practice.

Chen [1] introduced a new lifetime distribution that incorporates two parameters. The Chen distribution is often used in survival analysis. However, it has some limitations that researchers have noticed when comparing it to the Weibull distribution. One main issue is that the Chen distribution may be insufficient for some datasets (e.g., with complex hazard shapes or tail behavior), motivating extensions. It also cannot effectively model datasets with heavy or light tails. These problems make it hard to use the Chen distribution in more complex scenarios. As a result, there is a need to modify it to make it more adaptable.

In the literature, numerous generalized (–G) classes of distributions have been proposed. These families incorporate one or more additional shape parameters to enhance the flexibility of a given baseline distribution. Among them, the Chen distribution is notable for its hazard rate function (HRF), which can exhibit either a bathtub shape or a monotonic increasing pattern. Such features make it particularly suitable for modeling diverse types of data in reliability theory and survival analysis. To improve its flexibility and modeling capacity, several researchers have proposed extended forms of the Chen distribution. Among the notable extensions are the exponential generated modified Chen [2], Kumaraswamy exponentiated Chen [3], transmuted Chen [4], inverse Chen [5], additive Chen [6], additive Chen–Weibull [7] and Chen exponential [8], Lindley–Chen [9], logistic Chen [10], modified Chen [11], and truncated Nadarajah–Haghighi Chen [12] distributions.

In this paper, we introduce a new form of the Chen distribution, referred to as the odd inverse Pareto–Chen (OIPC) distribution. This model is developed within the framework of the odd inverse Pareto (OIP-G) family by Aldahlan et al. [13] and is designed to offer more flexibility for modeling complex real-world data from different applied fields. Relative to previously proposed extensions of the Chen distribution, the OIPC model offers several compelling features that justify its introduction. First, its hazard rate function exhibits remarkable flexibility, accommodating a wide spectrum of shapes such as increasing, decreasing, bathtub, unimodal, J-shaped, and reversed J-shaped patterns. Second, the proposed distribution effectively captures asymmetric and skewed data structures that may not be adequately described by the classical Chen model. Third, its adaptability makes it suitable for applications in diverse domains, including engineering reliability, insurance risk assessment, financial modeling, and biomedical studies. Finally, empirical analyses based on four real-life datasets indicate that the OIPC distribution outperforms several established Chen-type extensions in terms of goodness-of-fit measures. Collectively, these properties highlight both the practical relevance and the theoretical significance of the proposed model within the framework of parametric survival analysis.

A primary objective of this study is to investigate the performance of several frequentist estimation procedures for the proposed OIPC distribution under various parameter configurations and sample sizes. In particular, we examine and compare the weighted least squares (WLS), maximum likelihood (ML), least squares (LS), maximum product of spacings (MPS), percentile (PC), Anderson–Darling (AD), Cramér–von Mises (CRVM), and right-tail Anderson–Darling (RAD) methods.

The comparative evaluation is conducted through extensive Monte Carlo simulations, where estimator performance is assessed using absolute bias, mean squared error, and mean relative error. In addition, the results are systematically ranked using both partial and overall ranking criteria to identify the most reliable estimation approach for the OIPC parameters.

This comprehensive comparison provides practical guidance for selecting an appropriate estimation technique and offers useful insights for applied researchers dealing with lifetime and reliability data.

Another key motivation of this study is to illustrate actuarial/risk measures for a (non-negative) loss variable, potential relevance to finance/insurance risk management. Accordingly, a further objective is to derive several important actuarial and risk measures for the OIPC distribution, including value-at-Risk (VaR), tail value-at-risk (TVaR), tail variance (TV), and tail variance premium (TVP). These measures are essential tools for quantifying risk and supporting portfolio optimization decisions under uncertainty.

This paper is organized as follows. Section 2 introduces the OIPC distribution. Section 3 presents its main statistical properties. Section 4 derives important actuarial and risk measures of the OIPC distribution. Section 5 discusses the estimation of the unknown parameters using eight classical estimation methods. The simulation study, conducted to evaluate and compare the performance of these estimation procedures, is presented in Section 6. Section 7 demonstrates the applicability of the OIPC model to real lifetime datasets. Finally, Section 8 concludes the paper with a summary of the main findings.

2. The OIPC distribution

This section derives the OIPC distribution by combining the OIP–G generator with the Chen baseline.

The cumulative distribution function (CDF) of the Chen distribution is given by

$$G(x;\zeta,\omega) = 1 - \exp\left\{\zeta\left(1 - e^{x^{\omega}}\right)\right\}, \qquad x > 0,\ \zeta,\omega > 0. \tag{2.1}$$

The corresponding probability density function (PDF) of (2.1) reduces to

$$g(x;\zeta,\omega) = \zeta\omega x^{\omega-1}\exp\left\{x^{\omega} + \zeta\left(1 - e^{x^{\omega}}\right)\right\}, \tag{2.2}$$

where ζ and ω are shape parameters.

The HRF corresponding to (2.1) has the form

$$h(x;\zeta,\omega) = \zeta\omega x^{\omega-1}e^{x^{\omega}}. \tag{2.3}$$

Let $G(x;\boldsymbol{\varphi})$ denote a baseline CDF with parameter vector $\boldsymbol{\varphi}$. Then, the CDF of the OIP–G family, introduced by [13], is defined as

$$F(x;\kappa,\nu,\boldsymbol{\varphi}) = \frac{G(x;\boldsymbol{\varphi})^{\kappa}}{[1 - G(x;\boldsymbol{\varphi})]^{\kappa}}\left[\nu + \frac{G(x;\boldsymbol{\varphi})}{1 - G(x;\boldsymbol{\varphi})}\right]^{-\kappa}, \qquad x > 0,\ \kappa,\nu > 0, \tag{2.4}$$

where κ and ν are shape parameters.

The OIP–G density can be written as

$$f(x;\kappa,\nu,\boldsymbol{\varphi}) = \kappa\nu g(x;\boldsymbol{\varphi})G(x;\boldsymbol{\varphi})^{\kappa-1}\left[\nu + (1-\nu)\,G(x;\boldsymbol{\varphi})\right]^{-\kappa-1}, \tag{2.5}$$

where κ and ν are two positive shape parameters, and $g(x;\boldsymbol{\varphi}) = dG(x;\boldsymbol{\varphi})/dx$.

To this end, we derive the CDF of the OIPC distribution with parameter vector $\boldsymbol{\psi} = (\kappa, \nu, \zeta, \omega)$ as follows:

$$F(x;\boldsymbol{\psi}) = \left\{\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]-1\right\}^{\kappa}\left\{\nu+\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]-1\right\}^{-\kappa}, \qquad x>0. \tag{2.6}$$

Differentiating (2.6) with respect to x, the corresponding PDF is obtained as

$$f(x;\boldsymbol{\psi}) = \kappa\nu\zeta\omega x^{\omega-1}e^{x^{\omega}}\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]\left\{\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]-1\right\}^{\kappa-1}\left\{\nu+\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]-1\right\}^{-\kappa-1}, \tag{2.7}$$

where κ, ν, ζ, and ω are positive shape parameters.

The corresponding HRF is obtained from the ratio of the PDF to the survival function (SF) as follows:

$$h(x;\boldsymbol{\psi}) = \frac{\kappa\nu\zeta\omega x^{\omega-1}e^{x^{\omega}}\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]\left\{\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]-1\right\}^{\kappa-1}\left\{\nu+\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]-1\right\}^{-\kappa-1}}{1-\left\{\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]-1\right\}^{\kappa}\left\{\nu+\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]-1\right\}^{-\kappa}}. \tag{2.8}$$

Figures 1–4 illustrate the flexibility of the OIPC distribution through the behavior of its PDF and HRF under various systematically selected parameter settings. To improve visual clarity and facilitate interpretation, the original plots have been expanded into multipanel figures, allowing a more detailed comparison of how changes in the parameters κ, ν, ζ, and ω influence the density and hazard structures.

Figures 1 and 2 present the PDF of the OIPC distribution under multiple parameter configurations, revealing a broad range of shapes, including right-skewed, highly right-skewed, left-skewed, near-symmetric, unimodal, and heavy-tailed forms with varying degrees of peakedness. The expanded multipanel design provides a clear visualization of how specific parameter changes influence skewness, concentration, spread, and tail decay.

In particular, the parameters κ and ν primarily govern the overall shape and tail weight, whereas ζ and ω mainly regulate the concentration around the mode and the overall dispersion of the distribution. For certain parameter combinations, the density exhibits pronounced heavy right tails, whereas for others, it becomes sharply peaked with rapid decay. This wide variety of possible shapes further demonstrates the remarkable flexibility of the OIPC model in accommodating diverse real-life data patterns.

Similarly, Figures 3 and 4 display the corresponding HRF shapes, demonstrating the ability of the OIPC model to capture several important failure rate behaviors, including decreasing, increasing, J-shaped, reversed J-shaped, bathtub-shaped, and upside-down bathtub (unimodal) forms. The revised multipanel visual layout provides a clear understanding of how each parameter influences the hazard dynamics and transition patterns over time.

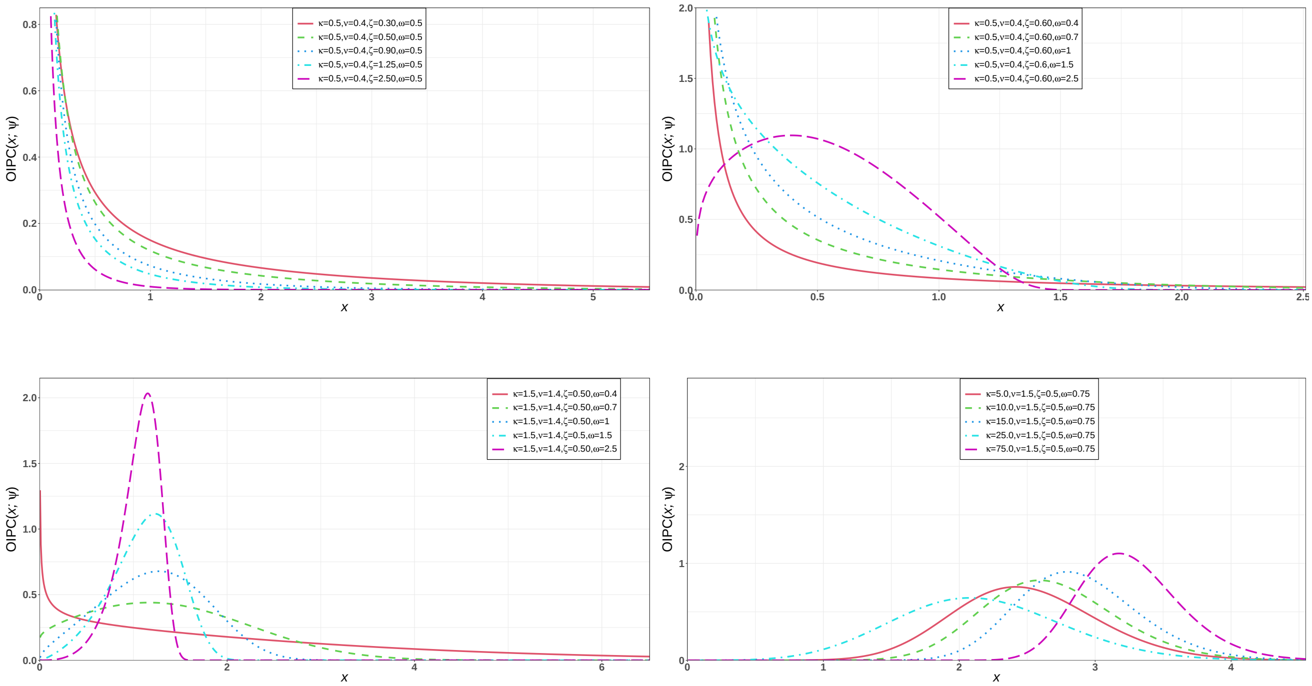


Figure 1. Plots of the OIPC PDF for different parametric values.

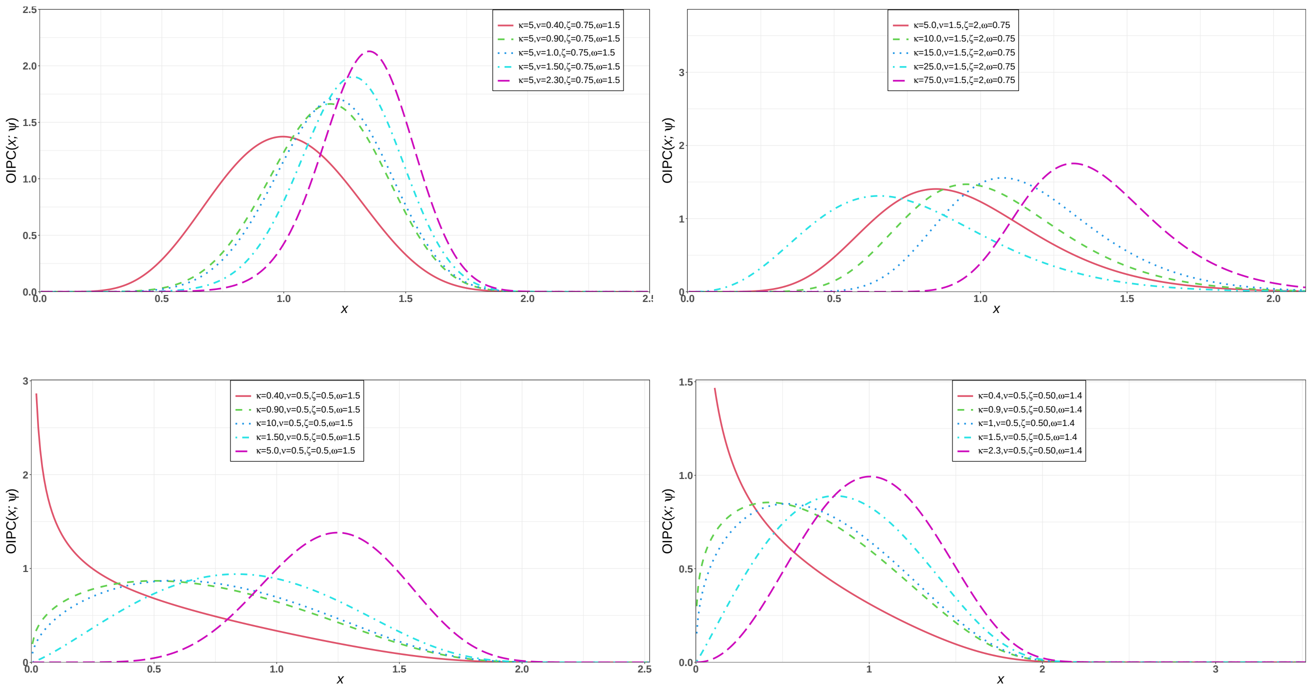


Figure 2. Plots of the OIPC PDF for different parametric values.

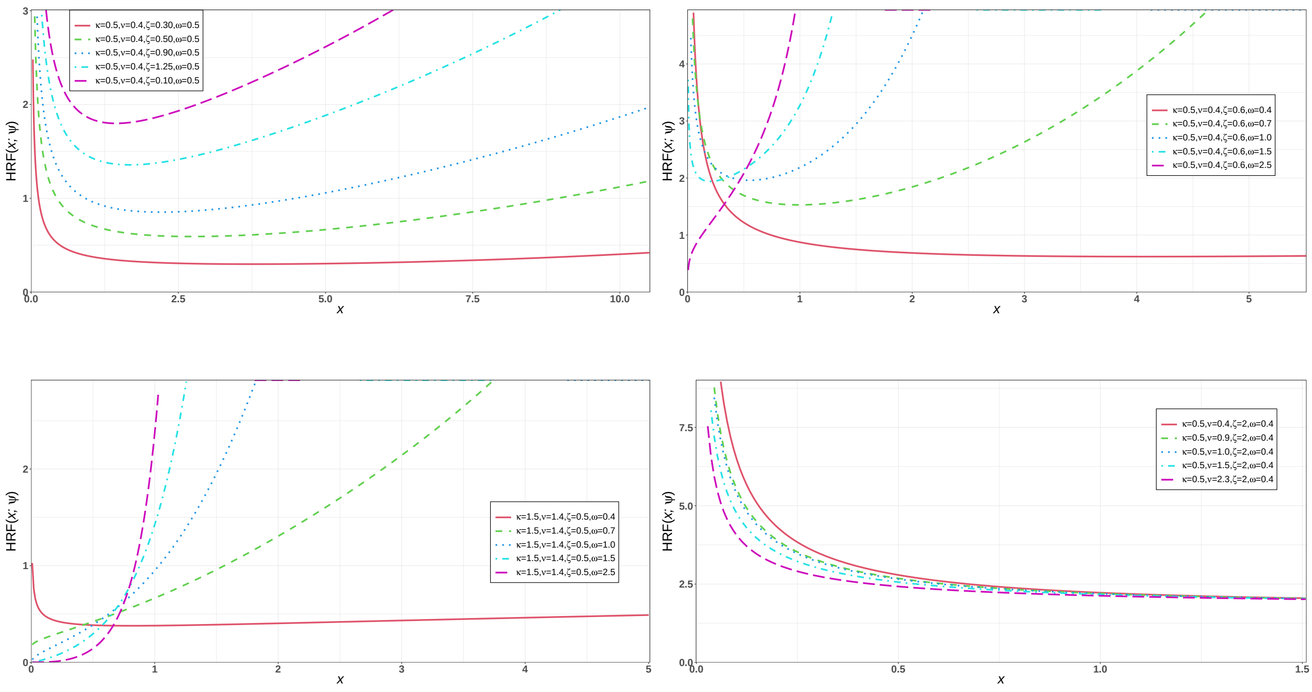


Figure 3. Plots of the OIPC HRF for different parametric values.

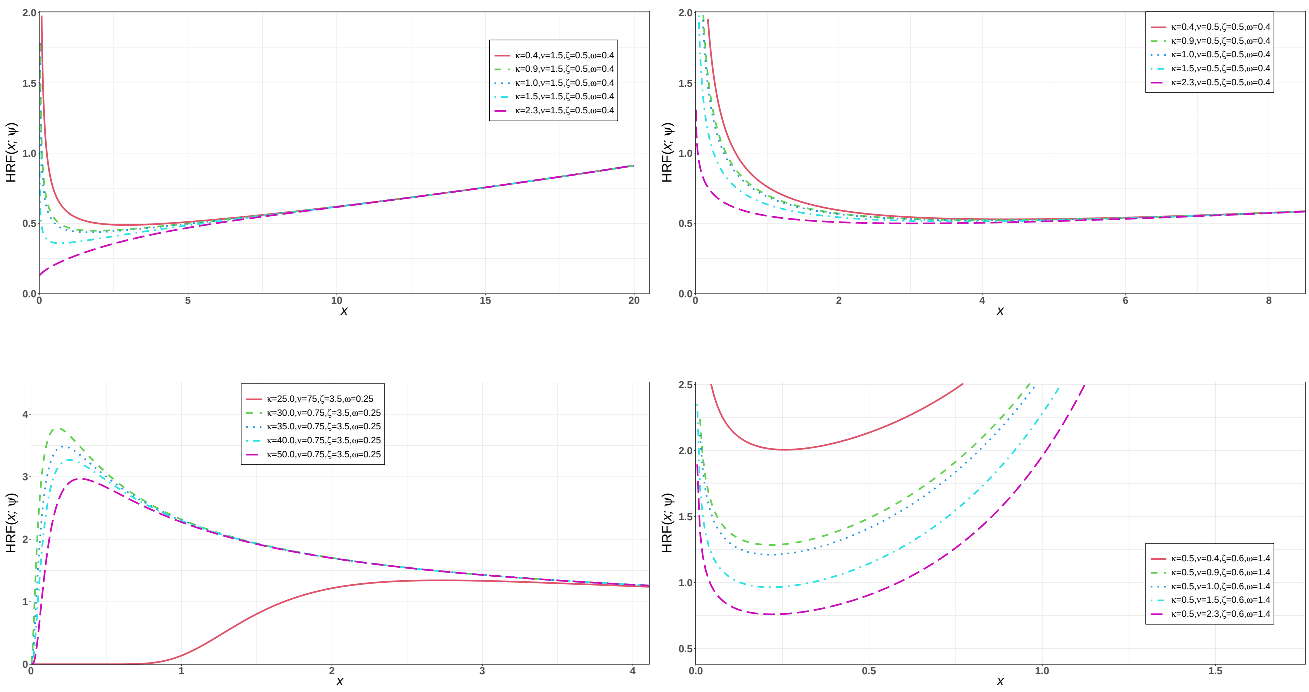


Figure 4. Plots of the OIPC HRF for different parametric values.

In particular, certain parameter configurations generate initially high failure rates that gradually decrease, whereas others produce monotonic increasing trends or more complex nonmonotonic structures. Such versatility is highly desirable in reliability analysis, survival studies, and engineering applications, where different physical mechanisms may govern early-life failures, random failures, and wear-out periods.

Overall, the expanded graphical analysis further confirms the remarkable flexibility of the OIPC distribution for modeling diverse lifetime and reliability data.

3. Properties of the OIPC distribution

In this section, we focus on the most practically relevant properties of the OIPC distribution, with particular emphasis on quantities that provide direct insight into its shape, tail behavior, and reliability interpretation, such as the quantile function (QF) and the mean residual life (MRL). To improve the readability of the manuscript, the more technical derivations, including the linear representation, higher-order moments, entropy measures, and order statistics, are provided in Appendix A.

3.1. Quantile function

The QF of the OIPC distribution can be obtained by solving the equation $F(Q(u)) = u$ for $Q(u)$, $0 < u < 1$ using Eq (2.6). This yields

$$Q(u) = \left\{\log\left(1 - \zeta^{-1}\log\left[1 - \left(\frac{\nu u^{\frac{1}{\kappa}}}{1-(1-\nu)u^{\frac{1}{\kappa}}}\right)\right]\right)\right\}^{\frac{1}{\omega}}. \tag{3.1}$$

The first quartile, median, and third quartile are obtained by substituting $u = 0.25$, 0.50, and 0.75, respectively, into (3.1).

The QF in (3.1) can be used to study the relationships between the parameters κ, ν, ζ, and ω and the distributional shape measures, namely skewness and kurtosis. In particular, Galton's skewness (GS) [14] and Moors' kurtosis (MK) [15] depend on the QF.

The GS and MK are given by

$$GS = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}$$

and

$$MK = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}. \tag{3.2}$$

Figure 5 displays the behavior of GS and MK of the OIPC distribution for different parameter combinations. The left panel shows that GS generally decreases as ν increases, indicating that the distribution becomes more symmetric for larger values of ν. For small values of κ and ω, the distribution exhibits higher positive skewness, whereas larger parameter values lead to near-symmetric shapes. The right panel illustrates that MK exhibits a more complex and sometimes nonmonotonic pattern, particularly for small κ and extreme ζ values. Overall, MK stabilizes for moderate and large ν, demonstrating that the OIPC distribution is capable of modeling a wide range of tail behaviors, from moderately light- to relatively heavy-tailed shapes, depending on the parameter configuration.

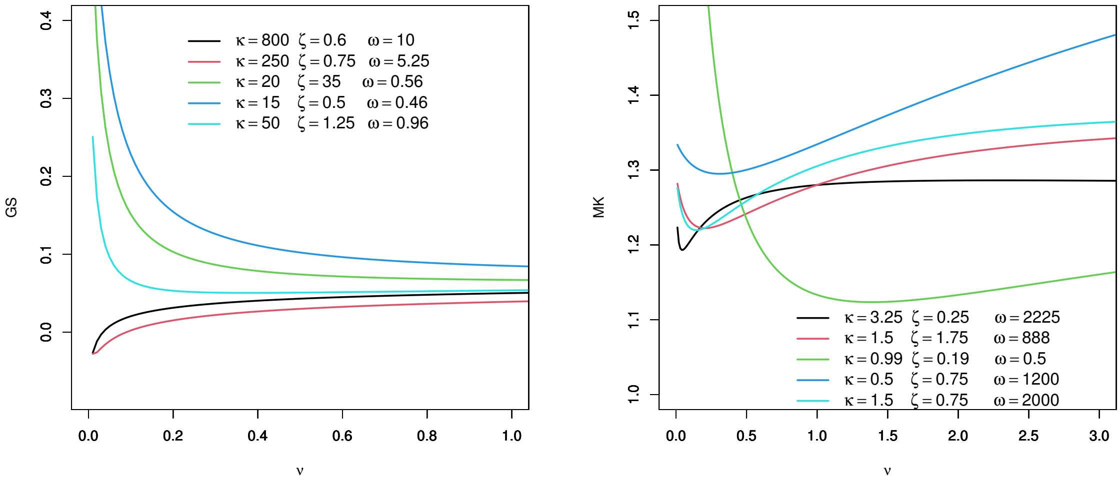


Figure 5. GS and MK measures of the OIPC distribution for some parameter values.

3.2. Numerical shape characteristics

The rth ordinary moment of the OIPC distribution is given by

$$\mu'_r = \omega \sum_{p,q,z,j\geq 0} d_{z,j}(z+j+\kappa)(-1)^{\frac{2r}{\omega}+p} \frac{(z+j+\kappa)_p\left(\frac{r}{\omega}\right)(z+j+\kappa)_q\left(\frac{r}{\omega}+p\right)}{\zeta^{\frac{2r}{\omega}+p}\left[\omega(z+j+\kappa+p+q)+r\right]}. \tag{3.3}$$

The detailed derivation of Eq (3.3) is provided in Appendix A.

Tables 1 and 2 report numerical values of the mean (μ_X), variance (σ_X^2), skewness ($\xi_1(X)$), and kurtosis ($\xi_2(X)$) of the OIPC distribution for $\zeta = 1$ under a systematically ordered parameter design. Specifically, ω is varied over increasing values within each block, whereas ν changes across blocks, and κ progresses across the two tables. This structured arrangement enables a clear assessment of how each parameter influences the distributional moments.

Table 1. Values of the μ_X, σ_X^2, $\xi_1(X)$, and $\xi_2(X)$ of the OIPC distribution for $\zeta = 1$ and selected values of κ, ν, and ω.

κ	ν	ω	μ_X	σ_X^2	$\xi_1(X)$	$\xi_2(X)$
0.50	0.50	0.50	0.1893	0.1732	3.953	23.79
		0.75	0.2310	0.1258	2.455	10.14
		1.50	0.3609	0.1007	1.038	3.504
		2.00	0.4305	0.0916	0.6419	2.663
		2.25	0.4604	0.0873	0.4978	2.465
0.50	0.75	0.50	0.2524	0.2353	3.311	17.33
		0.75	0.2926	0.1612	2.045	7.663
		1.50	0.4201	0.1162	0.8044	2.906
		2.00	0.4863	0.1017	0.4433	2.341
		2.25	0.5145	0.0954	0.3102	2.224
0.50	1.50	0.50	0.3962	0.3778	2.452	10.51
		0.75	0.4219	0.2327	1.470	4.966
		1.50	0.5304	0.1406	0.4410	2.276
		2.00	0.5869	0.1156	0.1218	2.053
		2.25	0.6106	0.1060	0.0015	2.044
0.50	2.00	0.50	0.4703	0.4507	2.163	8.676
		0.75	0.4842	0.2652	1.268	4.229
		1.50	0.5788	0.1492	0.3007	2.129
		2.00	0.6296	0.1198	-0.0067	2.017
		2.25	0.6510	0.1089	-0.1236	2.045
0.50	2.25	0.50	0.5032	0.4828	2.054	8.046
		0.75	0.5111	0.2788	1.190	3.975
		1.50	0.5989	0.1524	0.2447	2.085
		2.00	0.6472	0.1212	-0.0587	2.016
		2.25	0.6676	0.1098	-0.1744	2.058
0.75	0.50	0.50	0.2686	0.2356	3.246	16.94
		0.75	0.3168	0.1587	1.981	7.476
		1.50	0.4592	0.1059	0.7506	2.944
		2.00	0.5296	0.0883	0.3987	2.448
		2.25	0.5588	0.0811	0.2705	2.356

Table 2. Values of the μ_X, σ_X^2, $\xi_1(X)$, and $\xi_2(X)$ of the OIPC distribution for $\zeta = 1$ and selected values of κ, ν, and ω.

κ	ν	ω	μ_X	σ_X^2	$\xi_1(X)$	$\xi_2(X)$
1.50	0.50	0.50	0.4659	0.3670	2.336	10.13
		0.75	0.5098	0.2095	1.368	4.920
		1.50	0.6409	0.0991	0.3897	2.585
		2.00	0.6989	0.0714	0.1080	2.446
		2.25	0.7216	0.0619	0.0066	2.456
1.50	0.75	0.50	0.5997	0.4582	1.943	7.832
		0.75	0.6220	0.2404	1.090	4.039
		1.50	0.7226	0.0998	0.1888	2.477
		2.00	0.7683	0.0689	-0.0802	2.492
		2.25	0.7862	0.0588	-0.1782	2.555
1.50	1.50	0.50	0.8814	0.6218	1.406	5.409
		0.75	0.8379	0.2814	0.6898	3.168
		1.50	0.8624	0.0941	-0.1271	2.575
		2.00	0.8829	0.0606	-0.3852	2.829
		2.25	0.8914	0.0505	-0.4808	2.981
1.50	2.00	0.50	1.017	0.6866	1.223	4.775
		0.75	0.9341	0.2921	0.5467	2.976
		1.50	0.9189	0.0897	-0.2483	2.708
		2.00	0.9279	0.0562	-0.5048	3.060
		2.25	0.9324	0.0463	-0.6005	3.250
1.50	2.25	0.50	1.075	0.7119	1.153	4.562
		0.75	0.9743	0.2953	0.4915	2.919
		1.50	0.9416	0.0876	-0.2961	2.777
		2.00	0.9458	0.0543	-0.5525	3.170
		2.25	0.9485	0.0446	-0.6483	3.376
2.00	0.50	0.50	0.5733	0.4256	2.047	8.447
		0.75	0.6053	0.2244	1.175	4.329
		1.50	0.7166	0.0918	0.2840	2.564
		2.00	0.7649	0.0626	0.0296	2.507
		2.25	0.7836	0.0531	-0.0613	2.536

The numerical trends reveal that the mean generally increases with κ, ν, and ω, indicating strong control over the location of the distribution. In contrast, the variance exhibits a pronounced decreasing trend as ω increases, showing that larger ω values produce more concentrated distributions around the mean. For fixed ω, the variance generally increases with κ and ν, reflecting greater dispersion under larger shape parameter values.

A similar systematic behavior is observed for the higher-order moments. As ω increases, both skewness and kurtosis decrease substantially, reflecting a transition from highly right-skewed and heavy-tailed forms toward nearly symmetric and moderately light-tailed shapes. Meanwhile, larger values of κ and ν tend to shift the distribution from positive skewness toward near-symmetry or mild left-skewness. The kurtosis values also move from strongly leptokurtic levels toward near-mesokurtic behavior as ω increases. These synthesized trends clearly demonstrate the remarkable flexibility of the OIPC model in controlling central tendency, dispersion, asymmetry, and tail behavior.

The numerical summaries reported below are computed using the ordinary moments derived in Appendix A.

3.3. Mean residual life

The MRL is $m(x) = E[X - x \mid X > x]$, that is, the expected remaining lifetime given survival past x. In life testing, the MRL indicates the expected additional lifetime of a component that has lasted until that point. Because the MRL function is the expected remaining life, x must be subtracted.

The MRL of the OIPC distribution is given by

$$m(x) = \omega \sum_{p,q,z,j \geq 0} \frac{(z+j+\kappa) d_{z,j} (-1)^{\frac{2}{\omega}+p} (z+j+\kappa)_p \left(\frac{1}{\omega}\right) (z+j+\kappa)_q \left(\frac{1}{\omega}+p\right) \exp\left[-\zeta\left(e^{x^{\omega}}-1\right)\right]}{\zeta^{\frac{2}{\omega}+p}\left[\omega(z+j+\kappa+p+q)+1\right]\left(1-\left\{1-\exp\left[-\zeta\left(e^{x^{\omega}}-1\right)\right]\right\}^{z+j+\kappa}\right)} - x. \quad (3.4)$$

Equation (3.4) is derived using Eqs (A.1) and (A.5) in Appendix A.

4. Actuarial measures

Probability distributions help us understand risk exposure, which we measure using key risk indicators (KRIs) derived from models. Actuaries and risk managers use these KRIs to assess their companies exposure to risks from changes in stock prices, interest rates, or exchange rates.

This section discusses important risk measures including VaR, TVaR, TV, and TVP related to the OIPC distribution, all of which are key for optimizing investment portfolios under uncertainty.

4.1. VaR measure

VaR, also known as the quantile risk measure, is defined at a specified confidence level (typically 90%, 95%, or 99%) and represents the corresponding quantile of the loss distribution. Risk managers use VaR to assess the likelihood of adverse outcomes and to determine the amount of capital required to withstand such events. This measure is particularly important for investors, regulators, and rating agencies concerned with a firm's ability to manage financial risk.

The VaR of a random variable X at level $q \in (0, 1)$ is defined as the qth quantile of its CDF, denoted by VaR_q, and is given by

$$\mathrm{VaR}_q = Q(q),$$

where $Q(\cdot)$ is the QF (see Artzner et al. [16]).

If X has the PDF given in (2.7), then its VaR can be obtained explicitly from the QF as

$$VaR_q = \left[\log\left(1 - \zeta^{-1} \log\left\{1 - \left[\frac{\nu q^{\frac{1}{\kappa}}}{1-(1-\nu)q^{\frac{1}{\kappa}}}\right]\right\}\right)\right]^{\frac{1}{\omega}}.$$

4.2. TVaR measure

Another important risk measure is the TVaR, also referred to as the conditional tail expectation (CTE), conditional VaR (CVaR), or expected shortfall (ES). Unlike VaR, which specifies a threshold loss at a given confidence level, TVaR measures the expected loss conditional on losses exceeding that threshold. Thus, TVaR provides information about the severity of extreme losses and is widely used in risk management and actuarial science.

Formally, the TVaR of a random variable X at level $q \in (0, 1)$ is defined as

$$TVaR_q = E\Big(X \mid X > \mathrm{VaR}_q\Big) = \frac{1}{(1-q)} \int_{VaR_q}^{\infty} x f(x) dx.$$

The TVaR of the OIPC distribution is defined by

$$TVaR_q = \frac{\kappa\nu\zeta\omega}{1-q} \int_{VaR_q}^{\infty} x^{\omega} \frac{e^{x^{\omega}} \exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]\left\{\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]-1\right\}^{\kappa-1}}{\left\{\nu+\exp\left[\zeta\left(e^{x^{\omega}}-1\right)\right]-1\right\}^{\kappa+1}} dx. \tag{4.1}$$

Hence, it follows as

$$TVaR_q = \frac{\kappa\nu\omega}{1-q} \sum_{j,i,r,m=0}^{\infty} \frac{\zeta^{r+1}(j+1)^r r^m (-1)^{j+r} e^{(j+1)\zeta}}{r!m!\nu^{\kappa+i+1}(1-\nu)^{-i}} \binom{-\kappa-1}{i} \binom{\kappa+i-1}{j} \int_{VaR_q}^{\infty} x^{m\omega+\omega-1} e^{x^{\omega}} dx. \tag{4.2}$$

After some algebra, the TVaR of the OIPC distribution follows as

$$TVaR_q = \frac{\kappa\nu}{1-q} \sum_{j,i,r,m=0}^{\infty} \frac{\zeta^{r+1}(j+1)^r r^m (-1)^{j+r} e^{(j+1)\zeta}}{r!m!\nu^{\kappa+i+1}(1-\nu)^{-i}} \binom{-\kappa-1}{i} \binom{\kappa+i-1}{j} \Gamma\left(m+1, \left(VaR_q\right)^{\omega}\right). \tag{4.3}$$

4.3. TV measure

Landsman [17] introduced the TV risk measure, defined as the variance of the loss distribution conditional on exceeding a specified threshold. In particular, it measures the variability of losses beyond the VaR level and thus provides additional information about tail risk for the OIPC distribution.

Formally, the TV at level $q \in (0, 1)$ is defined as

$$TV_q = E(X^2 | X > VaR_q) - (TVaR_q)^2 = \frac{1}{(1-q)} \int_{VaR_q}^{\infty} x^2 f(x) dx - (TVaR_q)^2. \tag{4.4}$$

By using Eq (4.3), we get the TV of OIPC distribution.

4.4. TVP measure

The TVP is an important measure in insurance science. The TVP of the OIPC distribution takes the form

$$TVP_q = TVaR_q + \lambda TV_q, \qquad 0 < \lambda < 1. \tag{4.5}$$

The TVP of the OIPC distribution can be obtained by substituting Eqs (4.3) and (4.4) into Eq (4.5).

4.5. *Numerical simulations for risk measures*

In this subsection, numerical results for the four previously discussed risk measures of the OIPC and Chen distributions are presented under various parameter settings. These results are obtained based on a random sample of size $n = 100$ generated from the OIPC and Chen distributions, with the model parameters estimated using the ML method. The simulation study is repeated 1000 times in order to compute the corresponding risk measures.

The simulation study has been substantially revised to enhance its statistical relevance and clarity. Instead of relying on arbitrary parameter choices, three representative parameter configurations were selected to capture distinct and meaningful shapes of the HRF and PDF. This allows for a more comprehensive assessment of the model under different distributional scenarios.

To avoid redundancy, the detailed numerical results have been moved to Appendix A (Table 16), whereas the main text focuses on graphical illustrations. Specifically, the simulation results for the VaR, TVaR, TV, and TVP measures of the OIPC and Chen distributions are presented in Table 16 and are visualized in Figures 6–8.

The simulation is conducted for both distributions across the selected parameter configurations. It is well known that larger values of VaR, TVaR, TV, and TVP indicate heavier tail behavior. The results in Table 16, together with the corresponding plots, consistently show that the proposed OIPC model yields higher values of these risk measures compared to the C distribution.

These findings provide strong evidence that the OIPC distribution exhibits heavier tail behavior and is therefore more suitable for modeling heavy-tailed data in practical applications.

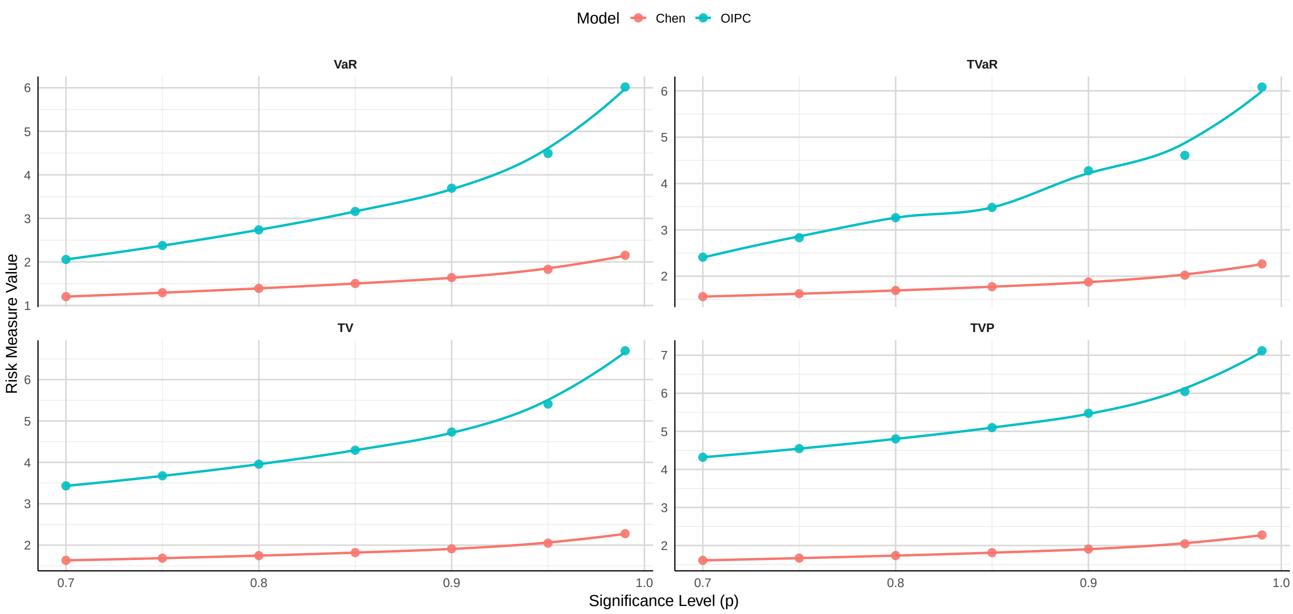


Figure 6. Plots of the four risk measures for the OIPC and Chen distributions for $\kappa = 0.5$, $\nu = 0.4$, $\zeta = 0.1$, and $\omega = 0.5$.

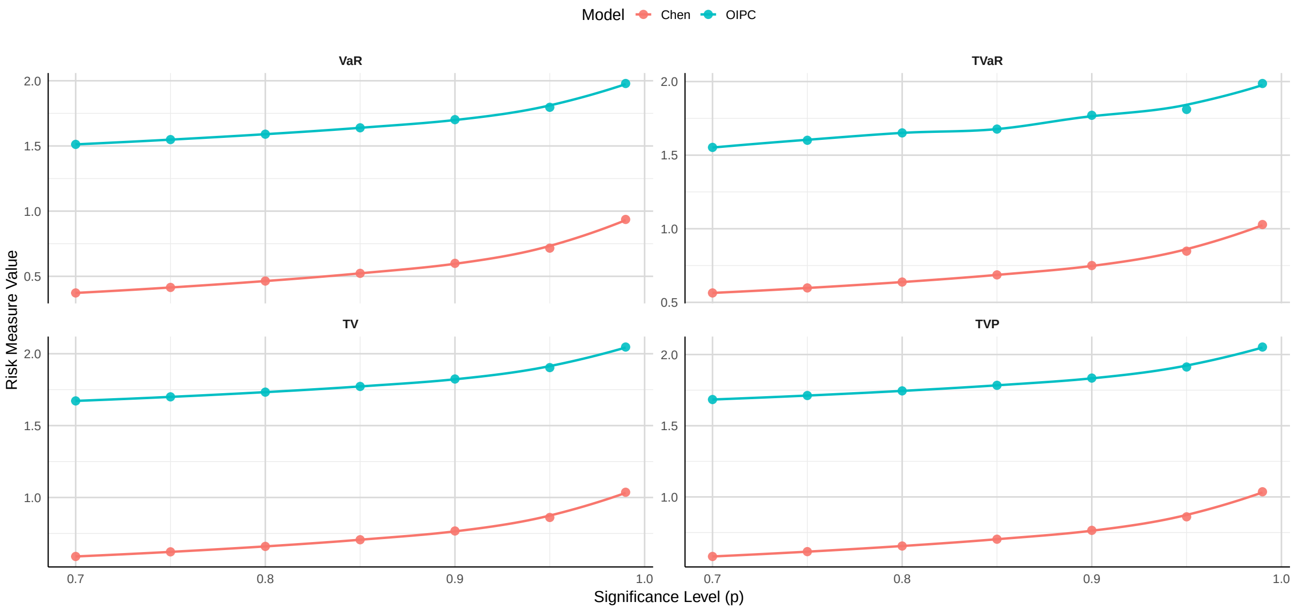


Figure 7. Plots of the four risk measures for the OIPC and Chen distributions for $\kappa = 75$, $\nu = 1.5$, $\zeta = 2$, and $\omega = 0.75$.

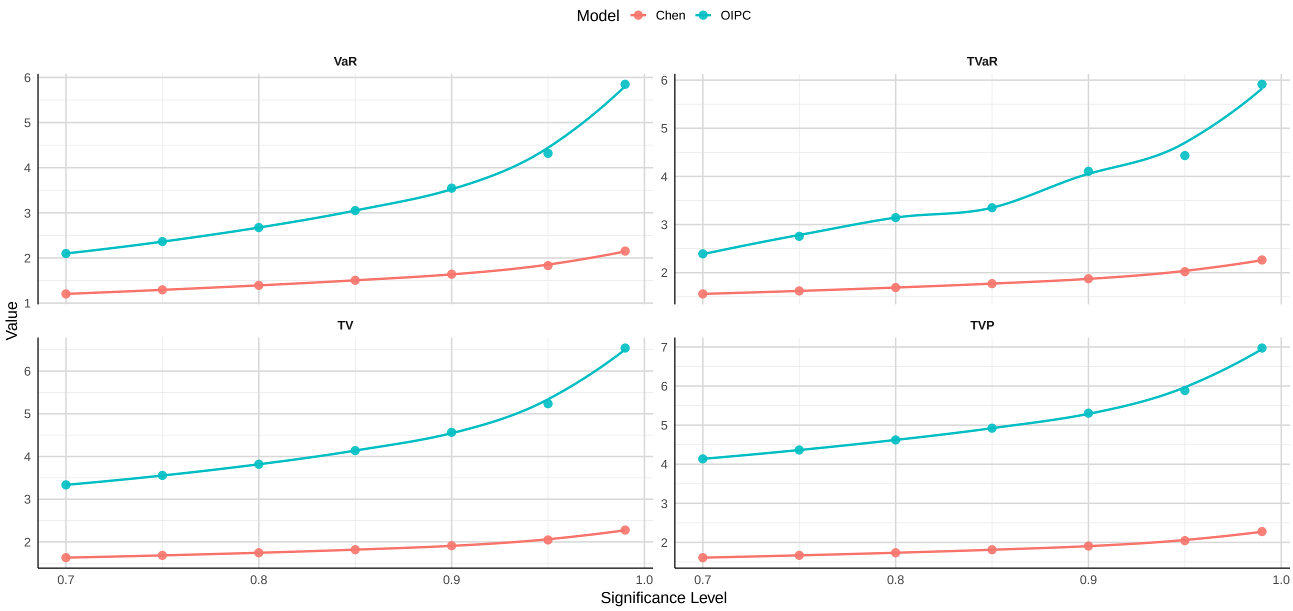


Figure 8. Plots of the four risk measures for the OIPC and Chen distributions for $\kappa = 2.3$, $\nu = 0.5$, $\zeta = 0.5$, and $\omega = 0.4$.

5. Estimation approaches

The OIPC parameters are estimated by using eight classical estimators including the WLS estimators (WLSEs), ML estimators (MLEs), LS estimators (LSEs), MPS estimators (MPSEs), PC estimators (PCEs), AD estimators (ADEs), CRVM estimators (CRVMEs), and RAD estimators

(RADEs). Estimation of the parameters of generalized distributions using classical estimation approaches has been addressed in the literature by many scholars. The reader can see, for example, [18–20].

The LSEs of the OIPC parameters can also be found by minimizing the function

$$V(\boldsymbol{\psi}) = \sum_{i=1}^{n} \left[F(x_{(i)}) - \frac{i}{n+1} \right]^2 .$$

They can also be determined by solving the nonlinear equations

$$\sum_{i=1}^{n} \left[F(x_{(i)}) - \frac{i}{n+1} \right] \Delta_s(x_{(i)}) = 0, \quad s = 1, 2, 3, 4,$$

where

$$\begin{aligned}
\Delta_1(x_{(i)}) &= \frac{\partial F(x_{(i)})}{\partial \kappa} = \left(\nu + \wp_{(i)}\right)^{-\kappa} \wp_{(i)}^{\kappa} \left[\ln \wp_{(i)} - \ln\left(\nu + \wp_{(i)}\right)\right], \\
\Delta_2(x_{(i)}) &= \frac{\partial F(x_{(i)})}{\partial \nu} = -\kappa \wp_{(i)}^{\kappa} \left(\nu + \wp_{(i)}\right)^{-\kappa-1}, \\
\Delta_3(x_{(i)}) &= \frac{\partial F(x_{(i)})}{\partial \zeta} = \kappa \left(\wp_{(i)}^{\kappa} - 1\right)\left(1 - e^{x_{(i)}^{\omega}}\right) \left\{ \frac{\wp_{(i)}^{\kappa}}{\left(\nu + \wp_{(i)}\right)^{\kappa+1}} - \frac{\wp_{(i)}^{\kappa-1}}{\left(\nu + \wp_{(i)}\right)^{\kappa}} \right\},
\end{aligned}$$

and

$$\Delta_4(x_{(i)}) = \frac{\partial F(x_{(i)})}{\partial \omega} \quad = \quad \kappa \zeta e^{x_{(i)}^{\omega}} \left(\wp_{(i)}^{\kappa} - 1\right) x_{(i)}^{\omega} \ln x_{(i)} \left\{ \frac{\wp_{(i)}^{\kappa}}{\left(\nu + \wp_{(i)}\right)^{\kappa+1}} - \frac{\wp_{(i)}^{\kappa-1}}{\left(\nu + \wp_{(i)}\right)^{\kappa}} \right\},$$

where $\wp_{(i)} = \exp\left\{\zeta \left[e^{x_{(i)}^{\omega}} - 1\right]\right\} - 1$.

The WLSEs of the OIPC parameters minimize

$$W(\boldsymbol{\psi}) = \sum_{i=1}^{n} \frac{(n+2)(n+1)^2}{i(n-i+1)} \left[F(x_{(i)}) - \frac{i}{n+1} \right]^2,$$

which follow by solving (for $s = 1, 2, 3, 4$):

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{(i)}) - \frac{i}{n+1} \right] \Delta_s(x_{(i)}) = 0.$$

The MLEs of the OIPC parameters are obtained by maximizing the log-likelihood function, denoted by ℓ, which is derived from Eq (2.7). The maximization can be carried out using standard iterative numerical methods. The log-likelihood function is given by

$$\begin{aligned}
\ell \quad = \quad & n\ln(\kappa) + n\ln(\nu) + n\ln(\zeta) + n\ln(\omega) + (\omega - 1)\sum_{i=1}^{n} \ln(x_i) + (\kappa - 1)\sum_{i=1}^{n} \ln(\aleph_i) \\
& + \sum_{i=1}^{n} x_i^{\omega} - \zeta\left(e^{x_i^{\omega}} - 1\right) - (\kappa + 1)\sum_{i=1}^{n} \ln\left(\nu + (1 - \nu)\aleph_i\right), \qquad (5.1)
\end{aligned}$$

where $\aleph_i = 1 - \exp\left[-\zeta\left(e^{x_i^{\omega}} - 1\right)\right]$.

The likelihood equations are given by

$$\frac{\partial \ell}{\partial \kappa} = \frac{n}{\kappa} + \sum_{i=1}^{n} \ln(\aleph_i) - \sum_{i=1}^{n} \ln(\nu + (1-\nu)\aleph_i) = 0,$$

$$\frac{\partial \ell}{\partial \nu} = \frac{n}{\nu} - (\kappa + 1) \sum_{i=1}^{n} \frac{1 - \aleph_i}{\nu + (1-\nu)\aleph_i} = 0,$$

$$\frac{\partial \ell}{\partial \zeta} = \frac{n}{\zeta} - \sum_{i=1}^{n} \left(e^{x_i^{\omega}} - 1\right) + (\kappa - 1) \sum_{i=1}^{n} \frac{\left(e^{x_i^{\omega}} - 1\right)(1 - \aleph_i)}{\aleph_i} - (\kappa + 1)(1-\nu) \sum_{i=1}^{n} \frac{\left(e^{x_i^{\omega}} - 1\right)(1 - \aleph_i)}{\nu + (1-\nu)\aleph_i} = 0,$$

and

$$\frac{\partial \ell}{\partial \omega} = \frac{n}{\omega} + \sum_{i=1}^{n} \ln(x_i) + \sum_{i=1}^{n} \left[x_i^{\omega} \ln(x_i) - \zeta e^{x_i^{\omega}} x_i^{\omega} \ln(x_i)\right] + (\kappa - 1) \sum_{i=1}^{n} \frac{\zeta e^{x_i^{\omega}} x_i^{\omega} (1 - \aleph_i) \ln(x_i)}{\aleph_i} - (\kappa + 1)(1-\nu) \sum_{i=1}^{n} \frac{\zeta e^{x_i^{\omega}} x_i^{\omega} (1 - \aleph_i) \ln(x_i)}{\nu + (1-\nu)\aleph_i} = 0.$$

Consider the uniform spacings defined by

$$D_i = D_i(\boldsymbol{\psi}) = F(x_{(i)}) - F(x_{(i-1)}), i = 1, \ldots, n+1,$$

where $F(x_{(0)}) = 0$, and $F(x_{(n+1)}) = 1$. Additionally, it follows that $\sum_{i=1}^{n+1} D_i = 1$.

The MPSEs of the OIPC parameters maximize the quantity

$$H(\boldsymbol{\psi}) = \sum_{i=1}^{n+1} \log[D_i],$$

which can be obtained from the following nonlinear equations (for $s = 1, 2, 3, 4$)

$$\sum_{i=1}^{n+1} \frac{1}{D_i} \left[\Delta_s(x_{(i)}) - \Delta_s(x_{(i-1)})\right] = 0.$$

The CRVMEs of the OIPC parameters minimize

$$C(\boldsymbol{\psi}) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{(i)}) - \frac{2i-1}{2n}\right]^2,$$

which also follow by solving

$$\sum_{i=1}^{n} \left[F(x_{(i)}) - \frac{2i-1}{2n}\right] \Delta_s(x_{(i)}) = 0, \quad s = 1, 2, 3, 4.$$

The PC method was originally proposed by [21]. Let $u_i = i/(n+1)$ be an unbiased estimator of $F(x_{(i)}|\boldsymbol{\psi})$. Then, the PCEs of the OIPC parameters are obtained by minimizing the following objective function:

$$P(\boldsymbol{\psi}) = \sum_{i=1}^{n}\left[x_{(i)} + \left\{\log\left(1 - \zeta^{-1}\log\left[1 - \left(\frac{\nu u_i^{\frac{1}{\kappa}}}{1-(1-\nu)u_i^{\frac{1}{\kappa}}}\right)\right]\right)\right\}^{\frac{1}{\omega}}\right]^2.$$

The ADEs of the OIPC parameters minimize

$$A(\boldsymbol{\psi}) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{(i)}) + \log\left(1 - F(x_{(n+1-i)})\right)\right],$$

which can be determined as solutions of the following system:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_s(x_{(i)})}{F(x_{(i)})} - \frac{\Delta_s(x_{(n+1-i)})}{S(x_{(n+1-i)})}\right] = 0, \quad s = 1,2,3,4.$$

The RADEs of the OIPC parameters minimize

$$R(\boldsymbol{\psi}) = \frac{n}{2} - 2\sum_{i=1}^{n}F(x_{(i)}) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S(x_{(n+1-i)}).$$

The RADEs can also be found from the nonlinear equations

$$-2\sum_{i=1}^{n}\Delta_s(x_{(i)}) + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\Delta_s(x_{(n+1-i)})}{S(x_{(n+1-i)})} = 0, \quad s = 1,2,3,4.$$

6. Simulation analysis

This section presents a Monte Carlo simulation study to compare the performance of the eight estimation methods. To obtain a comprehensive assessment of the proposed OIPC model, we consider eight well-established estimation techniques, including likelihood-based, least-squares-based, and goodness-of-fit-based approaches. Because no single method is optimal under all settings, this comparative framework allows us to evaluate the finite-sample performance, stability, and efficiency of the proposed estimators under different estimation philosophies. Such multimethod comparisons are standard in the literature on lifetime distributions and provide practical guidance for implementation, while also helping to identify the most efficient estimation method.

The comparison is based on the averages of three accuracy measures: absolute biases (AB), $AB(\widehat{\boldsymbol{\psi}}) = 1/N\sum_{i=1}^{N}|\widehat{\boldsymbol{\psi}}_i - \boldsymbol{\psi}|$; mean squared errors (MSE), MSE $= 1/N\sum_{i=1}^{N}(\widehat{\boldsymbol{\psi}}_i - \boldsymbol{\psi})^2$; and mean relative errors (MRE), MRE $= 1/N\sum_{i=1}^{N}|\widehat{\boldsymbol{\psi}}_i - \boldsymbol{\psi}|/\boldsymbol{\psi}$, where N denotes the number of Monte Carlo replications and $\widehat{\boldsymbol{\psi}}_i$ is the estimate obtained from the ith replication.

Random observations from the OIPC distribution are generated using the QF given in Eq (3.1). Specifically, for each n, we generate $N = 5000$ Monte Carlo samples $\{x_1,\ldots,x_n\}$ with sample sizes $n = 50, 150, 350$, and 500. In particular, the selected parameter settings are designed to represent a diverse range of distributional behaviors, including different shapes of the HRF and PDF, such as increasing, decreasing, reversed J-shaped, and unimodal patterns. These behaviors are induced by systematic variation of the model parameters to ensure meaningful and distinct distributional scenarios. The data are simulated from the OIPC model under the following parameter settings: $\kappa = \{0.50, 0.75, 1.25, 2.25, 2.50, 10\}$, $\nu = \{0.50, 0.75, 1.25, 2.25\}$, $\zeta = \{0.25, 0.75, 1.25, 1.75, 3.5, 5\}$ and $\omega = \{0.25, 0.5, 0.75, 1.25, 2.5, 3\}$.

All simulations are conducted using the R software (R Core Team, 2023, version 4.3.1).

For selected combinations of parameter values and sample sizes, the OIPC parameters are estimated using the eight estimation methods under consideration. For each method, we compute the AB, the MSE, and the MRE based on the $N = 5000$ Monte Carlo replications.

The simulation results for the eight methods are summarized in Tables 3–10. In these tables, the numerical values reported in each row correspond to the performance measures of the competing estimators, and the superscripts indicate the rank of each method relative to the others (with smaller values receiving better ranks). The quantity $\sum Ranks$ represents the partial sum of the ranks across the reported criteria, providing an overall measure of comparative performance.

The results clearly indicate that all estimation methods exhibit the consistency property. Specifically, for all considered parameter combinations, both the MSE and MRE decrease as the sample size n increases, confirming the asymptotic improvement in estimator accuracy.

Table 11 presents the overall rankings of the eight estimators. The results indicate that the LSEs achieve the best performance, attaining the smallest overall score of 79 for the OIPC distribution. In general, the findings demonstrate that the LSEs and CRVMEs outperform the competing methods and can be regarded as the most efficient estimation procedures for the OIPC model under the considered simulation settings.

Furthermore, the simulation results reported in Tables 3–10 are visualized in Figures 9–32, which illustrate the performance of the proposed estimation procedures in terms of AB, MSE, and MRE under different combinations of model parameters and sample sizes. The graphical representations clearly demonstrate the effect of increasing sample size on improving estimator performance, showing a systematic reduction in estimation errors and supporting the consistency of all considered methods. Because the MSE accounts for both bias and variance components, its decreasing trend with increasing sample size further confirms the simultaneous reduction in both estimation bias and variability.

Overall, the results indicate that all estimators exhibit improved performance as the sample size increases, as reflected by the decreasing patterns of AB, MSE, and MRE. In addition, the estimators perform satisfactorily across all considered parameter configurations, confirming the robustness of the proposed model under different distributional shapes and parameter regime.

Table 3. Simulation results for $\boldsymbol{\psi} = (\kappa = 0.75, \nu = 0.5, \zeta = 1.25, \omega = 0.25)$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	AB	$\hat{\kappa}$	0.22623 [8]	0.11737 [2]	0.13796 [5]	0.12056 [3]	0.21886 [7]	0.08207 [1]	0.13132 [4]	0.14347 [6]
		$\hat{\nu}$	0.18834 [8]	0.15411 [7]	0.15265 [6]	0.15166 [5]	0.09494 [1]	0.12101 [2]	0.14937 [4]	0.14931 [3]
		$\hat{\zeta}$	0.18248 [8]	0.11650 [1]	0.12069 [3]	0.12476 [6]	0.15910 [7]	0.12058 [2]	0.12134 [5]	0.12097 [4]
		$\hat{\omega}$	0.06351 [7]	0.04128 [2]	0.04205 [3]	0.04309 [4]	0.05220 [6]	0.10208 [8]	0.03961 [1]	0.04514 [5]
	MSE	$\hat{\kappa}$	0.08037 [8]	0.02309 [2]	0.03124 [5]	0.02398 [3]	0.08033 [7]	0.01243 [1]	0.02962 [4]	0.03327 [6]
		$\hat{\nu}$	0.05975 [7]	0.03921 [6]	0.03868 [5]	0.03721 [4]	0.07210 [8]	0.02305 [1]	0.03688 [3]	0.03632 [2]
		$\hat{\zeta}$	0.07297 [8]	0.02530 [2]	0.02765 [4]	0.03040 [6]	0.05199 [7]	0.02455 [1]	0.02781 [5]	0.02760 [3]
		$\hat{\omega}$	0.00877 [7]	0.00290 [2]	0.00316 [3]	0.00335 [4]	0.00531 [6]	0.02090 [8]	0.00283 [1]	0.00361 [5]
	MRE	$\hat{\kappa}$	0.30163 [8]	0.15650 [2]	0.18394 [5]	0.16075 [3]	0.29181 [7]	0.10943 [1]	0.17510 [4]	0.19129 [6]
		$\hat{\nu}$	0.37667 [7]	0.30822 [6]	0.30530 [5]	0.30331 [4]	0.40279 [8]	0.24201 [1]	0.29873 [3]	0.29863 [2]
		$\hat{\zeta}$	0.14598 [8]	0.09320 [1]	0.09655 [3]	0.09981 [6]	0.12728 [7]	0.09646 [2]	0.09707 [5]	0.09678 [4]
		$\hat{\omega}$	0.25402 [7]	0.16514 [2]	0.16819 [3]	0.17235 [4]	0.20879 [6]	0.40832 [8]	0.15845 [1]	0.18057 [5]
		$\sum$ *Ranks*	91 [8]	35 [1]	50 [4]	52 [6]	77 [7]	36 [2]	40 [3]	51 [5]
150	AB	$\hat{\kappa}$	0.14068 [8]	0.07549 [2]	0.08724 [5]	0.07588 [3]	0.13816 [7]	0.06507 [1]	0.07948 [4]	0.09491 [6]
		$\hat{\nu}$	0.12011 [7]	0.09391 [5]	0.09347 [4]	0.09134 [2]	0.12681 [8]	0.08272 [1]	0.09246 [3]	0.09525 [6]
		$\hat{\zeta}$	0.11046 [8]	0.07960 [1]	0.08133 [3]	0.08519 [5]	0.09889 [7]	0.09687 [6]	0.08200 [4]	0.07975 [2]
		$\hat{\omega}$	0.03216 [7]	0.02349 [2]	0.02375 [3]	0.02382 [4]	0.02926 [6]	0.06018 [8]	0.02192 [1]	0.02540 [5]
	MSE	$\hat{\kappa}$	0.03270 [7]	0.01051 [2]	0.01400 [5]	0.01058 [3]	0.03311 [8]	0.00855 [1]	0.01200 [4]	0.01604 [6]
		$\hat{\nu}$	0.02469 [7]	0.01549 [5]	0.01491 [4]	0.01452 [2]	0.02828 [8]	0.01069 [1]	0.01481 [3]	0.01638 [6]
		$\hat{\zeta}$	0.02307 [8]	0.01155 [1]	0.01269 [3]	0.01338 [5]	0.01940 [7]	0.01660 [6]	0.01271 [4]	0.01188 [2]
		$\hat{\omega}$	0.00211 [7]	0.00098 [2]	0.00102 [3.5]	0.00102 [3.5]	0.00141 [6]	0.00584 [8]	0.00085 [1]	0.00107 [5]
	MRE	$\hat{\kappa}$	0.18757 [8]	0.10066 [2]	0.11632 [5]	0.10117 [3]	0.18422 [7]	0.08676 [1]	0.10598 [4]	0.12655 [6]
		$\hat{\nu}$	0.24022 [7]	0.18782 [5]	0.18694 [4]	0.18268 [2]	0.25363 [8]	0.16543 [1]	0.18493 [3]	0.19050 [6]
		$\hat{\zeta}$	0.08837 [8]	0.06368 [1]	0.06506 [3]	0.06815 [5]	0.07911 [7]	0.07750 [6]	0.06560 [4]	0.06380 [2]
		$\hat{\omega}$	0.12863 [7]	0.09394 [2]	0.09500 [3]	0.09526 [4]	0.11706 [6]	0.24074 [8]	0.08769 [1]	0.10160 [5]
		$\sum$ *Ranks*	89 [8]	30 [1]	45.5 [4]	41.5 [3]	85 [7]	48 [5]	36 [2]	57 [6]
350	AB	$\hat{\kappa}$	0.09740 [7]	0.05205 [1]	0.05958 [4]	0.05263 [2]	0.10210 [8]	0.06292 [5]	0.05346 [3]	0.06549 [6]
		$\hat{\nu}$	0.09068 [7]	0.06774 [5]	0.06600 [2]	0.06539 [1]	0.09494 [8]	0.06989 [6]	0.06638 [3]	0.06729 [4]
		$\hat{\zeta}$	0.08704 [7]	0.06512 [3]	0.06484 [2]	0.06832 [5]	0.08275 [6]	0.09240 [8]	0.06568 [4]	0.06414 [1]
		$\hat{\omega}$	0.02065 [6]	0.01629 [3]	0.01643 [4]	0.01622 [2]	0.02161 [7]	0.04714 [8]	0.01522 [1]	0.01805 [5]
	MSE	$\hat{\kappa}$	0.01640 [7]	0.00536 [1]	0.00715 [4]	0.00545 [2]	0.01799 [8]	0.00802 [5]	0.00563 [3]	0.00806 [6]
		$\hat{\nu}$	0.01438 [7]	0.00829 [4]	0.00770 [2]	0.00756 [1]	0.01600 [8]	0.00843 [6]	0.00785 [3]	0.00837 [5]
		$\hat{\zeta}$	0.01452 [6.5]	0.00766 [3]	0.00754 [1]	0.00839 [5]	0.01452 [6.5]	0.01672 [8]	0.00774 [4]	0.00755 [2]
		$\hat{\omega}$	0.00077 [7]	0.00047 [3]	0.00048 [4]	0.00046 [2]	0.00075 [6]	0.00351 [8]	0.00040 [1]	0.00053 [5]
	MRE	$\hat{\kappa}$	0.12987 [7]	0.06940 [1]	0.07944 [4]	0.07017 [2]	0.13613 [8]	0.08389 [5]	0.07128 [3]	0.08732 [6]
		$\hat{\nu}$	0.18137 [7]	0.13548 [5]	0.13200 [2]	0.13077 [1]	0.18988 [8]	0.13978 [6]	0.13275 [3]	0.13459 [4]
		$\hat{\zeta}$	0.06963 [7]	0.05209 [3]	0.05188 [2]	0.05466 [5]	0.06620 [6]	0.07392 [8]	0.05254 [4]	0.05132 [1]
		$\hat{\omega}$	0.08260 [6]	0.06516 [3]	0.06571 [4]	0.06487 [2]	0.08642 [7]	0.18854 [8]	0.06088 [1]	0.07220 [5]
		$\sum$ *Ranks*	81.5 [7]	35 [3.5]	35 [3.5]	30 [1]	86.5 [8]	81 [6]	33 [2]	50 [5]
500	AB	$\hat{\kappa}$	0.07733 [7]	0.03907 [1]	0.04229 [4]	0.03923 [2]	0.08052 [8]	0.05969 [6]	0.04156 [3]	0.04920 [5]
		$\hat{\nu}$	0.07196 [7]	0.05371 [5]	0.05178 [2]	0.05038 [1]	0.07806 [8]	0.06179 [6]	0.05343 [4]	0.05280 [3]
		$\hat{\zeta}$	0.07484 [7]	0.05772 [3]	0.05421 [2]	0.05921 [5]	0.07412 [6]	0.08569 [8]	0.05800 [4]	0.05347 [1]
		$\hat{\omega}$	0.01612 [6]	0.01289 [4]	0.01251 [3]	0.01248 [2]	0.01749 [7]	0.03838 [8]	0.01218 [1]	0.01370 [5]
	MSE	$\hat{\kappa}$	0.01024 [7]	0.00300 [1]	0.00375 [4]	0.00309 [2]	0.01159 [8]	0.00716 [6]	0.00347 [3]	0.00475 [5]
		$\hat{\nu}$	0.00919 [7]	0.00497 [4]	0.00485 [2]	0.00452 [1]	0.01097 [8]	0.00684 [6]	0.00495 [3]	0.00505 [5]
		$\hat{\zeta}$	0.01085 [6]	0.00579 [3]	0.00510 [2]	0.00620 [5]	0.01195 [7]	0.01423 [8]	0.00588 [4]	0.00500 [1]
		$\hat{\omega}$	0.00046 [6]	0.00029 [4]	0.00028 [2.5]	0.00028 [2.5]	0.00049 [7]	0.00238 [8]	0.00025 [1]	0.00031 [5]
	MRE	$\hat{\kappa}$	0.10311 [7]	0.05210 [1]	0.05639 [4]	0.05231 [2]	0.10735 [8]	0.07959 [6]	0.05541 [3]	0.06559 [5]
		$\hat{\nu}$	0.14393 [7]	0.10742 [5]	0.10356 [2]	0.10075 [1]	0.15612 [8]	0.12358 [6]	0.10686 [4]	0.10561 [3]
		$\hat{\zeta}$	0.05987 [7]	0.04617 [3]	0.04337 [2]	0.04737 [5]	0.05930 [6]	0.06855 [8]	0.04640 [4]	0.04278 [1]
		$\hat{\omega}$	0.06447 [6]	0.05157 [4]	0.05006 [3]	0.04994 [2]	0.06994 [7]	0.15351 [8]	0.04872 [1]	0.05480 [5]
		$\sum$ *Ranks*	80 [6]	38 [4]	32.5 [2]	30.5 [1]	88 [8]	84 [7]	35 [3]	44 [5]

Table 4. Simulation results for $\boldsymbol{\psi} = (\kappa = 0.5, \nu = 0.5, \zeta = 0.25, \omega = 0.25)$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	AB	$\hat{\kappa}$	0.12613 $^{\{8\}}$	0.07599 $^{\{2\}}$	0.08770 $^{\{4\}}$	0.07911 $^{\{3\}}$	0.11800 $^{\{7\}}$	0.05647 $^{\{1\}}$	0.08876 $^{\{5\}}$	0.09254 $^{\{6\}}$
		$\hat{\nu}$	0.13463 $^{\{8\}}$	0.08285 $^{\{3\}}$	0.09228 $^{\{6\}}$	0.08291 $^{\{4\}}$	0.06637 $^{\{2\}}$	0.05094 $^{\{1\}}$	0.09259 $^{\{7\}}$	0.08856 $^{\{5\}}$
		$\hat{\zeta}$	0.05550 $^{\{7\}}$	0.04229 $^{\{1\}}$	0.04476 $^{\{3\}}$	0.04447 $^{\{2\}}$	0.05215 $^{\{6\}}$	0.06843 $^{\{8\}}$	0.04499 $^{\{4\}}$	0.04583 $^{\{5\}}$
		$\hat{\omega}$	0.04199 $^{\{7\}}$	0.03678 $^{\{4\}}$	0.03524 $^{\{3\}}$	0.03879 $^{\{6\}}$	0.03719 $^{\{5\}}$	0.04570 $^{\{8\}}$	0.03134 $^{\{1\}}$	0.03170 $^{\{2\}}$
	MSE	$\hat{\kappa}$	0.02558 $^{\{8\}}$	0.00965 $^{\{2\}}$	0.01257 $^{\{4\}}$	0.01029 $^{\{3\}}$	0.02314 $^{\{7\}}$	0.00609 $^{\{1\}}$	0.01342 $^{\{5\}}$	0.01527 $^{\{6\}}$
		$\hat{\nu}$	0.03050 $^{\{8\}}$	0.01193 $^{\{3\}}$	0.01504 $^{\{5\}}$	0.01192 $^{\{2\}}$	0.02687 $^{\{7\}}$	0.00525 $^{\{1\}}$	0.01510 $^{\{6\}}$	0.01377 $^{\{4\}}$
		$\hat{\zeta}$	0.00523 $^{\{7\}}$	0.00301 $^{\{1\}}$	0.00343 $^{\{3\}}$	0.00332 $^{\{2\}}$	0.00472 $^{\{6\}}$	0.00883 $^{\{8\}}$	0.00347 $^{\{4\}}$	0.00357 $^{\{5\}}$
		$\hat{\omega}$	0.00363 $^{\{8\}}$	0.00240 $^{\{4\}}$	0.00223 $^{\{3\}}$	0.00281 $^{\{6\}}$	0.00259 $^{\{5\}}$	0.00342 $^{\{7\}}$	0.00174 $^{\{1\}}$	0.00191 $^{\{2\}}$
	MRE	$\hat{\kappa}$	0.25227 $^{\{8\}}$	0.15199 $^{\{2\}}$	0.17540 $^{\{4\}}$	0.15822 $^{\{3\}}$	0.23600 $^{\{7\}}$	0.11294 $^{\{1\}}$	0.17751 $^{\{5\}}$	0.18508 $^{\{6\}}$
		$\hat{\nu}$	0.26926 $^{\{8\}}$	0.16570 $^{\{2\}}$	0.18456 $^{\{5\}}$	0.16582 $^{\{3\}}$	0.24943 $^{\{7\}}$	0.10187 $^{\{1\}}$	0.18517 $^{\{6\}}$	0.17712 $^{\{4\}}$
		$\hat{\zeta}$	0.22200 $^{\{7\}}$	0.16917 $^{\{1\}}$	0.17906 $^{\{3\}}$	0.17787 $^{\{2\}}$	0.20860 $^{\{6\}}$	0.27371 $^{\{8\}}$	0.17997 $^{\{4\}}$	0.18331 $^{\{5\}}$
		$\hat{\omega}$	0.16795 $^{\{7\}}$	0.14714 $^{\{4\}}$	0.14097 $^{\{3\}}$	0.15515 $^{\{6\}}$	0.14877 $^{\{5\}}$	0.18279 $^{\{8\}}$	0.12538 $^{\{1\}}$	0.12682 $^{\{2\}}$
		$\sum Ranks$	91 $^{\{8\}}$	29 $^{\{1\}}$	46 $^{\{3\}}$	42 $^{\{2\}}$	70 $^{\{7\}}$	53 $^{\{6\}}$	49 $^{\{4\}}$	52 $^{\{5\}}$
150	AB	$\hat{\kappa}$	0.06918 $^{\{7\}}$	0.05128 $^{\{2\}}$	0.05827 $^{\{6\}}$	0.05216 $^{\{3\}}$	0.07032 $^{\{8\}}$	0.04879 $^{\{1\}}$	0.05708 $^{\{4\}}$	0.05770 $^{\{5\}}$
		$\hat{\nu}$	0.08334 $^{\{7\}}$	0.05962 $^{\{3\}}$	0.06472 $^{\{6\}}$	0.05847 $^{\{2\}}$	0.08528 $^{\{8\}}$	0.04518 $^{\{1\}}$	0.06302 $^{\{5\}}$	0.06114 $^{\{4\}}$
		$\hat{\zeta}$	0.03038 $^{\{6\}}$	0.02741 $^{\{1\}}$	0.02836 $^{\{5\}}$	0.02797 $^{\{2\}}$	0.03146 $^{\{7\}}$	0.04871 $^{\{8\}}$	0.02798 $^{\{3\}}$	0.02822 $^{\{4\}}$
		$\hat{\omega}$	0.01956 $^{\{3\}}$	0.02176 $^{\{6\}}$	0.02004 $^{\{5\}}$	0.02187 $^{\{7\}}$	0.01959 $^{\{4\}}$	0.02802 $^{\{8\}}$	0.01864 $^{\{2\}}$	0.01807 $^{\{1\}}$
	MSE	$\hat{\kappa}$	0.00791 $^{\{7\}}$	0.00487 $^{\{1\}}$	0.00594 $^{\{5\}}$	0.00498 $^{\{2\}}$	0.00832 $^{\{8\}}$	0.00508 $^{\{3\}}$	0.00582 $^{\{4\}}$	0.00625 $^{\{6\}}$
		$\hat{\nu}$	0.01237 $^{\{7\}}$	0.00680 $^{\{3\}}$	0.00787 $^{\{6\}}$	0.00643 $^{\{2\}}$	0.01315 $^{\{8\}}$	0.00436 $^{\{1\}}$	0.00734 $^{\{4\}}$	0.00750 $^{\{5\}}$
		$\hat{\zeta}$	0.00171 $^{\{6\}}$	0.00135 $^{\{1\}}$	0.00147 $^{\{4\}}$	0.00143 $^{\{2\}}$	0.00190 $^{\{7\}}$	0.00478 $^{\{8\}}$	0.00144 $^{\{3\}}$	0.00149 $^{\{5\}}$
		$\hat{\omega}$	0.00071 $^{\{5\}}$	0.00078 $^{\{6\}}$	0.00068 $^{\{4\}}$	0.00084 $^{\{7\}}$	0.00062 $^{\{3\}}$	0.00130 $^{\{8\}}$	0.00057 $^{\{2\}}$	0.00054 $^{\{1\}}$
	MRE	$\hat{\kappa}$	0.13836 $^{\{7\}}$	0.10256 $^{\{2\}}$	0.11654 $^{\{6\}}$	0.10432 $^{\{3\}}$	0.14064 $^{\{8\}}$	0.09758 $^{\{1\}}$	0.11417 $^{\{4\}}$	0.11540 $^{\{5\}}$
		$\hat{\nu}$	0.16669 $^{\{7\}}$	0.11923 $^{\{3\}}$	0.12944 $^{\{6\}}$	0.11694 $^{\{2\}}$	0.17055 $^{\{8\}}$	0.09036 $^{\{1\}}$	0.12604 $^{\{5\}}$	0.12229 $^{\{4\}}$
		$\hat{\zeta}$	0.12150 $^{\{6\}}$	0.10965 $^{\{1\}}$	0.11343 $^{\{5\}}$	0.11190 $^{\{2\}}$	0.12583 $^{\{7\}}$	0.19486 $^{\{8\}}$	0.11192 $^{\{3\}}$	0.11287 $^{\{4\}}$
		$\hat{\omega}$	0.07823 $^{\{3\}}$	0.08706 $^{\{6\}}$	0.08016 $^{\{5\}}$	0.08748 $^{\{7\}}$	0.07834 $^{\{4\}}$	0.11207 $^{\{8\}}$	0.07457 $^{\{2\}}$	0.07229 $^{\{1\}}$
		$\sum Ranks$	71 $^{\{7\}}$	35 $^{\{1\}}$	63 $^{\{6\}}$	41 $^{\{2.5\}}$	80 $^{\{8\}}$	56 $^{\{5\}}$	41 $^{\{2.5\}}$	45 $^{\{4\}}$
350	AB	$\hat{\kappa}$	0.04761 $^{\{7\}}$	0.03597 $^{\{1\}}$	0.04081 $^{\{5\}}$	0.03665 $^{\{2\}}$	0.04938 $^{\{8\}}$	0.04343 $^{\{6\}}$	0.04068 $^{\{4\}}$	0.04054 $^{\{3\}}$
		$\hat{\nu}$	0.06260 $^{\{7\}}$	0.04635 $^{\{4\}}$	0.05127 $^{\{6\}}$	0.04505 $^{\{2\}}$	0.06637 $^{\{8\}}$	0.04089 $^{\{1\}}$	0.04876 $^{\{5\}}$	0.04582 $^{\{3\}}$
		$\hat{\zeta}$	0.02282 $^{\{6\}}$	0.02060 $^{\{4\}}$	0.02000 $^{\{1\}}$	0.02041 $^{\{3\}}$	0.02421 $^{\{7\}}$	0.03753 $^{\{8\}}$	0.02025 $^{\{2\}}$	0.02110 $^{\{5\}}$
		$\hat{\omega}$	0.01255 $^{\{1\}}$	0.01522 $^{\{6\}}$	0.01394 $^{\{5\}}$	0.01526 $^{\{7\}}$	0.01380 $^{\{4\}}$	0.02032 $^{\{8\}}$	0.01327 $^{\{3\}}$	0.01282 $^{\{2\}}$
	MSE	$\hat{\kappa}$	0.00381 $^{\{6\}}$	0.00260 $^{\{1\}}$	0.00307 $^{\{3\}}$	0.00270 $^{\{2\}}$	0.00418 $^{\{8\}}$	0.00402 $^{\{7\}}$	0.00311 $^{\{4\}}$	0.00324 $^{\{5\}}$
		$\hat{\nu}$	0.00745 $^{\{7\}}$	0.00432 $^{\{4\}}$	0.00501 $^{\{6\}}$	0.00406 $^{\{2\}}$	0.00857 $^{\{8\}}$	0.00355 $^{\{1\}}$	0.00459 $^{\{5\}}$	0.00413 $^{\{3\}}$
		$\hat{\zeta}$	0.00105 $^{\{6\}}$	0.00075 $^{\{2\}}$	0.00074 $^{\{1\}}$	0.00076 $^{\{3\}}$	0.00129 $^{\{7\}}$	0.00294 $^{\{8\}}$	0.00077 $^{\{4\}}$	0.00084 $^{\{5\}}$
		$\hat{\omega}$	0.00028 $^{\{2\}}$	0.00039 $^{\{6\}}$	0.00032 $^{\{5\}}$	0.00040 $^{\{7\}}$	0.00030 $^{\{4\}}$	0.00071 $^{\{8\}}$	0.00029 $^{\{3\}}$	0.00027 $^{\{1\}}$
	MRE	$\hat{\kappa}$	0.09522 $^{\{7\}}$	0.07195 $^{\{1\}}$	0.08161 $^{\{5\}}$	0.07330 $^{\{2\}}$	0.09875 $^{\{8\}}$	0.08685 $^{\{6\}}$	0.08137 $^{\{4\}}$	0.08107 $^{\{3\}}$
		$\hat{\nu}$	0.12521 $^{\{7\}}$	0.09271 $^{\{4\}}$	0.10254 $^{\{6\}}$	0.09010 $^{\{2\}}$	0.13273 $^{\{8\}}$	0.08179 $^{\{1\}}$	0.09751 $^{\{5\}}$	0.09164 $^{\{3\}}$
		$\hat{\zeta}$	0.09128 $^{\{6\}}$	0.08239 $^{\{4\}}$	0.08000 $^{\{1\}}$	0.08165 $^{\{3\}}$	0.09686 $^{\{7\}}$	0.15013 $^{\{8\}}$	0.08100 $^{\{2\}}$	0.08438 $^{\{5\}}$
		$\hat{\omega}$	0.05020 $^{\{1\}}$	0.06088 $^{\{6\}}$	0.05574 $^{\{5\}}$	0.06105 $^{\{7\}}$	0.05521 $^{\{4\}}$	0.08129 $^{\{8\}}$	0.05309 $^{\{3\}}$	0.05130 $^{\{2\}}$
		$\sum Ranks$	63 $^{\{6\}}$	43 $^{\{3\}}$	49 $^{\{5\}}$	42 $^{\{2\}}$	81 $^{\{8\}}$	70 $^{\{7\}}$	44 $^{\{4\}}$	40 $^{\{1\}}$
500	AB	$\hat{\kappa}$	0.03674 $^{\{6\}}$	0.02720 $^{\{2\}}$	0.03020 $^{\{3\}}$	0.02633 $^{\{1\}}$	0.03756 $^{\{7\}}$	0.04035 $^{\{8\}}$	0.03021 $^{\{4.5\}}$	0.03021 $^{\{4.5\}}$
		$\hat{\nu}$	0.05142 $^{\{7\}}$	0.03814 $^{\{3\}}$	0.04189 $^{\{6\}}$	0.03550 $^{\{1\}}$	0.05351 $^{\{8\}}$	0.03888 $^{\{4\}}$	0.03920 $^{\{5\}}$	0.03742 $^{\{2\}}$
		$\hat{\zeta}$	0.01850 $^{\{6\}}$	0.01625 $^{\{2\}}$	0.01568 $^{\{1\}}$	0.01628 $^{\{3\}}$	0.02087 $^{\{7\}}$	0.03041 $^{\{8\}}$	0.01650 $^{\{5\}}$	0.01635 $^{\{4\}}$
		$\hat{\omega}$	0.00973 $^{\{1.5\}}$	0.01165 $^{\{7\}}$	0.01019 $^{\{3\}}$	0.01161 $^{\{6\}}$	0.01063 $^{\{5\}}$	0.01587 $^{\{8\}}$	0.01023 $^{\{4\}}$	0.00973 $^{\{1.5\}}$
	MSE	$\hat{\kappa}$	0.00222 $^{\{6\}}$	0.00151 $^{\{2\}}$	0.00177 $^{\{3.5\}}$	0.00147 $^{\{1\}}$	0.00236 $^{\{7\}}$	0.00351 $^{\{8\}}$	0.00177 $^{\{3.5\}}$	0.00188 $^{\{5\}}$
		$\hat{\nu}$	0.00493 $^{\{7\}}$	0.00303 $^{\{4\}}$	0.00336 $^{\{6\}}$	0.00261 $^{\{1\}}$	0.00565 $^{\{8\}}$	0.00327 $^{\{5\}}$	0.00298 $^{\{3\}}$	0.00288 $^{\{2\}}$
		$\hat{\zeta}$	0.00065 $^{\{6\}}$	0.00048 $^{\{2.5\}}$	0.00046 $^{\{1\}}$	0.00048 $^{\{2.5\}}$	0.00096 $^{\{7\}}$	0.00192 $^{\{8\}}$	0.00051 $^{\{4\}}$	0.00052 $^{\{5\}}$
		$\hat{\omega}$	0.00016 $^{\{2\}}$	0.00022 $^{\{6.5\}}$	0.00018 $^{\{4.5\}}$	0.00022 $^{\{6.5\}}$	0.00018 $^{\{4.5\}}$	0.00043 $^{\{8\}}$	0.00017 $^{\{3\}}$	0.00015 $^{\{1\}}$
	MRE	$\hat{\kappa}$	0.07348 $^{\{6\}}$	0.05440 $^{\{2\}}$	0.06039 $^{\{3\}}$	0.05266 $^{\{1\}}$	0.07512 $^{\{7\}}$	0.08070 $^{\{8\}}$	0.06042 $^{\{4.5\}}$	0.06042 $^{\{4.5\}}$
		$\hat{\nu}$	0.10283 $^{\{7\}}$	0.07628 $^{\{3\}}$	0.08378 $^{\{6\}}$	0.07099 $^{\{1\}}$	0.10702 $^{\{8\}}$	0.07776 $^{\{4\}}$	0.07840 $^{\{5\}}$	0.07483 $^{\{2\}}$
		$\hat{\zeta}$	0.07401 $^{\{6\}}$	0.06501 $^{\{2\}}$	0.06274 $^{\{1\}}$	0.06512 $^{\{3\}}$	0.08349 $^{\{7\}}$	0.12163 $^{\{8\}}$	0.06599 $^{\{5\}}$	0.06540 $^{\{4\}}$
		$\hat{\omega}$	0.03893 $^{\{1.5\}}$	0.04661 $^{\{7\}}$	0.04077 $^{\{3\}}$	0.04643 $^{\{6\}}$	0.04253 $^{\{5\}}$	0.06350 $^{\{8\}}$	0.04091 $^{\{4\}}$	0.03893 $^{\{1.5\}}$
		$\sum Ranks$	62 $^{\{6\}}$	43 $^{\{4\}}$	41 $^{\{3\}}$	33 $^{\{1\}}$	80.5 $^{\{7\}}$	85 $^{\{8\}}$	50.5 $^{\{5\}}$	37 $^{\{2\}}$

Table 5. Simulation results for $\boldsymbol{\psi} = (\kappa = 1.25, \nu = 0.5, \zeta = 1.25, \omega = 0.25)$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	AB	$\hat{\kappa}$	0.31810 [8]	0.12778 [2]	0.15803 [6]	0.12824 [3]	0.30649 [7]	0.07731 [1]	0.15449 [5]	0.14747 [4]
		$\hat{\nu}$	0.21001 [8]	0.13820 [6]	0.14840 [7]	0.13512 [5]	0.10977 [2]	0.09629 [1]	0.13466 [4]	0.12770 [3]
		$\hat{\zeta}$	0.17842 [8]	0.11185 [3]	0.11296 [4]	0.11795 [6]	0.15547 [7]	0.10772 [2]	0.11570 [5]	0.10736 [1]
		$\hat{\omega}$	0.04747 [7]	0.03908 [3]	0.03766 [2]	0.04205 [6]	0.03972 [5]	0.07930 [8]	0.03573 [1]	0.03958 [4]
	MSE	$\hat{\kappa}$	0.16089 [8]	0.03046 [2]	0.04471 [6]	0.03078 [3]	0.14991 [7]	0.01166 [1]	0.04393 [5]	0.04002 [4]
		$\hat{\nu}$	0.07514 [7]	0.03383 [5]	0.03801 [6]	0.03216 [3]	0.09264 [8]	0.01464 [1]	0.03279 [4]	0.02860 [2]
		$\hat{\zeta}$	0.05946 [8]	0.02422 [3]	0.02448 [4]	0.02904 [6]	0.04484 [7]	0.01934 [1]	0.02577 [5]	0.02176 [2]
		$\hat{\omega}$	0.00468 [7]	0.00263 [3]	0.00244 [2]	0.00333 [6]	0.00276 [4]	0.01119 [8]	0.00222 [1]	0.00284 [5]
	MRE	$\hat{\kappa}$	0.25448 [8]	0.10223 [2]	0.12642 [6]	0.10259 [3]	0.24519 [7]	0.06185 [1]	0.12359 [5]	0.11798 [4]
		$\hat{\nu}$	0.42003 [7]	0.27640 [5]	0.29681 [6]	0.27025 [4]	0.46184 [8]	0.19258 [1]	0.26932 [3]	0.25540 [2]
		$\hat{\zeta}$	0.14273 [8]	0.08948 [3]	0.09037 [4]	0.09436 [6]	0.12438 [7]	0.08617 [2]	0.09256 [5]	0.08589 [1]
		$\hat{\omega}$	0.18989 [7]	0.15630 [3]	0.15062 [2]	0.16819 [6]	0.15888 [5]	0.31719 [8]	0.14294 [1]	0.15832 [4]
		$\sum Ranks$	91 [8]	40 [3]	55 [5]	57 [6]	74 [7]	35 [1]	44 [4]	36 [2]
150	AB	$\hat{\kappa}$	0.21422 [8]	0.08786 [3]	0.10354 [6]	0.08480 [2]	0.20181 [7]	0.06505 [1]	0.09839 [4]	0.10268 [5]
		$\hat{\nu}$	0.14196 [7]	0.08703 [5]	0.08921 [6]	0.08329 [2.5]	0.15106 [8]	0.07051 [1]	0.08329 [2.5]	0.08454 [4]
		$\hat{\zeta}$	0.11202 [8]	0.07401 [2]	0.07458 [3]	0.07469 [4]	0.10342 [7]	0.09238 [6]	0.07559 [5]	0.06807 [1]
		$\hat{\omega}$	0.02336 [7]	0.02241 [5]	0.02033 [2]	0.02229 [4]	0.02305 [6]	0.05003 [8]	0.01985 [1]	0.02151 [3]
	MSE	$\hat{\kappa}$	0.07773 [8]	0.01550 [3]	0.02092 [6]	0.01464 [2]	0.07211 [7]	0.00851 [1]	0.01969 [4]	0.02032 [5]
		$\hat{\nu}$	0.03469 [7]	0.01467 [6]	0.01447 [5]	0.01328 [3]	0.04168 [8]	0.00799 [1]	0.01326 [2]	0.01333 [4]
		$\hat{\zeta}$	0.02368 [8]	0.01042 [2]	0.01044 [3]	0.01052 [4]	0.02194 [7]	0.01594 [6]	0.01076 [5]	0.00895 [1]
		$\hat{\omega}$	0.00099 [7]	0.00084 [5]	0.00068 [2]	0.00084 [5]	0.00084 [5]	0.00402 [8]	0.00066 [1]	0.00074 [3]
	MRE	$\hat{\kappa}$	0.17138 [8]	0.07029 [3]	0.08284 [6]	0.06784 [2]	0.16145 [7]	0.05204 [1]	0.07871 [4]	0.08214 [5]
		$\hat{\nu}$	0.28393 [7]	0.17406 [5]	0.17842 [6]	0.16658 [2.5]	0.30213 [8]	0.14101 [1]	0.16658 [2.5]	0.16908 [4]
		$\hat{\zeta}$	0.08962 [8]	0.05921 [2]	0.05966 [3]	0.05975 [4]	0.08274 [7]	0.07391 [6]	0.06047 [5]	0.05445 [1]
		$\hat{\omega}$	0.09346 [7]	0.08965 [5]	0.08133 [2]	0.08916 [4]	0.09218 [6]	0.20010 [8]	0.07941 [1]	0.08604 [3]
		$\sum Ranks$	90 [8]	46 [4]	50 [6]	39 [2.5]	83 [7]	48 [5]	37 [1]	39 [2.5]
350	AB	$\hat{\kappa}$	0.15445 [8]	0.06237 [2]	0.07218 [5]	0.06099 [1]	0.15290 [7]	0.06371 [3]	0.06721 [4]	0.07408 [6]
		$\hat{\nu}$	0.10394 [7]	0.06119 [5]	0.06339 [6]	0.05722 [1]	0.10977 [8]	0.05902 [3]	0.05760 [2]	0.06041 [4]
		$\hat{\zeta}$	0.08584 [6]	0.05669 [2]	0.05850 [4]	0.06025 [5]	0.08619 [7]	0.08753 [8]	0.05758 [3]	0.05100 [1]
		$\hat{\omega}$	0.01653 [6]	0.01533 [4]	0.01417 [2]	0.01544 [5]	0.01808 [7]	0.03886 [8]	0.01405 [1]	0.01510 [3]
	MSE	$\hat{\kappa}$	0.04279 [7]	0.00801 [2]	0.01124 [6]	0.00765 [1]	0.04358 [8]	0.00830 [3]	0.01018 [4]	0.01102 [5]
		$\hat{\nu}$	0.01961 [7]	0.00741 [5]	0.00786 [6]	0.00638 [2]	0.02239 [8]	0.00624 [1]	0.00656 [3]	0.00726 [4]
		$\hat{\zeta}$	0.01438 [6]	0.00605 [3]	0.00620 [4]	0.00679 [5]	0.01591 [8]	0.01496 [7]	0.00603 [2]	0.00485 [1]
		$\hat{\omega}$	0.00047 [6]	0.00039 [4.5]	0.00033 [2]	0.00039 [4.5]	0.00051 [7]	0.00245 [8]	0.00032 [1]	0.00036 [3]
	MRE	$\hat{\kappa}$	0.12356 [8]	0.04990 [2]	0.05774 [5]	0.04879 [1]	0.12232 [7]	0.05097 [3]	0.05376 [4]	0.05927 [6]
		$\hat{\nu}$	0.20787 [7]	0.12238 [5]	0.12678 [6]	0.11445 [1]	0.21955 [8]	0.11804 [3]	0.11520 [2]	0.12081 [4]
		$\hat{\zeta}$	0.06867 [6]	0.04535 [2]	0.04680 [4]	0.04820 [5]	0.06896 [7]	0.07002 [8]	0.04606 [3]	0.04080 [1]
		$\hat{\omega}$	0.06613 [6]	0.06132 [4]	0.05667 [2]	0.06176 [5]	0.07231 [7]	0.15543 [8]	0.05620 [1]	0.06041 [3]
		$\sum Ranks$	80 [7]	40.5 [3]	52 [5]	36.5 [2]	89 [8]	63 [6]	30 [1]	41 [4]
500	AB	$\hat{\kappa}$	0.11934 [7]	0.04733 [1]	0.05232 [4]	0.04867 [2]	0.12141 [8]	0.06213 [6]	0.05119 [3]	0.05567 [5]
		$\hat{\nu}$	0.08060 [7]	0.04503 [3]	0.04676 [5]	0.04520 [4]	0.08792 [8]	0.05259 [6]	0.04379 [1]	0.04383 [2]
		$\hat{\zeta}$	0.07421 [6]	0.04840 [3]	0.04891 [4]	0.05199 [5]	0.07673 [7]	0.08036 [8]	0.04839 [2]	0.04211 [1]
		$\hat{\omega}$	0.01339 [6]	0.01210 [4]	0.01097 [2]	0.01213 [5]	0.01511 [7]	0.03167 [8]	0.01086 [1]	0.01199 [3]
	MSE	$\hat{\kappa}$	0.02641 [7]	0.00465 [1]	0.00614 [4]	0.00491 [2]	0.02804 [8]	0.00787 [6]	0.00560 [3]	0.00647 [5]
		$\hat{\nu}$	0.01192 [7]	0.00410 [3.5]	0.00427 [5]	0.00410 [3.5]	0.01465 [8]	0.00506 [6]	0.00373 [1]	0.00397 [2]
		$\hat{\zeta}$	0.01113 [6]	0.00431 [4]	0.00423 [3]	0.00491 [5]	0.01304 [8]	0.01267 [7]	0.00413 [2]	0.00314 [1]
		$\hat{\omega}$	0.00031 [6]	0.00025 [5]	0.00019 [1.5]	0.00024 [4]	0.00036 [7]	0.00163 [8]	0.00019 [1.5]	0.00023 [3]
	MRE	$\hat{\kappa}$	0.09547 [7]	0.03786 [1]	0.04186 [4]	0.03893 [2]	0.09713 [8]	0.04970 [6]	0.04095 [3]	0.04454 [5]
		$\hat{\nu}$	0.16120 [7]	0.09005 [3]	0.09353 [5]	0.09041 [4]	0.17584 [8]	0.10519 [6]	0.08759 [1]	0.08765 [2]
		$\hat{\zeta}$	0.05937 [6]	0.03872 [3]	0.03913 [4]	0.04160 [5]	0.06139 [7]	0.06429 [8]	0.03871 [2]	0.03369 [1]
		$\hat{\omega}$	0.05357 [6]	0.04840 [4]	0.04389 [2]	0.04852 [5]	0.06044 [7]	0.12669 [8]	0.04346 [1]	0.04797 [3]
		$\sum Ranks$	78 [6]	35.5 [3]	43.5 [4]	46.5 [5]	91 [8]	83 [7]	21.5 [1]	33 [2]

Table 6. Simulation results for $\boldsymbol{\psi} = (\kappa = 2.5, \nu = 1.25, \zeta = 1.75, \omega = 0.75)$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	AB	$\hat{\kappa}$	0.67273 {8}	0.19357 {1}	0.26401 {4}	0.20038 {2}	0.62165 {7}	0.27989 {6}	0.26941 {5}	0.21190 {3}
		$\hat{\nu}$	0.41417 {8}	0.18849 {2}	0.21164 {6}	0.19052 {3}	0.24341 {7}	0.19140 {4}	0.20745 {5}	0.16300 {1}
		$\hat{\zeta}$	0.17290 {8}	0.11685 {4}	0.11543 {2}	0.12512 {6}	0.15932 {7}	0.10205 {1}	0.12125 {5}	0.11590 {3}
		$\hat{\omega}$	0.14162 {8}	0.11045 {5}	0.10104 {2}	0.11460 {6}	0.11543 {7}	0.10655 {4}	0.09512 {1}	0.10109 {3}
	MSE	$\hat{\kappa}$	0.73104 {8}	0.07506 {1}	0.13959 {4}	0.08041 {2}	0.63286 {7}	0.16347 {6}	0.14474 {5}	0.09787 {3}
		$\hat{\nu}$	0.28685 {7}	0.06907 {4}	0.08748 {5}	0.06616 {2}	0.39666 {8}	0.06755 {3}	0.08776 {6}	0.04837 {1}
		$\hat{\zeta}$	0.05248 {8}	0.02512 {3}	0.02452 {2}	0.03002 {6}	0.04539 {7}	0.01917 {1}	0.02665 {5}	0.02518 {4}
		$\hat{\omega}$	0.03979 {8}	0.02113 {5}	0.01700 {2}	0.02484 {7}	0.02116 {6}	0.01762 {3}	0.01525 {1}	0.01826 {4}
	MRE	$\hat{\kappa}$	0.26909 {8}	0.07743 {1}	0.10560 {4}	0.08015 {2}	0.24866 {7}	0.11195 {6}	0.10776 {5}	0.08476 {3}
		$\hat{\nu}$	0.33134 {7}	0.15079 {2}	0.16931 {6}	0.15242 {3}	0.37638 {8}	0.15312 {4}	0.16596 {5}	0.13040 {1}
		$\hat{\zeta}$	0.09880 {8}	0.06677 {4}	0.06596 {2}	0.07150 {6}	0.09104 {7}	0.05832 {1}	0.06929 {5}	0.06623 {3}
		$\hat{\omega}$	0.18882 {8}	0.14726 {5}	0.13472 {2}	0.15280 {6}	0.15390 {7}	0.14206 {4}	0.12683 {1}	0.13479 {3}
		$\sum$ *Ranks*	94 {8}	37 {2}	41 {3}	51 {6}	85 {7}	43 {4}	49 {5}	32 {1}
150	AB	$\hat{\kappa}$	0.45629 {8}	0.13878 {1}	0.18851 {5}	0.14644 {2}	0.43634 {7}	0.21850 {6}	0.18489 {4}	0.15106 {3}
		$\hat{\nu}$	0.28705 {7}	0.12633 {2}	0.13802 {5}	0.12664 {3}	0.33705 {8}	0.14097 {6}	0.13578 {4}	0.11113 {1}
		$\hat{\zeta}$	0.11846 {7}	0.07375 {3}	0.07455 {4}	0.07612 {5}	0.12135 {8}	0.07120 {2}	0.07657 {6}	0.07003 {1}
		$\hat{\omega}$	0.07451 {7}	0.06304 {5}	0.05630 {2}	0.06045 {4}	0.07585 {8}	0.06753 {6}	0.05504 {1}	0.05873 {3}
	MSE	$\hat{\kappa}$	0.36599 {8}	0.03885 {1}	0.07207 {4}	0.04350 {2}	0.33069 {7}	0.09948 {6}	0.07224 {5}	0.04756 {3}
		$\hat{\nu}$	0.15166 {7}	0.03290 {2}	0.04111 {4}	0.03459 {3}	0.21975 {8}	0.04180 {5}	0.04316 {6}	0.02411 {1}
		$\hat{\zeta}$	0.02584 {7}	0.00929 {2}	0.00958 {3}	0.00999 {4}	0.02778 {8}	0.01005 {5}	0.01025 {6}	0.00888 {1}
		$\hat{\omega}$	0.00968 {8}	0.00627 {5}	0.00506 {2}	0.00586 {4}	0.00907 {7}	0.00674 {6}	0.00476 {1}	0.00536 {3}
	MRE	$\hat{\kappa}$	0.18252 {8}	0.05551 {1}	0.07540 {5}	0.05857 {2}	0.17454 {7}	0.08740 {6}	0.07395 {4}	0.06043 {3}
		$\hat{\nu}$	0.22964 {7}	0.10107 {2}	0.11041 {5}	0.10131 {3}	0.26964 {8}	0.11278 {6}	0.10862 {4}	0.08890 {1}
		$\hat{\zeta}$	0.06769 {7}	0.04214 {3}	0.04260 {4}	0.04350 {5}	0.06934 {8}	0.04069 {2}	0.04375 {6}	0.04002 {1}
		$\hat{\omega}$	0.09935 {7}	0.08406 {5}	0.07506 {2}	0.08060 {4}	0.10114 {8}	0.09004 {6}	0.07339 {1}	0.07831 {3}
		$\sum$ *Ranks*	88 {7}	32 {2}	45 {4}	41 {3}	92 {8}	62 {6}	48 {5}	24 {1}
350	AB	$\hat{\kappa}$	0.33688 {8}	0.10946 {1}	0.14457 {5}	0.11387 {2}	0.33080 {7}	0.17655 {6}	0.13894 {4}	0.12026 {3}
		$\hat{\nu}$	0.21300 {7}	0.09357 {3}	0.10275 {5}	0.09216 {2}	0.24341 {8}	0.11312 {6}	0.09929 {4}	0.09159 {1}
		$\hat{\zeta}$	0.10032 {7}	0.05883 {3}	0.05944 {4}	0.06017 {6}	0.10288 {8}	0.05800 {2}	0.05981 {5}	0.05291 {1}
		$\hat{\omega}$	0.05679 {7}	0.04531 {5}	0.04156 {2}	0.04458 {4}	0.06027 {8}	0.04988 {6}	0.04101 {1}	0.04269 {3}
	MSE	$\hat{\kappa}$	0.20731 {8}	0.02336 {1}	0.04341 {5}	0.02591 {2}	0.19669 {7}	0.06555 {6}	0.04161 {4}	0.02953 {3}
		$\hat{\nu}$	0.08608 {7}	0.01799 {3}	0.02266 {5}	0.01791 {2}	0.11552 {8}	0.02745 {6}	0.02217 {4}	0.01660 {1}
		$\hat{\zeta}$	0.01878 {7}	0.00573 {2}	0.00596 {4}	0.00587 {3}	0.02006 {8}	0.00651 {6}	0.00625 {5}	0.00489 {1}
		$\hat{\omega}$	0.00551 {7}	0.00315 {5}	0.00273 {2}	0.00308 {4}	0.00567 {8}	0.00379 {6}	0.00264 {1}	0.00281 {3}
	MRE	$\hat{\kappa}$	0.13475 {8}	0.04378 {1}	0.05783 {5}	0.04555 {2}	0.13232 {7}	0.07062 {6}	0.05558 {4}	0.04810 {3}
		$\hat{\nu}$	0.17040 {7}	0.07486 {3}	0.08220 {5}	0.07373 {2}	0.19473 {8}	0.09050 {6}	0.07944 {4}	0.07327 {1}
		$\hat{\zeta}$	0.05733 {7}	0.03361 {3}	0.03396 {4}	0.03438 {6}	0.05879 {8}	0.03314 {2}	0.03418 {5}	0.03023 {1}
		$\hat{\omega}$	0.07572 {7}	0.06041 {5}	0.05541 {2}	0.05945 {4}	0.08037 {8}	0.06651 {6}	0.05468 {1}	0.05692 {3}
		$\sum$ *Ranks*	87 {7}	35 {2}	48 {5}	39 {3}	93 {8}	64 {6}	42 {4}	24 {1}
500	AB	$\hat{\kappa}$	0.26442 {8}	0.09248 {1}	0.11287 {4}	0.09372 {2}	0.25897 {7}	0.14624 {6}	0.11310 {5}	0.10422 {3}
		$\hat{\nu}$	0.16919 {7}	0.07643 {3}	0.08023 {4}	0.07290 {1}	0.19612 {8}	0.09281 {6}	0.08073 {5}	0.07440 {2}
		$\hat{\zeta}$	0.08895 {7}	0.04995 {2}	0.05360 {5}	0.05145 {3}	0.09208 {8}	0.05385 {6}	0.05310 {4}	0.04447 {1}
		$\hat{\omega}$	0.04772 {7}	0.03527 {5}	0.03312 {2}	0.03506 {4}	0.05118 {8}	0.04056 {6}	0.03284 {1}	0.03382 {3}
	MSE	$\hat{\kappa}$	0.12974 {8}	0.01602 {1}	0.02616 {4}	0.01725 {2}	0.12550 {7}	0.04443 {6}	0.02786 {5}	0.02156 {3}
		$\hat{\nu}$	0.05448 {7}	0.01149 {3}	0.01296 {4}	0.01019 {1}	0.07430 {8}	0.01780 {6}	0.01361 {5}	0.01093 {2}
		$\hat{\zeta}$	0.01482 {7}	0.00402 {2}	0.00484 {5}	0.00431 {3}	0.01558 {8}	0.00564 {6}	0.00481 {4}	0.00349 {1}
		$\hat{\omega}$	0.00393 {7}	0.00193 {5}	0.00174 {2}	0.00192 {4}	0.00411 {8}	0.00255 {6}	0.00172 {1}	0.00184 {3}
	MRE	$\hat{\kappa}$	0.10577 {8}	0.03699 {1}	0.04515 {4}	0.03749 {2}	0.10359 {7}	0.05849 {6}	0.04524 {5}	0.04169 {3}
		$\hat{\nu}$	0.13535 {7}	0.06114 {3}	0.06418 {4}	0.05832 {1}	0.15690 {8}	0.07425 {6}	0.06458 {5}	0.05952 {2}
		$\hat{\zeta}$	0.05083 {7}	0.02854 {2}	0.03063 {5}	0.02940 {3}	0.05261 {8}	0.03077 {6}	0.03034 {4}	0.02541 {1}
		$\hat{\omega}$	0.06363 {7}	0.04702 {5}	0.04416 {2}	0.04675 {4}	0.06824 {8}	0.05409 {6}	0.04378 {1}	0.04510 {3}
		$\sum$ *Ranks*	87 {7}	33 {3}	45 {4.5}	30 {2}	93 {8}	72 {6}	45 {4.5}	27 {1}

Table 7. Simulation results for $\boldsymbol{\psi} = (\kappa = 0.5, \nu = 2.25, \zeta = 3.5, \omega = 2.5)$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	AB	$\hat{\kappa}$	0.15499 $^{\{7\}}$	0.10389 $^{\{1\}}$	0.10732 $^{\{3\}}$	0.11284 $^{\{5\}}$	0.15721 $^{\{8\}}$	0.10418 $^{\{2\}}$	0.11024 $^{\{4\}}$	0.12543 $^{\{6\}}$
		$\hat{\nu}$	0.85789 $^{\{8\}}$	0.41606 $^{\{1\}}$	0.46604 $^{\{4\}}$	0.41632 $^{\{2\}}$	0.54252 $^{\{7\}}$	0.47377 $^{\{6\}}$	0.47284 $^{\{5\}}$	0.42154 $^{\{3\}}$
		$\hat{\zeta}$	0.71035 $^{\{8\}}$	0.36751 $^{\{1\}}$	0.39121 $^{\{3\}}$	0.38575 $^{\{2\}}$	0.57985 $^{\{7\}}$	0.39775 $^{\{4\}}$	0.39926 $^{\{5\}}$	0.41560 $^{\{6\}}$
		$\hat{\omega}$	0.69451 $^{\{8\}}$	0.26320 $^{\{1\}}$	0.28293 $^{\{3\}}$	0.26495 $^{\{2\}}$	0.62253 $^{\{7\}}$	0.30132 $^{\{5\}}$	0.29929 $^{\{4\}}$	0.32838 $^{\{6\}}$
	MSE	$\hat{\kappa}$	0.04288 $^{\{7\}}$	0.01924 $^{\{2\}}$	0.02079 $^{\{3\}}$	0.02301 $^{\{5\}}$	0.04482 $^{\{8\}}$	0.01905 $^{\{1\}}$	0.02214 $^{\{4\}}$	0.02876 $^{\{6\}}$
		$\hat{\nu}$	1.39700 $^{\{7\}}$	0.31678 $^{\{1\}}$	0.38869 $^{\{4\}}$	0.31760 $^{\{2\}}$	1.74602 $^{\{8\}}$	0.40373 $^{\{5\}}$	0.40546 $^{\{6\}}$	0.33472 $^{\{3\}}$
		$\hat{\zeta}$	1.11432 $^{\{8\}}$	0.24506 $^{\{1\}}$	0.26288 $^{\{2\}}$	0.26959 $^{\{3\}}$	0.65037 $^{\{7\}}$	0.27806 $^{\{4\}}$	0.28275 $^{\{5\}}$	0.30347 $^{\{6\}}$
		$\hat{\omega}$	0.77993 $^{\{8\}}$	0.13146 $^{\{2\}}$	0.14150 $^{\{3\}}$	0.12726 $^{\{1\}}$	0.56778 $^{\{7\}}$	0.16421 $^{\{5\}}$	0.15493 $^{\{4\}}$	0.18653 $^{\{6\}}$
	MRE	$\hat{\kappa}$	0.30998 $^{\{7\}}$	0.20778 $^{\{1\}}$	0.21463 $^{\{3\}}$	0.22568 $^{\{5\}}$	0.31442 $^{\{8\}}$	0.20835 $^{\{2\}}$	0.22048 $^{\{4\}}$	0.25086 $^{\{6\}}$
		$\hat{\nu}$	0.38128 $^{\{7\}}$	0.18492 $^{\{1\}}$	0.20713 $^{\{4\}}$	0.18503 $^{\{2\}}$	0.42597 $^{\{8\}}$	0.21057 $^{\{6\}}$	0.21015 $^{\{5\}}$	0.18735 $^{\{3\}}$
		$\hat{\zeta}$	0.20296 $^{\{8\}}$	0.10500 $^{\{1\}}$	0.11177 $^{\{3\}}$	0.11021 $^{\{2\}}$	0.16567 $^{\{7\}}$	0.11364 $^{\{4\}}$	0.11407 $^{\{5\}}$	0.11874 $^{\{6\}}$
		$\hat{\omega}$	0.27780 $^{\{8\}}$	0.10528 $^{\{1\}}$	0.11317 $^{\{3\}}$	0.10598 $^{\{2\}}$	0.24901 $^{\{7\}}$	0.12053 $^{\{5\}}$	0.11972 $^{\{4\}}$	0.13135 $^{\{6\}}$
		$\sum$ *Ranks*	91 $^{\{8\}}$	14 $^{\{1\}}$	38 $^{\{3\}}$	33 $^{\{2\}}$	89 $^{\{7\}}$	49 $^{\{4\}}$	55 $^{\{5\}}$	63 $^{\{6\}}$
150	AB	$\hat{\kappa}$	0.09721 $^{\{7\}}$	0.06014 $^{\{3\}}$	0.05920 $^{\{1\}}$	0.05984 $^{\{2\}}$	0.10525 $^{\{8\}}$	0.06093 $^{\{5\}}$	0.06059 $^{\{4\}}$	0.06738 $^{\{6\}}$
		$\hat{\nu}$	0.61147 $^{\{7\}}$	0.29555 $^{\{2\}}$	0.32896 $^{\{5\}}$	0.28322 $^{\{1\}}$	0.69888 $^{\{8\}}$	0.33374 $^{\{6\}}$	0.32480 $^{\{4\}}$	0.30014 $^{\{3\}}$
		$\hat{\zeta}$	0.42717 $^{\{8\}}$	0.24081 $^{\{1\}}$	0.25362 $^{\{4\}}$	0.24973 $^{\{3\}}$	0.36809 $^{\{7\}}$	0.24694 $^{\{2\}}$	0.25654 $^{\{5\}}$	0.26579 $^{\{6\}}$
		$\hat{\omega}$	0.46883 $^{\{8\}}$	0.15992 $^{\{1\}}$	0.17752 $^{\{3\}}$	0.16210 $^{\{2\}}$	0.46567 $^{\{7\}}$	0.18876 $^{\{5\}}$	0.17960 $^{\{4\}}$	0.20868 $^{\{6\}}$
	MSE	$\hat{\kappa}$	0.01549 $^{\{7\}}$	0.00627 $^{\{2\}}$	0.00624 $^{\{1\}}$	0.00635 $^{\{3\}}$	0.01887 $^{\{8\}}$	0.00685 $^{\{4\}}$	0.00689 $^{\{5\}}$	0.00764 $^{\{6\}}$
		$\hat{\nu}$	0.65605 $^{\{7\}}$	0.15512 $^{\{2\}}$	0.19530 $^{\{4\}}$	0.14461 $^{\{1\}}$	0.89869 $^{\{8\}}$	0.19996 $^{\{6\}}$	0.19742 $^{\{5\}}$	0.16450 $^{\{3\}}$
		$\hat{\zeta}$	0.35327 $^{\{8\}}$	0.10287 $^{\{1\}}$	0.11629 $^{\{4\}}$	0.11041 $^{\{3\}}$	0.24133 $^{\{7\}}$	0.10824 $^{\{2\}}$	0.11878 $^{\{5\}}$	0.12344 $^{\{6\}}$
		$\hat{\omega}$	0.36671 $^{\{8\}}$	0.05198 $^{\{2\}}$	0.06122 $^{\{3\}}$	0.05144 $^{\{1\}}$	0.33198 $^{\{7\}}$	0.06943 $^{\{5\}}$	0.06382 $^{\{4\}}$	0.08629 $^{\{6\}}$
	MRE	$\hat{\kappa}$	0.19442 $^{\{7\}}$	0.12028 $^{\{3\}}$	0.11840 $^{\{1\}}$	0.11968 $^{\{2\}}$	0.21051 $^{\{8\}}$	0.12187 $^{\{5\}}$	0.12118 $^{\{4\}}$	0.13476 $^{\{6\}}$
		$\hat{\nu}$	0.27176 $^{\{7\}}$	0.13136 $^{\{2\}}$	0.14621 $^{\{5\}}$	0.12588 $^{\{1\}}$	0.31061 $^{\{8\}}$	0.14833 $^{\{6\}}$	0.14435 $^{\{4\}}$	0.13340 $^{\{3\}}$
		$\hat{\zeta}$	0.12205 $^{\{8\}}$	0.06880 $^{\{1\}}$	0.07246 $^{\{4\}}$	0.07135 $^{\{3\}}$	0.10517 $^{\{7\}}$	0.07055 $^{\{2\}}$	0.07330 $^{\{5\}}$	0.07594 $^{\{6\}}$
		$\hat{\omega}$	0.18753 $^{\{8\}}$	0.06397 $^{\{1\}}$	0.07101 $^{\{3\}}$	0.06484 $^{\{2\}}$	0.18627 $^{\{7\}}$	0.07550 $^{\{5\}}$	0.07184 $^{\{4\}}$	0.08347 $^{\{6\}}$
		$\sum$ *Ranks*	90 $^{\{7.5\}}$	21 $^{\{1\}}$	38 $^{\{3\}}$	24 $^{\{2\}}$	90 $^{\{7.5\}}$	53 $^{\{4.5\}}$	53 $^{\{4.5\}}$	63 $^{\{6\}}$
350	AB	$\hat{\kappa}$	0.06905 $^{\{7\}}$	0.04016 $^{\{2\}}$	0.04146 $^{\{4\}}$	0.03978 $^{\{1\}}$	0.07401 $^{\{8\}}$	0.04196 $^{\{5\}}$	0.04046 $^{\{3\}}$	0.04754 $^{\{6\}}$
		$\hat{\nu}$	0.48208 $^{\{7\}}$	0.23801 $^{\{3\}}$	0.26616 $^{\{6\}}$	0.23194 $^{\{2\}}$	0.54252 $^{\{8\}}$	0.26056 $^{\{4\}}$	0.26068 $^{\{5\}}$	0.23167 $^{\{1\}}$
		$\hat{\zeta}$	0.31087 $^{\{8\}}$	0.19339 $^{\{3\}}$	0.20455 $^{\{6\}}$	0.19324 $^{\{1\}}$	0.28145 $^{\{7\}}$	0.19329 $^{\{2\}}$	0.20208 $^{\{5\}}$	0.19833 $^{\{4\}}$
		$\hat{\omega}$	0.34034 $^{\{8\}}$	0.10821 $^{\{2\}}$	0.12511 $^{\{4\}}$	0.10516 $^{\{1\}}$	0.33627 $^{\{7\}}$	0.13005 $^{\{5\}}$	0.12229 $^{\{3\}}$	0.14060 $^{\{6\}}$
	MSE	$\hat{\kappa}$	0.00794 $^{\{7\}}$	0.00288 $^{\{2\}}$	0.00315 $^{\{4\}}$	0.00282 $^{\{1\}}$	0.00963 $^{\{8\}}$	0.00325 $^{\{5\}}$	0.00301 $^{\{3\}}$	0.00390 $^{\{6\}}$
		$\hat{\nu}$	0.40738 $^{\{7\}}$	0.10196 $^{\{3\}}$	0.13214 $^{\{6\}}$	0.09713 $^{\{2\}}$	0.56001 $^{\{8\}}$	0.12447 $^{\{4\}}$	0.12846 $^{\{5\}}$	0.09672 $^{\{1\}}$
		$\hat{\zeta}$	0.17800 $^{\{8\}}$	0.06550 $^{\{2\}}$	0.07607 $^{\{6\}}$	0.06627 $^{\{3\}}$	0.13543 $^{\{7\}}$	0.06751 $^{\{4\}}$	0.07475 $^{\{5\}}$	0.06541 $^{\{1\}}$
		$\hat{\omega}$	0.20461 $^{\{8\}}$	0.02414 $^{\{2\}}$	0.03181 $^{\{4\}}$	0.02247 $^{\{1\}}$	0.18143 $^{\{7\}}$	0.03551 $^{\{5\}}$	0.03006 $^{\{3\}}$	0.04190 $^{\{6\}}$
	MRE	$\hat{\kappa}$	0.13809 $^{\{7\}}$	0.08032 $^{\{2\}}$	0.08292 $^{\{4\}}$	0.07957 $^{\{1\}}$	0.14802 $^{\{8\}}$	0.08391 $^{\{5\}}$	0.08092 $^{\{3\}}$	0.09507 $^{\{6\}}$
		$\hat{\nu}$	0.21426 $^{\{7\}}$	0.10578 $^{\{3\}}$	0.11829 $^{\{6\}}$	0.10309 $^{\{2\}}$	0.24112 $^{\{8\}}$	0.11581 $^{\{4\}}$	0.11586 $^{\{5\}}$	0.10296 $^{\{1\}}$
		$\hat{\zeta}$	0.08882 $^{\{8\}}$	0.05525 $^{\{3\}}$	0.05844 $^{\{6\}}$	0.05521 $^{\{1\}}$	0.08041 $^{\{7\}}$	0.05523 $^{\{2\}}$	0.05774 $^{\{5\}}$	0.05667 $^{\{4\}}$
		$\hat{\omega}$	0.13614 $^{\{8\}}$	0.04328 $^{\{2\}}$	0.05005 $^{\{4\}}$	0.04206 $^{\{1\}}$	0.13451 $^{\{7\}}$	0.05202 $^{\{5\}}$	0.04892 $^{\{3\}}$	0.05624 $^{\{6\}}$
		$\sum$ *Ranks*	90 $^{\{7.5\}}$	29 $^{\{2\}}$	60 $^{\{6\}}$	17 $^{\{1\}}$	90 $^{\{7.5\}}$	50 $^{\{5\}}$	48 $^{\{3.5\}}$	48 $^{\{3.5\}}$
500	AB	$\hat{\kappa}$	0.05272 $^{\{7\}}$	0.03189 $^{\{4\}}$	0.03182 $^{\{3\}}$	0.03131 $^{\{2\}}$	0.05602 $^{\{8\}}$	0.03245 $^{\{5\}}$	0.03100 $^{\{1\}}$	0.03763 $^{\{6\}}$
		$\hat{\nu}$	0.38304 $^{\{7\}}$	0.20433 $^{\{2\}}$	0.21728 $^{\{5\}}$	0.19660 $^{\{1\}}$	0.43689 $^{\{8\}}$	0.21388 $^{\{4\}}$	0.21754 $^{\{6\}}$	0.20561 $^{\{3\}}$
		$\hat{\zeta}$	0.24194 $^{\{8\}}$	0.16422 $^{\{2\}}$	0.17057 $^{\{6\}}$	0.16833 $^{\{4\}}$	0.22282 $^{\{7\}}$	0.15852 $^{\{1\}}$	0.16672 $^{\{3\}}$	0.16996 $^{\{5\}}$
		$\hat{\omega}$	0.25421 $^{\{7\}}$	0.08321 $^{\{2\}}$	0.09650 $^{\{5\}}$	0.08142 $^{\{1\}}$	0.25568 $^{\{8\}}$	0.09638 $^{\{4\}}$	0.09105 $^{\{3\}}$	0.10706 $^{\{6\}}$
	MSE	$\hat{\kappa}$	0.00485 $^{\{7\}}$	0.00180 $^{\{3\}}$	0.00184 $^{\{4\}}$	0.00174 $^{\{1\}}$	0.00555 $^{\{8\}}$	0.00193 $^{\{5\}}$	0.00178 $^{\{2\}}$	0.00242 $^{\{6\}}$
		$\hat{\nu}$	0.26603 $^{\{7\}}$	0.07479 $^{\{3\}}$	0.08691 $^{\{6\}}$	0.06760 $^{\{1\}}$	0.36210 $^{\{8\}}$	0.08298 $^{\{4\}}$	0.08657 $^{\{5\}}$	0.07289 $^{\{2\}}$
		$\hat{\zeta}$	0.10251 $^{\{8\}}$	0.04839 $^{\{3\}}$	0.05200 $^{\{6\}}$	0.04993 $^{\{5\}}$	0.08368 $^{\{7\}}$	0.04412 $^{\{1\}}$	0.04933 $^{\{4\}}$	0.04750 $^{\{2\}}$
		$\hat{\omega}$	0.11751 $^{\{8\}}$	0.01391 $^{\{2\}}$	0.01893 $^{\{4\}}$	0.01324 $^{\{1\}}$	0.11186 $^{\{7\}}$	0.01912 $^{\{5\}}$	0.01696 $^{\{3\}}$	0.02461 $^{\{6\}}$
	MRE	$\hat{\kappa}$	0.10545 $^{\{7\}}$	0.06378 $^{\{4\}}$	0.06364 $^{\{3\}}$	0.06261 $^{\{2\}}$	0.11204 $^{\{8\}}$	0.06490 $^{\{5\}}$	0.06201 $^{\{1\}}$	0.07526 $^{\{6\}}$
		$\hat{\nu}$	0.17024 $^{\{7\}}$	0.09081 $^{\{2\}}$	0.09657 $^{\{5\}}$	0.08738 $^{\{1\}}$	0.19418 $^{\{8\}}$	0.09506 $^{\{4\}}$	0.09669 $^{\{6\}}$	0.09138 $^{\{3\}}$
		$\hat{\zeta}$	0.06912 $^{\{8\}}$	0.04692 $^{\{2\}}$	0.04873 $^{\{6\}}$	0.04810 $^{\{4\}}$	0.06366 $^{\{7\}}$	0.04529 $^{\{1\}}$	0.04763 $^{\{3\}}$	0.04856 $^{\{5\}}$
		$\hat{\omega}$	0.10168 $^{\{7\}}$	0.03328 $^{\{2\}}$	0.03860 $^{\{5\}}$	0.03257 $^{\{1\}}$	0.10227 $^{\{8\}}$	0.03855 $^{\{4\}}$	0.03642 $^{\{3\}}$	0.04283 $^{\{6\}}$
		$\sum$ *Ranks*	88 $^{\{7\}}$	31 $^{\{2\}}$	58 $^{\{6\}}$	24 $^{\{1\}}$	92 $^{\{8\}}$	43 $^{\{4\}}$	40 $^{\{3\}}$	56 $^{\{5\}}$

Table 8. Simulation results for $\boldsymbol{\psi} = (\kappa = 10, \nu = 0.5, \zeta = 5, \omega = 3)$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	AB	$\hat{\kappa}$	1.17032 {8}	0.88503 {4}	0.88647 {5}	0.91858 {6}	1.09205 {7}	0.83743 {2}	0.81443 {1}	0.87665 {3}
		$\hat{\nu}$	0.17542 {8}	0.11764 {5}	0.11506 {4}	0.12178 {6}	0.10112 {1}	0.11141 {2}	0.12889 {7}	0.11279 {3}
		$\hat{\zeta}$	0.74052 {8}	0.60428 {1}	0.63381 {3}	0.63872 {5}	0.67036 {6}	0.63849 {4}	0.61435 {2}	0.70378 {7}
		$\hat{\omega}$	0.33157 {7}	0.18624 {1}	0.18841 {2}	0.19093 {3}	0.34995 {8}	0.19782 {5}	0.19233 {4}	0.23424 {6}
	MSE	$\hat{\kappa}$	2.84397 {8}	1.56777 {5}	1.52581 {4}	1.75325 {6}	2.34085 {7}	1.31113 {1}	1.36083 {2}	1.43352 {3}
		$\hat{\nu}$	0.05477 {8}	0.02423 {2}	0.02523 {4}	0.02807 {5}	0.05184 {7}	0.02165 {1}	0.02914 {6}	0.02443 {3}
		$\hat{\zeta}$	1.04889 {8}	0.62932 {1}	0.71098 {5}	0.76815 {6}	0.69882 {4}	0.66100 {2}	0.66639 {3}	0.90967 {7}
		$\hat{\omega}$	0.20772 {7}	0.07735 {1}	0.07922 {3}	0.08303 {5}	0.20842 {8}	0.08223 {4}	0.07819 {2}	0.12374 {6}
	MRE	$\hat{\kappa}$	0.11703 {8}	0.08850 {4}	0.08865 {5}	0.09186 {6}	0.10921 {7}	0.08374 {2}	0.08144 {1}	0.08766 {3}
		$\hat{\nu}$	0.35085 {8}	0.23527 {4}	0.23012 {3}	0.24356 {5}	0.34610 {7}	0.22282 {1}	0.25777 {6}	0.22558 {2}
		$\hat{\zeta}$	0.14810 {8}	0.12086 {1}	0.12676 {3}	0.12774 {5}	0.13407 {6}	0.12770 {4}	0.12287 {2}	0.14076 {7}
		$\hat{\omega}$	0.11052 {7}	0.06208 {1}	0.06280 {2}	0.06364 {3}	0.11665 {8}	0.06594 {5}	0.06411 {4}	0.07808 {6}
		$\sum$ *Ranks*	93 {8}	30 {1}	43 {4}	61 {6}	76 {7}	33 {2}	40 {3}	56 {5}
150	AB	$\hat{\kappa}$	0.89319 {8}	0.72117 {6}	0.65367 {4}	0.69609 {5}	0.82768 {7}	0.62304 {2}	0.61172 {1}	0.62346 {3}
		$\hat{\nu}$	0.11980 {7}	0.07346 {4}	0.06428 {3}	0.07536 {5}	0.12639 {8}	0.06269 {2}	0.08534 {6}	0.06136 {1}
		$\hat{\zeta}$	0.39394 {8}	0.37451 {3}	0.37656 {4}	0.38939 {5}	0.39267 {6}	0.36702 {2}	0.36267 {1}	0.39357 {7}
		$\hat{\omega}$	0.20262 {7}	0.09753 {3}	0.09007 {1}	0.09668 {2}	0.22789 {8}	0.09888 {4}	0.10992 {5}	0.11043 {6}
	MSE	$\hat{\kappa}$	1.56043 {8}	0.97658 {6}	0.80191 {4}	0.91444 {5}	1.29329 {7}	0.74951 {3}	0.72593 {2}	0.70086 {1}
		$\hat{\nu}$	0.02402 {7}	0.00965 {4}	0.00743 {3}	0.01046 {5}	0.02644 {8}	0.00704 {1}	0.01264 {6}	0.00708 {2}
		$\hat{\zeta}$	0.24993 {7}	0.22680 {4}	0.22587 {3}	0.24932 {6}	0.23729 {5}	0.20772 {1}	0.21446 {2}	0.25843 {8}
		$\hat{\omega}$	0.07275 {7}	0.02235 {3}	0.01985 {1}	0.02264 {4}	0.08961 {8}	0.02034 {2}	0.02530 {5}	0.02745 {6}
	MRE	$\hat{\kappa}$	0.08932 {8}	0.07212 {6}	0.06537 {4}	0.06961 {5}	0.08277 {7}	0.06230 {2}	0.06117 {1}	0.06235 {3}
		$\hat{\nu}$	0.23959 {7}	0.14691 {4}	0.12856 {3}	0.15072 {5}	0.25279 {8}	0.12538 {2}	0.17068 {6}	0.12273 {1}
		$\hat{\zeta}$	0.07879 {8}	0.07490 {3}	0.07531 {4}	0.07788 {5}	0.07853 {6}	0.07340 {2}	0.07253 {1}	0.07871 {7}
		$\hat{\omega}$	0.06754 {7}	0.03251 {3}	0.03002 {1}	0.03223 {2}	0.07596 {8}	0.03296 {4}	0.03664 {5}	0.03681 {6}
		$\sum$ *Ranks*	89 {8}	49 {4}	35 {2}	54 {6}	86 {7}	27 {1}	41 {3}	51 {5}
350	AB	$\hat{\kappa}$	0.73951 {8}	0.56656 {6}	0.50557 {4}	0.56138 {5}	0.72207 {7}	0.49137 {1}	0.50148 {3}	0.49442 {2}
		$\hat{\nu}$	0.09829 {7}	0.05457 {5}	0.04277 {2}	0.05324 {4}	0.10112 {8}	0.04203 {1}	0.06747 {6}	0.04404 {3}
		$\hat{\zeta}$	0.27546 {6}	0.28151 {7}	0.26252 {2}	0.27543 {5}	0.27460 {4}	0.25956 {1}	0.26769 {3}	0.28279 {8}
		$\hat{\omega}$	0.16143 {7}	0.06462 {4}	0.05410 {1}	0.06173 {2}	0.16971 {8}	0.06452 {3}	0.07816 {6}	0.07274 {5}
	MSE	$\hat{\kappa}$	1.03167 {8}	0.59165 {6}	0.48832 {4}	0.58572 {5}	0.98079 {7}	0.47235 {3}	0.45302 {2}	0.42746 {1}
		$\hat{\nu}$	0.01641 {7}	0.00548 {5}	0.00324 {2}	0.00509 {4}	0.01762 {8}	0.00322 {1}	0.00775 {6}	0.00355 {3}
		$\hat{\zeta}$	0.12159 {6}	0.12602 {7}	0.10772 {2}	0.12109 {5}	0.11805 {4}	0.10464 {1}	0.11327 {3}	0.12782 {8}
		$\hat{\omega}$	0.04756 {7}	0.01109 {4}	0.00745 {1}	0.00958 {3}	0.05127 {8}	0.00922 {2}	0.01259 {6}	0.01158 {5}
	MRE	$\hat{\kappa}$	0.07395 {8}	0.05666 {6}	0.05056 {4}	0.05614 {5}	0.07221 {7}	0.04914 {1}	0.05015 {3}	0.04944 {2}
		$\hat{\nu}$	0.19658 {7}	0.10914 {5}	0.08554 {2}	0.10649 {4}	0.20224 {8}	0.08405 {1}	0.13494 {6}	0.08808 {3}
		$\hat{\zeta}$	0.05509 {5.5}	0.05630 {7}	0.05250 {2}	0.05509 {5.5}	0.05492 {4}	0.05191 {1}	0.05354 {3}	0.05656 {8}
		$\hat{\omega}$	0.05381 {7}	0.02154 {4}	0.01803 {1}	0.02058 {2}	0.05657 {8}	0.02151 {3}	0.02605 {6}	0.02425 {5}
		$\sum$ *Ranks*	83.5 {8}	66 {6}	27 {2}	49.5 {3}	81 {7}	19 {1}	53 {4.5}	53 {4.5}
500	AB	$\hat{\kappa}$	0.64964 {8}	0.48541 {6}	0.41205 {3}	0.48040 {5}	0.62669 {7}	0.38788 {1}	0.40882 {2}	0.42884 {4}
		$\hat{\nu}$	0.08234 {7}	0.04172 {4.5}	0.03276 {2}	0.04172 {4.5}	0.08657 {8}	0.03173 {1}	0.05463 {6}	0.03568 {3}
		$\hat{\zeta}$	0.21081 {4}	0.22122 {6}	0.20556 {3}	0.22301 {8}	0.21306 {5}	0.19940 {1}	0.20400 {2}	0.22281 {7}
		$\hat{\omega}$	0.13240 {7}	0.04525 {4}	0.03826 {1}	0.04361 {2}	0.14325 {8}	0.04464 {3}	0.06123 {6}	0.05435 {5}
	MSE	$\hat{\kappa}$	0.76524 {8}	0.44196 {6}	0.31650 {3}	0.42747 {5}	0.74479 {7}	0.28008 {1}	0.28586 {2}	0.31732 {4}
		$\hat{\nu}$	0.01140 {7}	0.00305 {4}	0.00185 {2}	0.00308 {5}	0.01278 {8}	0.00175 {1}	0.00509 {6}	0.00230 {3}
		$\hat{\zeta}$	0.07079 {4}	0.07605 {6}	0.06563 {2}	0.07793 {7}	0.07139 {5}	0.06085 {1}	0.06632 {3}	0.07848 {8}
		$\hat{\omega}$	0.03286 {7}	0.00583 {4.5}	0.00396 {1}	0.00517 {3}	0.03643 {8}	0.00478 {2}	0.00786 {6}	0.00583 {4.5}
	MRE	$\hat{\kappa}$	0.06496 {8}	0.04854 {6}	0.04121 {3}	0.04804 {5}	0.06267 {7}	0.03879 {1}	0.04088 {2}	0.04288 {4}
		$\hat{\nu}$	0.16467 {7}	0.08344 {5}	0.06551 {2}	0.08343 {4}	0.17314 {8}	0.06346 {1}	0.10926 {6}	0.07136 {3}
		$\hat{\zeta}$	0.04216 {4}	0.04424 {6}	0.04111 {3}	0.04460 {8}	0.04261 {5}	0.03988 {1}	0.04080 {2}	0.04456 {7}
		$\hat{\omega}$	0.04413 {7}	0.01508 {4}	0.01275 {1}	0.01454 {2}	0.04775 {8}	0.01488 {3}	0.02041 {6}	0.01812 {5}
		$\sum$ *Ranks*	78 {7}	62 {6}	26 {2}	58.5 {5}	84 {8}	17 {1}	49 {3}	57.5 {4}

Table 9. Simulation results for $\boldsymbol{\psi} = (\kappa = 1.25, \nu = 0.75, \zeta = 1.25, \omega = 1.25)$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	AB	$\hat{\kappa}$	0.41321 {8}	0.18794 {1}	0.22742 {3}	0.19507 {2}	0.38987 {7}	0.26051 {6}	0.23712 {5}	0.23492 {4}
		$\hat{\nu}$	0.29380 {8}	0.20030 {5}	0.20938 {6}	0.19511 {4}	0.19001 {2}	0.19374 {3}	0.21391 {7}	0.18469 {1}
		$\hat{\zeta}$	0.19464 {8}	0.09252 {1}	0.10005 {5}	0.09979 {4}	0.16866 {7}	0.09529 {2}	0.10466 {6}	0.09800 {3}
		$\hat{\omega}$	0.31821 {8}	0.22657 {5}	0.21780 {2}	0.23455 {6}	0.25147 {7}	0.21958 {3}	0.20472 {1}	0.22172 {4}
	MSE	$\hat{\kappa}$	0.25687 {8}	0.05528 {1}	0.08035 {3}	0.05995 {2}	0.23708 {7}	0.10545 {6}	0.08659 {5}	0.08657 {4}
		$\hat{\nu}$	0.13341 {7}	0.06111 {4}	0.06742 {5}	0.05814 {2}	0.17513 {8}	0.05965 {3}	0.07064 {6}	0.05374 {1}
		$\hat{\zeta}$	0.06492 {8}	0.01569 {1}	0.01795 {4}	0.01886 {5}	0.05013 {7}	0.01651 {2}	0.01952 {6}	0.01756 {3}
		$\hat{\omega}$	0.19903 {8}	0.08513 {4}	0.08094 {3}	0.09598 {6}	0.10886 {7}	0.07803 {2}	0.07242 {1}	0.08706 {5}
	MRE	$\hat{\kappa}$	0.33057 {8}	0.15035 {1}	0.18194 {3}	0.15606 {2}	0.31189 {7}	0.20841 {6}	0.18969 {5}	0.18794 {4}
		$\hat{\nu}$	0.39174 {7}	0.26707 {4}	0.27918 {5}	0.26014 {3}	0.43672 {8}	0.25832 {2}	0.28521 {6}	0.24626 {1}
		$\hat{\zeta}$	0.15571 {8}	0.07401 {1}	0.08004 {5}	0.07983 {4}	0.13493 {7}	0.07623 {2}	0.08373 {6}	0.07840 {3}
		$\hat{\omega}$	0.25457 {8}	0.18125 {5}	0.17424 {2}	0.18764 {6}	0.20118 {7}	0.17566 {3}	0.16377 {1}	0.17737 {4}
		$\sum$ *Ranks*	94 {8}	33 {1}	46 {4.5}	46 {4.5}	81 {7}	40 {3}	55 {6}	37 {2}
150	AB	$\hat{\kappa}$	0.27029 {8}	0.13289 {1}	0.17019 {5}	0.13591 {2}	0.26665 {7}	0.19022 {6}	0.16780 {4}	0.16706 {3}
		$\hat{\nu}$	0.20757 {7}	0.13702 {4}	0.14214 {5}	0.13348 {2}	0.24361 {8}	0.13426 {3}	0.14676 {6}	0.12572 {1}
		$\hat{\zeta}$	0.12887 {7}	0.05806 {1}	0.06254 {4}	0.06099 {2}	0.13148 {8}	0.06512 {6}	0.06482 {5}	0.06165 {3}
		$\hat{\omega}$	0.16792 {8}	0.13108 {5}	0.12864 {4}	0.13473 {6}	0.16043 {7}	0.12653 {3}	0.12129 {1}	0.12550 {2}
	MSE	$\hat{\kappa}$	0.11804 {8}	0.02921 {1}	0.04727 {5}	0.03037 {2}	0.11720 {7}	0.05899 {6}	0.04545 {3}	0.04605 {4}
		$\hat{\nu}$	0.06998 {7}	0.03174 {4}	0.03406 {5}	0.02955 {2}	0.10517 {8}	0.03162 {3}	0.03533 {6}	0.02742 {1}
		$\hat{\zeta}$	0.03086 {7}	0.00636 {1}	0.00734 {4}	0.00701 {2}	0.03345 {8}	0.00829 {6}	0.00763 {5}	0.00732 {3}
		$\hat{\omega}$	0.05260 {8}	0.02872 {5}	0.02766 {4}	0.03150 {6}	0.03925 {7}	0.02507 {2}	0.02431 {1}	0.02680 {3}
	MRE	$\hat{\kappa}$	0.21623 {8}	0.10631 {1}	0.13615 {5}	0.10873 {2}	0.21332 {7}	0.15217 {6}	0.13424 {4}	0.13365 {3}
		$\hat{\nu}$	0.27675 {7}	0.18269 {4}	0.18952 {5}	0.17798 {2}	0.32482 {8}	0.17901 {3}	0.19569 {6}	0.16762 {1}
		$\hat{\zeta}$	0.10310 {7}	0.04645 {1}	0.05003 {4}	0.04879 {2}	0.10518 {8}	0.05209 {6}	0.05186 {5}	0.04932 {3}
		$\hat{\omega}$	0.13433 {8}	0.10486 {5}	0.10291 {4}	0.10778 {6}	0.12834 {7}	0.10123 {3}	0.09703 {1}	0.10040 {2}
		$\sum$ *Ranks*	90 {7.5}	33 {2}	54 {6}	36 {3}	90 {7.5}	53 {5}	47 {4}	29 {1}
350	AB	$\hat{\kappa}$	0.18931 {7}	0.10098 {2}	0.12459 {4}	0.10085 {1}	0.19132 {8}	0.15138 {6}	0.12351 {3}	0.13014 {5}
		$\hat{\nu}$	0.15675 {7}	0.09810 {4}	0.10538 {6}	0.09807 {3}	0.19001 {8}	0.09799 {2}	0.10488 {5}	0.09647 {1}
		$\hat{\zeta}$	0.09977 {7}	0.04452 {2}	0.04845 {5}	0.04437 {1}	0.11001 {8}	0.05283 {6}	0.04807 {4}	0.04547 {3}
		$\hat{\omega}$	0.11582 {7}	0.09189 {5}	0.08720 {2}	0.09078 {4}	0.12286 {8}	0.09348 {6}	0.08478 {1}	0.08963 {3}
	MSE	$\hat{\kappa}$	0.05858 {7}	0.01774 {2}	0.02580 {4}	0.01749 {1}	0.06169 {8}	0.04010 {6}	0.02540 {3}	0.02887 {5}
		$\hat{\nu}$	0.04461 {7}	0.01698 {1}	0.01955 {6}	0.01729 {3}	0.06613 {8}	0.01753 {4}	0.01939 {5}	0.01711 {2}
		$\hat{\zeta}$	0.02018 {7}	0.00373 {2}	0.00429 {5}	0.00365 {1}	0.02420 {8}	0.00568 {6}	0.00426 {4}	0.00392 {3}
		$\hat{\omega}$	0.02330 {8}	0.01413 {6}	0.01246 {2}	0.01409 {5}	0.02294 {7}	0.01367 {4}	0.01181 {1}	0.01301 {3}
	MRE	$\hat{\kappa}$	0.15145 {7}	0.08079 {2}	0.09967 {4}	0.08068 {1}	0.15306 {8}	0.12110 {6}	0.09881 {3}	0.10411 {5}
		$\hat{\nu}$	0.20900 {7}	0.13080 {4}	0.14051 {6}	0.13075 {3}	0.25334 {8}	0.13065 {2}	0.13984 {5}	0.12863 {1}
		$\hat{\zeta}$	0.07982 {7}	0.03561 {2}	0.03876 {5}	0.03549 {1}	0.08801 {8}	0.04227 {6}	0.03845 {4}	0.03638 {3}
		$\hat{\omega}$	0.09266 {7}	0.07351 {5}	0.06976 {2}	0.07262 {4}	0.09829 {8}	0.07479 {6}	0.06782 {1}	0.07170 {3}
		$\sum$ *Ranks*	85 {7}	37 {2.5}	51 {5}	28 {1}	95 {8}	60 {6}	39 {4}	37 {2.5}
500	AB	$\hat{\kappa}$	0.15424 {8}	0.07895 {2}	0.09626 {4}	0.07871 {1}	0.15291 {7}	0.11884 {6}	0.09571 {3}	0.10068 {5}
		$\hat{\nu}$	0.13119 {7}	0.07496 {2}	0.08123 {5}	0.07553 {3}	0.15423 {8}	0.07823 {4}	0.08202 {6}	0.07354 {1}
		$\hat{\zeta}$	0.08634 {7}	0.03774 {3}	0.04007 {4}	0.03756 {2}	0.09810 {8}	0.04456 {6}	0.04127 {5}	0.03720 {1}
		$\hat{\omega}$	0.09307 {7}	0.07010 {5}	0.06681 {1}	0.06960 {4}	0.10204 {8}	0.07394 {6}	0.06716 {3}	0.06692 {2}
	MSE	$\hat{\kappa}$	0.03817 {7}	0.01104 {2}	0.01598 {4}	0.01097 {1}	0.03992 {8}	0.02473 {6}	0.01597 {3}	0.01822 {5}
		$\hat{\nu}$	0.03079 {7}	0.01048 {2}	0.01205 {6}	0.01072 {3}	0.04649 {8}	0.01182 {4}	0.01201 {5}	0.01032 {1}
		$\hat{\zeta}$	0.01506 {7}	0.00265 {3}	0.00296 {4}	0.00256 {1}	0.01980 {8}	0.00390 {6}	0.00308 {5}	0.00257 {2}
		$\hat{\omega}$	0.01462 {7}	0.00842 {5}	0.00731 {2}	0.00820 {4}	0.01645 {8}	0.00847 {6}	0.00741 {3}	0.00726 {1}
	MRE	$\hat{\kappa}$	0.12339 {8}	0.06316 {2}	0.07701 {4}	0.06297 {1}	0.12232 {7}	0.09507 {6}	0.07657 {3}	0.08054 {5}
		$\hat{\nu}$	0.17492 {7}	0.09995 {2}	0.10831 {5}	0.10071 {3}	0.20564 {8}	0.10430 {4}	0.10936 {6}	0.09805 {1}
		$\hat{\zeta}$	0.06907 {7}	0.03020 {3}	0.03206 {4}	0.03005 {2}	0.07848 {8}	0.03565 {6}	0.03302 {5}	0.02976 {1}
		$\hat{\omega}$	0.07446 {7}	0.05608 {5}	0.05344 {1}	0.05568 {4}	0.08163 {8}	0.05916 {6}	0.05373 {3}	0.05354 {2}
		$\sum$ *Ranks*	86 {7}	36 {3}	44 {4}	29 {2}	94 {8}	66 {6}	50 {5}	27 {1}

Table 10. Simulation results for $\boldsymbol{\psi} = (\kappa = 2.25, \nu = 0.5, \zeta = 0.75, \omega = 0.5)$.

n	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
50	AB	$\hat{\kappa}$	0.43399 {8}	0.10871 {1}	0.15208 {6}	0.10948 {2}	0.41460 {7}	0.12256 {4}	0.15069 {5}	0.11051 {3}
		$\hat{\nu}$	0.20096 {8}	0.07726 {3}	0.08859 {5}	0.07476 {2}	0.12372 {7}	0.07809 {4}	0.09215 {6}	0.06991 {1}
		$\hat{\zeta}$	0.11101 {7}	0.05194 {2}	0.05489 {3}	0.05171 {1}	0.12078 {8}	0.06064 {5}	0.06191 {6}	0.06000 {4}
		$\hat{\omega}$	0.06087 {4}	0.06130 {5}	0.05515 {3}	0.06161 {7}	0.06135 {6}	0.06178 {8}	0.05207 {2}	0.05186 {1}
	MSE	$\hat{\kappa}$	0.34640 {8}	0.02267 {1}	0.04903 {6}	0.02321 {2}	0.33396 {7}	0.03034 {4}	0.04796 {5}	0.02711 {3}
		$\hat{\nu}$	0.07702 {7}	0.01181 {4}	0.01735 {5}	0.01127 {3}	0.12012 {8}	0.01084 {2}	0.01819 {6}	0.00997 {1}
		$\hat{\zeta}$	0.02358 {7}	0.00492 {1}	0.00575 {3}	0.00510 {2}	0.02940 {8}	0.00742 {6}	0.00703 {5}	0.00636 {4}
		$\hat{\omega}$	0.00660 {8}	0.00611 {6}	0.00493 {3}	0.00650 {7}	0.00572 {4}	0.00607 {5}	0.00420 {1}	0.00422 {2}
	MRE	$\hat{\kappa}$	0.19289 {8}	0.04831 {1}	0.06759 {6}	0.04866 {2}	0.18427 {7}	0.05447 {4}	0.06697 {5}	0.04911 {3}
		$\hat{\nu}$	0.40192 {7}	0.15452 {3}	0.17718 {5}	0.14951 {2}	0.48153 {8}	0.15617 {4}	0.18430 {6}	0.13983 {1}
		$\hat{\zeta}$	0.14801 {7}	0.06926 {2}	0.07319 {3}	0.06895 {1}	0.16104 {8}	0.08086 {5}	0.08255 {6}	0.07999 {4}
		$\hat{\omega}$	0.12174 {4}	0.12260 {5}	0.11029 {3}	0.12322 {7}	0.12270 {6}	0.12357 {8}	0.10414 {2}	0.10371 {1}
		$\sum$ *Ranks*	83 {7}	34 {2}	51 {4}	38 {3}	84 {8}	59 {6}	55 {5}	28 {1}
150	AB	$\hat{\kappa}$	0.32158 {8}	0.08870 {1}	0.12100 {6}	0.08920 {2}	0.30253 {7}	0.10780 {4}	0.10852 {5}	0.09361 {3}
		$\hat{\nu}$	0.13153 {7}	0.04157 {2}	0.04594 {4}	0.03929 {1}	0.16272 {8}	0.05166 {5}	0.05571 {6}	0.04469 {3}
		$\hat{\zeta}$	0.08451 {7}	0.03613 {1}	0.03799 {3}	0.03651 {2}	0.09749 {8}	0.03998 {4}	0.04574 {6}	0.04226 {5}
		$\hat{\omega}$	0.03586 {7}	0.03556 {6}	0.02982 {1}	0.03422 {4}	0.04169 {8}	0.03533 {5}	0.03059 {2}	0.03073 {3}
	MSE	$\hat{\kappa}$	0.19502 {8}	0.01395 {2}	0.02886 {6}	0.01370 {1}	0.18342 {7}	0.02405 {5}	0.02394 {4}	0.01686 {3}
		$\hat{\nu}$	0.03272 {7}	0.00361 {2}	0.00496 {5}	0.00331 {1}	0.05226 {8}	0.00488 {4}	0.00642 {6}	0.00380 {3}
		$\hat{\zeta}$	0.01455 {7}	0.00223 {1}	0.00256 {3}	0.00225 {2}	0.02034 {8}	0.00333 {6}	0.00332 {5}	0.00295 {4}
		$\hat{\omega}$	0.00216 {7}	0.00196 {5}	0.00139 {1}	0.00185 {4}	0.00269 {8}	0.00198 {6}	0.00144 {2.5}	0.00144 {2.5}
	MRE	$\hat{\kappa}$	0.14292 {8}	0.03942 {1}	0.05378 {6}	0.03964 {2}	0.13446 {7}	0.04791 {4}	0.04823 {5}	0.04160 {3}
		$\hat{\nu}$	0.26307 {7}	0.08314 {2}	0.09188 {4}	0.07857 {1}	0.32545 {8}	0.10333 {5}	0.11141 {6}	0.08939 {3}
		$\hat{\zeta}$	0.11268 {7}	0.04817 {1}	0.05065 {3}	0.04868 {2}	0.12999 {8}	0.05330 {4}	0.06098 {6}	0.05635 {5}
		$\hat{\omega}$	0.07171 {7}	0.07112 {6}	0.05963 {1}	0.06844 {4}	0.08338 {8}	0.07066 {5}	0.06118 {2}	0.06146 {3}
		$\sum$ *Ranks*	87 {7}	30 {2}	43 {4}	26 {1}	93 {8}	57 {6}	55.5 {5}	40.5 {3}
350	AB	$\hat{\kappa}$	0.23897 {8}	0.08013 {1}	0.10584 {6}	0.08057 {2}	0.23216 {7}	0.09863 {5}	0.08494 {4}	0.08493 {3}
		$\hat{\nu}$	0.09846 {7}	0.02800 {2}	0.03124 {3}	0.02753 {1}	0.12372 {8}	0.03849 {5}	0.04063 {6}	0.03437 {4}
		$\hat{\zeta}$	0.07217 {7}	0.02929 {3}	0.03144 {4}	0.02907 {2}	0.08832 {8}	0.02888 {1}	0.04055 {6}	0.03681 {5}
		$\hat{\omega}$	0.02752 {7}	0.02489 {6}	0.02157 {1}	0.02433 {5}	0.03456 {8}	0.02415 {4}	0.02334 {3}	0.02328 {2}
	MSE	$\hat{\kappa}$	0.11291 {8}	0.00998 {1}	0.02045 {6}	0.01011 {2}	0.10711 {7}	0.01871 {5}	0.01300 {4}	0.01283 {3}
		$\hat{\nu}$	0.01872 {7}	0.00172 {2}	0.00225 {4}	0.00168 {1}	0.02942 {8}	0.00256 {5}	0.00311 {6}	0.00224 {3}
		$\hat{\zeta}$	0.01035 {7}	0.00141 {2}	0.00163 {3}	0.00140 {1}	0.01531 {8}	0.00169 {4}	0.00236 {6}	0.00210 {5}
		$\hat{\omega}$	0.00128 {7}	0.00096 {6}	0.00071 {1}	0.00092 {4.5}	0.00186 {8}	0.00092 {4.5}	0.00083 {2.5}	0.00083 {2.5}
	MRE	$\hat{\kappa}$	0.10621 {8}	0.03561 {1}	0.04704 {6}	0.03581 {2}	0.10318 {7}	0.04383 {5}	0.03775 {3.5}	0.03775 {3.5}
		$\hat{\nu}$	0.19693 {7}	0.05601 {2}	0.06248 {3}	0.05507 {1}	0.24744 {8}	0.07699 {5}	0.08126 {6}	0.06874 {4}
		$\hat{\zeta}$	0.09623 {7}	0.03905 {3}	0.04192 {4}	0.03876 {2}	0.11775 {8}	0.03850 {1}	0.05406 {6}	0.04909 {5}
		$\hat{\omega}$	0.05504 {7}	0.04978 {6}	0.04314 {1}	0.04866 {5}	0.06911 {8}	0.04829 {4}	0.04668 {3}	0.04656 {2}
		$\sum$ *Ranks*	87 {7}	35 {2}	42 {3.5}	28.5 {1}	93 {8}	48.5 {5}	56 {6}	42 {3.5}
500	AB	$\hat{\kappa}$	0.18989 {7}	0.07496 {2}	0.09147 {5}	0.07506 {3}	0.19044 {8}	0.09532 {6}	0.07478 {1}	0.08009 {4}
		$\hat{\nu}$	0.08143 {7}	0.02075 {1}	0.02402 {3}	0.02076 {2}	0.09991 {8}	0.03142 {5}	0.03513 {6}	0.02856 {4}
		$\hat{\zeta}$	0.06631 {7}	0.02494 {2}	0.02797 {4}	0.02568 {3}	0.08120 {8}	0.02344 {1}	0.03676 {6}	0.03375 {5}
		$\hat{\omega}$	0.02397 {7}	0.01992 {6}	0.01734 {1}	0.01917 {5}	0.02982 {8}	0.01828 {2}	0.01866 {3}	0.01875 {4}
	MSE	$\hat{\kappa}$	0.07284 {7}	0.00821 {1}	0.01416 {5}	0.00822 {2}	0.07318 {8}	0.01777 {6}	0.00950 {3}	0.01038 {4}
		$\hat{\nu}$	0.01286 {7}	0.00090 {1}	0.00128 {3}	0.00092 {2}	0.01899 {8}	0.00182 {5}	0.00215 {6}	0.00156 {4}
		$\hat{\zeta}$	0.00865 {7}	0.00097 {1}	0.00121 {4}	0.00103 {2}	0.01206 {8}	0.00118 {3}	0.00180 {6}	0.00170 {5}
		$\hat{\omega}$	0.00100 {7}	0.00061 {6}	0.00046 {1}	0.00057 {5}	0.00139 {8}	0.00055 {3}	0.00055 {3}	0.00055 {3}
	MRE	$\hat{\kappa}$	0.08440 {7}	0.03331 {2}	0.04065 {5}	0.03336 {3}	0.08464 {8}	0.04237 {6}	0.03323 {1}	0.03560 {4}
		$\hat{\nu}$	0.16285 {7}	0.04151 {1}	0.04804 {3}	0.04152 {2}	0.19982 {8}	0.06284 {5}	0.07025 {6}	0.05711 {4}
		$\hat{\zeta}$	0.08842 {7}	0.03325 {2}	0.03730 {4}	0.03424 {3}	0.10827 {8}	0.03125 {1}	0.04901 {6}	0.04499 {5}
		$\hat{\omega}$	0.04793 {7}	0.03984 {6}	0.03467 {1}	0.03834 {5}	0.05963 {8}	0.03656 {2}	0.03731 {3}	0.03750 {4}
		$\sum$ *Ranks*	84 {7}	31 {1}	39 {3}	37 {2}	96 {8}	45 {4}	50 {5.5}	50 {5.5}

Table 11. Partial and overall ranks of all methods of estimation for several parametric values.

ψ	n	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
$(\hat{\kappa}=0.75, \hat{\nu}=0.5, \hat{\zeta}=1.25, \hat{\omega}=0.25)$	50	8	1	4	6	7	2	3	5
	150	8	1	4	3	7	5	2	6
	350	7	3.5	3.5	1	8	6	2	5
	500	6	4	2	1	8	7	3	5
$(\hat{\kappa}=0.5, \hat{\nu}=0.5, \hat{\zeta}=0.25, \hat{\omega}=0.25)$	50	8	1	3	2	7	6	4	5
	150	7	1	6	2.5	8	5	2.5	4
	350	6	3	5	2	8	7	4	1
	500	6	4	3	1	7	8	5	2
$(\hat{\kappa}=1.25, \hat{\nu}=0.5, \hat{\zeta}=1.25, \hat{\omega}=0.25)$	50	8	3	5	6	7	1	4	2
	150	8	4	6	2.5	7	5	1	2.5
	350	7	3	5	2	8	6	1	4
	500	6	3	4	5	8	7	1	2
$(\hat{\kappa}=2.5, \hat{\nu}=1.25, \hat{\zeta}=1.75, \hat{\omega}=0.75)$	50	8	2	3	6	7	4	5	1
	150	7	2	4	3	8	6	5	1
	350	7	2	5	3	8	6	4	1
	500	7	3	4.5	2	8	6	4.5	1
$(\hat{\kappa}=0.5, \hat{\nu}=2.25, \hat{\zeta}=3.5, \hat{\omega}=2.5)$	50	8	1	3	2	7	4	5	6
	150	7.5	1	3	2	7.5	4.5	4.5	6
	350	7.5	2	6	1	7.5	5	3.5	3.5
	500	7	2	6	1	8	4	3	5
$(\hat{\kappa}=10, \hat{\nu}=0.5, \hat{\zeta}=5, \hat{\omega}=3)$	50	8	1	4	6	7	2	3	5
	150	8	4	2	6	7	1	3	5
	350	8	6	2	3	7	1	4.5	4.5
	500	7	6	2	5	8	1	3	4
$(\hat{\kappa}=1.25, \hat{\nu}=0.75, \hat{\zeta}=1.25, \hat{\omega}=1.25)$	50	8	1	4.5	4.5	7	3	6	2
	150	7.5	2	6	3	7.5	5	4	1
	350	7	2.5	5	1	8	6	4	2.5
	500	7	3	4	2	8	6	5	1
$(\hat{\kappa}=2.25, \hat{\nu}=0.5, \hat{\zeta}=0.75, \hat{\omega}=0.5)$	50	7	2	4	3	8	6	5	1
	150	7	2	4	1	8	6	5	3
	350	7	2	3.5	1	8	5	6	3.5
	500	7	1	3	2	8	4	5.5	5.5
$\sum Ranks$		232.5	79	129	91.5	242.5	150.5	121	106
Overall Rank		7	1	5	2	8	6	4	3

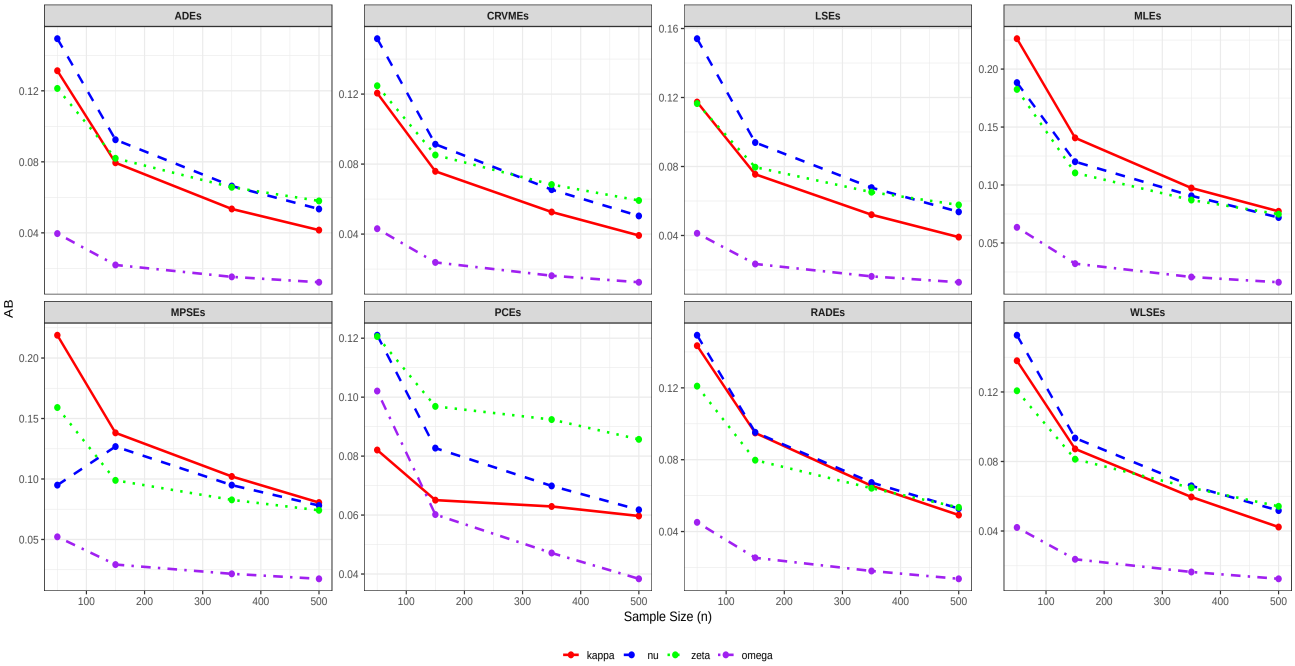


Figure 9. The AB of the OIPC parameters ($\kappa = 0.75, \nu = 0.5, \zeta = 1.25, \omega = 0.25$) for various sample sizes.

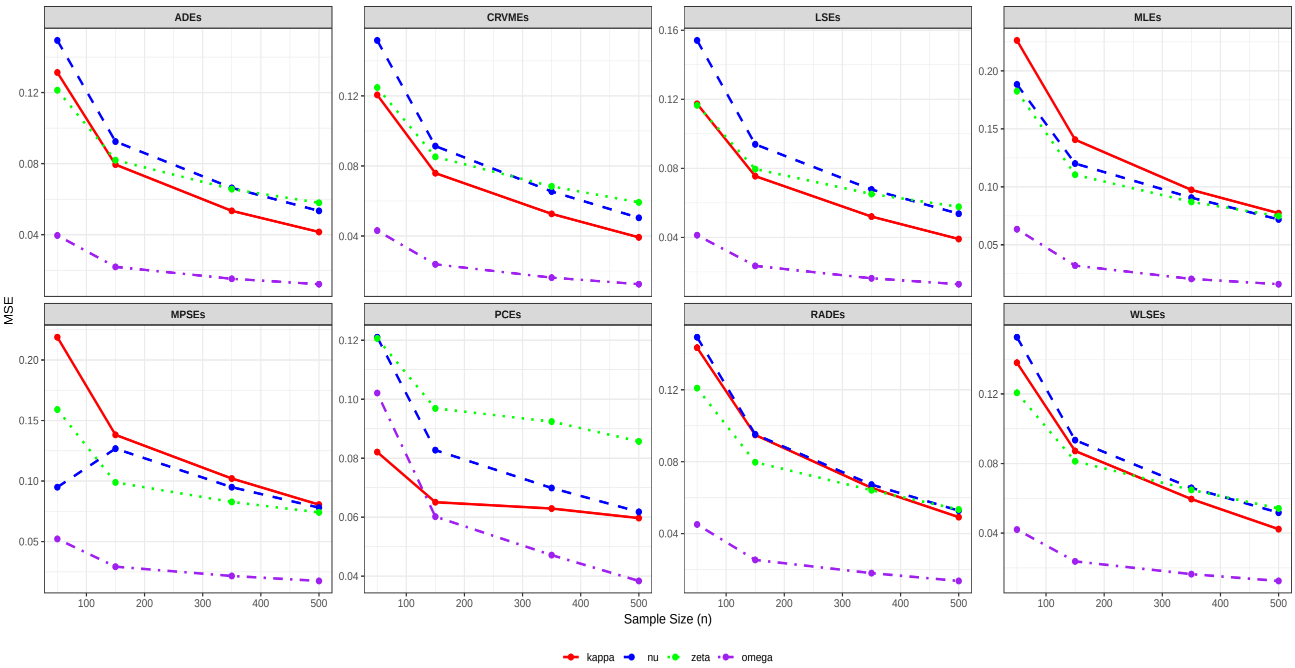


Figure 10. The MSE of the OIPC parameters ($\kappa = 0.75, \nu = 0.5, \zeta = 1.25, \omega = 0.25$) for various sample sizes.

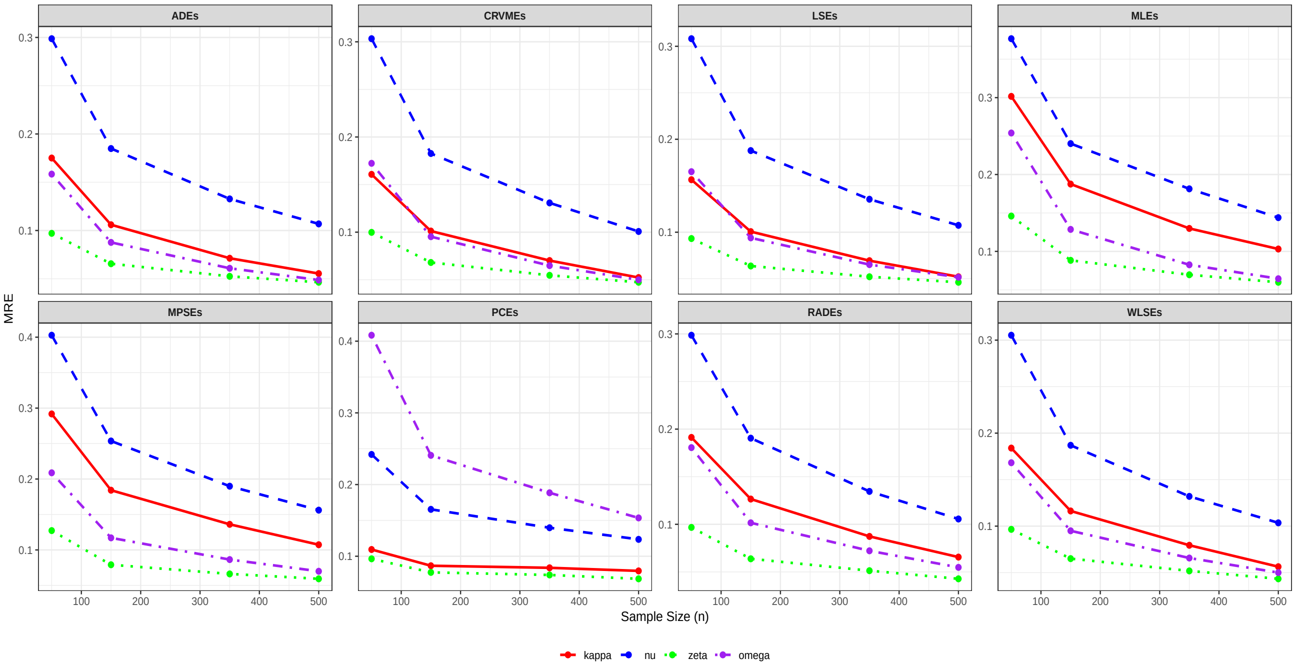


Figure 11. The MRE of the OIPC parameters ($\kappa = 0.75, \nu = 0.5, \zeta = 1.25, \omega = 0.25$) for various sample sizes.

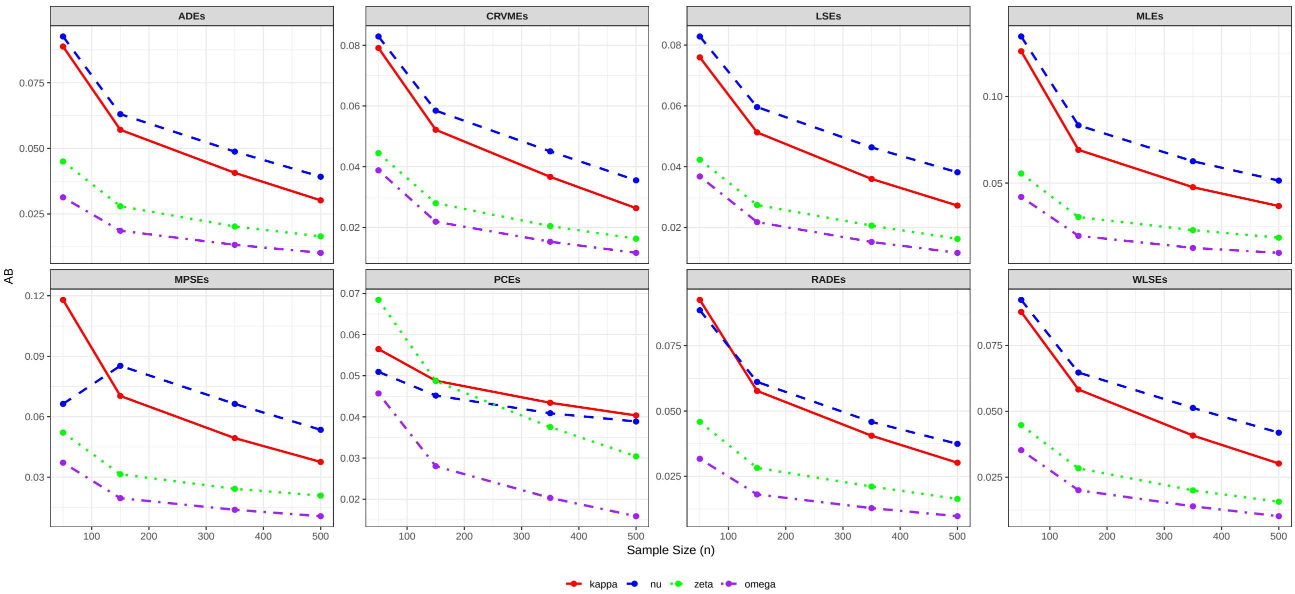


Figure 12. The AB of the OIPC parameters ($\kappa = 0.5, \nu = 0.5, \zeta = 0.25, \omega = 0.25$) for various sample sizes.

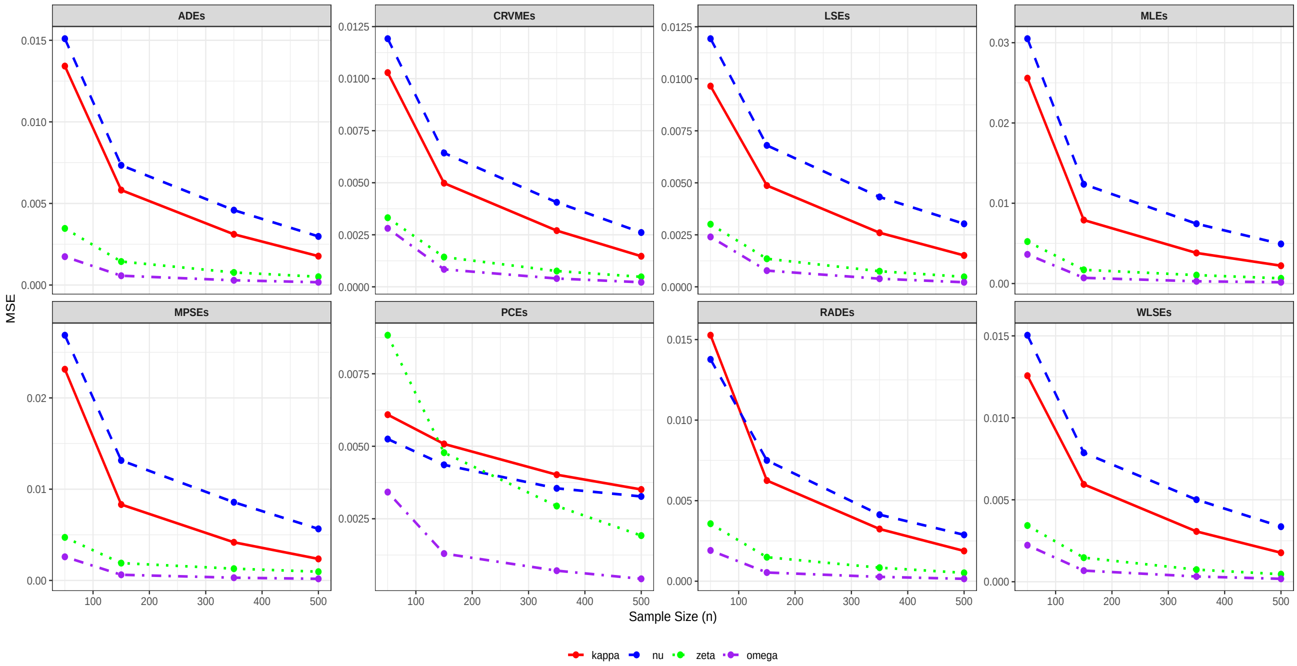


Figure 13. The MSE of the OIPC parameters ($\kappa = 0.5, \nu = 0.5, \zeta = 0.25, \omega = 0.25$) for various sample sizes.

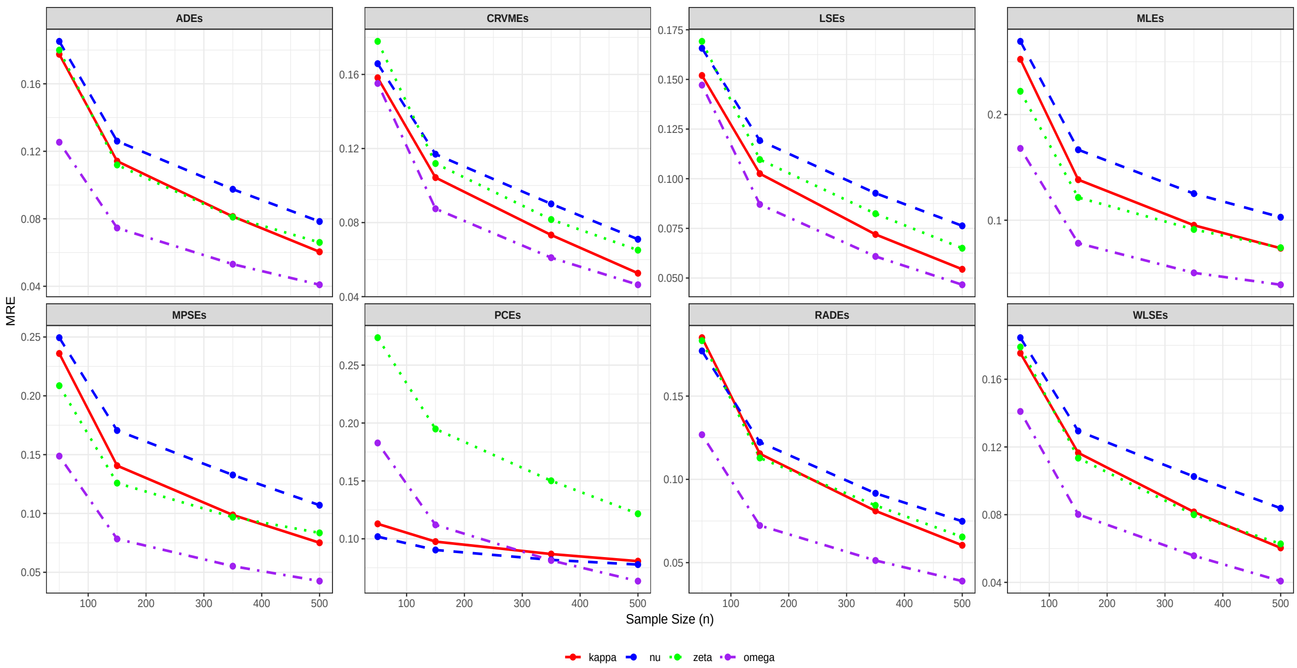


Figure 14. The MRE of the OIPC parameters ($\kappa = 0.5, \nu = 0.5, \zeta = 0.25, \omega = 0.25$) for various sample sizes.

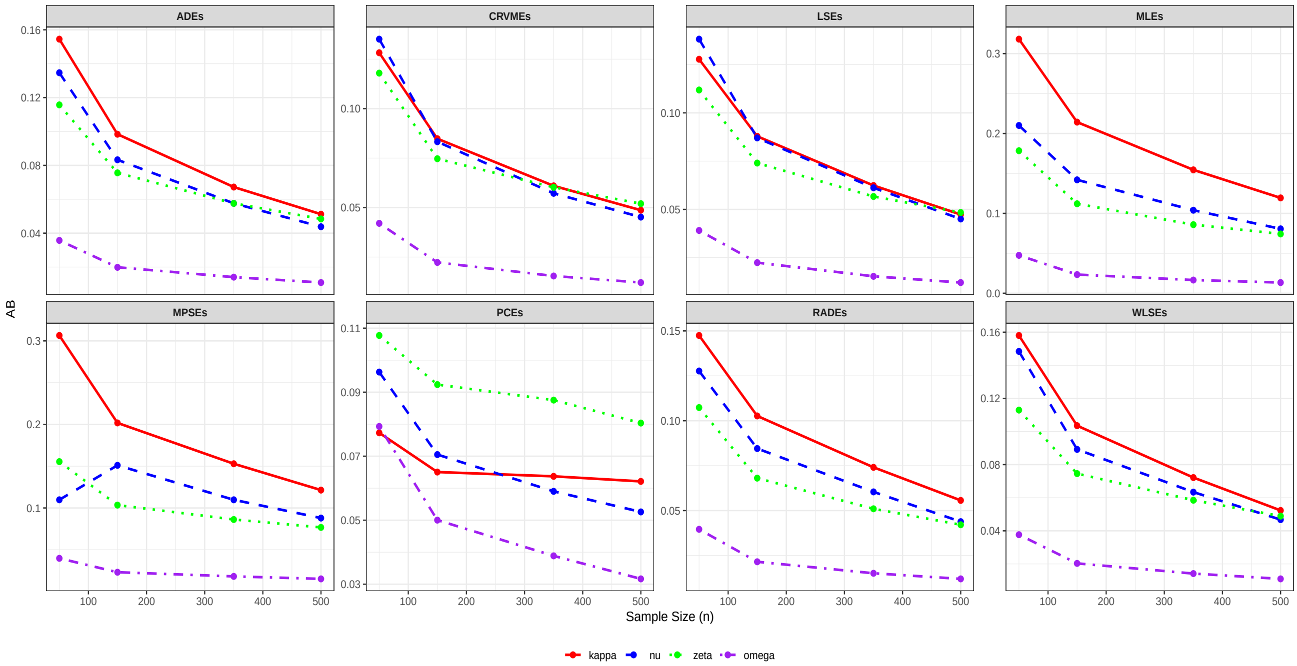


Figure 15. The AB of the OIPC parameters ($\kappa = 1.25, \nu = 0.5, \zeta = 1.25, \omega = 0.25$) for various sample sizes.

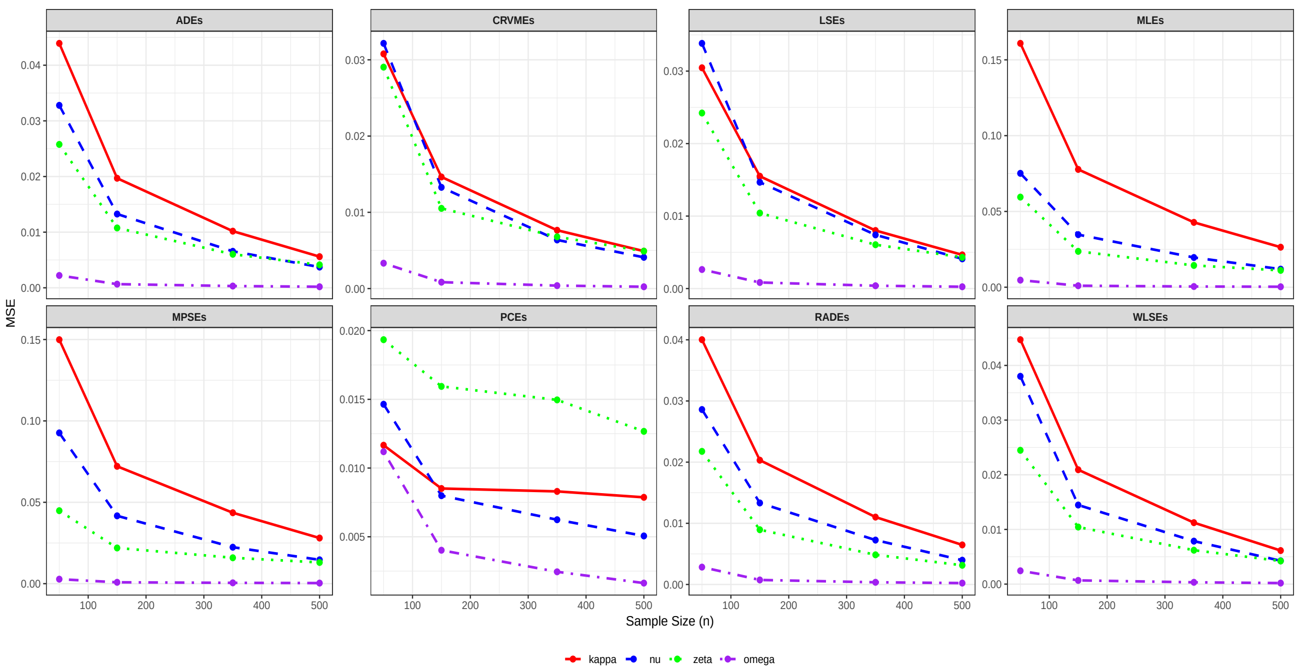


Figure 16. The MSE of the OIPC parameters ($\kappa = 1.25, \nu = 0.5, \zeta = 1.25, \omega = 0.25$) for various sample sizes.

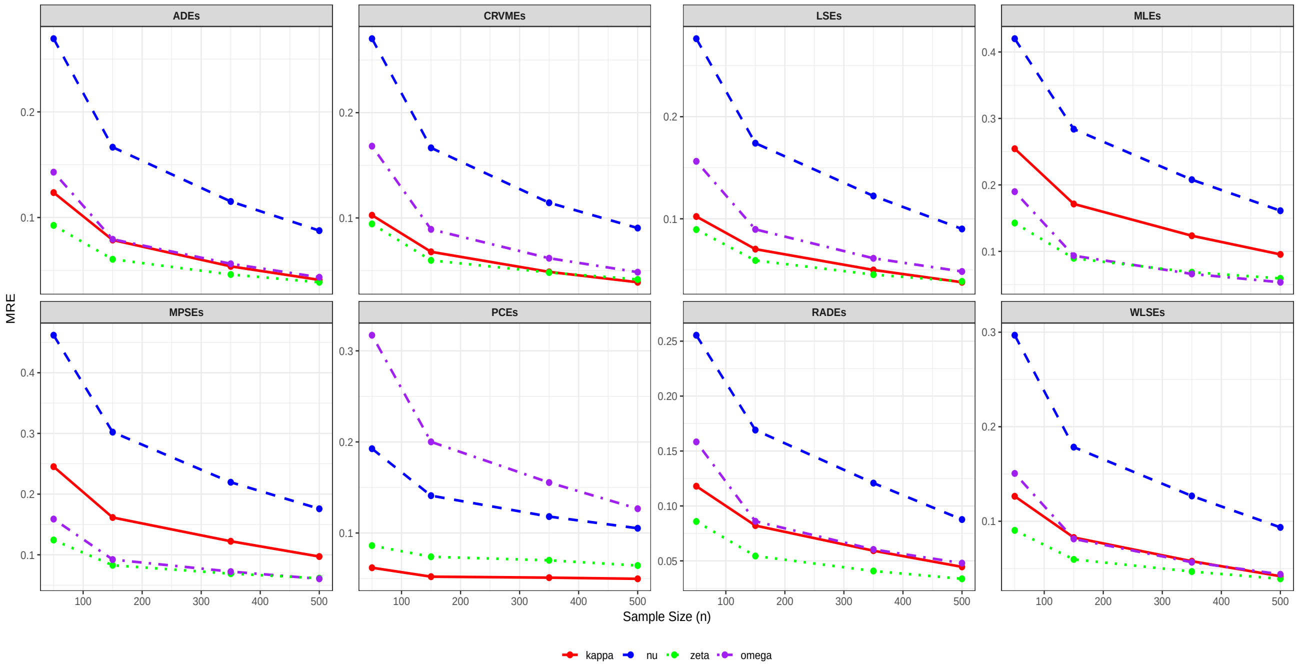


Figure 17. The MRE of the OIPC parameters ($\kappa = 1.25, \nu = 0.5, \zeta = 1.25, \omega = 0.25$) for various sample sizes.

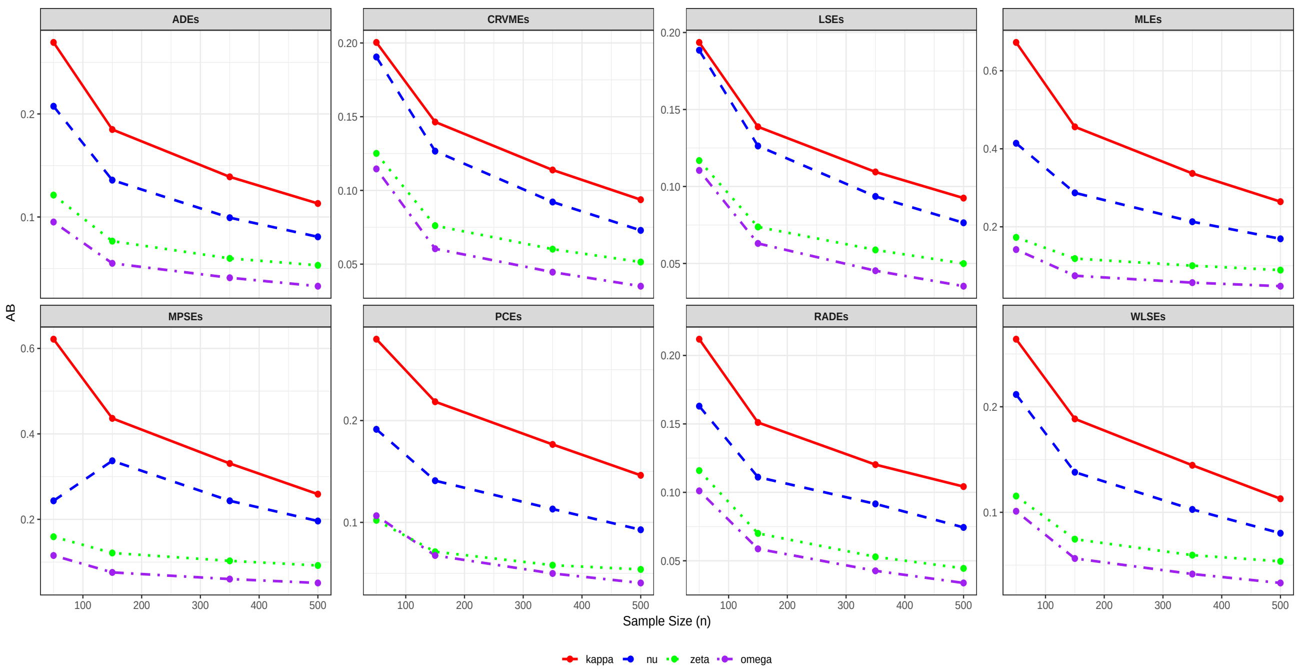


Figure 18. The AB of the OIPC parameters ($\kappa = 2.5, \nu = 1.25, \zeta = 1.75, \omega = 0.75$) for various sample sizes.

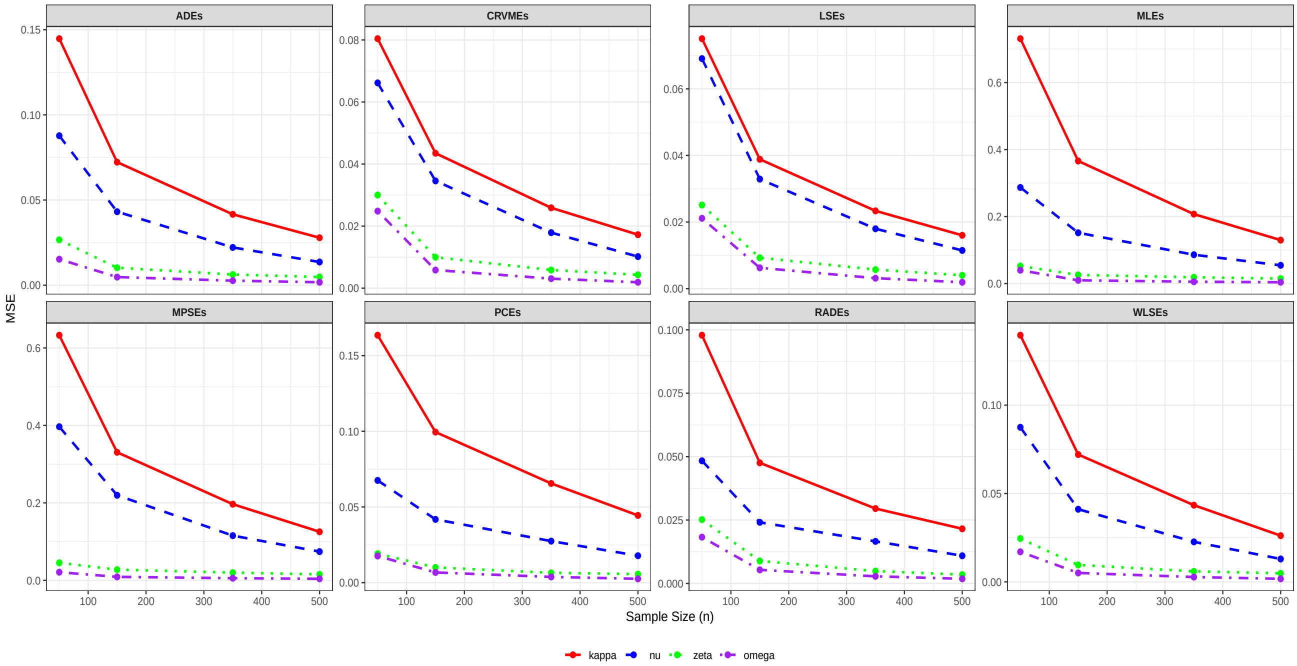


Figure 19. The MSE of the OIPC parameters ($\kappa = 2.5, \nu = 1.25, \zeta = 1.75, \omega = 0.75$) for various sample sizes.

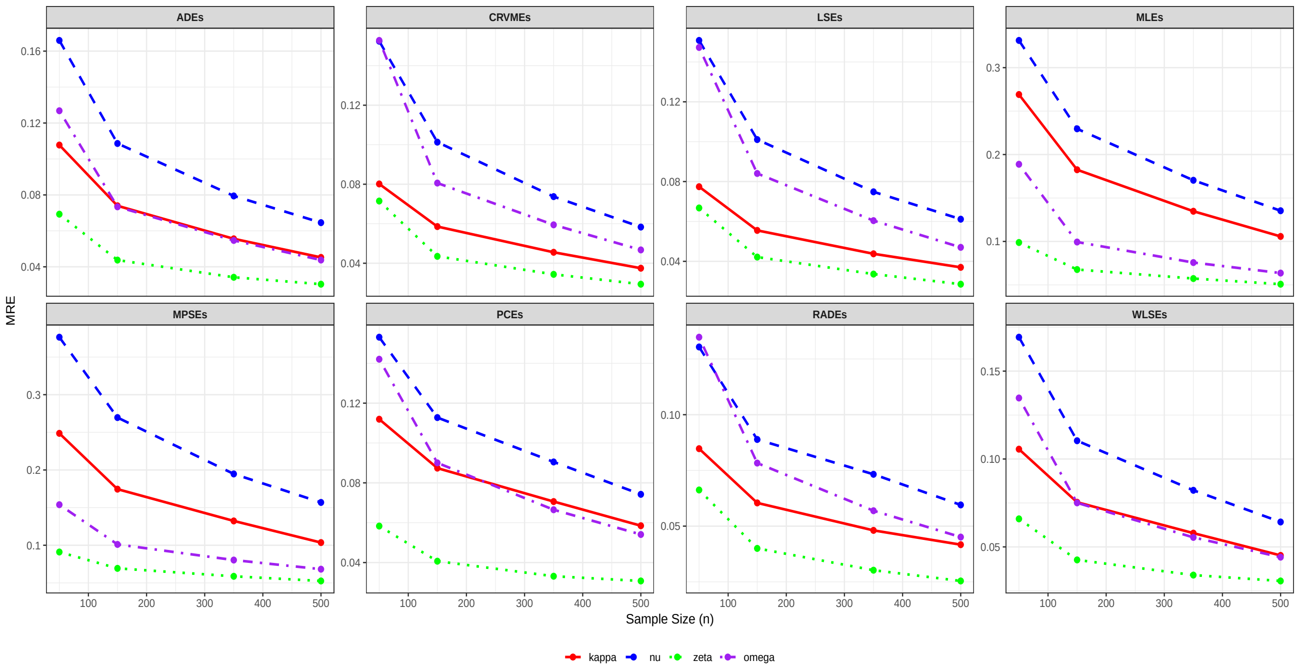


Figure 20. The MRE of the OIPC parameters ($\kappa = 2.5, \nu = 1.25, \zeta = 1.75, \omega = 0.75$) for various sample sizes.

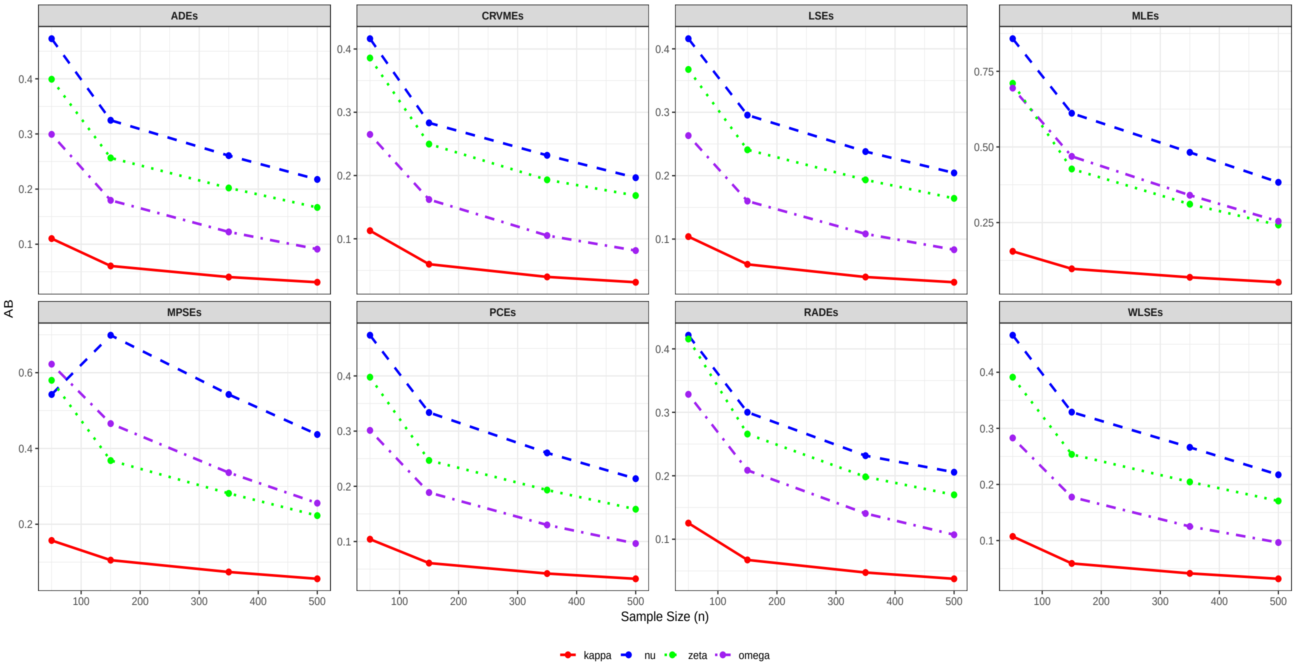


Figure 21. The AB of the OIPC parameters ($\kappa = 0.5, \nu = 2.25, \zeta = 3.5, \omega = 2.5$) for various sample sizes.

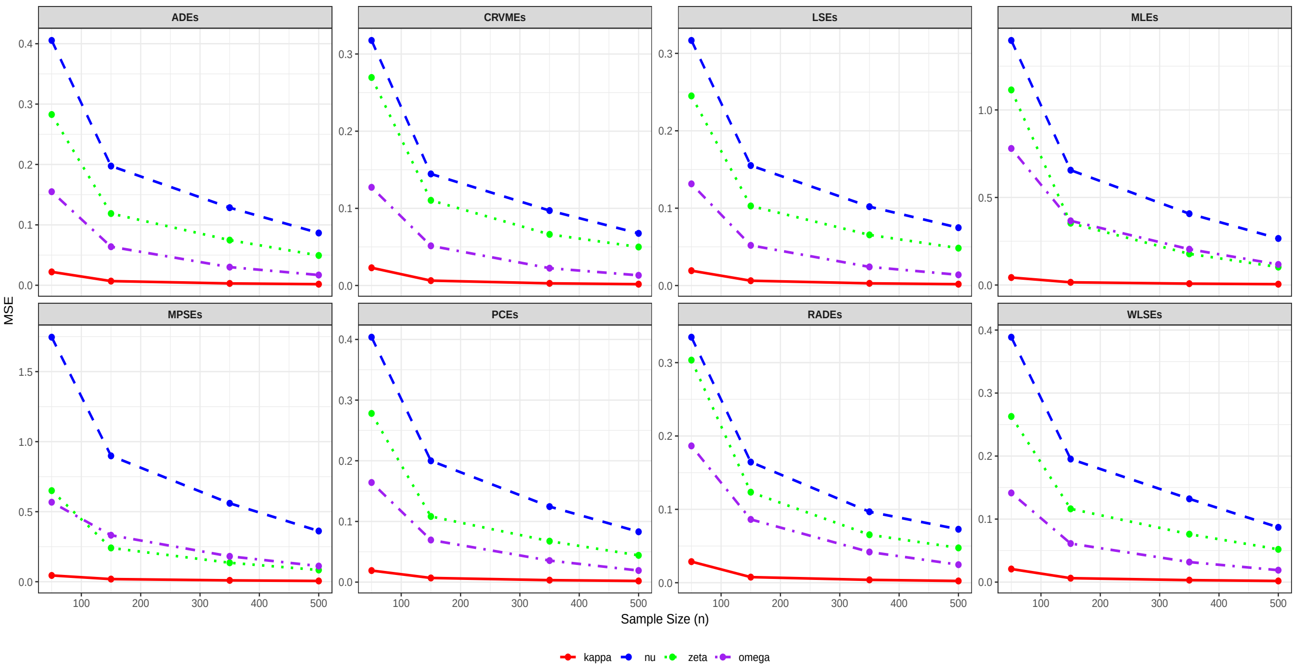


Figure 22. The MSE of the OIPC parameters ($\kappa = 0.5, \nu = 2.25, \zeta = 3.5, \omega = 2.5$) for various sample sizes.

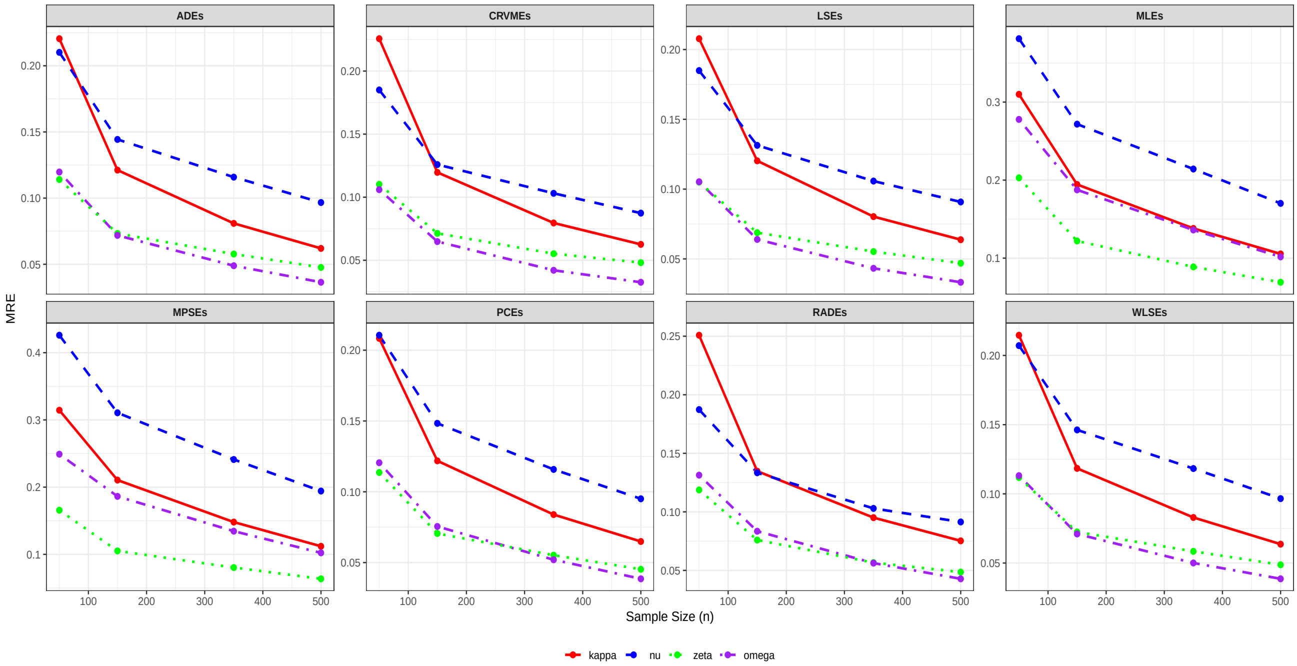


Figure 23. The MRE of the OIPC parameters ($\kappa = 0.5, \nu = 2.25, \zeta = 3.5, \omega = 2.5$) for various sample sizes.

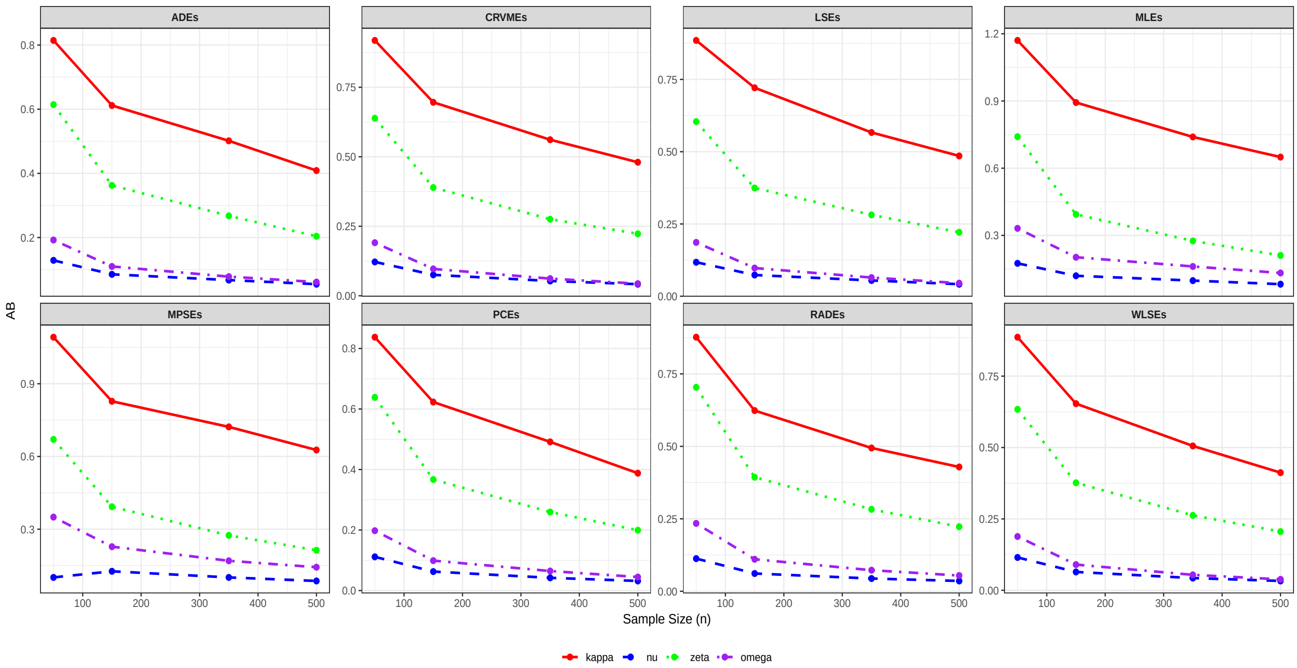


Figure 24. The AB of the OIPC parameters ($\kappa = 10, \nu = 0.5, \zeta = 5, \omega = 3$) for various sample sizes.

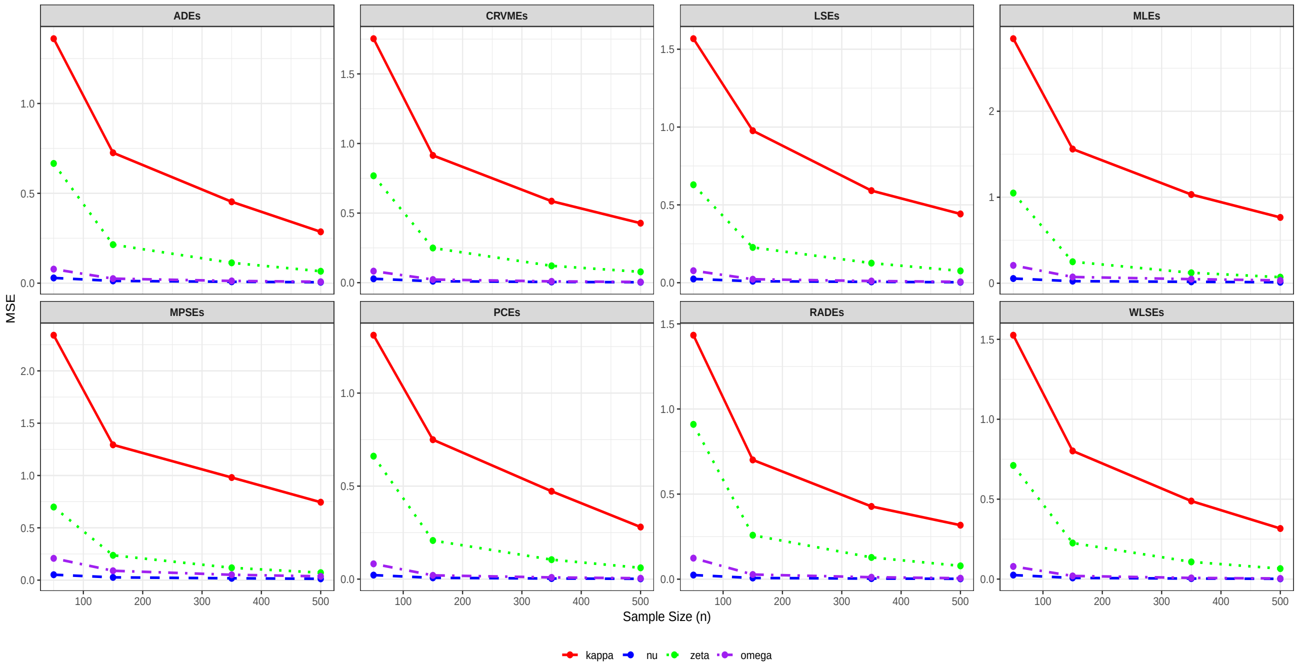


Figure 25. The MSE of the OIPC parameters ($\kappa = 10, \nu = 0.5, \zeta = 5, \omega = 3$) for various sample sizes.

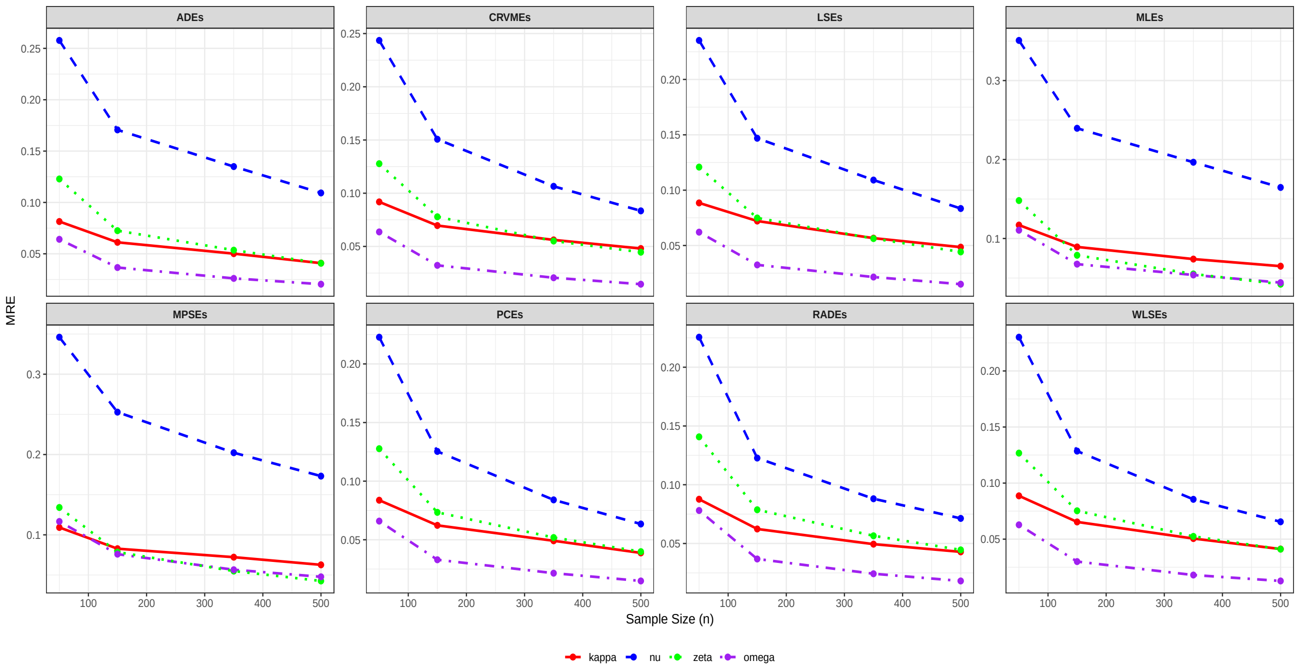


Figure 26. The MRE of the OIPC parameters ($\kappa = 10, \nu = 0.5, \zeta = 5, \omega = 3$) for various sample sizes.

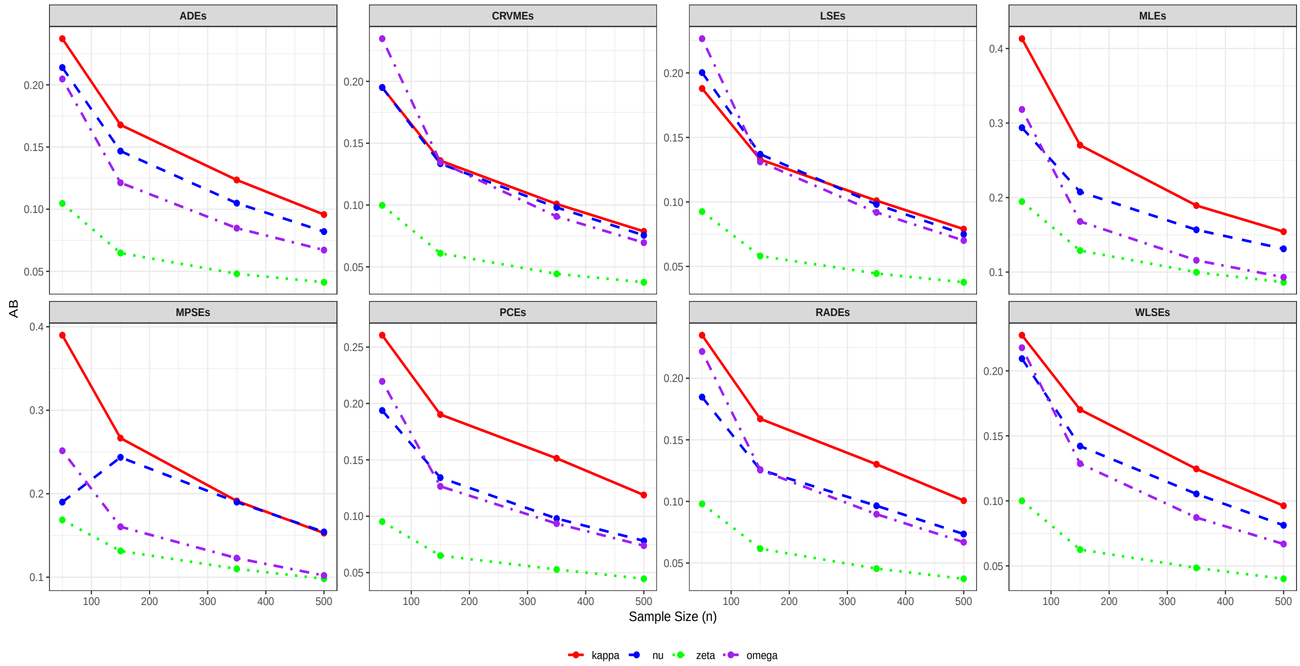


Figure 27. The AB of the OIPC parameters ($\kappa = 1.25, \nu = 0.75, \zeta = 1.25, \omega = 1.25$) for various sample sizes.

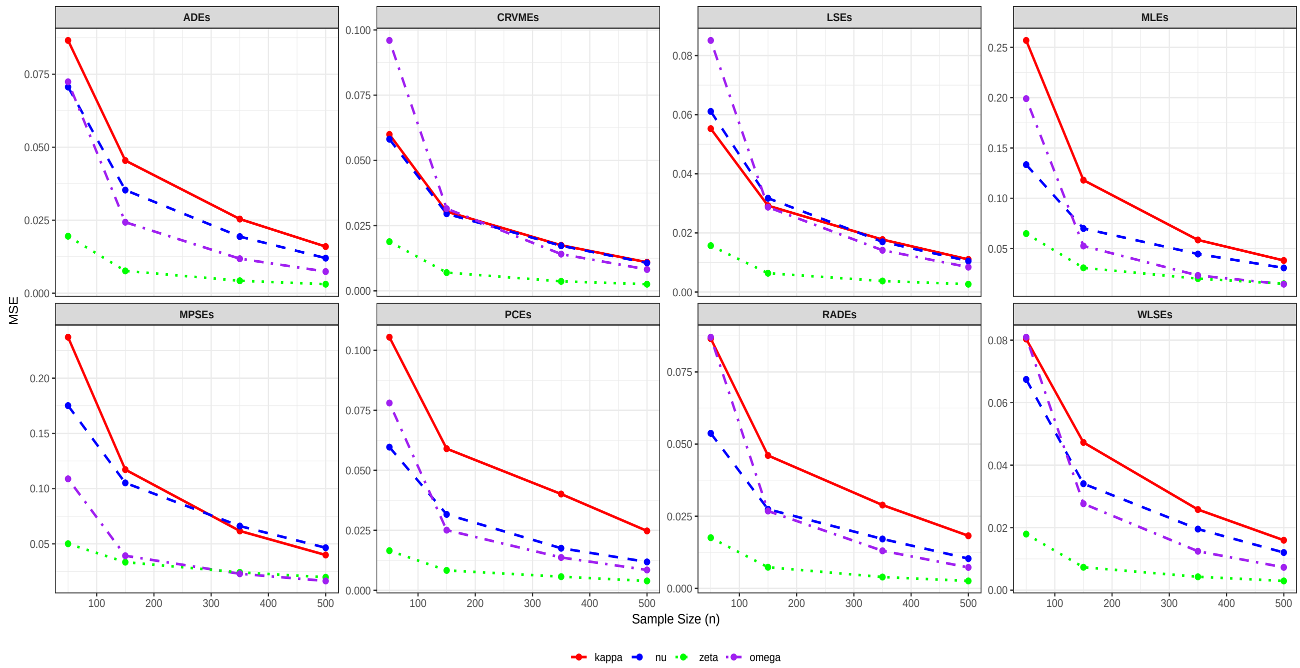


Figure 28. The MSE of the OIPC parameters ($\kappa = 1.25, \nu = 0.75, \zeta = 1.25, \omega = 1.25$) for various sample sizes.

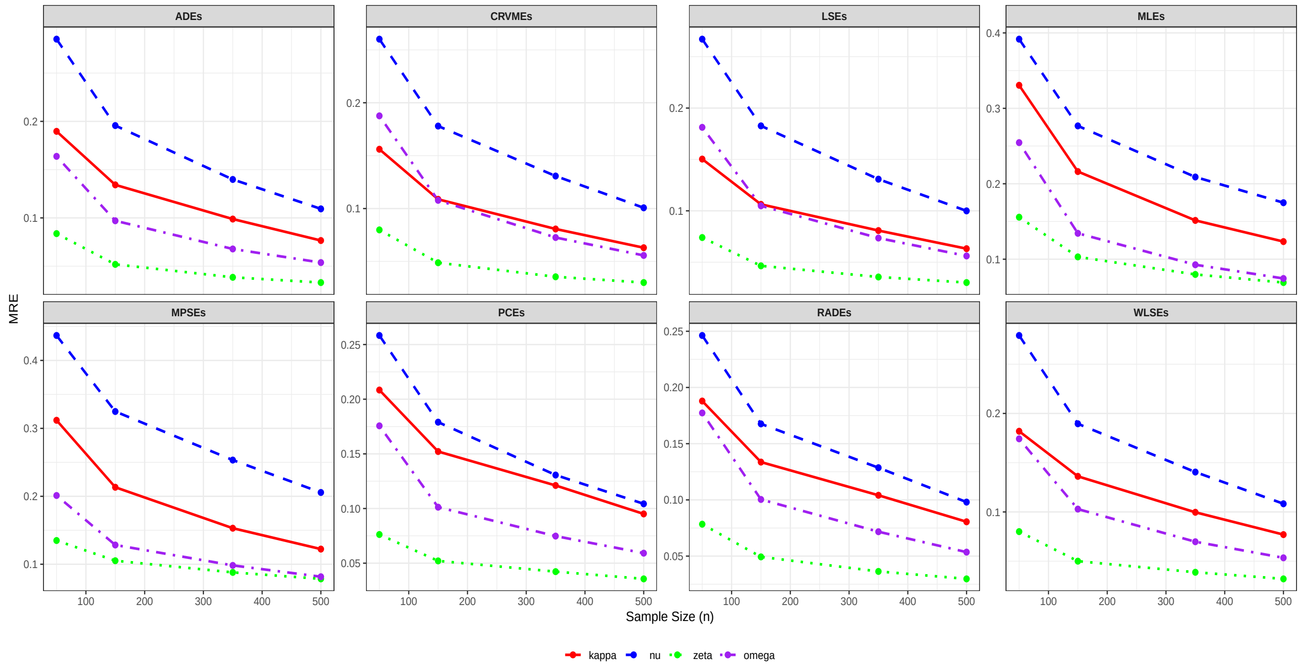


Figure 29. The MRE of the OIPC parameters ($\kappa = 1.25, \nu = 0.75, \zeta = 1.25, \omega = 1.25$) for various sample sizes.

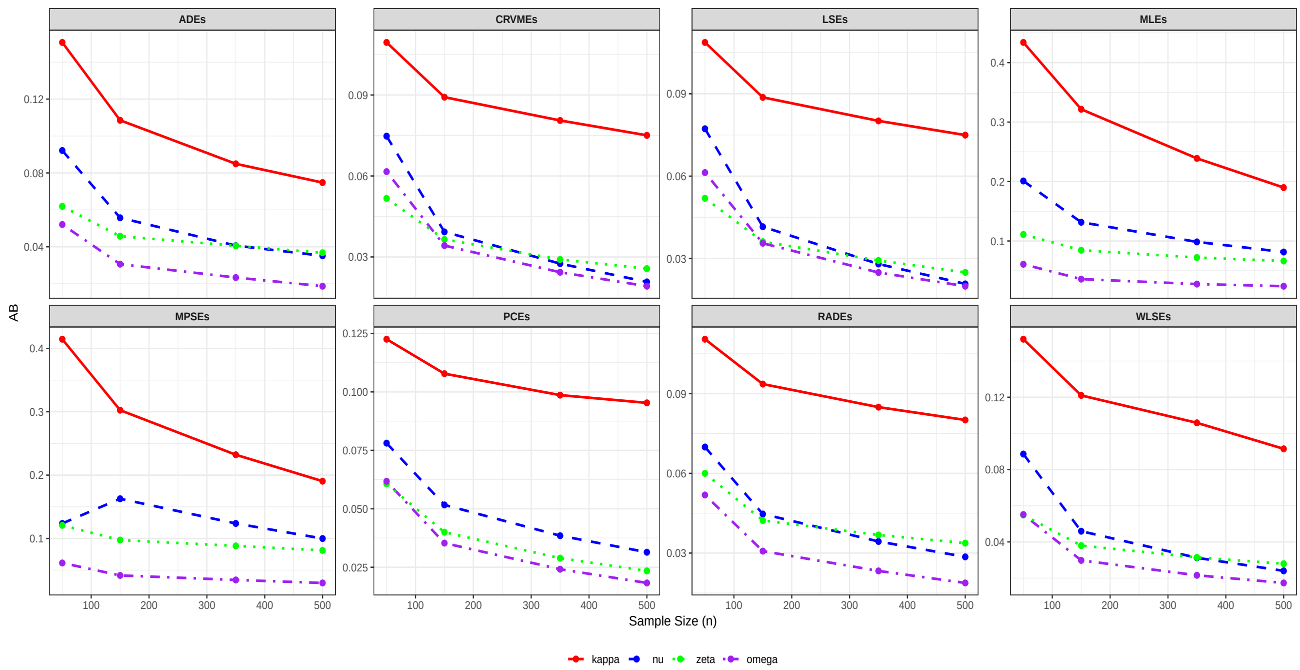


Figure 30. The AB of the OIPC parameters ($\kappa = 2.25, \nu = 0.5, \zeta = 0.75, \omega = 0.5$) for various sample sizes.

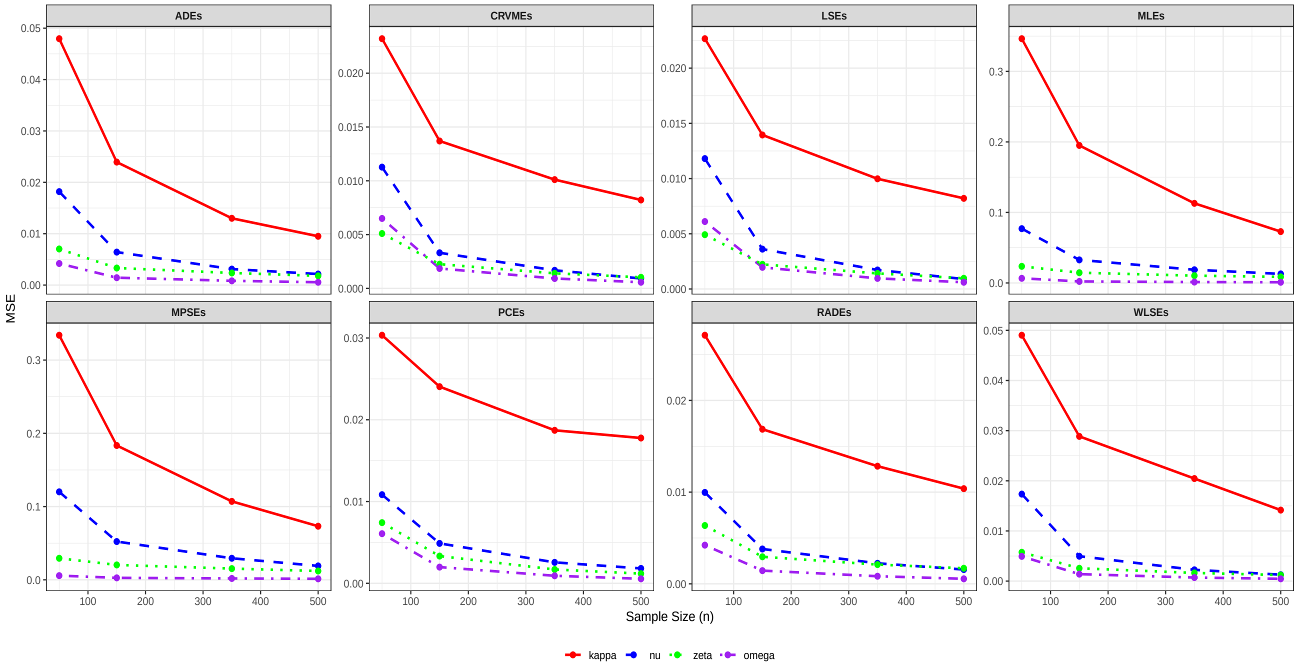


Figure 31. The MSE of the OIPC parameters ($\kappa = 2.25, \nu = 0.5, \zeta = 0.75, \omega = 0.5$) for various sample sizes.

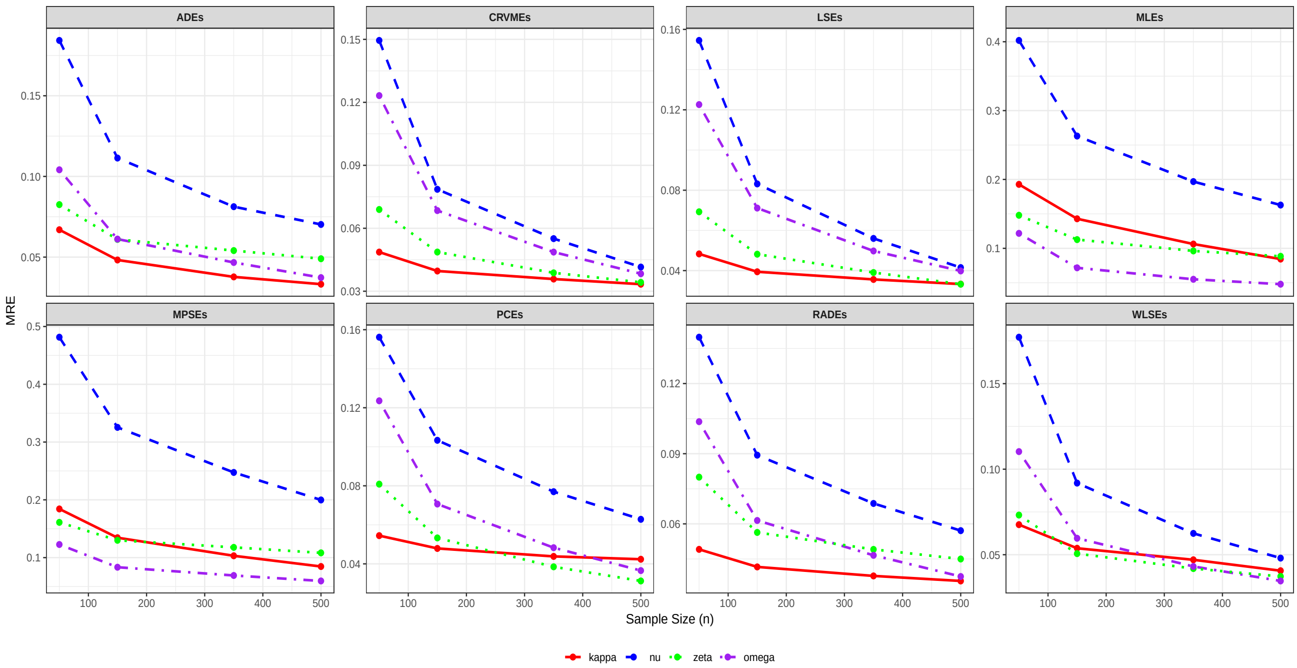


Figure 32. The MRE of the OIPC parameters ($\kappa = 2.25, \nu = 0.5, \zeta = 0.75, \omega = 0.5$) for various sample sizes.

7. Four real-life data applications

This section illustrates the practical relevance, flexibility, and empirical usefulness of the proposed OIPC distribution through applications to four real-world datasets. To provide a comprehensive assessment, the model is fitted to four datasets arising from diverse applied domains and compared with the Chen baseline distribution as well as several established competing models. These comparisons allow us to evaluate the ability of the proposed model to capture different forms of non-negative continuous data and to demonstrate its flexibility and competitive fitting performance in practice.

The datasets considered in this section were deliberately selected to reflect both practical significance and statistical diversity. They originate from different fields, including engineering and medical studies, where modeling non-negative lifetime-type observations is of central importance. Moreover, the datasets exhibit a variety of distributional characteristics, including different levels of skewness, tail heaviness, and hazard rate behaviors. This diversity provides an informative basis for examining the adaptability and robustness of the OIPC model relative to the baseline and competing distributions. In addition, some of these datasets have been previously analyzed in the literature, which facilitates meaningful benchmarking and enhances the reproducibility and comparability of the reported findings.

For the four real-data applications, parameter estimation is performed using the ML method for all fitted models. Although the simulation study indicates that some alternative estimation methods may outperform ML under certain finite-sample scenarios, the objective of this section is to ensure a unified and fair comparison across all competing distributions within a common likelihood-based framework. In this context, ML provides the most appropriate and coherent basis for empirical comparison, particularly because several of the adopted model selection criteria are naturally defined under the likelihood framework. Accordingly, to compare the fitted competing distributions, we consider the Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), Hannan–Quinn information criterion (HQIC), Cramér–von Mises statistic (W^*), Anderson–Darling statistic (A^*), Kolmogorov–Smirnov (KS) statistic, and its corresponding p-value.

The first dataset represents the remission times (in months) of 128 bladder cancer patients [22]. The observations are 22.69, 3.31, 2.69, 6.94, 0.9, 43.01, 4.5, 10.06, 11.64, 1.05, 13.8, 5.71, 3.52, 36.66, 3.02, 4.34, 2.23, 7.63, 10.66, 1.4, 21.73, 2.62, 14.83, 3.88, 3.82, 4.18, 5.62, 7.59, 46.12, 20.28, 6.54, 2.46, 10.34, 3.7, 14.76, 23.63, 0.08, 14.77, 1.76, 4.4, 1.26, 4.23, 1.19, 7.66, 2.87, 16.62, 4.26, 15.96, 9.47, 18.1, 5.17, 2.54, 12.03, 8.66, 12.02, 6.76, 8.26, 3.57, 2.64, 0.81, 7.32, 2.75, 17.14, 17.36, 6.93, 4.51, 2.07, 4.87, 13.11, 34.26, 11.98, 2.83, 0.2, 25.82, 12.07, 3.48, 7.26, 2.02, 79.05, 8.53, 6.25, 5.32, 3.25, 3.36, 5.41, 5.32, 2.26, 7.39, 6.97, 5.09, 9.74, 7.62, 3.36, 11.79, 7.28, 13.29, 17.12, 11.25, 10.75, 4.33, 4.98, 1.46, 5.49, 5.34, 5.85, 2.09, 7.93, 7.09, 19.13, 3.64, 2.69, 1.35, 2.02, 8.37, 8.65, 0.4, 9.02, 12.63, 5.06, 14.24, 5.41, 26.31, 7.87, 0.5, 0.51, 32.15, 9.22, and 25.74.

The second dataset consists of measurements of the strength of 1.5cm glass fibers obtained at the National Physical Laboratory in England. Unfortunately, the original source does not specify the units of measurement [23]. These data were previously analyzed by [24] using the Weibull Fréchet distribution. The observations are 1.59, 1.52, 1.04, 1.69, 1.11, 2.01, 1.61, 1.55, 1.68, 1.48, 1.27, 1.5, 1.64, 1.63, 1.66, 2, 1.49, 1.62, 0.55, 1.29, 1.68, 1.24, 1.66, 1.13, 1.55, 1.66, 1.77, 1.49, 1.61, 1.5, 0.93, 1.53, 1.7, 1.39, 1.7, 1.54, 0.74, 1.61, 1.36, 1.81, 1.58, 1.61, 1.84, 1.78, 1.48, 1.76, 1.6, 1.67, 1.73, 1.28, 1.89, 1.82, 2.24, 1.42, 1.62, 1.51, 1.25, 1.3, 0.77, 1.76, 1.84, 0.81, and 0.84.

The third dataset consists of the COVID-19 data from the United Kingdom collected over a 76-day period, from April 15 to June 30, 2020. The data were obtained from `https://covid19.who.int/` and were previously analyzed by [25]. The observations are 0.1652, 1.8392, 0.3553, 0.6197, 0.5696, 7.4456, 1.0438, 0.3926, 3.3609, 5.45, 0.0587, 1.8164, 0.1277, 0.2395, 2.4153, 2.3987, 5.3664, 10.187, 0.4633, 5.7522, 1.6998, 0.859, 0.2845, 0.3188, 0.2079, 1.8721, 2.5225, 0.3446, 2.7946, 0.3622, 6.4241, 0.469, 11.1429, 1.5709, 0.3926, 1.1533, 0.4954, 0.7193, 1.2707, 1.3423, 3.784, 0.2992, 1.0602, 3.9042, 11.2019, 0.2751, 0.7096, 9.6315, 4.3451, 1.226, 0.3317, 11.4584, 1.6324, 0.5139, 2.7087, 1.6083, 1.1305, 0.1247, 0.411, 7.0657, 1.9844, 0.7444, 2.136, 4.6477, 4.4627, 0.0863, 0.1303, 0.5837, 1.6017, 1.4149, 1.1468, 0.6365, 8.2307, 3.3715, 0.1165, and 4.1969.

The fourth dataset comprises 106 observations representing mortality rates during the COVID-19 pandemic in Mexico, recorded between March 4 and July 20, 2020. For analytical convenience, all observations were rescaled by dividing them by five. This linear transformation does not affect the inferential conclusions regarding the model parameters. These data were previously introduced by [26]. The corresponding observations are 0.679, 2.9208, 1.3936, 0.414, 1.568, 2.0766, 0.6998, 1.3312, 1.526, 1.0286, 0.6514, 1.221, 1.2364, 1.4534, 0.7502, 0.6596, 1.0918, 2.644, 1.312, 2.9942, 1.2824, 0.7264, 0.643, 0.6654, 0.6456, 0.241, 1.0784, 2.007, 0.7556, 1.1708, 0.4116, 0.6672, 1.8782, 0.8848, 0.3846, 2.333, 0.6058, 1.987, 0.6466, 1.4972, 0.9394, 0.9296, 0.7884, 1.7626, 1.3394, 0.7074, 1.3628, 0.9114, 0.4652, 0.36, 1.8568, 0.6054, 3.2996, 1.0634, 0.6572, 1.325, 1.307, 0.9898, 0.5052, 0.363, 2.0086, 1.452, 0.2082, 0.6436, 1.274, 0.5202, 1.0484, 0.4876, 1.5708, 1.7392, 0.413, 1.91, 0.5, 2.171, 0.946, 1.5806, 1.7652, 1.197, 1.6216, 0.824, 0.4704, 0.9322, 1.665, 0.5852, 1.228, 1.4302, 1.0884, 0.2804, 0.8178, 0.6438, 0.4154, 2.137, 0.9818, 1.0812, 0.688, 1.2244, 1.712, 0.8584, 1.203, 0.8668, 1.2654, 2.5756, 0.6718, 2.0316, 0.5976, and 2.4084.

The fit of the OIPC distributions is compared with some rival distributions, namely: exponential generated modified Chen (EGMC) [2], Kumaraswamy exponentiated Chen (KwEC) [3], new extended Chen (NEC) [27], transmuted Chen (TC) [4], inverse Chen (IC) [5], and Chen (C) [1] distributions.

To further justify the complexity of the proposed model, we augmented the comparative analysis with several simpler classical distributions, including the Fréchet (Fr) [28, 29], Pareto (Pa) [30], Burr XII (BXII) [31], and Weibull (W) [32] models.

Tables 12–15 present the ML estimates along with their standard errors (SEs), as well as several goodness-of-fit measures for the OIPC distribution and competing models across four real-world datasets. The results indicate that the proposed OIPC distribution is a competitive and flexible alternative to existing Chen-type models and provides superior overall fitting performance for the analyzed datasets according to most goodness-of-fit criteria. In particular, the OIPC model generally yields smaller values of AIC, CAIC, HQIC, and goodness-of-fit (W^*, A^*, and KS) statistics, together with larger KS p-values than the competing models. Although the W distribution attains slightly smaller BIC values for the four datasets, this behavior is expected due to the stronger penalty imposed by the BIC on models with additional parameters, which naturally favors simpler models. Nevertheless, the improved performance of the OIPC model under the remaining fit measures suggests that the additional parameters contribute meaningful flexibility in capturing complex distributional characteristics. Overall, these findings demonstrate that the OIPC distribution is an effective and adaptable model for analyzing skewed, heavy-tailed, and heterogeneous real-world data.

Table 12. The ML estimates, SEs, and goodness-of-fit statistics for the competing models fitted to the first dataset.

Model	Param.	Estimates	(SEs)	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	KS p-value
OIPC	$\hat{\kappa}$	1.786353	(1.176967)	826.941300	827.266500	838.349400	831.576500	0.014900	0.094800	0.029900	0.999843
	$\hat{\nu}$	295.734610	(1353.225000)								
	$\hat{\zeta}$	2.506496	(1.936282)								
	$\hat{\omega}$	0.130812	(0.053891)								
EGMC	$\hat{\alpha}$	8.176506	(3.700384)	829.669300	829.994500	841.077400	834.304500	0.047200	0.311000	0.045900	0.949919
	$\hat{\beta}$	4.776463	(18.678758)								
	$\hat{\tau}$	0.150392	(0.587103)								
	$\hat{\eta}$	0.220676	(0.027626)								
KwEC	$\hat{a}$	9.755048	(184.048380)	831.210800	831.702600	845.470900	837.004700	0.042600	0.279900	0.045200	0.956129
	$\hat{b}$	3.813939	(10.967520)								
	$\hat{\alpha}$	1.320883	(24.913433)								
	$\hat{\tau}$	0.753889	(0.477853)								
	$\hat{\eta}$	0.146295	(0.153159)								
NEC	$\hat{\alpha}$	0.384977	(0.012697)	859.148400	859.341900	867.704500	862.624800	0.379710	2.233848	0.128287	0.029600
	$\hat{\tau}$	0.041374	(0.007008)								
	$\hat{\eta}$	0.097247	(0.082970)								
TC	$\hat{\lambda}$	0.844131	(0.122306)	860.861200	861.054700	869.417300	864.337600	0.399800	2.348600	0.132300	0.022597
	$\hat{\tau}$	0.056439	(0.008939)								
	$\hat{\eta}$	0.374336	(0.012804)								
IC	$\hat{\tau}$	1.135632	(0.100381)	949.219400	949.315400	954.923400	951.537000	1.288500	7.574400	0.239800	0.000000
	$\hat{\eta}$	0.411452	(0.020105)								
C	$\hat{\tau}$	0.111429	(0.015530)	866.325100	866.421100	872.029100	868.642700	0.451100	2.658400	0.142900	0.010709
	$\hat{\eta}$	0.355027	(0.012264)								
Fr	$\hat{\alpha}$	0.752080	(0.042424)	892.001500	892.097500	897.705600	894.319100	0.744300	4.546400	0.140800	0.012502
	$\hat{\lambda}$	2.431095	(0.219281)								
Pa	$\hat{\beta}$	0.280000	(0.000000)	1081.046000	1081.142000	1086.750000	1083.364000	3.379800	18.531500	0.420400	0.000000
	$\hat{\theta}$	0.233690	(0.020655)								
BXII	$\hat{\tau}$	2.334808	(0.354047)	911.033200	911.129200	916.737200	913.350800	0.749700	4.554900	0.250700	0.000000
	$\hat{\eta}$	0.233753	(0.039964)								
W	$\hat{\beta}$	1.047835	(0.067577)	832.173800	832.269800	837.877800	834.491300	0.131374	0.786483	0.070017	0.556965
	$\hat{\alpha}$	9.560699	(0.852901)								

Table 13. The ML estimates, SEs, and goodness-of-fit statistics for the competing models fitted to the second dataset.

Model	Param.	Estimates	(SEs)	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	KS p-value
OIPC	$\hat{\kappa}$	0.398408	(0.090990)	31.129000	31.818650	39.701540	34.500620	0.068185	0.400239	0.098484	0.574193
	$\hat{\gamma}$	28340180	(28.716302)								
	$\hat{\xi}$	6.060708	(0.612246)								
	$\hat{\omega}$	0.545841	(0.087912)								
EGMC	$\hat{\alpha}$	1.948958	(0.663836)	36.546520	37.236170	45.119060	39.918140	0.171673	0.964459	0.133432	0.211958
	$\hat{\beta}$	0.683504	(10.478340)								
	$\hat{\tau}$	0.252515	(3.870186)								
	$\hat{\eta}$	1.683137	(0.177198)								
KwEC	$\hat{a}$	41.528736	(96.336930)	38.255380	39.308010	48.971050	42.469900	0.168159	0.943475	0.133146	0.213997
	$\hat{b}$	4.498721	(17.584115)								
	$\hat{\alpha}$	0.051096	(0.119707)								
	$\hat{\tau}$	0.088999	(0.137364)								
	$\hat{\eta}$	1.486329	(0.574876)								
NEC	$\hat{\alpha}$	1.553526	(0.170373)	33.020380	33.427160	39.749790	35.549100	0.125014	0.717675	0.116750	0.356975
	$\hat{\tau}$	0.272738	(0.102738)								
	$\hat{\eta}$	3.709409	(1.303141)								
TC	$\hat{\lambda}$	0.856614	(0.149414)	35.549430	35.956210	41.978830	38.078140	0.135171	0.804190	0.130794	0.231333
	$\hat{\tau}$	0.034633	(0.008512)								
	$\hat{\eta}$	2.065899	(0.097130)								
IC	$\hat{\tau}$	0.958033	(0.121316)	124.957800	125.157800	129.244100	126.643600	1.508429	7.808474	0.298016	0.000028
	$\hat{\eta}$	1.581039	(0.115314)								
C	$\hat{\tau}$	0.072052	(0.016229)	36.922670	37.122670	41.208940	38.608480	0.161440	0.961901	0.137700	0.183000
	$\hat{\eta}$	1.960390	(0.093981)								
Fr	$\hat{\alpha}$	2.887556	(0.234430)	97.706630	97.906630	101.992900	99.392440	1.225456	6.486581	0.244434	0.001080
	$\hat{\lambda}$	1.968518	(0.248528)								
Pa	$\hat{\beta}$	0.549450	(0.000076)	175.455000	175.655000	179.741300	177.140800	1.709036	8.861022	0.421375	0.000000
	$\hat{a}$	1.020514	(0.128574)								
BXII	$\hat{\lambda}$	7.482073	(1.285056)	101.442400	101.642400	105.728700	103.128200	1.132462	6.128199	0.330327	0.000002
	$\hat{\theta}$	0.320662	(0.065453)								
W	$\hat{\beta}$	5.780651	(0.576091)	34.413680	34.613680	38.699950	36.099490	0.237240	1.303711	0.152238	0.107826
	$\hat{\alpha}$	1.628113	(0.037095)								

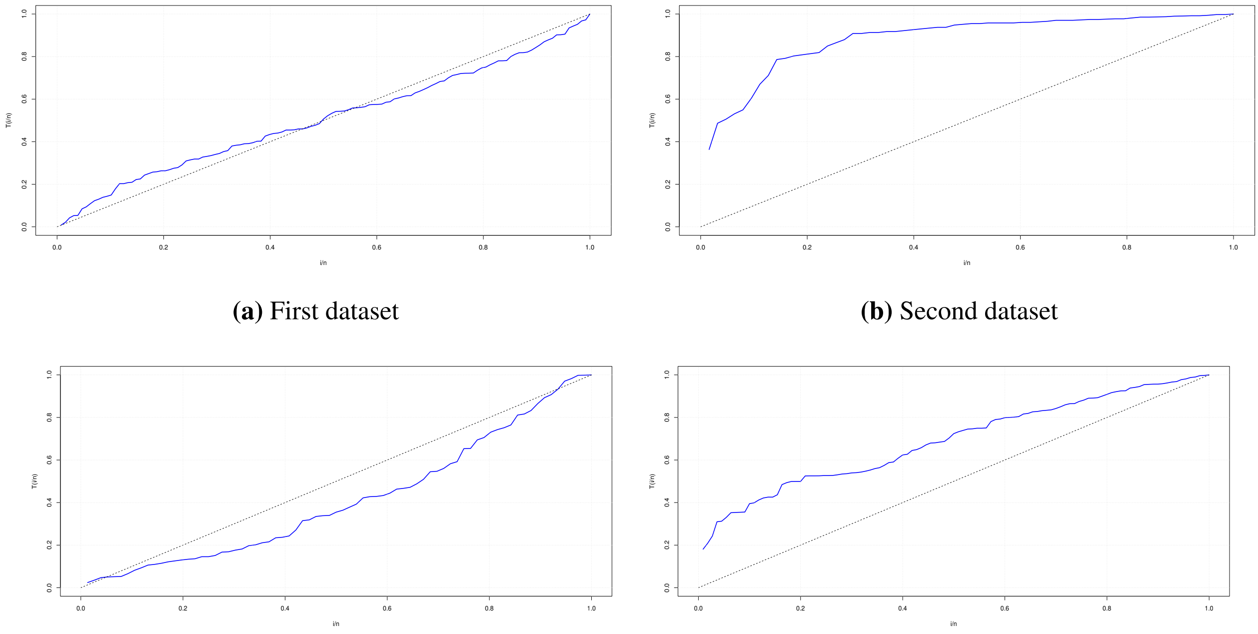
Table 14. The ML estimates, SEs, and goodness-of-fit statistics for the competing models fitted to the third dataset.

Model	Param.	Estimates	(SEs)	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	KS p-value
OIPC	$\hat{\kappa}$	16.234639	(19.066599)	282.932600	283.496000	292.255500	286.658500	0.029438	0.213195	0.047583	0.995349
	$\hat{\gamma}$	0.009535	(0.006716)								
	$\hat{\xi}$	0.099187	(0.101144)								
	$\hat{\omega}$	0.469177	(0.074976)								
EGMC	$\hat{\alpha}$	16.694285	(21.126465)	287.255100	287.818500	296.578100	290.981000	0.048173	0.340470	0.056987	0.965956
	$\hat{\beta}$	4.090482	(44.200293)								
	$\hat{\tau}$	0.432229	(4.669502)								
	$\hat{\eta}$	0.168576	(0.060237)								
KwEC	$\hat{a}$	57.811374	(309.659021)	289.67800	290.535800	301.332300	294.336000	0.050659	0.358124	0.059778	0.948799
	$\hat{b}$	10.501337	(30.319937)								
	$\hat{\alpha}$	6.899368	(36.942409)								
	$\hat{\tau}$	2.860432	(3.285486)								
	$\hat{\eta}$	0.039329	(0.070399)								
NEC	$\hat{\alpha}$	0.466700	(0.030279)	298.557700	298.891100	305.549900	301.352100	0.237288	1.524547	0.100950	0.420837
	$\hat{\tau}$	0.159515	(0.047742)								
	$\hat{\eta}$	0.330662	(0.247131)								
TC	$\hat{\lambda}$	0.550400	(0.297275)	299.296400	299.629700	306.288600	302.090800	0.245961	1.576659	0.103465	0.390003
	$\hat{\tau}$	0.195848	(0.049751)								
	$\hat{\eta}$	0.451027	(0.029511)								
IC	$\hat{\tau}$	0.384085	(0.052687)	311.868100	312.032500	316.529600	313.731000	0.311541	2.030039	0.163878	0.033700
	$\hat{\eta}$	0.394036	(0.025551)								
C	$\hat{\tau}$	0.293212	(0.044530)	298.982200	299.146600	303.643700	300.845200	0.266445	1.702620	0.109072	0.326000
	$\hat{\eta}$	0.422863	(0.028143)								
Fr	$\hat{\alpha}$	0.789624	(0.066940)	294.344500	294.508900	299.005900	296.207400	0.151690	1.021070	0.102154	0.406000
	$\hat{\lambda}$	0.670082	(0.091667)								
Pa	$\hat{\beta}$	0.058641	(0.000076)	345.679400	345.843800	350.340900	347.542300	0.371432	2.490381	0.285329	0.000008
	$\hat{a}$	0.334539	(0.038374)								
BXII	$\hat{\lambda}$	1.332127	(0.141896)	288.974400	289.138800	293.635800	290.837300	0.085618	0.573216	0.068534	0.868000
	$\hat{\theta}$	0.901745	(0.117066)								
W	$\hat{\beta}$	0.846592	(0.074347)	287.503700	287.668100	292.165200	289.366700	0.113331	0.765320	0.080560	0.707394
	$\hat{\alpha}$	2.220008	(0.318469)								

Table 15. The ML estimates, SEs, and goodness-of-fit statistics for the competing models fitted to the fourth dataset.

Model	Param.	Estimates	(SEs)	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	KS p-value
OIPC	$\hat{\kappa}$	25.447652	(177.172800)	187.939700	188.335700	198.593400	192.257700	0.033645	0.203404	0.047388	0.971164
	$\hat{\gamma}$	0.007061	(0.003705)								
	$\hat{\xi}$	0.128470	(0.765024)								
	$\hat{\omega}$	1.034892	(0.948517)								
EGMC	$\hat{\alpha}$	11.219554	(8.965016)	189.906100	190.302200	200.559900	194.224200	0.050933	0.281551	0.065805	0.748290
	$\hat{\beta}$	0.882685	(18.505520)								
	$\hat{\tau}$	1.820883	(38.175780)								
	$\hat{\eta}$	0.435745	(0.108863)								
KwEC	$\hat{a}$	2.533525	(0.148222)	190.145900	190.745900	203.463100	195.543400	0.034503	0.204620	0.054035	0.916329
	$\hat{b}$	0.193528	(0.024618)								
	$\hat{\alpha}$	5.262509	(NaN)								
	$\hat{\tau}$	3.465633	(0.043877)								
	$\hat{\eta}$	0.683389	(0.035257)								
NEC	$\hat{\alpha}$	1.064068	(0.056659)	207.941300	208.176600	215.931700	211.179900	0.239743	1.546970	0.104064	0.201155
	$\hat{\tau}$	0.159085	(0.032251)								
	$\hat{\eta}$	0.188192	(0.144968)								
TC	$\hat{\lambda}$	0.723928	(0.194660)	209.098500	209.333800	217.088800	212.337000	0.251431	1.619263	0.107218	0.174710
	$\hat{\tau}$	0.202055	(0.035872)								
	$\hat{\eta}$	1.025405	(0.055749)								
IC	$\hat{\tau}$	0.369735	(0.042554)	244.075800	244.192300	249.402700	246.234800	0.551816	3.457843	0.171641	0.003880
	$\hat{\eta}$	0.771993	(0.038452)								
C	$\hat{\tau}$	0.339937	(0.041039)	210.773300	210.889800	216.100200	212.932300	0.277364	1.793901	0.115802	0.116000
	$\hat{\eta}$	0.955111	(0.052285)								
Fr	$\hat{\alpha}$	1.683150	(0.116280)	210.017700	210.134200	215.344600	212.176700	0.272920	1.746980	0.092660	0.323000
	$\hat{\lambda}$	0.603081	(0.072083)								
Pa	$\hat{\beta}$	0.207992	(0.000076)	309.947200	310.063700	315.274100	312.106200	0.570651	3.742933	0.332650	0.000000
	$\hat{a}$	0.638849	(0.062050)								
BXII	$\hat{\lambda}$	2.946975	(0.256841)	191.862000	191.978500	198.688900	194.021000	0.107755	0.611492	0.080757	0.494000
	$\hat{\theta}$	1.021696	(0.108693)								
W	$\hat{\beta}$	1.919709	(0.140808)	191.388800	191.505300	196.715600	193.547800	0.102244	0.656982	0.069244	0.689647
	$\hat{\alpha}$	1.319015	(0.070631)								

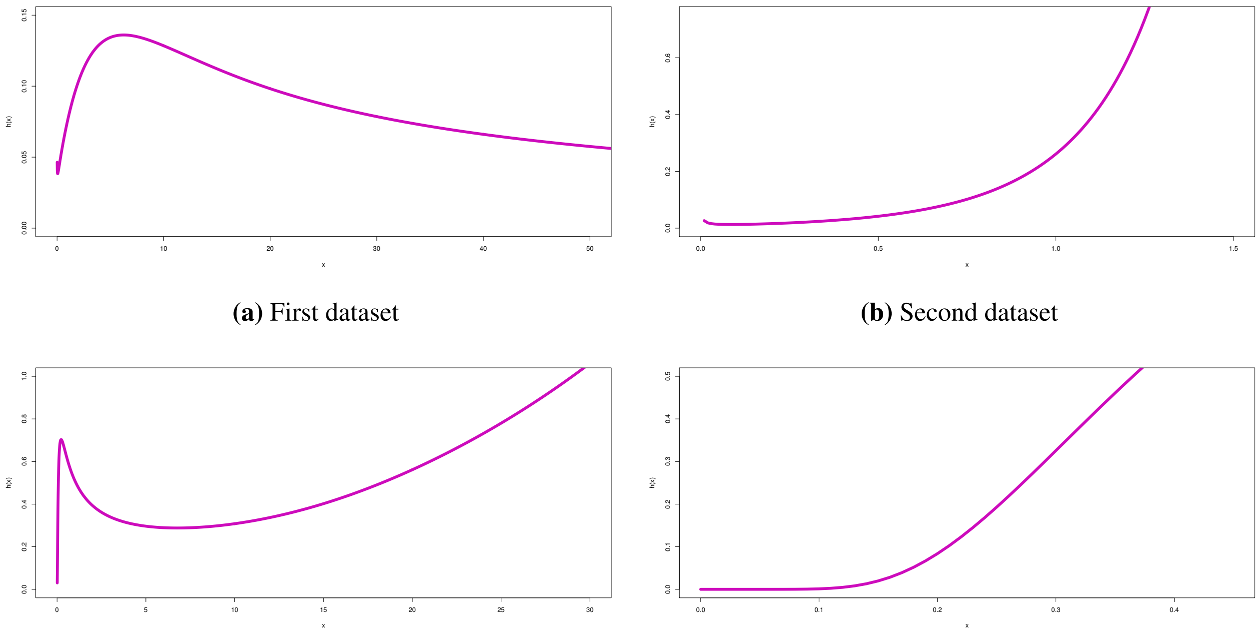
The total time on test (TTT) plots for the four datasets are displayed in Figure 33, while the corresponding HRF plots of the OIPC distribution are presented in Figure 34. These plots indicate that the analyzed datasets exhibit diverse failure rate patterns, including unimodal (first dataset), increasing (second and fourth datasets), and modified bathtub-shaped (third dataset) behaviors. This empirical evidence highlights the flexibility of the proposed OIPC distribution in capturing various hazard rate structures commonly encountered in real-life reliability data.



(a) First dataset **(b)** Second dataset

(c) Third dataset **(d)** Fourth dataset

Figure 33. The TTT plots for the four real-life datasets.



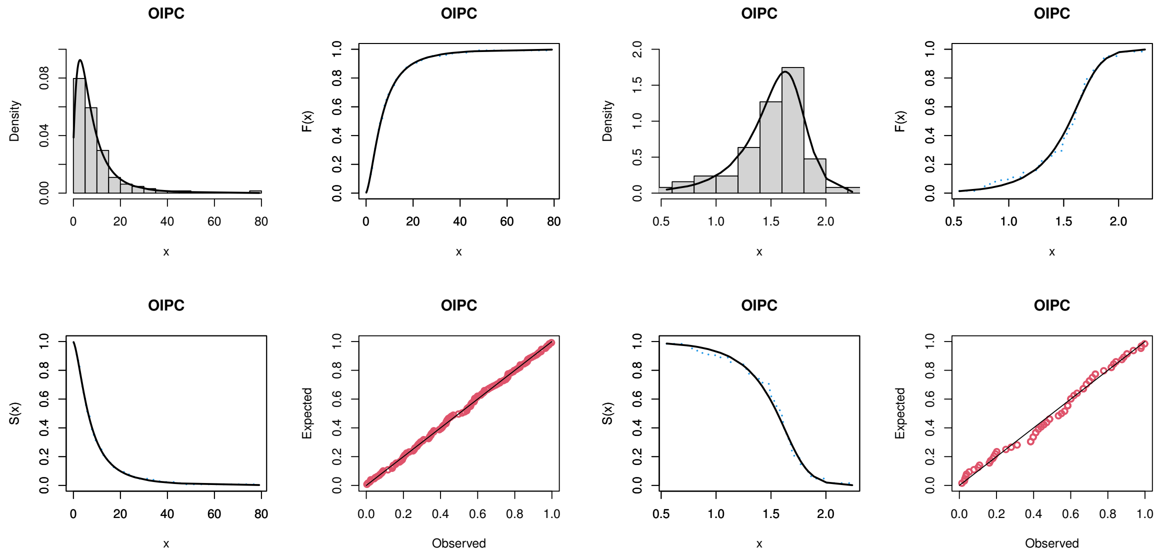
(a) First dataset **(b)** Second dataset

(c) Third dataset **(d)** Fourth dataset

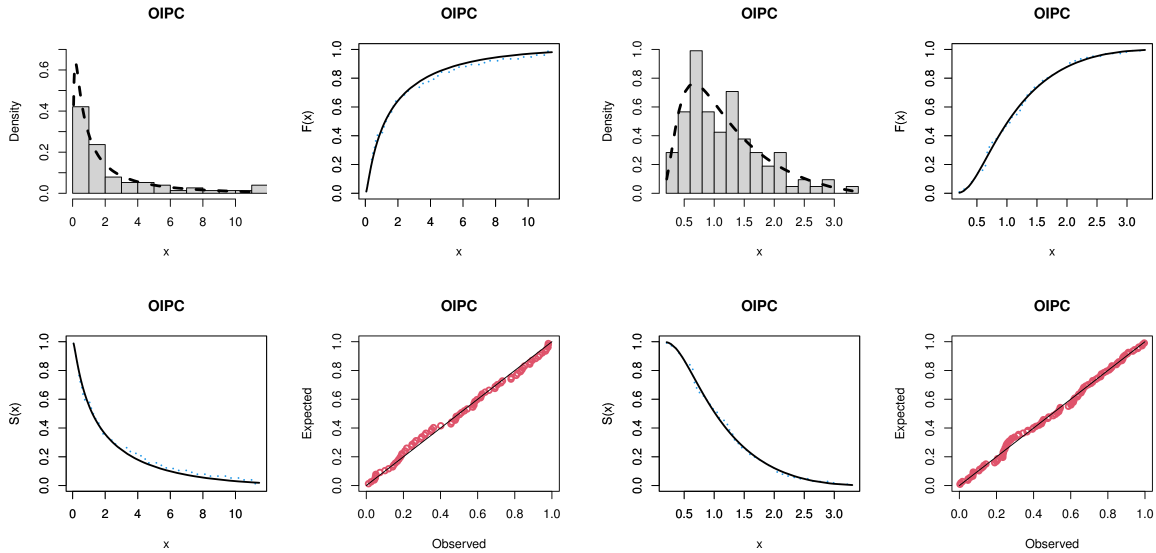
Figure 34. The HRF plots of the OIPC distribution for the four real-life datasets.

Furthermore, the numerical results reported in Tables 12–15 are complemented by graphical illustrations in Figures 35–37 to further assess the adequacy of the proposed model. These figures present the fitted PDFs and CDFs of the OIPC distribution, the probability–probability (PP) plots, and histograms with the fitted densities of the competing models for the four analyzed datasets.

Overall, the graphical diagnostics support the strong fitting performance of the OIPC distribution. The model demonstrates considerable flexibility in capturing diverse data structures arising in medical and engineering studies. The agreement across multiple diagnostic tools highlights the robustness and practical relevance of the proposed distribution for real-world data analysis.

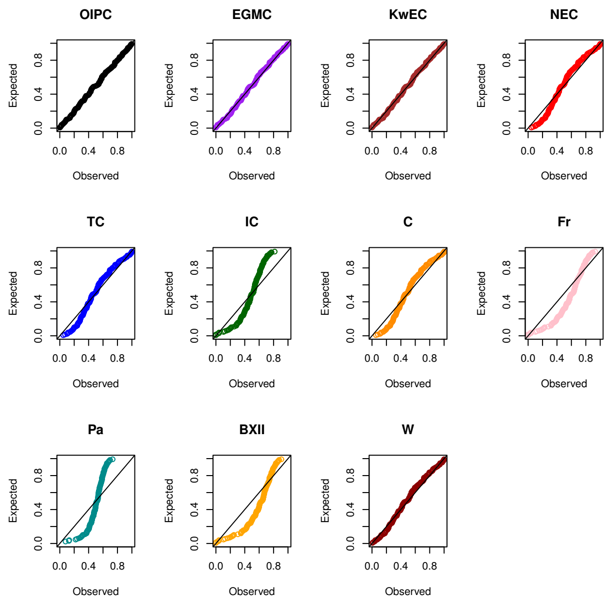


(a) First dataset **(b)** Second dataset

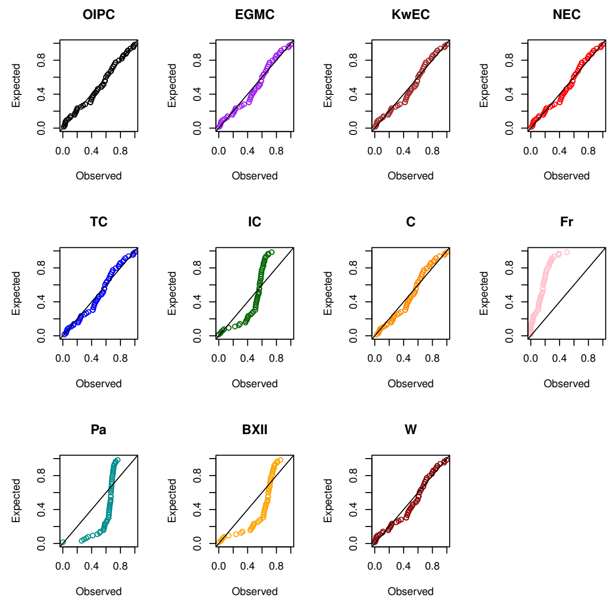


(c) Third dataset **(d)** Fourth dataset

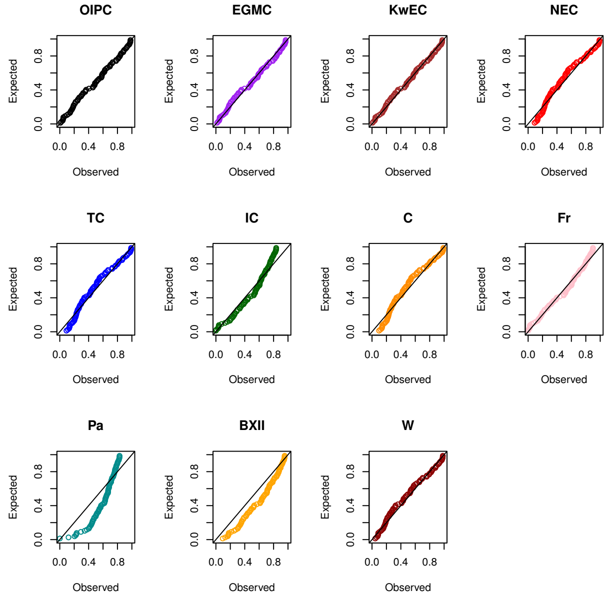
Figure 35. Plots of the fitted OIPC distribution functions for the four real-life datasets.



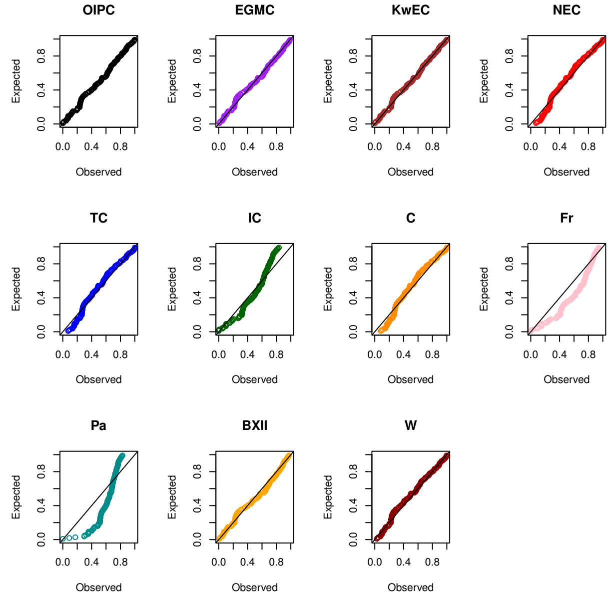
(a) First dataset



(b) Second dataset

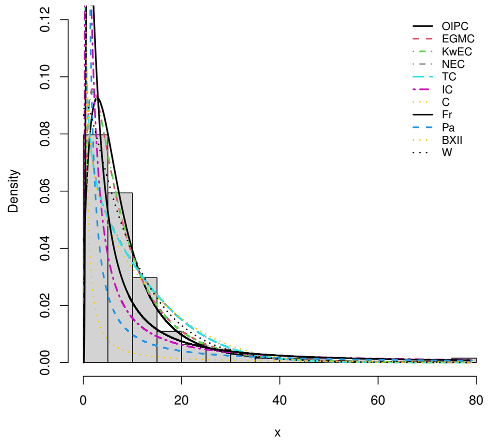


(c) Third dataset

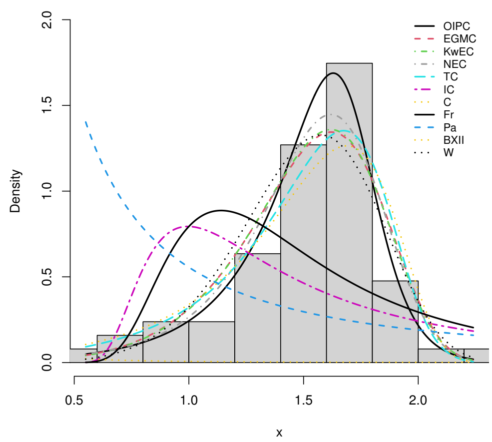


(d) Fourth dataset

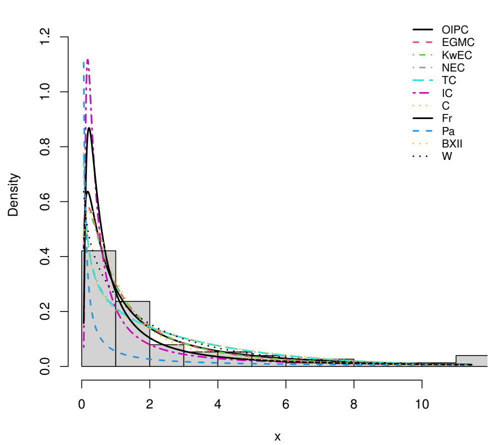
Figure 36. PP plots of the OIPC distribution and other competing models for the four real-life datasets.



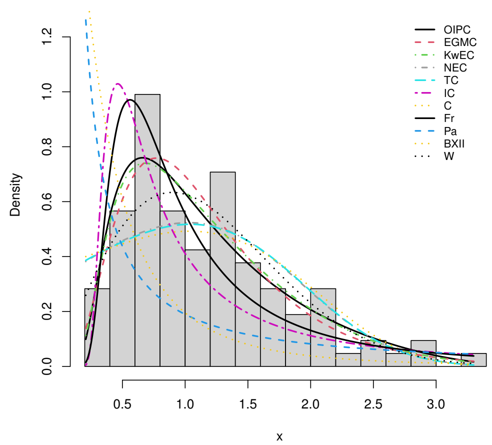
(a) First dataset



(b) Second dataset



(c) Third dataset



(d) Fourth dataset

Figure 37. Fitted densities of the OIPC distribution and other competing models for the four real-life datasets.

8. Concluding remarks

In this paper, we introduce a flexible extension of the Chen model, termed the OIPC distribution. The OIPC density can be expressed as a mixture of exponentiated Chen densities. Its failure rate exhibits a variety of shapes, including decreasing, J-shaped, increasing, reversed-J, bathtub-shaped, unimodal, and modified bathtub forms, and its density can take right-skewed, highly right-skewed, near-symmetric, and unimodal forms with varying degrees of peakedness. The mathematical properties of the OIPC distribution are derived in detail. The parameters of the OIPC distribution are estimated using eight different estimation methods. Extensive simulation studies are conducted to assess the accuracy of these approaches, revealing that the least-squares and Cramér–von Mises methods outperform the others, achieving overall scores of 79 and 91.5, respectively. Based on these results, these two methods perform best in the simulation settings considered and are recommended as default choices. Finally, the applications to four real-life datasets from medicine and engineering demonstrate the practical utility and flexibility of the OIPC distribution in modeling complex real-world data.

An important extension of the present work is the incorporation of censored data, which are frequently encountered in survival analysis and reliability studies. Although the current analysis is based on complete samples to clearly establish the statistical properties of the proposed OIPC model, the developed estimation procedures can be extended to accommodate different censoring mechanisms by suitable modifications of the underlying likelihood or objective functions. This includes, but is not limited to, right-censoring and other common censoring schemes. Future research may focus on studying the performance of the proposed model under censored data and exploring its applications in more complex survival settings.

Author contributions

Fatimah M. Alghamdi: Methodology, software, validation, writing-review and editing; Ekramy A. Hussein: Conceptualization, methodology, resources, writing-original draft preparation, writing-review and editing; Mohammed Alqawba: Validation, formal analysis, investigation, resources, writing-review and editing; Hassan M. Aljohani: Conceptualization, methodology, resources, writing-review and editing; Ahmed Z. Afify: Conceptualization, methodology, software, project administration, writing-original draft preparation, writing-review and editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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A. Appendix A

This appendix provides the detailed mathematical derivations and additional theoretical properties of the OIPC distribution that support the main results presented in Section 3. To improve the readability and application-oriented flow of the manuscript, the more technical developments, including the mixture representation, moment derivations, mean deviations, entropy measures, and order statistics, are collected here.

A.1. Linear representation

We present a highly effective linear representation of the OIPC density. Aldahlan et al. [13] derived a mixture representation of the OIP-G density as follows:

$$f(x) = \sum_{z,j=0}^{\infty} d_{z,j} h_{z+j+\kappa}(x), \tag{A.1}$$

where $d_{z,j} = \kappa(-1)^j/(z+j+\kappa)\nu^{z+\kappa}\binom{-\kappa-1}{z}\binom{-z-\kappa-1}{j}$ is a constant term, and $h_{z+j+\kappa}(x) = (z+j+\kappa)g(x)G(x)^{(z+j+\kappa)-1}$ denotes the Exp-G density with power parameter $(z+j+\kappa) > 0$.

Using the PDF and CDF of the Chen distribution, the last equation can be rewritten as

$$f(x) = \sum_{z,j=0}^{\infty} d_{z,j}\zeta\omega(z+j+\kappa)x^{\omega-1}\exp\left[x^{\omega} - \zeta\left(e^{x^{\omega}}-1\right)\right]\left\{1-\exp\left[-\zeta\left(e^{x^{\omega}}-1\right)\right]\right\}^{z+j+\kappa-1}. \tag{A.2}$$

Equation (A.2) corresponds to the exponentiated Chen (EC) distribution with shape parameter $(z + j + \kappa)$. In our setting, the EC power parameter is $\alpha = z + j + \kappa$; replace α by $z + j + \kappa$ in the formulas below.

Dey et al. [33] discussed the behavior of the EC distribution and presented the following properties: The rth ordinary moment is derived as

$$\mu_r' = \alpha\omega \sum_{p,q\geq 0} (-1)^{\frac{2r}{\omega}+p} \frac{\alpha_p\left(\frac{r}{\omega}\right)\alpha_q\left(\frac{r}{\omega}+p\right)}{\zeta^{\frac{2r}{\omega}+p}\left[\omega(\alpha+p+q)+r\right]}, \tag{A.3}$$

where $\alpha_p\,(r/\omega)$ is the coefficient of $\left[(1/\zeta)\log(1-[G(x)]^{1/\alpha})\right]^{2r/\omega+p}$ in the expansion of $\left[\sum_{i\geq1}(1/\zeta\, i)\log\left(1-[G(x)]^{1/\alpha}\right)\right]^{r/\omega}$, and $\alpha_q\,(r/\omega+p)$ is the coefficient of $[G(x)]^{(1/\alpha)(r/\omega+p+q)}$ in the expansion of $\left[\sum_{j\geq1}(1/j)[G(x)]^{j/\alpha}\right]^{r/\omega+p}$.

The conditional moments of the EC takes the form

$$E(X^r|X > x) = \alpha\omega \sum_{p,q\geq 0} (-1)^{\frac{2r}{\omega}+p} \frac{\alpha_p\left(\frac{r}{\omega}\right)\alpha_q\left(\frac{r}{\omega}+p\right)\exp\left[-\zeta\left(e^{x^{\omega}}-1\right)\right]}{\zeta^{\frac{2r}{\omega}+p}\left[\omega(\alpha+p+q)+r\right]\left(1-\left\{1-\exp\left[-\zeta\left(e^{x^{\omega}}-1\right)\right]\right\}^{\alpha}\right)}. \tag{A.4}$$

The MRL of the EC distribution is given by

$$m(x) = \alpha\omega \sum_{p,q\geq 0} (-1)^{\frac{2}{\omega}+p} \frac{\alpha_p\left(\frac{1}{\omega}\right)\alpha_q\left(\frac{1}{\omega}+p\right)\exp\left[-\zeta\left(e^{x^{\omega}}-1\right)\right]}{\zeta^{\frac{2}{\omega}+p}\left[\omega(\alpha+p+q)+1\right]\left(1-\left\{1-\exp\left[-\zeta\left(e^{x^{\omega}}-1\right)\right]\right\}^{\alpha}\right)} - x. \tag{A.5}$$

The mean deviation about the mean is defined as

$$D(\mu) = E|X-\mu| = \int_0^{\infty} |x-\mu| f(x)\,dx. \tag{A.6}$$

Hence, for the EC distribution (see Dey et al. [33]), it can be expressed as

$$D(\mu) = 2\mu F(\mu) - 2\mu + 2\alpha\omega \sum_{p,q\geq 0} (-1)^{\frac{2}{\omega}+p} \frac{\alpha_p\left(\frac{1}{\omega}\right)\alpha_q\left(\frac{1}{\omega}+p\right)\left\{1-\exp\left[-\zeta\left(e^{x^{\omega}}-1\right)\right]\right\}^{\alpha+p+q+\frac{1}{\omega}}}{\zeta^{\frac{\omega+p}{\omega}}\left[\omega(\alpha+p+q)+1\right]}. \tag{A.7}$$

The mean deviation about the median is defined as

$$D(m) = E|X-m| = \int_0^\infty |x-m| f(x)\,dx. \tag{A.8}$$

For the EC distribution, it can be expressed as

$$D(m) = 2mF(m) - m - \mu + 2\alpha\omega \sum_{p,q\geq 0} (-1)^{\frac{2}{\omega}+p} \frac{\alpha_p\left(\frac{1}{\omega}\right)\alpha_q\left(\frac{1}{\omega}+p\right)\left\{1-\exp\left[-\zeta\left(e^{x^{\omega}}-1\right)\right]\right\}^{\alpha+p+q+\frac{1}{\omega}}}{\zeta^{\frac{\omega+p}{\omega}}\left[\omega(\alpha+p+q)+1\right]}. \tag{A.9}$$

A.2. Ordinary and incomplete moments

In this subsection, we provide the rth noncentral and the moment generating function (MGF) for the OIPC distribution.

Based on (A.1) and (A.3), the rth moment of the OIPC distribution is defined by

$$\mu'_r = \sum_{z,j\geq 0} d_{z,j} \int_0^\infty x^r h_{z+j+\kappa}(x)dx.$$

Hence, it reduces to

$$\mu'_r = \omega \sum_{p,q,z,j\geq 0} d_{z,j}\,(z+j+\kappa)\,(-1)^{\frac{2r}{\omega}+p} \frac{(z+j+\kappa)_p\left(\frac{r}{\omega}\right)(z+j+\kappa)_q\left(\frac{r}{\omega}+p\right)}{\zeta^{\frac{2r}{\omega}+p}\left[\omega(z+j+\kappa+p+q)+r\right]}. \tag{A.10}$$

Setting $r = 1$ in Eq (A.10), we get the mean of x,

$$\mu'_1 = \mu_X = \omega \sum_{p,q,z,j\geq 0} d_{z,j}(z+j+\kappa)(-1)^{\frac{2}{\omega}+p} \frac{(z+j+\kappa)_p\left(\frac{1}{\omega}\right)(z+j+\kappa)_q\left(\frac{1}{\omega}+p\right)}{\zeta^{\frac{2}{\omega}+p}\left[\omega(z+j+\kappa+p+q)+1\right]}. \tag{A.11}$$

Using Eqs (A.1) and (A.10), the rth incomplete moment can be defined as

$$\begin{aligned} m_r(s) &= \sum_{z,j\geq 0} d_{z,j} \int_0^s x^r h_{z+j+\kappa}(x)dx \\ &= \sum_{z,j\geq 0} d_{z,j} \int_0^\infty x^r h_{z+j+\kappa}(x)dx - \sum_{z,j\geq 0} d_{z,j} \int_s^\infty x^r h_{z+j+\kappa}(x)dx \\ &= \mu'_r - \sum_{z,j\geq 0} d_{z,j} \int_s^\infty x^r h_{z+j+\kappa}(x)dx. \end{aligned}$$

Then, we obtain

$$m_r(s) = \omega \sum_{p,q,z,j\geq 0} d_{z,j}(z+j+\kappa)(-1)^{\frac{2r}{\omega}+p} \frac{(z+j+\kappa)_p\left(\frac{r}{\omega}\right)(z+j+\kappa)_q\left(\frac{r}{\omega}+p\right)}{\zeta^{\frac{2r}{\omega}+p}\left[\omega(z+j+\kappa+p+q)+r\right]} \times \left\{1-\exp\left[-\zeta\left(e^{x^{\omega}}-1\right)\right]\right\}.$$

The MGF of a random variable is typically expressed in the form of

$$M_X(t) = E\left(e^{tX}\right) = \sum_{h=0}^{\infty} \frac{E\left(X^h\right) t^h}{h!} = \sum_{h=0}^{\infty} \frac{\mu'_h\, t^h}{h!}.$$

Set $r = h$ in Eq (A.10) and sum over h. The MGF of the OIPC distribution then reduces to

$$M_X(t) = \omega \sum_{p,q,z,j,h\geq 0} d_{z,j}\,(z+j+\kappa)\,t^h(-1)^{\frac{2r}{\omega}+p} \frac{(z+j+\kappa)_p\left(\frac{r}{\omega}\right)(z+j+\kappa)_q\left(\frac{r}{\omega}+p\right)}{\zeta^{\frac{2r}{\omega}+p} h!\,[\omega(z+j+\kappa+p+q)+r]}.$$

A.3. Mean deviations

The mean deviations about the mean and the median are useful measures for assessing the dispersion of a distribution around its central tendency. Let $\mu = E(X)$ denote the mean and m denote the median. Using Eqs (A.1), (A.7), and (A.9), the mean deviations of the OIPC distribution with respect to the mean and the median can be derived.

The mean deviation about the mean of the OIPC distribution is given by

$$D(\mu) = 2\mu F(\mu) - 2\mu + 2\,\omega \sum_{p,q,z,j\geq 0} \frac{(-1)^{-\frac{2}{\omega}-p}\,(z+j+\kappa)_p\left(\frac{1}{\omega}\right)(z+j+\kappa)_q\left(\frac{1}{\omega}+p\right)\delta(x)}{(z+j+\kappa)^{-1}\,\zeta^{\frac{\omega+p}{\omega}}\,[\omega(z+j+\kappa+p+q)+1]}. \tag{A.12}$$

The mean deviation about the median of the OIPC distribution takes the form

$$D(m) = 2mF(m) - m - \mu + 2\,\omega \sum_{p,q,z,j\geq 0} \frac{(-1)^{-\frac{2}{\omega}-p}\,(z+j+\kappa)_p\left(\frac{1}{\omega}\right)(z+j+\kappa)_q\left(\frac{1}{\omega}+p\right)\delta(x)}{(z+j+\kappa)^{-1}\,\zeta^{\frac{\omega+p}{\omega}}\,[\omega(z+j+\kappa)+p+q+1]}, \tag{A.13}$$

where $\delta(x) = \left\{1 - \exp\left[-\zeta\left(e^{x^\omega}-1\right)\right]\right\}^{z+j+\kappa+p+q+\frac{1}{\omega}}$.

A.4. Rényi entropy

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. It is defined by

$$I_\rho(X) = \frac{1}{1-\rho}\log\left(J_\rho(X)\right),\ \rho > 0 \text{ and } \rho \neq 1,$$

where

$$J_\rho(X) = \int_0^\infty f^\rho(x)dx.$$

Using Eq (2.7), we can write

$$\begin{aligned} f^\rho(x) \;&=\; \kappa^\rho\nu^\rho\zeta^\rho\omega^\rho x^{\rho(\omega-1)}\exp\left[\rho x^\omega - \rho\zeta\left(e^{x^\omega}-1\right)\right]\left\{1-\exp\left[-\zeta\left(e^{x^\omega}-1\right)\right]\right\}^{\rho(\kappa-1)} \\ &\quad\times\left\{1-(1-\nu)\exp\left[-\zeta\left(e^{x^\omega}-1\right)\right]\right\}^{-\rho(\kappa+1)}. \end{aligned}$$

After some simplifications, we have that the Rényi entropy of the OIPC model reduces to

$$I_\rho(X) = \frac{1}{1-\rho}\log\left\{\sum_{m=0}^{\infty}\varrho_m\,\Gamma\left[m+\rho\left(1-\frac{1}{\omega}\right)+\frac{1}{\omega}\right]\right\},$$

where

$$\varrho_m = \kappa^\rho\,\omega^{\rho+1}\sum_{j,i,r=0}^{\infty}\frac{\zeta^{r+\rho}\,(j+\rho)^r\,r^m(-1)^{j+r+\rho\left(1-\frac{1}{\omega}\right)+m+\frac{1}{\omega}}}{r!m!\nu^{\rho\kappa+i}\,(1-\nu)^{-i}\,e^{-(j+\rho)\zeta}}\binom{-\rho(\kappa+1)}{i}\binom{\rho(\kappa-1)+i}{j}.$$

A.5. Order statistics

Let $X_1, X_2, \ldots X_n$ be a random sample from (2.7) and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote the associated order statistics. It is well-known that the PDF and CDF of the of ith order statistic, say $X_{(i)}$, are given by

$$\begin{aligned} f_{X_{(i)}}(x) &= \frac{n!}{(i-1)!(n-i)!}\,F(x)^{i-1}\,[1-F(x)]^{n-i}\,f(x) \\ &= \frac{n!}{(i-1)!\,(n-i)!}\sum_{u=0}^{n-i}(-1)^u\binom{n-i}{u}F(x)^{i-1+u}\,f(x) \end{aligned} \tag{A.14}$$

and

$$F_{X_{(i)}}(x) = \sum_{l=i}^{n}\binom{n}{l}F(x)^l\,[1-F(x)]^{n-l} = \sum_{l=i}^{n}\sum_{u=0}^{n-i}(-1)^u\binom{n}{l}\binom{n-i}{u}F(x)^{l+u}, \tag{A.15}$$

for $i = 1, 2, \ldots, n$.

It follows from Eqs (A.14) and (A.15) that the PDF and CDF of the ith order statistic from the OIPC distribution are given, respectively, by

$$\begin{aligned} f_{X_{(i)}}(x) = \frac{n!\kappa\nu\zeta\omega}{(i-1)!(n-i)!}&\sum_{u=0}^{n-i}(-1)^u\binom{n-i}{u}x^{\omega-1}\exp\left[x^\omega-\zeta\left(e^{x^\omega}-1\right)\right] \\ &\times\left\{1-\exp\left[-\zeta\left(e^{x^\omega}-1\right)\right]\right\}^{\kappa(i+u)-1}\left\{1-(1-\nu)\exp\left[-\zeta\left(e^{x^\omega}-1\right)\right]\right\}^{-\kappa(i+u)-1} \end{aligned}$$

and

$$F_{X_{(i)}}(x) = \sum_{l=i}^{n}\sum_{u=0}^{n-i}(-1)^u\binom{n}{l}\binom{n-i}{u}\left\{\exp\left[\zeta\left(e^{x^\omega}-1\right)\right]-1\right\}^{\kappa(u+1)}\left\{\nu+\exp\left[\zeta\left(e^{x^\omega}-1\right)\right]-1\right\}^{-\kappa(u+1)}.$$

Simulation results for the VaR, TVaR, TV, and TVP of the OIPC and C distributions follow (Table 16).

Table 16. Simulation results for the VaR, TVaR, TV, and TVP measures of the OIPC and Chen distributions.

Distribution	Parameters	Sig. Level	VaR	TVaR	TV	TVP
		0.70	2.056889	2.410522	3.431363	4.318738
		0.75	2.377399	2.830679	3.674750	4.547292
	$\kappa = 0.5$	0.80	2.737550	3.262298	3.955069	4.801130
OIPC	$\nu = 0.4$	0.85	3.159673	3.483204	4.292750	5.097867
	$\zeta = 0.1$	0.90	3.693340	4.275098	4.732287	5.475197
	$\omega = 0.5$	0.95	4.490879	4.606879	5.408252	6.046019
		0.99	6.017979	6.082239	6.696794	7.117800
		0.70	1.203597	1.559534	1.631939	1.610217
		0.75	1.293979	1.621794	1.685206	1.669353
		0.80	1.392668	1.691625	1.746191	1.735277
Chen	$\zeta = 0.1$	0.85	1.504666	1.773083	1.818760	1.811908
	$\omega = 0.5$	0.90	1.640449	1.874701	1.911103	1.907463
		0.95	1.830591	2.021116	2.046938	2.045647
		0.99	2.151990	2.264504	2.277662	2.277530
		0.70	1.512070	1.552653	1.671750	1.684076
		0.75	1.548851	1.601038	1.700069	1.712263
	$\kappa = 75$	0.80	1.590279	1.651199	1.732815	1.744711
OIPC	$\nu = 1.5$	0.85	1.639223	1.677083	1.772438	1.783817
	$\zeta = 2$	0.90	1.701819	1.770832	1.824181	1.834712
	$\omega = 0.75$	0.95	1.796571	1.810428	1.903513	1.912529
		0.99	1.979089	1.986739	2.046567	2.052713
		0.70	0.372445	0.563904	0.589698	0.581960
		0.75	0.414813	0.598050	0.621962	0.615984
		0.80	0.463793	0.637902	0.659776	0.655401
Chen	$\zeta = 2$	0.85	0.522863	0.686445	0.706046	0.703106
	$\omega = 0.75$	0.90	0.599485	0.750075	0.766995	0.765303
		0.95	0.715963	0.847742	0.861082	0.860415
		0.99	0.936353	1.028618	1.036376	1.036298
		0.70	2.098585	2.392384	3.335822	4.137258
		0.75	2.364392	2.755819	3.557287	4.365654
	$\kappa = 2.3$	0.80	2.673869	3.144505	3.817924	4.621658
OIPC	$\nu = 0.5$	0.85	3.050713	3.349047	4.138835	4.922524
	$\zeta = 0.5$	0.90	3.546438	4.106255	4.565744	5.305849
	$\omega = 0.4$	0.95	4.317805	4.432231	5.236497	5.885431
		0.99	5.848142	5.913226	6.539413	6.972866
		0.70	1.203597	1.559534	1.631939	1.610217
		0.75	1.293979	1.621794	1.685206	1.669353
		0.80	1.392668	1.691625	1.746191	1.735277
C	$\zeta = 0.5$	0.85	1.504666	1.773083	1.818760	1.811908
	$\omega = 0.4$	0.90	1.640449	1.874701	1.911103	1.907463
		0.95	1.830591	2.021116	2.046938	2.045647
		0.99	2.151990	2.264504	2.277662	2.277530