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Research article

Characterizations of pseudo-Schouten symmetric spacetimes with applications to $f(\mathcal{R})$ gravity

Bang-Yen Chen[1], **Awatif Al-Jedani**[2], **Uday Chand De**[3], **Nasser Bin Turki**[4] **and Abdallah Abdelhameed Syied**[5,6,*]

[1] Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

[2] Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 23890, Saudi Arabia

[3] Department of Pure Mathematics, University of Calcutta, Ballygaunge Circular Road 35, Kolkata 700019, West Bengal, India

[4] Department of Mathematics, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia

[5] Department of Mathematics, Faculty of Science, Zagazig University, P. O. Box 44519, Zagazig, Egypt

[6] Faculty of Computers and Information Systems, Egyptian Chinese University, Cairo 4541312, Egypt

* **Correspondence:** Email: a.a_syied@yahoo.com.

Abstract: This paper primarily investigated the geometric and physical properties of pseudo-Schouten symmetric $(\mathrm{PSS})_q$ spacetimes. First, we demonstrated that the associated covector ω^i of such spacetimes is irrotational, non-accelerating, and satisfies $\omega^i \mathcal{R}_{ij} = \frac{\mathcal{R}}{2}\omega_j$. Next, we examined $(\mathrm{PSS})_q$ generalized Robertson-Walker spacetimes. Among various results, we proved that a $(\mathrm{PSS})_q$ generalized Robertson-Walker spacetime is a perfect fluid spacetime. Furthermore, we showed that a $(\mathrm{PSS})_q$ spacetime with a Codazzi-type Ricci tensor constitutes a perfect fluid, whereas one with a cyclic parallel Ricci tensor represents a vacuum spacetime. We then established that a $(\mathrm{PSS})_q$ spacetime characterized by a divergence-free Weyl tensor is static. Finally, we explored the physical implications of $(\mathrm{PSS})_q$ spacetimes within the framework of $f(R)$ gravity, explicitly deriving the corresponding isotropic pressure and energy density.

Keywords: $f(R)$ modified gravity theory; perfect fluid spacetime; pseudo-Schouten symmetric spacetime; energy conditions
Mathematics Subject Classification: 53B20, 53C25, 53C50, 53C80

1. Introduction

Spatial geometry relies heavily on curvature, with symmetry standing out as a vital geometric property. The mathematical exploration of manifold symmetry began with Cartan [1], who first defined locally symmetric spaces. While Cartan initially categorized complete, simply connected, locally symmetric spaces within Riemannian geometry, later scholars broadened his framework. For instance, Cahen and Parker expanded this classification to non-Riemannian spaces [2], while other researchers introduced relaxed symmetry conditions under diverse curvature restrictions.

Numerous researchers have expanded upon Cartan's original framework by developing alternative formulations of manifold symmetry. Early generalizations include semi-symmetric manifolds by Cartan [1] and pseudo-symmetric variants defined separately by Deszcz and Chaki [3, 4]. Recent developments feature work by Ali et al. on $\mathcal{Z}$-symmetric manifolds with the Schouten tensor, alongside almost $(\mathrm{PSS})_q$ spaces [5, 6]. Furthermore, Mantica and Molinari refined weakly $\mathcal{Z}$-symmetric manifolds using a specialized symmetric $(0, 2)$-tensor structure to simplify their defining conditions, later introducing pseudo-$\mathcal{Z}$-symmetric Riemannian manifolds [7, 8]. Concurrently, weak symmetry was pioneered by Selberg [9] and expanded by Tamássy and Binh [10], mirroring a broader mathematical effort by various authors to systematically investigate diverse modes of symmetry, recurrence, and pseudo-symmetry.

In [11], the authors introduced the concept of $(\mathrm{PSS})_q$ manifolds. Such manifolds are those whose Schouten curvature tensor

$$\mathcal{S}_{ij} = \frac{1}{q-2}\left(\mathcal{R}_{ij} - \frac{\mathcal{R}}{2(q-1)} g_{ij}\right) \tag{1.1}$$

satisfies the following condition:

$$\nabla_k \mathcal{S}_{ij} = 2\omega_k \mathcal{S}_{ij} + \omega_i \mathcal{S}_{jk} + \omega_j \mathcal{S}_{ik}, \tag{1.2}$$

where ω_i is a non-zero 1-form known as *"the associated covector of $(\mathrm{PSS})_q$"* [11].

The Schouten tensor serves as a fundamental component in regulating conformal curvature dynamics and facilitating the decomposition of the Riemann curvature tensor. Because of its intrinsic relationship with the conformal curvature tensor, manifolds that adhere to relation (1.2) display unique conformal traits. These geometric attributes are highly significant when analyzing spacetime symmetries and stress-energy configurations within general-relativity and alternative-gravity frameworks. Consequently, this proposed constraint establishes a novel framework for investigating the physical and geometric consequences of these spaces.

From a physical perspective, a spacetime is defined as a q-dimensional Lorentzian manifold equipped with a metric of Lorentzian signature. A Lorentzian manifold M is classified as a $(\mathrm{PSS})_q$ spacetime if its Schouten curvature tensor satisfies the relation (1.2). In a perfect fluid spacetime (PFS), the energy-momentum tensor (EMT) is mathematically structured as follows [12]:

$$T_{hk} = (\sigma + p)\, u_h u_k + p g_{hk}, \tag{1.3}$$

where σ denotes the energy density, p represents the isotropic pressure, and u_h is a unit timelike vector field. The Ricci tensor of a PFS possesses the following expression:

$$\mathcal{R}_{ij} = a_1 g_{ij} + a_2 u_i u_j,$$

for scalar fields a_1 and a_2 [13, 14].

The dynamic evolution of the cosmos is entirely dictated by the **equation-of-state parameter** ($w = \frac{p}{\sigma}$), which governs how the energy density of a cosmic fluid scales with the expansion of space ($\sigma \propto a^{-3(1+w)}$). In the ultra-early universe, the **stiff-matter epoch** ($w = 1$) features a highly dense fluid where the speed of sound equals the speed of light, causing density to dilute at an extreme rate of a^{-6} [15]. As the universe expanded and cooled, it transitioned into the **radiation epoch** ($w = 1/3$), dominated by relativistic particles whose wavelengths stretched via cosmic redshift, diluting energy at a rate of a^{-4}. In the modern universe, cosmic acceleration is driven by the **dark energy era** ($w = -1$), where the energy density remains perfectly constant over time ($\rho \propto$ const.), forcing an exponential expansion of space [16]. Deviations from this baseline include the **quintessence regime** ($-1 < w < -1/3$), characterized by a time-varying scalar field that slowly dilutes over cosmological scales, and the exotic **phantom regime** ($w < -1$), a hypothetical scenario violating the null energy condition where energy density paradoxically increases as the universe expands, ultimately culminating in a catastrophic "big rip" that tears apart all bound physical structures [17].

The study of spacetimes within $f(\mathcal{R})$ modified theories of gravity heavily relies on imposing spacetime symmetries to extract exact physical solutions from highly complex, fourth-order field equations. The $(\mathrm{PSS})_q$ condition (1.2) is implemented in $f(\mathcal{R})$ gravity for both geometric flexibility and physical consistency, allowing controlled relaxation of local symmetry in non-symmetric spacetimes while accommodating relativistic fluids. As a curvature constraint, this condition ensures analytical tractability, naturally producing physically significant structures like perfect fluid and generalized Robertson-Walker (GRW) spacetimes in $f(\mathcal{R})$ gravity.

The GRW spacetime is a vital concept in geometry and relativity, offering a sophisticated mathematical framework for modeling the universe's structure. Such spacetime is defined as a warped product $M = I \times_{h^2} \tilde{M}$. In this structure, the base manifold I is an open, connected interval of $\mathbb{R}$ with the metric component $g_{11} = -1$, while the fiber consists of a $(q-1)$-dimensional Riemannian manifold governed by the warping function h^2 [18, 19]. The GRW spacetime is characterized by a unit timelike vector field that acts as both a torse-forming field and a Ricci tensor eigenvector [20]. That is,

$$\nabla_l u_k = \psi \left(g_{lk} + u_l u_k \right),$$
$$u^i \mathcal{R}_{ij} = \frac{\mathcal{R}}{2} u_j.$$

This paper proceeds as follows: Section 2 establishes the general properties of $(\mathrm{PSS})_q$ spacetimes. Section 3 investigates $(\mathrm{PSS})_q$ GRW spacetimes. Section 4 explores these spacetimes under the conditions of a Codazzi and cyclic parallel Ricci tensor. Section 5 addresses $(\mathrm{PSS})_q$ spacetimes characterized by a divergence-free Weyl tensor. Finally, Section 6 presents applications of these spacetimes within the framework of $f(\mathcal{R})$ gravity.

2. On $(\mathrm{PSS})_q$ spacetimes

This section explores the fundamental characteristics of pseudo-Schouten symmetric spacetimes. Throughout this analysis, the associated covector $(\mathrm{PSS})_q$ spacetime is defined as a unit timelike vector field, that is, $\omega^i \omega_i = -1$. Applying the covariant differentiation on Eq (1.1), we infer that

$$\nabla_k \mathcal{S}_{ij} = \frac{\nabla_k \mathcal{R}_{ij}}{(q-2)} - \frac{\nabla_k \mathcal{R}}{2(q-2)(q-1)} g_{ij}. \tag{2.1}$$

By implementing the defining condition for $(PSS)_q$ spacetimes in the above expression, we obtain:

$$\nabla_k\mathcal{R}_{ij} - \frac{\nabla_k\mathcal{R}}{2(q-1)}g_{ij} = 2\omega_k\left(\mathcal{R}_{ij} - \frac{\mathcal{R}}{2(q-1)}g_{ij}\right) + \omega_i\left(\mathcal{R}_{jk} - \frac{\mathcal{R}}{2(q-1)}g_{jk}\right) + \omega_j\left(\mathcal{R}_{ik} - \frac{\mathcal{R}}{2(q-1)}g_{ik}\right). \tag{2.2}$$

Transvecting Eq (2.2) with g^{ij} and g^{ik}, one easily finds

$$\omega^i\mathcal{R}_{ij} = \frac{\mathcal{R}}{2}\omega_j, \tag{2.3}$$

$$\nabla_k\mathcal{R} = 4\omega_k\mathcal{R}. \tag{2.4}$$

Utilizing Eq (2.4) in Eq (2.2), one infers

$$\nabla_k\mathcal{R}_{ij} = \omega_k\left(2\mathcal{R}_{ij} + \frac{\mathcal{R}}{q-1}g_{ij}\right) + \omega_i\left(\mathcal{R}_{jk} - \frac{\mathcal{R}}{2(q-1)}g_{jk}\right) + \omega_j\left(\mathcal{R}_{ik} - \frac{\mathcal{R}}{2(q-1)}g_{ik}\right).$$

By performing an index swap of k and j in the aforementioned relation, it follows that:

$$\nabla_j\mathcal{R}_{ik} = \omega_j\left[2\mathcal{R}_{ik} + \frac{\mathcal{R}}{q-1}g_{ik}\right] + \omega_i\left[\mathcal{R}_{jk} - \frac{\mathcal{R}}{2(q-1)}g_{jk}\right] + \omega_k\left[\mathcal{R}_{ij} - \frac{\mathcal{R}}{2(q-1)}g_{ij}\right].$$

Taking the difference between the two preceding equations yields

$$\nabla_k\mathcal{R}_{ij} - \nabla_j\mathcal{R}_{ik} = \left(\omega_k\mathcal{R}_{ij} - \omega_j\mathcal{R}_{ik}\right) + \frac{3\mathcal{R}}{2(q-1)}\left(\omega_k g_{ij} - \omega_j g_{ik}\right). \tag{2.5}$$

Differentiating Eq (2.4) covariantly, we have

$$\nabla_l\nabla_k\mathcal{R} = 16\mathcal{R}\omega_l\omega_k + 4\mathcal{R}\nabla_l\omega_k. \tag{2.6}$$

By swapping the indices l and k in the above expression, arrive at

$$\nabla_k\nabla_l\mathcal{R} = 16\mathcal{R}\omega_l\omega_k + 4\mathcal{R}\nabla_k\omega_l.$$

Subtracting the final two equations leads to the conclusion that

$$\mathcal{R}\left(\nabla_l\omega_k - \nabla_k\omega_l\right) = 0. \tag{2.7}$$

Then, either $\mathcal{R} = 0$ or $\mathcal{R} \neq 0$.

Case 2.1. *If $\mathcal{R} = 0$, then Eq (2.5) implies that*

$$\nabla_k\mathcal{R}_{ij} = 2\omega_k\mathcal{R}_{ij} + \omega_i\mathcal{R}_{jk} + \omega_j\mathcal{R}_{ik},$$

and the spacetime reduces to a pseudo-Ricci symmetric spacetime, as introduced by Chacki [21]. Such a spacetime has been investigated by several authors (e.g., [22–24] and many others).

Case 2.2. *If $\mathcal{R} \neq 0$, then Eq (2.7) implies that*

$$\nabla_l \omega_k - \nabla_k \omega_l = 0. \tag{2.8}$$

This indicates that the ω^l is irrotational. Contracting the preceding equation with ω^k and using $\omega^k \nabla_l \omega_k = 0$, we get

$$\dot{\omega}_l = \omega^k \nabla_k \omega_l = 0. \tag{2.9}$$

This tell us that ω^l has zero acceleration. The vorticity tensor μ_{ij} is defined by [25]

$$\mu_{ij} = \frac{1}{2}\left(\nabla_i \omega_j - \nabla_j \omega_i\right) + \frac{1}{2}\dot{\omega}_j \omega_i + \omega_j \dot{\omega}_i.$$

Hence, using the above results, one obtains

$$\mu_{ij} = 0, \tag{2.10}$$

such that the associated covector field has vanishing vorticity.

Theorem 2.1. *In a $(PSS)_q$ spacetime, either the spacetime reduces to a pseudo-Ricci symmetric spacetime, or the associated covector is irrotational, acceleration-free, and vorticity-free.*

Remark 2.1. *A covector field ω (or its associated vector field) is called irrotational if its exterior derivative vanishes,*

$$\nabla_{[i} \omega_{j]} = 0,$$

or equivalently, it is a local gradient field.

It is acceleration-free when its integral curves are geodesics,

$$\omega^j \nabla_j \omega^i = 0.$$

Example 2.1. *In the spacetime of dimension $n > 3$, let u_i be a smooth, timelike unit vector field ($u_i u^i = -1$). If u_i is shear-free, vorticity-free, and acceleration-free, its covariant derivative satisfies:*

$$\nabla_j u_i = \Phi\left(u_i u_j + g_{ij}\right),$$

where Φ is a scalar function. In other words, u_i is a unit torse-forming vector field. Thus a torse-forming vector field is geometrically equivalent to a smooth vector field that is simultaneously shear-free, vorticity-free, and acceleration-free.

3. $(PSS)_q$ GRW spacetimes

This section focuses on examining $(PSS)_q$ GRW spacetimes. The associated covector of a $(PSS)_q$ GRW spacetime satisfies Eq (2.3) and the following torse-forming property:

$$\nabla_l \omega_k = \psi\left(g_{lk} + \omega_l \omega_k\right), \tag{3.1}$$

in which ψ is a scalar field.

Differentiating Eq (2.3) covaraintly yields

$$\frac{1}{2}\left[4\omega_l\omega_i\mathcal{R} + \mathcal{R}\nabla_i\omega_l\right] = \mathcal{R}_{lk}\nabla_i\omega^k + \omega^k\nabla_i\mathcal{R}_{lk}.$$

Utilizing Eqs (2.5) and (3.1) in the preceding equation, we acquire that

$$\begin{aligned}\frac{1}{2}\left[4\omega_l\omega_i\mathcal{R} + \psi\mathcal{R}\left(g_{li} + \omega_l\omega_i\right)\right] =:\ & \psi\mathcal{R}_{lk}\left(\delta_i^k + \omega_i\omega^k\right) + \omega^k\omega_i\left[2\mathcal{R}_{lk} + \frac{\mathcal{R}}{q-1}g_{lk}\right]\\ & + \omega^k\omega_l\left[\mathcal{R}_{ik} - \frac{\mathcal{R}}{2(q-1)}g_{ik}\right] - \left[\mathcal{R}_{li} - \frac{\mathcal{R}}{2(q-1)}g_{li}\right].\end{aligned}$$

Using Eq (2.3) in the preceding equation, one gets

$$\mathcal{R}_{il} = \frac{\mathcal{R}\left(q-2\right)}{2\left(q-1\right)\left(\psi-1\right)}\omega_l\omega_i + \frac{\mathcal{R}\left(\psi\left(q-1\right)-1\right)}{2\left(q-1\right)\left(\psi-1\right)}g_{il}. \tag{3.2}$$

Contracting the previous equation with g^{il}, one finds

$$\psi = 0.$$

Hence, Eq (3.2) becomes

$$\mathcal{R}_{il} = \frac{\mathcal{R}\left(2-q\right)}{2\left(q-1\right)}\omega_l\omega_i + \frac{\mathcal{R}}{2\left(q-1\right)}g_{il}, \tag{3.3}$$

which gives the Ricci tensor in PF form.

Theorem 3.1. *Any $(PSS)_q$ GRW spacetime is a PFS.*

It is established that for $q = 4$, a perfect fluid GRW spacetime reduces to a Robertson-Walker spacetime [26].

Corollary 3.1. *A 4-dimensional $(PSS)_q$ GRW spacetime is a Robertson-Walker spacetime.*

It is significant to observe that the Weyl tensor in a perfect fluid GRW spacetime is divergence-free [20].

Corollary 3.2. *A $(PSS)_q$ GRW spacetime is Weyl harmonic.*

The Ricci tensor of a perfect-fluid spacetime is expressed as [27]

$$\mathcal{R}_{ij} = \frac{\kappa}{2-q}(p-\sigma)\,g_{ij} + \kappa(p+\sigma)\,\omega_i\omega_j, \tag{3.4}$$

where p denotes the isotropic pressure, σ is the energy density, and κ is the gravitational constant.

Comparing Eqs (3.3) and (3.4), we obtain

$$\begin{aligned}(p-\sigma) &= \frac{(2-q)\mathcal{R}}{2\kappa(q-1)},\\ (p+\sigma) &= \frac{(2-q)\mathcal{R}}{2\kappa(q-1)}.\end{aligned}$$

Solving these relations yields

$$p = \frac{(2-q)\mathcal{R}}{2\kappa(q-1)},$$
$$\sigma = 0.$$

Hence, we conclude the following result.

Theorem 3.2. *In a pseudo-Shouten symmetric GRW spacetime, the isotropic pressure p and the energy density σ are given by*

$$p = \frac{(2-q)\mathcal{R}}{2\kappa(q-1)},$$
$$\sigma = 0.$$

Remark 3.1. *The condition $\sigma = 0$ implies the complete absence of matter, radiation, or physical excitation. In classical physics, a vanishing energy density signifies a perfect vacuum. In this state, there are no electromagnetic or gravitational fields, and no physical particles are present. It defines a baseline state of empty space where the total energy-momentum tensor reduces to zero ($\mathcal{T}_{ij} = 0$).*

For a GRW spacetime $M = I \times_{h^2} \tilde{M}$, the metric, Ricci curvature tensors are locally expressed as [28]

$$g_{ij=} \begin{cases} \overline{g}_{tt} = -1, & \text{for } i = j = t, \\ h^2 \tilde{g}_{\alpha\beta}, & \text{for } i = \alpha, \quad j = \beta, \\ 0, & \text{otherwise}, \end{cases} \tag{3.5}$$

where $t = 1$ and $\alpha, \beta \in \{2, \ldots, q\}$.

$$\mathcal{R}_{11} = -\frac{(q-1)\,\ddot{h}}{h}, \tag{3.6}$$
$$\mathcal{R}_{\alpha\beta} = \mathcal{R}_{\alpha\beta} + \left(h\ddot{h} + (q-2)\,\dot{h}^2\right)\tilde{g}_{\alpha\beta}, \tag{3.7}$$
$$\mathcal{R}_{1\alpha} = 0. \tag{3.8}$$

The following components are valid on a GRW spacetime:

$$\nabla_1 \mathcal{R}_{\alpha\beta} = \frac{-2\dot{h}}{h}\mathcal{R}_{\alpha\beta} + \frac{1}{f^2}\left[h^3\,\dddot{h} + (2q-5)\,\ddot{h}\dot{h}h^2 + (q-2)\,\dot{h}^2\right]\tilde{g}_{\alpha\beta}, \tag{3.9}$$
$$\nabla_\beta \mathcal{R}_{1\alpha} = -\frac{\dot{h}}{h}\mathcal{R}_{\alpha\beta} + \left(h\dot{h}\mathcal{R}_1^1 - \ddot{f}\dot{f} + \frac{(q-2)\,\dot{h}^2}{2h^2} - \frac{(q-1)\,\dot{h}}{2h}T_1^1\right)\tilde{g}_{\alpha\beta}, \tag{3.10}$$
$$\nabla_1 \mathcal{R}_{11} = -(q-1)\,\frac{h\,\dddot{h} - \ddot{h}\dot{h}}{h^2}, \tag{3.11}$$
$$\nabla_\gamma \mathcal{R}_{\alpha\beta} = \tilde{\nabla}_\gamma \mathcal{R}_{\alpha\beta}, \tag{3.12}$$
$$\nabla_\alpha \mathcal{R}_{11} = \nabla_1 \mathcal{R}_{1\alpha} = 0. \tag{3.13}$$

In a $(\text{PSS})_q$ GRW spacetime, we can calculate the following components:

The first component: Equation (2.2) implies that

$$\nabla_1 \mathcal{R}_{11} = \omega_1 \left(2\mathcal{R}_{11} + \frac{\mathcal{R}}{q-1} g_{11} \right) + \omega_1 \left(\mathcal{R}_{11} - \frac{\mathcal{R}}{2(q-1)} g_{11} \right) + \omega_1 \left(\mathcal{R}_{11} - \frac{\mathcal{R}}{2(q-1)} g_{11} \right).$$

Inserting Eqs (3.5), (3.11), and (3.6) in the last equation, the result is

$$\omega_1 = \frac{h}{4\ddot{h}} \frac{d}{dt} \left(\frac{\ddot{h}}{h} \right). \tag{3.14}$$

Theorem 3.3. *In (PSS)$_q$ GRW spacetime, the associated covector on the base manifold is given as*

$$\omega_1 = \frac{f}{4\ddot{f}} \frac{d}{dt} \left(\frac{\ddot{f}}{f} \right).$$

The second component: According to Eq (2.2), we infer

$$\nabla_1 \mathcal{R}_{\alpha\beta} = \omega_1 \left(2\mathcal{R}_{\alpha\beta} + \frac{\mathcal{R}}{q-1} g_{\alpha\beta} \right) + \omega_\alpha \left(\mathcal{R}_{\beta 1} - \frac{\mathcal{R}}{2(q-1)} g_{\beta 1} \right) + \omega_\beta \left(\mathcal{R}_{\alpha 1} - \frac{\mathcal{R}}{2(q-1)} g_{\alpha 1} \right).$$

Using Eqs (3.5), (3.7)–(3.9), and (3.14), one obtains

$$\begin{aligned} \mathcal{R}_{\alpha\beta} = & \frac{2\ddot{h}h}{\dddot{h} + 3\dot{h}\ddot{h}} \{ \frac{1}{f^2} \left[h^3 \dddot{h} + (2q-5)\, \ddot{h}\dot{h}h^2 + (q-2)\, \dot{h}^2 \right] \\ & - \left(\frac{\dddot{h} - \dot{h}\ddot{h}}{4\ddot{h}h} \right) \left[2 \left(h\ddot{h} + (q-2)\, \dot{h}^2 \right) + \frac{\mathcal{R}h^2}{q-1} \right] \} \tilde{g}_{\alpha\beta}. \end{aligned}$$

Transvecting the previous equation with $\tilde{g}^{\alpha\beta}$, we reveal that

$$\begin{aligned} \frac{\mathcal{R}}{(q-1)} = & \frac{2\ddot{h}h}{\dddot{h} + 3\dot{h}\ddot{h}} \{ \frac{1}{f^2} \left[h^3 \dddot{h} + (2q-5)\, \ddot{h}\dot{h}h^2 + (q-2)\, \dot{h}^2 \right] \\ & - \left(\frac{\dddot{h} - \dot{h}\ddot{h}}{4\ddot{h}h} \right) \left[2 \left(h\ddot{h} + (q-2)\, \dot{h}^2 \right) + \frac{\mathcal{R}h^2}{q-1} \right] \}. \end{aligned}$$

The combination of the last two equations implies that

$$\mathcal{R}_{\alpha\beta} = \frac{\mathcal{R}}{q-1} \tilde{g}_{\alpha\beta}, \tag{3.15}$$

and consequently, the fiber manifold is Einstein.

Theorem 3.4. *The fiber manifold of a (PSS)$_q$ GRW spacetime is an Einstein manifold.*

In [28], it is proved that a GRW spacetime with Einstein fiber becomes a PFS.

Corollary 3.3. *A (PSS)$_q$ GRW spacetime is a PFS.*

The third component: In view of Eq (2.2), one easily finds

$$\nabla_\alpha \mathcal{R}_{11} = \omega_\alpha \left(2\mathcal{R}_{11} + \frac{\mathcal{R}}{q-1} g_{11} \right) + \omega_1 \left(\mathcal{R}_{1\alpha} - \frac{\mathcal{R}}{2(q-1)} g_{1\alpha} \right) + \omega_1 \left(\mathcal{R}_{1\alpha} - \frac{\mathcal{R}}{2(q-1)} g_{1\alpha} \right).$$

Inserting Eqs (3.5), (3.6), (3.8), and (3.13) in the foregoing equation it produces

$$\omega_\alpha\left(-\frac{2\,(q-1)\,\ddot{h}}{h}-\frac{\mathcal{R}}{q-1}\right)=0.$$

A contraction with ω^α yields

$$\omega^\alpha\omega_\alpha\left(-\frac{2\,(q-1)\,\ddot{h}}{h}-\frac{\mathcal{R}}{q-1}\right)=0.$$

If $\omega^\alpha\omega_\alpha\neq 0$, then

$$\mathcal{R}=-2\,(q-1)^2\,\frac{\ddot{h}}{h}.$$

Theorem 3.5. *In a $(PSS)_q$ GRW spacetime, the scalar curvature is given by*

$$\mathcal{R}=-2\,(q-1)^2\,\frac{\ddot{h}}{h}.$$

The fourth component: In virtue of Eq (2.2), we have

$$\nabla_\alpha\mathcal{R}_{\beta\gamma}=\omega_\alpha\left(2\mathcal{R}_{\beta\gamma}+\frac{\mathcal{R}}{q-1}g_{\beta\gamma}\right)+\omega_\beta\left(\mathcal{R}_{\gamma\alpha}-\frac{\mathcal{R}}{2(q-1)}g_{\gamma\alpha}\right)+\omega_\gamma\left(\mathcal{R}_{\beta\alpha}-\frac{\mathcal{R}}{2(q-1)}g_{\beta\alpha}\right).$$

Utilizing Eqs (3.5), (3.7), and (3.12), one finds

$$\begin{aligned}\tilde{\nabla}_\alpha\mathcal{R}_{\beta\gamma}=&\,\omega_\alpha\left(2\mathcal{R}_{\gamma\beta}+2\left(h\ddot{h}+(q-2)\,\dot{h}^2\right)\tilde{g}_{\gamma\beta}+\frac{h^2\mathcal{R}}{q-1}\tilde{g}_{\beta\gamma}\right)\\&+\omega_\beta\left(\mathcal{R}_{\alpha\gamma}+\left(h\ddot{h}+(q-2)\,\dot{h}^2\right)\tilde{g}_{\alpha\gamma}-\frac{h^2\mathcal{R}}{2(q-1)}\tilde{g}_{\gamma\alpha}\right)\\&+\omega_\gamma\left(\mathcal{R}_{\alpha\beta}+\left(h\ddot{h}+(q-2)\,\dot{h}^2\right)\tilde{g}_{\alpha\beta}-\frac{h^2\mathcal{R}}{2(q-1)}\tilde{g}_{\beta\alpha}\right).\end{aligned}$$

In view of Eq (3.15), the fiber manifold is Einstein. As is known, every Einstein manifold is of constant scalar curvature, that is,

$$\tilde{\nabla}_\alpha\mathcal{R}=0.$$

Consequently,

$$\tilde{\nabla}_\alpha\mathcal{R}_{\beta\gamma}=0.$$

It follows that

$$\begin{aligned}&\left(2\frac{\mathcal{R}}{q-1}+2\left(h\ddot{h}+(q-2)\,\dot{h}^2\right)+\frac{h^2\mathcal{R}}{q-1}\right)\omega_\alpha\tilde{g}_{\beta\gamma}+\left(\frac{\mathcal{R}}{q-1}+\left(h\ddot{h}+(q-2)\,\dot{h}^2\right)-\frac{h^2\mathcal{R}}{2(q-1)}\right)\omega_\beta\tilde{g}_{\alpha\gamma}\\&+\left(\frac{\mathcal{R}}{q-1}+\left(h\ddot{h}+(q-2)\,\dot{h}^2\right)-\frac{h^2\mathcal{R}}{2(q-1)}\right)\omega_\gamma\tilde{g}_{\beta\alpha}=0.\end{aligned}$$

Executing a contraction with $\tilde{g}^{\alpha\beta}$, one infers

$$\left[\frac{4\mathcal{R}}{(q-1)}+(q+2)\left(h\ddot{h}+(q-2)\,\dot{h}^2\right)\right]\omega_\gamma=0.$$

A multiplication with ω^{γ} implies that

$$\mathcal{R} = -\frac{(q-1)(q+2)\left(h\ddot{h} + (q-2)\dot{h}^2\right)}{4},$$

where $\omega^{\gamma}\omega_{\gamma} \neq 0$.

Theorem 3.6. *The scalar curvature of a (PSS)$_q$ GRW spacetime satisfies*

$$\mathcal{R} = -\frac{(q-1)(q+2)\left(h\ddot{h} + (q-2)\dot{h}^2\right)}{4}.$$

Example 3.1. *For a Euclidean space $\mathbb{R}^3\backslash\{0\}$, with respect to spherical coordinates, the metric is given by*

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

This space can be identified as a warped product $\mathbb{R}^+ \times_r S^2$. The base space is a ray from the origin ($\mathbb{R}$) and the fibers are spheres $S^2(r)$ of radius $r > 0$. Since the sphere S^2 is a space of constant curvature, it is an Einstein space.

Example 3.2. *A surface of revolution serves as another standard example of a warped product.*

Example 3.3. *The Schwarzschild spacetime is expressed as the warped product $P \times_r S^2$, where the fiber S^2 is the unit sphere (which is an Einstein space) and the base space $P = \mathbb{R} \times \mathbb{R}^+$ represents a half-plane where $r > 0$.*

4. (PSS)$_q$ spacetime whose Ricci tensor is Codazzi and cyclic parallel

This section examines the class of (PSS)$_q$ spacetimes admitting a Ricci tensor that fulfills both the Codazzi identity and the cyclic parallel condition. The Ricci tensor $\mathcal{R}_{kl}$ is classified as of Codazzi type if its covariant derivative satisfies the symmetry condition [29]:

$$\nabla_k\mathcal{R}_{ij} = \nabla_j\mathcal{R}_{ik}, \tag{4.1}$$

whereas it is considered cyclic parallel if the sum of its covariant derivatives under cyclic permutation of indices vanish [29]

$$\nabla_k\mathcal{R}_{ij} + \nabla_i\mathcal{R}_{jk} + \nabla_j\mathcal{R}_{ik} = 0. \tag{4.2}$$

4.1. (PSS)$_q$ spacetime with $\nabla_k\mathcal{R}_{ij} = \nabla_j\mathcal{R}_{ik}$

Inserting Eq (4.1) in Eq (2.5), one gets

$$\left(\omega_k\mathcal{R}_{ij} - \omega_j\mathcal{R}_{ik}\right) + \frac{3\mathcal{R}}{2(q-1)}\left(\omega_k g_{ij} - \omega_j g_{ik}\right) = 0.$$

Transvecting the aforementioned equation with ω^k, with the help of Eq (2.3), one infers

$$\mathcal{R}_{ij} = -\frac{3\mathcal{R}}{2(q-1)}g_{ij} - \frac{\mathcal{R}(q+2)}{2(q-1)}\omega_j\omega_i, \tag{4.3}$$

where the expression means that the Ricci tensor is in PF form.

Theorem 4.1. *A (PSS)$_q$ spacetime admitting the Codazzi type of $\mathcal{R}_{ij}$ is a PFS.*

In [30], Guilfoyle and Nolan named a Yang pure space as a 4-dimensional Lorentzian manifold whose metric solves Yang's equation:

$$\nabla_k \mathcal{R}_{ij} = \nabla_j \mathcal{R}_{ik}.$$

They proved that a 4-dimensional PFS is a Yang pure space if and only if the spacetime is an RW spacetime.

Hence, we can state the following theorem:

Corollary 4.1. *A 4-dimensional (PSS)$_q$ spacetime admitting Codazzi a type of Ricci tensor is an RW spacetime.*

Comparing Eqs (3.4) and (4.3), one infers

$$p = \frac{\mathcal{R}(q-4)}{2\kappa(q-1)}, \tag{4.4}$$

$$\sigma = \frac{-\mathcal{R}}{\kappa}. \tag{4.5}$$

Corollary 4.2. *In a (PSS)$_q$ spacetime admitting a Codazzi type of $\mathcal{R}_{ij}$, the isotropic pressure and the energy density are constants and given by Eqs (4.4) and (4.5).*

For $q = 4$, Eq (4.4) implies

$$p = 0,$$

which shows that the spacetime represents the dust matter era.

Corollary 4.3. *A 4-dimensional (PSS)$_q$ spacetime admitting a Codazzi type of $\mathcal{R}_{ij}$ represents the dust matter era.*

Since the Ricci tensor is of Codazzi type, the scalar curvature is constant. Therefore, Eqs (4.4) and (4.5) lead to

$$p + \sigma = -\frac{\mathcal{R}(q+2)}{2\kappa(q-1)}.$$

If $\mathcal{R} = 0$, then

$$p + \sigma = 0,$$

which shows that the spacetime represents the dark energy era [31].

Corollary 4.4. *A (PSS)$_q$ spacetime admitting a Codazzi type of $\mathcal{R}_{ij}$ represents the dark energy era.*

4.2. (PSS)$_q$ spacetime with cyclic parallel of the Ricci tensor

From Eq (2.5), it follows that

$$\nabla_k \mathcal{R}_{ij} + \nabla_i \mathcal{R}_{jk} + \nabla_j \mathcal{R}_{ik} = \omega_k \mathcal{R}_{ij} + \omega_i \mathcal{R}_{jk} + \omega_j \mathcal{R}_{ik}.$$

If the Ricci tensor is cyclic parallel, then the previous equation becomes

$$\omega_k \mathcal{R}_{ij} + \omega_i \mathcal{R}_{jk} + \omega_j \mathcal{R}_{ik} = 0.$$

Contracting the previous equation with ω^k and using Eq (2.3), it produces

$$\mathcal{R}_{ij} = \mathcal{R}\omega_i\omega_j.$$

A contraction with ω^i, with the aid of Eq (2.3), yields

$$\mathcal{R} = 0.$$

Consequently,

$$\mathcal{R}_{ij} = 0,$$

which indicates that the spacetime is Ricci flat.

Theorem 4.2. *A (PSS)$_q$ spacetime admitting a cyclic parallel of $\mathcal{R}_{ij}$ is a vacuum.*

Remark 4.1. *The conclusion of the previous theorem should be interpreted as a rigidity result. In the classical definition of a (PSS)$_q$ spacetime, the Ricci tensor is assumed to be non-zero. Therefore, the previous theorem implies that no non-trivial (PSS)$_q$ spacetime has a cyclic parallel of $\mathcal{R}_{ij}$. It is important to point out that the Riemann curvature tensor is not identically zero. Equivalently, the Weyl tensor is non-vanishing. Thus the spacetime is Ricci-flat but not flat. Consequently, the previous theorem is optimal at the level of the Ricci tensor: it forces the vanishing of the Ricci curvature, but does not imply the vanishing of the full curvature tensor.*

5. (PSS)$_q$ spacetimes with $\nabla^i C_{kijm} = 0$

This section investigates (PSS)$_q$ spacetimes with a divergence-free Weyl tensor. We prove that such spacetimes are both static and represent a perfect fluid. A static spacetime is defined by the presence of a unit timelike vector field that satisfies the following two conditions [31]:

$$\nabla_i\omega_j = -\omega_i\dot{\omega}_j, \tag{5.1}$$

$$\omega_j\ddot{\omega}_i - \omega_i\ddot{\omega}_j = 0. \tag{5.2}$$

It is established that the divergence of the Weyl tensor is [32]

$$\nabla^i C_{kijm} = -\frac{q-3}{q-2}\left[\nabla_k\mathcal{R}_{jm} - \nabla_j\mathcal{R}_{km} - \frac{g_{jm}\nabla_k\mathcal{R} - g_{km}\nabla_j\mathcal{R}}{2(q-1)}\right]. \tag{5.3}$$

Assume that $\nabla^i C_{kijm} = 0$, and it follows that

$$\nabla_k\mathcal{R}_{jm} - \nabla_j\mathcal{R}_{km} - \frac{g_{jm}\nabla_k\mathcal{R} - g_{km}\nabla_j\mathcal{R}}{2(q-1)} = 0.$$

The use of Eqs (2.4) and (2.5) implies that

$$\left(\omega_k\mathcal{R}_{ij} - \omega_j\mathcal{R}_{ik}\right) - \frac{\mathcal{R}}{2(q-1)}(\omega_k g_{jm} - \omega_j g_{km}) = 0.$$

Contracting with ω^k and using Eq (2.3), we have

$$\mathcal{R}_{jm} = \frac{\mathcal{R}}{2(q-1)}g_{jm} - \frac{(q-2)\mathcal{R}}{2(q-1)}\omega_j\omega_m, \tag{5.4}$$

which reveals the perfect fluid nature of $\mathcal{R}_{jm}$.

Theorem 5.1. *A $(PSS)_q$ spacetime with $\nabla^i C_{kijm} = 0$ is a PFS.*

Applying the operator of covariant derivative on Eq (5.4), one gets

$$\nabla_i \mathcal{R}_{jm} = \frac{\nabla_i \mathcal{R}}{2(q-1)} g_{jm} - \frac{(q-2)\,\mathcal{R}}{2(q-1)} \left[\omega_m \nabla_i \omega_j + \omega_j \nabla_i \omega_m\right] - \frac{(q-2)\,\nabla_i \mathcal{R}}{2(q-1)} \omega_j \omega_m.$$

The use of Eq (2.4) leads to

$$\nabla_i \mathcal{R}_{jm} = \frac{4\omega_i \mathcal{R}}{2(q-1)} g_{jm} - \frac{(q-2)\,\mathcal{R}}{2(q-1)} \left[\omega_m \nabla_i \omega_j + \omega_j \nabla_i \omega_m\right] - \frac{2\mathcal{R}\,(q-2)}{(q-1)} \omega_i \omega_j \omega_m.$$

Swapping $i \leftrightarrow j$, one finds

$$\nabla_j \mathcal{R}_{im} = \frac{4\omega_j \mathcal{R}}{2(q-1)} g_{im} - \frac{(q-2)\,\mathcal{R}}{2(q-1)} \left[\omega_m \nabla_j \omega_i + \omega_i \nabla_j \omega_m\right] - \frac{2\mathcal{R}\,(q-2)}{(q-1)} \omega_i \omega_j \omega_m.$$

Subtracting the last two equations, one infers

$$\begin{aligned}\nabla_i \mathcal{R}_{jm} - \nabla_j \mathcal{R}_{im} =& \frac{4\mathcal{R}}{2(q-1)} \left[g_{jm}\omega_i - g_{im}\omega_j\right] \\ &- \frac{(q-2)\,\mathcal{R}}{2(q-1)} \left[\omega_j \nabla_i \omega_m - \omega_i \nabla_j \omega_m + \omega_m \left(\nabla_i \omega_j - \nabla_j \omega_i\right)\right].\end{aligned}$$

Utilizing Eqs (2.5) and (2.8) in the previous equation, we reveal that

$$\begin{aligned}\omega_i \mathcal{R}_{jm} - \omega_j \mathcal{R}_{im} + \frac{3\mathcal{R}}{2(q-1)} \left(\omega_i g_{jm} - \omega_j g_{im}\right) =& \frac{4\mathcal{R}}{2(q-1)} \left[g_{jm}\omega_i - g_{im}\omega_j\right] \\ &- \frac{(q-2)\,\mathcal{R}}{2(q-1)} \left[\omega_j \nabla_i \omega_m - \omega_i \nabla_j \omega_m\right].\end{aligned}$$

Contracting the foregoing equation with ω^j, with the aid of Eqs (2.3) and (2.9), the result is

$$\frac{\mathcal{R}}{2} \omega_m \omega_i + \mathcal{R}_{im} = \frac{\mathcal{R}}{2(q-1)} \left[\omega_m \omega_i + g_{im}\right] + \frac{(q-2)\,\mathcal{R}}{2(q-1)} \nabla_i \omega_m.$$

Employing Eq (5.4) in the previous equation, we obtain

$$\nabla_i \omega_m = 0, \tag{5.5}$$

which establishes that ω^i is parallel. As a result, Eqs (5.1) and (5.2) are satisfied. Consequently, the spacetime is static.

Corollary 5.1. *A $(PSS)_q$ spacetime with $\nabla^i C_{kijm} = 0$ is a static spacetime.*

Remark 5.1. *A spherically symmetric vacuum solution is necessarily static. The classic example is the exterior Schwarzschild spacetime, which describes the gravitational field outside a perfectly spherical, non-rotating mass.*

Example 5.1. *The Schwarzschild spacetime is a classic example of a static spacetime. By Birkhoff's theorem, the Schwarzschild solution is the unique spherically symmetric vacuum solution to the Einstein field equations.*

6. Cosmological perfect fluid $(\mathrm{PSS})_q$ spacetimes in $f(\mathcal{R})$ gravity

In the field equations of $f(\mathcal{R})$ gravity, the inclusion of higher-order derivative terms, specifically the Hessian of the Ricci scalar $\nabla_k\nabla_l\mathcal{R}$ and the tensor product of its gradients $(\nabla_k\mathcal{R})(\nabla_l\mathcal{R})$, introduces geometric corrections that prevent the effective energy-momentum tensor from being expressed in the standard perfect fluid form. These terms represent non-trivial gravitational degrees of freedom that deviate from the p and σ characteristics of an ideal fluid. The geometry of $(\mathrm{PSS})_q$ spacetimes allows the EMT to be mathematically structured as a perfect fluid. The $f(\mathcal{R})$ gravity field equations are expressed as

$$\begin{aligned}\kappa T_{lk} =& f(\mathcal{R})_{\mathcal{R}}\mathcal{R}_{lk} - f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}}\nabla_l\mathcal{R}\nabla_k\mathcal{R} - f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\nabla_l\nabla_k\mathcal{R}\\ &+ g_{lk}\left[f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}}\nabla_k\mathcal{R}\nabla^k\mathcal{R} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\nabla^2\mathcal{R} - \frac{1}{2}f(\mathcal{R})\right],\end{aligned} \tag{6.1}$$

where κ is the gravitational coupling and $f(\mathcal{R})_{\mathcal{R}}$ is the derivative of $f(\mathcal{R})$ regarding $\mathcal{R}$ [33].

Utilizing Eq (5.5) in Eq (2.6), the result is

$$\nabla_l\nabla_k\mathcal{R} = 16\mathcal{R}\omega_l\omega_k. \tag{6.2}$$

From Eq (2.4), with the aid of Eq (5.5), one easily gets

$$\begin{aligned}\nabla_l\mathcal{R}\nabla_k\mathcal{R} &= 16\mathcal{R}^2\omega_l\omega_k,\\ \nabla_k\mathcal{R}\nabla^k\mathcal{R} &= -16\mathcal{R}^2,\\ \nabla^2\mathcal{R} &= -16\mathcal{R}.\end{aligned}$$

Utilizing the last four equations in Eq (6.1), one infers

$$\begin{aligned}\kappa T_{lk} =& \left[\frac{\mathcal{R}f(\mathcal{R})_{\mathcal{R}}}{2(q-1)} - 16\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) - \frac{f(\mathcal{R})}{2}\right]g_{lk}\\ &- \left[\frac{(q-2)\mathcal{R}f(\mathcal{R})_{\mathcal{R}}}{2(q-1)} + 16\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right)\right]\omega_l\omega_k.\end{aligned} \tag{6.3}$$

In a PFS, the EMT is given as

$$\kappa T_{lk} = \kappa p g_{ij} + \kappa(p+\sigma)\omega_l\omega_k. \tag{6.4}$$

Comparing Eqs (6.3) and (6.4), one finds

$$p = \frac{1}{\kappa}\left[\frac{\mathcal{R}f(\mathcal{R})_{\mathcal{R}}}{2(q-1)} - 16\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) - \frac{f(\mathcal{R})}{2}\right], \tag{6.5}$$

$$\sigma = \frac{1}{\kappa}\left[\frac{f(\mathcal{R}) - \mathcal{R}f(\mathcal{R})_{\mathcal{R}}}{2}\right]. \tag{6.6}$$

Hence, we can state:

Theorem 6.1. *In a $(\mathrm{PSS})_q$ spacetime obeying $f(\mathcal{R})$ gravity, p and σ are given by Eqs (6.5) and (6.6).*

In general relativity ($f(\mathcal{R}) = \mathcal{R}$), Eqs (6.5) and (6.6) become

$$p = -\frac{\mathcal{R}}{2\kappa}, \tag{6.7}$$

$$\sigma = 0. \tag{6.8}$$

Corollary 6.1. *In general relativity, the isotropic pressure and the energy density for a $(PSS)_q$ spacetime are given by Eqs (6.7) and (6.8).*

In the following part, we assume that the scalar curvature is constant. Consequently, Eq (2.4) implies

$$\mathcal{R} = 0.$$

It follows from Eq (5.4) that

$$\mathcal{R}_{ij} = 0.$$

We now study the consequences of this rigidity result in $f(\mathcal{R})$ gravity. In this setting, the field equations reduce to

$$T_{ij} = -\frac{f(0)}{2\kappa} g_{ij}. \tag{6.9}$$

Thus the matter tensor is proportional to the metric tensor and therefore represents a vacuum fluid (equivalently, a cosmological-constant-type source). Writing

$$T_{ij} = -\sigma\, g_{ij},$$

and comparing with (6.9), we obtain

$$\sigma = \frac{f(0)}{2\kappa}. \tag{6.10}$$

For a vacuum fluid, the pressure satisfies

$$p = -\sigma.$$

Hence

$$p = -\frac{f(0)}{2\kappa}. \tag{6.11}$$

Therefore, we obtain the following theorem.

Theorem 6.2. *Let M be a $(PSS)_q$ spacetime with constant scalar curvature. Then the matter content of $f(\mathcal{R})$ gravity is a vacuum fluid with constant pressure and energy density given by* (6.11) *and* (6.10)*, respectively.*

Corollary 6.2. *Under the hypotheses of the preceding theorem, if $f(0) \neq 0$, then the equation-of-state parameter is*

$$\omega = \frac{p}{\sigma} = -1,$$

which means that the $(PSS)_n$ spacetime represents the dark energy era.

Remark 6.1. *The constant scalar curvature condition imposes a strong rigidity on the corresponding $f(\mathcal{R})$-gravity model. When $f(0) \neq 0$, the only admissible effective matter source is a vacuum fluid, i.e., a cosmological-constant-type source. Thus, the geometric constraint completely determines the form of the effective energy-momentum tensor.*

6.1. *Energy conditions in $(PSS)_q$ spacetimes*

In what follows, we examine the null energy condition (**NEC**) , weak energy condition (**WEC**), strong energy condition (**SEC**), and dominant energy condition (**DEC**) for the perfect-fluid matter content derived in the previous theorem [34]. Using Eqs (6.5) and (6.6), a direct computation yields

$$\kappa(\sigma + p) = -\frac{(q-2)\,\mathcal{R}f(\mathcal{R})_{\mathcal{R}}}{2\,(q-1)} - 16\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right), \tag{6.12}$$

$$\kappa(\sigma - p) = f(\mathcal{R}) - \frac{q\mathcal{R}f(\mathcal{R})_{\mathcal{R}}}{2\,(q-1)} + 16\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right), \tag{6.13}$$

$$\kappa(\sigma + 3p) = -f(\mathcal{R}) + \frac{(4-q)\mathcal{R}f(\mathcal{R})_{\mathcal{R}}}{2\,(q-1)} - 48\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right). \tag{6.14}$$

In particular, for a 4-dimensional $(\mathrm{PSS})_q$ spacetime $(\mathrm{PSS})_4$, these expressions simplify to

$$\kappa(\sigma + p) = -\frac{\mathcal{R}f(\mathcal{R})_{\mathcal{R}}}{3} - 16\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right), \tag{6.15}$$

$$\kappa(\sigma - p) = f(\mathcal{R}) - \frac{2\mathcal{R}f(\mathcal{R})_{\mathcal{R}}}{3} + 16\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right), \tag{6.16}$$

$$\kappa(\sigma + 3p) = -f(\mathcal{R}) - 48\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right). \tag{6.17}$$

Combining these with the explicit form $\kappa\sigma = \frac{1}{2}(f(\mathcal{R}) - \mathcal{R}f(\mathcal{R})_{\mathcal{R}})$ of the energy density, we find the following characterization of the ECs in terms of $f(\mathcal{R})$.

Theorem 6.3. *In a* 4*-dimensional* $(PSS)_q$ *spacetime* $(PSS)_4$ *governed by* $f(\mathcal{R})$ *gravity, the four energy conditions translate into the following constraints on* f *(see Table 1):*

Table 1. Energy conditions.

The EC	The mathematical form	The condition in $(\mathrm{PSS})_4$ spacetime
NEC	$p + \sigma \geq 0$	$\mathcal{R}f(\mathcal{R})_{\mathcal{R}} + 48\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) \leq 0$
WEC	$\sigma \geq 0,\, p + \sigma \geq 0$	$\mathcal{R} \geq 0,\, \mathcal{R}f(\mathcal{R})_{\mathcal{R}} + 48\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) \leq 0$
DEC	$\sigma \pm p \geq 0,\, \sigma \geq 0$	$\mathcal{R} \geq 0,\, f(\mathcal{R}) - \frac{2}{3}\mathcal{R}f(\mathcal{R})_{\mathcal{R}} + 16\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) \geq 0$
SEC	$\sigma + 3p \geq 0,\, p + \sigma \geq 0$	$f(\mathcal{R}) + 48\mathcal{R}\left(\mathcal{R}f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) \leq 0$

Remark 6.2. *The pointwise energy conditions serve as coordinate-independent mathematical constraints imposed on the stress-energy tensor* $\mathcal{T}_{ij}$*. They mathematically formalize the intuitive classical concepts that local energy density should remain positive, and that gravitational interaction is fundamentally attractive [35, 36]:*

(1) NEC. Defined by $\mathcal{T}_{ij}k^ik^j \geq 0$ for any null vector k^i. Physically, it dictates that the energy density measured by an observer moving at the speed of light can never be negative. A local violation of the NEC implies the presence of "exotic matter", which is a theoretical requirement for the stability of traversable wormholes or warp drives.

(2) WEC. Defined by $\mathcal{T}_{ij}u^iu^j \geq 0$ for any timelike vector u^i. It requires that any local, physically realizable observer moving along a regular timelike trajectory will always measure a non-negative local energy density ($\sigma \geq 0$).

(3) DEC. Satisfies the WEC constraint and further requires that the energy-momentum flux vector $Y^i = T_i^j u^i$ is non-spacelike (causal). Physically, this condition enforces that energy-momentum cannot propagate or flow faster than the speed of light, ensuring strict local causality.

(4) SEC. Defined by $\mathcal{T}_{ij} - \frac{\mathcal{T}}{2} g_{ij})u^i u^j \geq 0$ for any timelike vector u^i. It physically captures the principle that matter gravitates toward other matter, keeping gravity purely attractive. Consequently, the SEC must be systematically violated to allow for periods of global cosmic acceleration, such as inflationary eras or the current dark-energy-dominated era.

6.2. *Equation of state and cosmological eras*

The equation-of-state parameter $w = p/\sigma$ classifies the cosmological epoch realized by the matter content of the spacetime. From Eqs (6.5) and (6.6), in the four-dimensional case,

$$w = \frac{p}{\sigma} = \frac{\mathcal{R} f(\mathcal{R})_{\mathcal{R}} - 96\mathcal{R}\left(\mathcal{R} f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) - 3f(\mathcal{R})}{3\left(f(\mathcal{R}) - \mathcal{R} f(\mathcal{R})_{\mathcal{R}}\right)}. \tag{6.18}$$

Conventional cosmological epochs correspond to the following standard values of w [16]:

$$\begin{array}{lll} w = 0 \text{ (dust)}, & w = \frac{1}{3} \text{ (radiation)}, & w = 1 \text{ (stiff matter)}, \\ w = -\frac{1}{3} \text{ (marginal acceleration)}, & & -1 < w < -\frac{1}{3} \text{ (quintessence)}, \\ w = -1 \text{ (dark energy)}, & & w < -1 \text{ (phantom)}. \end{array}$$

Theorem 6.4. *Let M be a 4-dimensional (PSS)$_q$ spacetime in $f(\mathcal{R})$ gravity with $\sigma \neq 0$. Then the matter content of M realizes:*

(1) The dust epoch ($w = 0$) *if and only if*

$$\mathcal{R} f(\mathcal{R})_{\mathcal{R}} - 96\mathcal{R}\left(\mathcal{R} f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) = 3f(\mathcal{R});$$

(2) The radiation epoch $\left(w = \frac{1}{3}\right)$ *if and only if*

$$\mathcal{R} f(\mathcal{R})_{\mathcal{R}} - 48\mathcal{R}\left(\mathcal{R} f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) = 2f(\mathcal{R});$$

(3) The stiff-matter epoch ($w = 1$) *if and only if*

$$2\mathcal{R} f(\mathcal{R})_{\mathcal{R}} - 48\mathcal{R}\left(\mathcal{R} f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) = 3f(\mathcal{R});$$

(4) The cosmological-constant (dark-energy) epoch ($w = -1$) *if and only if*

$$\mathcal{R} f(\mathcal{R})_{\mathcal{R}} + 48\mathcal{R}\left(\mathcal{R} f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) = 0;$$

(5) The quintessence regime $\left(-1 < w < -\frac{1}{3}\right)$ *if and only if*

$$\mathcal{R} f(\mathcal{R})_{\mathcal{R}} + 48\mathcal{R}\left(\mathcal{R} f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) < 0 \text{ and } f(\mathcal{R}) + 48\mathcal{R}\left(\mathcal{R} f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) > 0;$$

(6) The phantom regime ($w < -1$) *if and only if*

$$\mathcal{R} f(\mathcal{R})_{\mathcal{R}} + 48\mathcal{R}\left(\mathcal{R} f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} + f(\mathcal{R})_{\mathcal{R}\mathcal{R}}\right) > 0 \text{ and } f(\mathcal{R}) > \mathcal{R} f(\mathcal{R})_{\mathcal{R}}.$$

Example 6.1. *Physical realization: Quark-gluon plasma is the closest known physical substance to a perfect fluid. Perfect fluids are utilized in general relativity to model idealized distributions of matter, such as the interior of a star or an isotropic universe. In cosmological models, the equation of state of a perfect fluid is used in the Friedmann-Lemaitre-Robertson-Walker equations to describe the evolution of the universe.*

Example 6.2. *Observational evidence: Galaxy clusters contain hundreds or thousands of galaxies, each embedded in its own dark matter halo. The bullet cluster, in particular, provides some of the best empirical evidence for the existence of dark matter. Cosmological scale: Using fluctuations in the cosmic microwave background, astronomers have determined that dark matter constitutes approximately 27% of the total mass-energy budget of the universe.*

7. Applications to specific $f(\mathcal{R})$ gravity models

To demonstrate the structural and physical viability of the general expressions obtained in Theorem 11, we subject our formulations for isotropic pressure p and energy density σ (given by the Eqs (6.5) and (6.6)) to three distinct, physically motivated $f(\mathcal{R})$ gravity models. Throughout this analysis, we evaluate the specialized configurations within a four-dimensional spacetime setting ($q = 4$).

7.1. The quadratic Starobinsky model: $f(\mathcal{R}) = \mathcal{R} + \alpha\mathcal{R}^2$

The Starobinsky framework is highly valued in modern early-universe cosmology for driving cosmic inflation via geometric corrections without introducing a primary scalar inflaton field [37]. Computing the respective derivatives with respect to the scalar curvature yields:

$$f(\mathcal{R})_{\mathcal{R}} = 1 + 2\alpha\mathcal{R}, \quad f(\mathcal{R})_{\mathcal{R}\mathcal{R}} = 2\alpha, \quad f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} = 0.$$

Substituting these fields directly into Eqs (6.5) and (6.6) produces the exact profiles for the model:

$$\sigma = -\frac{\alpha\mathcal{R}^2}{2\kappa},$$
$$p = -\frac{2\mathcal{R} + 192\alpha\mathcal{R} + \alpha\mathcal{R}^2}{6\kappa}.$$

Physical interpretation: In an expanding cosmological domain where the scalar curvature is strictly positive ($\mathcal{R} > 0$) and the coupling constraint obeys $\alpha > 0$, the $(PSS)_q$ framework forces the effective energy density to become negative ($\sigma < 0$). This induces an immediate local violation of the WEC, meaning that standard Starobinsky cosmic inflation cannot proceed within a regular $(PSS)_4$ spacetime without assuming non-classical, exotic matter elements. If the strong rigidity constraint of a constant scalar curvature ($\mathcal{R} = 0$) is enforced as established in Theorem 12, the entire configuration drops cleanly into a true physical vacuum state ($\sigma = 0, p = 0$).

7.2. The linear expansion model: $f(\mathcal{R}) = \mathcal{R} - \frac{\mu^4}{\mathcal{R}}$

This functional model serves as a standard late-time cosmic expansion model designed to simulate the acceleration of dark energy through an inverse curvature term governed by the mass scale parameter

μ [38]. Its higher-order derivative profiles are given by:

$$f(\mathcal{R})_{\mathcal{R}} = 1 + \frac{\mu^4}{\mathcal{R}^2}, \quad f(\mathcal{R})_{\mathcal{R}\mathcal{R}} = -\frac{2\mu^4}{\mathcal{R}^3}, \quad f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} = \frac{6\mu^4}{\mathcal{R}^4}.$$

Applying these structural derivatives directly into our target perfect-fluid formulations isolates the following matter signatures:

$$\sigma = \frac{\mu^4}{\kappa\mathcal{R}},$$
$$p = \frac{384\mu^4 - \mathcal{R}^3 + 4\mu^4\mathcal{R}}{6\kappa\mathcal{R}^2}.$$

Physical interpretation: The effective energy density displays an inverse dependency on the global scalar curvature field ($\sigma \propto \mathcal{R}^{-1}$). Consequently, during highly curved early-universe regimes, the energy density remains tightly suppressed, whereas late-time cosmic epochs driving $\mathcal{R} \to 0$ force σ to grow rapidly. Crucially, the higher-derivative combinations within the isotropic pressure term push the system's global equation of state parameter ($w = p/\sigma$) deep into the phantom regime ($w < -1$). Hence, matching this model with the $(PSS)_4$ constraint means the universe must inevitably terminate in a chaotic "big rip" cosmic doomsday scenario.

7.3. The exponential gravity framework: $f(\mathcal{R}) = \mathcal{R} - \beta e^{-\alpha\mathcal{R}}$

This sophisticated model represents an adaptable framework for unified dark matter and dark energy, modifying general relativity smoothly at lower limits while maintaining stability under higher limits [39]. The corresponding derivatives are evaluated as:

$$f(\mathcal{R})_{\mathcal{R}} = 1 + \alpha\beta e^{-\alpha\mathcal{R}}, \quad f(\mathcal{R})_{\mathcal{R}\mathcal{R}} = -\alpha^2\beta e^{-\alpha\mathcal{R}}, \quad f(\mathcal{R})_{\mathcal{R}\mathcal{R}\mathcal{R}} = \alpha^3\beta e^{-\alpha\mathcal{R}}.$$

Evaluating the underlying expressions establishes the following specific fluid density and isotropic pressure equations:

$$\sigma = \frac{\beta e^{-\alpha\mathcal{R}}(1 + \alpha\mathcal{R})}{2\kappa},$$
$$p = \frac{1}{\kappa}\left[\frac{\mathcal{R}(1 + \alpha\beta e^{-\alpha\mathcal{R}})}{6} - 16\alpha^2\beta e^{-\alpha\mathcal{R}}\mathcal{R}(\alpha\mathcal{R} - 1) - \frac{\mathcal{R} - \beta e^{-\alpha\mathcal{R}}}{2}\right].$$

Physical interpretation: This model exhibits excellent asymptotic stability within our geometric framework. For high-curvature domains ($\mathcal{R} \gg 1/\alpha$), the exponential terms dilute to zero ($e^{-\alpha\mathcal{R}} \to 0$), causing the energy density to drop cleanly to $\sigma \approx 0$, which mirrors our paper's baseline general relativity scenario. Furthermore, because our $(PSS)_4$ geometric relation bounds the troublesome Hessian curvature gradients ($\nabla_l\nabla_k\mathcal{R} = 16\mathcal{R}\omega_l\omega_k$), the system is protected from the destructive high-frequency metric fluctuations that usually break modified exponential models. In the late-time universe, it tracks perfectly as a stable vacuum fluid source.

8. Conclusions

Ricci-symmetric manifolds ($\nabla_k R_{ij} = 0$) play a crucial role in differential geometry, general relativity, and cosmology, thanks to their abundant geometric properties, the simplifications they introduce in analytical computations, and their applicability to the study of exact solutions and symmetries. Furthermore, they provide significant insight into the intrinsic geometry of manifolds and contribute to our knowledge of gravitational phenomena as described by general relativity [1]. Such manifolds that are generalized by pseudo-Ricci symmetric manifolds are pseudo-Ricci symmetric manifolds [21]. Pseudo-Ricci symmetric manifolds provide a natural extension that retain essential geometric structure while offering greater flexibility for modeling realistic relativistic and cosmological scenarios. The concept of pseudo-Ricci symmetric manifolds is generalized by many researchers (see [7, 10]).

This work investigates $(\mathrm{PSS})_q$ spacetimes. The associated covector ω^i is found to be irrotational and non-accelerating, and obeys $\omega^i \mathcal{R}_{ij} = \frac{\mathcal{R}}{2}\omega_j$. $(\mathrm{PSS})_q$ GRW spacetimes are shown to be a PFS. Moreover, when the Ricci tensor is of Codazzi type, such spacetimes are a PFS, and when it is cyclically parallel, they are a vacuum. The physical interpretation of these spacetimes is also clarified using the $f(\mathcal{R})$ gravity framework.

In the future, we or possibly other researchers intend to investigate the properties of $(\mathrm{PSS})_q$ warped product manifolds, particularly in connection with cosmological models, general relativity, and modified theories such as $f(\mathcal{R}, T)$ and $f(\mathcal{R}, \mathcal{G})$ gravity [40, 41]. Furthermore, it would be worthwhile to extend the results of this study to multiply warped products, which represent a more general framework than ordinary warped products and allow for the construction of a wider variety of spacetime models [42, 43].

Author contributions

Bang-Yen Chen: conceptualization, validation, data curation, supervision, project administration; Awatif Al-Jedani: methodology, software, formal analysis, investigation, data curation, writing—original draft; Uday Chand De: methodology, validation, resources, project administration; Nasser Bin Turki: software, investigation, supervision; Abdallah Abdelhameed Syied: conceptualization, formal analysis, resources, writing—original draft. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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