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Research article

A flexible power generated Topp–Leone distribution for modelling bounded lifetime data: Theory, simulation, and applications

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Abstract: The increasing demand for flexible probabilistic models in medical survival analysis motivates the development of advanced extensions for bounded lifetime data. In this paper, we obtain a novel generalization of the Topp-Leone distribution using the power generalized Dinesh Uma Singh (PGDUS) transformation. The role of an additional shape parameter substantially enhances model flexibility, enabling it to capture diverse distributional shapes and various hazard rate behaviors, including monotonic and non-monotonic patterns, which has been discussed. Fundamental statistical properties of the proposed distribution are derived, including explicit expressions for the density and distribution functions, moments, quantile function, and key reliability measures. Parameter estimation is addressed through maximum likelihood and alternative methods, and their finite-sample performance is examined via Monte Carlo simulation. Measures of uncertainty and inequality, such as entropy and Bonferroni curves, are also investigated. The applicability and effectiveness of the proposed model are demonstrated using real medical survival dataset. Comparative results indicate better goodness-of-fit relative to some of the existing competing models. Overall, the PGDUS-generated Topp-Leone distribution provides a flexible framework for modelling bounded lifetime phenomena.

Keywords: PGDUS transformation; Topp-Leone distribution; maximum likelihood method;

bounded distribution; reliability
Mathematics Subject Classification: 62A86, 62F12, 62G07

1. Introduction

In modern scientific research, the accurate analysis and interpretation of data play a crucial role in improving system performance, supporting informed decision making, and enhancing practical outcomes across a wide range of applications. In fields such as medical and engineering sciences, data often exhibit uncertainty due to factors containing biological variation, manufacturing tolerances, varying operational conditions, and limitations in measurement processes. As a result, such data are inherently random and require appropriate statistical methodologies for effective analysis. Probability distribution models serve as a fundamental tool in this context, offering a structured approach to capture variability, quantify uncertainty, and facilitate reliable prediction and inference.

In recent years, substantial attention has been devoted to the development of flexible probability distributions capable of modelling complex lifetime and reliability data arising in engineering and medical sciences. For instance, El-Morshedy [1] introduced the exponentiated odd Chen-H Weibull distribution using the exponentiated odd Chen-H family of distributions and demonstrated its practical relevance through engineering data on the strength of 1.5cm glass fibres. Hassan et al. [2] proposed a novel extension of the power Lomax distribution by integrating the type-II exponentiated half -logistic class, validating the model using medical data related to relief experienced after analgesic administration as well as environmental data on acid rain. Shanker [3] studied a three-parameter generalized Lindley distribution and illustrated its applicability in modelling lifetime data from biomedical and engineering contexts. Similarly, Alsaggaf et al. [4] developed the exponential generalized inverse generalized Weibull distribution and applied it to several real-world medical and engineering datasets, including fatigue fracture data of Kevlar 373/epoxy composites and service times of aircraft windshields. Furthermore, Eliwa et al. [5] investigated a discrete exponential generalized-G family of distributions with applications to medical data analysis. These studies highlight the growing importance of distributional extensions and generator-based families in accurately capturing the variability and hazard behaviour inherent in engineering and medical data. Furthermore, numerous researchers have developed generalizations of classical probability distributions by employing various transformation techniques, such as weighted methods, length-biased approaches, and transmutation techniques for continuous unbounded probability distributions applied in various domains. Some notable contributions in this direction include the following: Saghir et al. [6] studied the length-biased weighted exponentiated inverted Weibull distribution and examined its statistical properties. Bhat et al. [7] proposed the weighted generalized interval cumulative residual entropy and explored its theoretical properties along with practical applications. Rather et al. [8] developed weighted Erlang-truncated exponential distribution and studied its properties with reference to system reliability optimization. Ahmad et al. [9] introduced a novel arc cosine- ψ class of distributions with theory and data evaluation related to corona virus. Qayoom et al. [10] proposed a new class of Lindley distributions with emphasis on system reliability analysis, simulation, and real data applications. A Linear combination of order statistics of exponentiated Nadarajah-Haghighi distribution and their applications was studied by Singh et al. [11]. Ahmad et al. [12] introduced a novel sin-G class of distributions with an illustration based on the Lomax distributions, focusing on theoretical properties and data analysis. Rashid et al. [13] presented a class of Weibull-Pareto distributions, emphasizing their structural properties. Cordeiro and Brito [14] proposed the beta power distribution. Ahmadini et al. [15] proposed truncated Lomax inverse Lomax distribution, whose density is represented as a linear combination of the inverse Lomax distribution and is capable of monotonically increasing, decreasing, reversed J-shaped, and upside-down-shaped hazard rates. Further, Tang et al. [16] derived a new beta power flexible Weibull distribution with an application to sports and reliability data. More recently, Yerneni and Rather [17] proposed a novel approach to the quasi Garima distribution, examining its properties and applications. Yerneni and Rather [18] also obtained a novel model called the weighted quasi Suja distribution and studied its properties and applications. Rather and Subramaniam [19] derived a new exponentiated distribution with engineering science applications. Ahmad et al. [20] developed a new statistical approach to COVID-19 variability using the Weibull inverse Nadarjah Haghighi distribution. Ahmad et al. [21] obtained a novel family of probability generating distributions and studied their properties along with data analysis. A novel extension of power Lindley [22] as well as Rayleigh [23] distribution was derived and studied by Qayoom et al.

While extensive work has focused on continuous distributions over unbounded intervals, bounded distributions on (0,1) have not been explored to the same extent. Since many practical data sets frequently involve bounded measurements such as proportions, rates, normalized indicators, and dimensionless quantities, it is essential to develop more flexible and robust models specifically tailored for bounded data. Among bounded probability distributions, the Topp-Leone distribution has been widely studied due to its analytical simplicity and ability to represent a variety of distributional shapes. Several studies have explored the development, properties, and applications of the Topp-Leone distribution and its extensions. Some notable contributions include, Topp and Leone [24] introducing a family of J-shaped frequency functions, laying the foundational framework for the Topp-Leone distribution. Francesca and Filippo [25] proposed the reflected generalized Topp-Leone power series distribution, extending its flexibility for modelling bounded data. Al-Zahrani [26] investigated goodness-of-fit procedures for the Topp-Leone distribution. A Topp-Leone generator of distributions, and its properties and inferences was studied by Sangsanit and Bodhisuwan [27]. Ahmad et al. [28] introduced the exponentiated Topp-Leone distribution and examined its statistical properties in detail. Kotz and Seier [29] analyzed the kurtosis characteristics of the Topp-Leone distribution, providing further insight into its shape behaviour. Tuoyo et al. [30] developed the Topp-Leone Weibull distribution and demonstrated its applicability to real-life data. Behairy et al. [31] proposed the Topp-Leone inverted Kumaraswamy distribution and studied its properties, parameter estimation techniques, and predictive performance. Sudsila et al. [32] introduced generalized distributions on the unit interval based on the T-Topp-Leone family, enhancing modelling flexibility for bounded datasets. More recently, Arshad and Jamal [33] focused on statistical inference for the Topp-Leone generated family of distributions using record values. Its bounded support makes it suitable for modelling both medical and engineering data defined on the unit interval. However, despite its usefulness, the classical Topp-Leone distribution offers limited flexibility in capturing complex skewness patterns and tail behaviours commonly observed in real-world datasets from both domains.

To address these limitations, motivated by the fact that many researchers have applied various transformation techniques to develop more robust and flexible models capable of effectively adapting to diverse data structures, in this article, we propose an extension of the Topp-Leone distribution based on the power generalized Dinesh Uma Singh (PGDUS) transformation originally introduced by Thomas [34]. However, it is important to note that the proposed PGDUS transformation on the Topp-

Leone distribution differs conceptually from existing Topp-Leone generalization families. Many existing models in the literature are constructed by developing new generator mechanisms based on the Topp-Leone family itself. In contrast, the present work applies the PGDUS generator transformation directly to the classical Topp-Leone distribution as a baseline model. Consequently, the PGDUS-transformed Topp-Leone distribution represents a distinct generator-based extension that provides additional flexibility in modelling bounded data on (0,1).

The PGDUS transformation has been widely employed to construct flexible and robust probability models for effective analysis and interpretation of real-world data. Several researchers have contributed to the development and application of PGDUS-based models, including Amruta and Chacko [35], with Thomas and Chacko [36] demonstrating its versatility and effectiveness across diverse applied settings. These studies motivate the present work, which aims to further enhance the modelling capability of the Topp-Leone distribution through the PGDUS framework, which is not yet explored.

This article is organized as follows. Section 2 introduces the PGDUS-Topp-Leone (PGDUS-TL) model. In Section 3, the statistical properties of the proposed distribution are derived. Sections 4 and 5 discuss entropy measures and the Bonferroni curve, respectively. In Section 6, various parameter estimation methods for the PGDUS-TL distribution are explained. Extensive simulation studies are presented in Section 7, to assess the performance of the proposed estimators. Section 8 illustrates the applicability and effectiveness of the PGDUS-TL model using real datasets from medical and engineering domains. Finally, concluding remarks are made in Section 9.

2. An extension of Topp-Leone model

The Topp-Leone distribution is a continuous probability distribution defined on the unit interval (0,1), originally introduced to model, J-shaped and reversed J-shaped data. It is characterised by a simple functional form with flexible shape behaviour, making it useful for modelling proportions, rates, and reliability data.

Let X be a random variable following the Topp-Leone distribution with parameter α, i.e., $X \sim TL(\alpha)$, then its probability density function (PDF) [27] is given by

$$f_X(x) = 2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}, \tag{1}$$

where $0 < x < 1$ and $\alpha > 0$. The corresponding cumulative distribution function (CDF) is given by

$$F_X(x) = x^{\alpha}(2-x)^{\alpha} . \tag{2}$$

If X is a random variable under consideration with its CDF as $F(x)$ and PDF as $f(x)$, then the CDF of the proposed PGDUS distribution [35, 34] is defined as

$$\Psi(x) = \left(\frac{e^{F(x)}-1}{e-1}\right)^{\beta}, \ \beta > 0, x > 0 \ , \tag{3}$$

and the corresponding PDF is

$$\psi(x) = \frac{\beta}{(e-1)^{\beta}}\left(e^{F(x)}-1\right)^{\beta-1} e^{F(x)} f(x), \ \beta > 0, x > 0. \tag{4}$$

Theorem 1. Validity of the PGDUS transformation

Let $F(x)$ be a CDF of a baseline distribution. Then the PGDUS transformation defined by

$$\Psi(x) = \left(\frac{e^{F(x)}-1}{e-1}\right)^{\beta}, \quad \beta > 0, x > 0,$$

is a valid CDF and defines a proper probability distribution.

Proof. F(x) is a CDF of any baseline probability distribution, i.e.,

$$F(-\infty) = 0, and\ F(\infty) = 1.$$

Now from the PGDUS transformation, we have

$$\Psi(x) = \left(\frac{e^{F(x)}-1}{e-1}\right)^{\beta}, \ \beta > 0, x > 0.$$

Then

$$\lim_{x\to-\infty} \Psi(x) = \lim_{x\to-\infty} \left(\frac{e^{F(x)}-1}{e-1}\right)^{\beta},$$

$$\lim_{x\to-\infty} \Psi(x) = \left(\frac{e^{0}-1}{e-1}\right)^{\beta},$$

$$\lim_{x\to-\infty} \Psi(x) = 0.$$

Similarly,

$$\lim_{x\to+\infty} \Psi(x) = \lim_{x\to+\infty} \left(\frac{e^{F(x)}-1}{e-1}\right)^{\beta},$$

$$\lim_{x\to+\infty} \Psi(x) = \left(\frac{e^{1}-1}{e-1}\right)^{\beta},$$

$$\lim_{x\to+\infty} \Psi(x) = 1.$$

Since $\lim_{x\to-\infty} \Psi(x) = 0$ and $\lim_{x\to+\infty} \Psi(x) = 1$, $\Psi(x)$ is non-decreasing and continuous. Therefore, $\Psi(x)$ is a valid CDF.

Now, differentiating $\Psi(x)$ with respect to x, we get the PDF as

$$\Psi'(x) = \frac{\beta}{(e-1)^{\beta}}\left(e^{F(x)} - 1\right)^{\beta-1} e^{F(x)} f(x),$$

where $f(x)$ is the baseline PDF. Since all terms are non-negative for $\beta > 0$, $\Psi'(x) \geq 0$.

Hence, $\Psi(x)$ is non-decreasing. Now, for $x_1 \leq x_2, F(x_1) \leq F(x_2)$ as $F(x)$ is a CDF. Since $\Psi(x)$ is non-decreasing, $\Psi(F(x_1)) \leq \Psi(F(x_2))$. Hence, $\Psi(x_1) \leq \Psi(x_2)$ for all $x_1 \leq x_2$. Therefore, $\Psi(x)$ satisfies the monotonicity property of a CDF. Hence, the PGDUS transformation defines a proper probability distribution. Considering the Topp-Leone distribution as a baseline distribution and substituting Eqs (1) and (2) in Eqs (3) and (4) accordingly, we get a CDF and PDF of the PGDUS-TL distribution as

$$\Psi(x) = \left(\frac{e^{x^{\alpha}(2-x)^{\alpha}}-1}{e-1}\right)^{\beta}, \quad \beta > 0, \alpha > 0, \ 0 < x < 1, \tag{5}$$

$$\psi(x) = \frac{2\alpha\beta}{(e-1)^{\beta}}\left(e^{x^{\alpha}(2-x)^{\alpha}} - 1\right)^{\beta-1} e^{x^{\alpha}(2-x)^{\alpha}} x^{\alpha-1}(1-x)(2-x)^{\alpha-1}, \tag{6}$$

$$\beta > 0, \alpha > 0, \ 0 < x < 1.$$

When we take the limit of Eq (5) for x tending to 1, it results to 1. Also, the CDF curve of the PGDUS-TL distribution in Figure 1 reveals that curve stretches in between 0 and 1, proving that Eqs (5) and (6) are valid probability functions, hence PGDUS-TL is a valid probability distribution. From the graphical representation of the CDF of the PGDUS-TL distribution, it can be seen that as α and β values increase, the curve shifts toward right. When α is fixed and β increases, the CDF rises more sharply. In contrast, for constant β, increasing α results in a slower rise of the curve, indicating greater spread and a reduced probability of X taking values below a given x.

The PDF plot of the PGDUS-TL distribution in Figure 2 indicates that the distribution exhibits slight negative skewness, with the curve extending more toward the left. For fixed values of α and varying β, the spread and peak of the curve remain unchanged. In contrast, when β is held constant and α increases, the curve becomes more dispersed and more peaked. These observations suggest that α acts as both scale and shape parameters of the distribution, whereas β primarily acts as a location parameter.

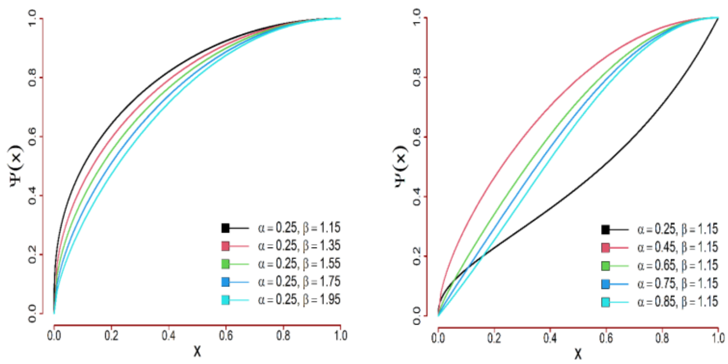

Figure 1. CDF plot of PGDUS-TL distribution.

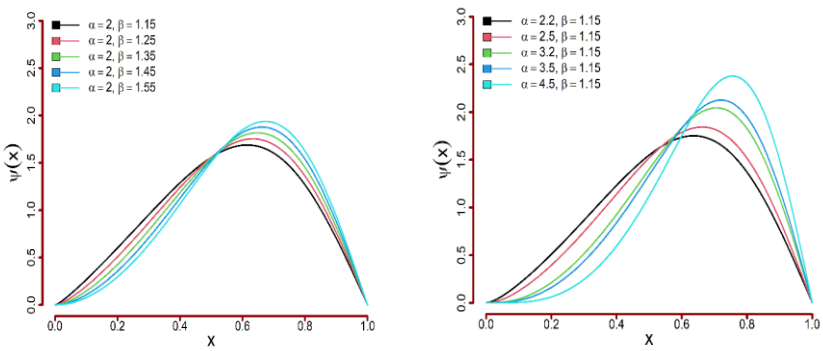

Figure 2. PDF plot of PGDUS-TL distribution.

3. Statistical properties

3.1. Quantile function

The quantile function, defined as the inverse of the CDF, provides a complete characterization of a probability distribution; it describes features such as location, dispersion, and tail behaviours of the distribution and remains well-defined even for distributions lacking finite moments. It is widely used in statistical inference, simulation via the inverse transform method, and reliability and risk analysis.

Given a CDF $F_X(x) = \mathrm{P}(X \le x) = \mathrm{u}$ such that $\mathrm{u} \in (0,1)$ of a continuous random variable X, then the quantile function denoted by $\mathrm{Q}(u)$ for $\mathrm{u} \in (0,1)$ is defined as $\mathrm{Q}(u) = F_X^{-1}(u)$.

For the PGDUS-TL distribution, the quantile function is obtained by determining the value of $X = x$ for which $\Psi_X(x) = \mathrm{u}$, that is,

$$\left[\frac{e^{(x^{\alpha}(2-x)^{\alpha})}-1}{e-1}\right]^{\beta} = \mathrm{u}.$$

Taking log on both sides,

$$\beta \ln\left[\frac{e^{x^{\alpha}(2-x)^{\alpha}}-1}{e-1}\right] = \ln u,\ \ln\left[\frac{e^{x^{\alpha}(2-x)^{\alpha}}-1}{e-1}\right] = \frac{\ln u}{\beta},\ \frac{e^{x^{\alpha}(2-x)^{\alpha}}-1}{e-1} = e^{\frac{\ln u}{\beta}},$$

$$[x(2-x)]^{\alpha} = \ln\left\{1 + (e-1)e^{\frac{\ln u}{\beta}}\right\},\ x(2-x) = \left(\ln\left\{1 + (e-1)e^{\frac{\ln u}{\beta}}\right\}\right)^{\frac{1}{\alpha}}.$$

On substituting,

$$\mathrm{z} = \left(\ln\left\{1 + (e-1)e^{\frac{\ln u}{\beta}}\right\}\right)^{\frac{1}{\alpha}},\ x^2 - 2x + z = 0.$$

On solving the above quadratic equation, we get $\mathrm{x} = 1 \pm \sqrt{1-z}$. Nevertheless, the root corresponding to the positive sign is not feasible since it results in $x > 1$, which is beyond the range of the distribution. Therefore,

$$\mathrm{x} = 1 - \sqrt{1-z},\ \mathrm{x} = 1 - \sqrt{1 - \left(\ln\left\{1 + (e-1)e^{\frac{\ln u}{\beta}}\right\}\right)^{\frac{1}{\alpha}}}.$$

3.2. Moments

Moments are important statistical measures that characterize the fundamental properties of a probability distribution. The first and second moments represent mean and variance, while higher-order moments account for skewness and kurtosis. These measures offer valuable insight into the shape and variability of the distribution.

We will use the following generalized binomial expansion and Taylor series expansion for deriving the moments and other properties.

$$(1-z)^{\alpha} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} z^j \quad |z| < 1,\ \alpha \text{ is any real number},$$

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!},$$

and the r^{th} moment about the origin of the given model is

$$\mu_r' = \int_0^1 x^r \psi(x) dx.$$

By substituting Eq (6) in the above expression, we get

$$\mu_r' = \int_0^1 x^r \frac{2\alpha\beta}{(e-1)^\beta} \left(e^{x^\alpha(2-x)^\alpha} - 1\right)^{\beta-1} e^{x^\alpha(2-x)^\alpha} x^{\alpha-1}(1-x)(2-x)^{\alpha-1} dx,$$

$$= \frac{2\alpha\beta}{(e-1)^\beta} \int_0^1 x^{r+\alpha-1}(1-x)(2-x)^{\alpha-1} e^{x^\alpha(2-x)^\alpha} e^{x^\alpha(2-x)^\alpha(\beta-1)} \left(1 - \frac{1}{e^{x^\alpha(2-x)^\alpha}}\right)^{\beta-1} dx,$$

$$= \frac{2\alpha\beta}{(e-1)^\beta} \int_0^1 x^{r+\alpha-1}(1-x)(2-x)^{\alpha-1} e^{x^\alpha(2-x)^\alpha} e^{x^\alpha(2-x)^\alpha(\beta-1)} \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \left(e^{x^\alpha(2-x)^\alpha}\right)^{-k} dx,$$

$$= \frac{2\alpha\beta}{(e-1)^\beta} \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \int_0^1 x^{r+\alpha-1}(1-x)(2-x)^{\alpha-1} e^{x^\alpha(2-x)^\alpha(\beta-k)} dx.$$

On substituting $y = x^\alpha(2-x)^\alpha,\ a = \beta - k$ for simplification purposes

$$= \frac{2\alpha\beta}{(e-1)^\beta} \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \int_0^1 x^{r+\alpha-1}(1-x)(2-x)^{\alpha-1} e^{ay} dx,$$

$$= \frac{2\alpha\beta}{(e-1)^\beta} \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \int_0^1 x^{r+\alpha-1}(1-x)(2-x)^{\alpha-1} \sum_{j=0}^{\infty} \frac{(ay)^j}{j!} dx,$$

$$= \frac{2\alpha\beta}{(e-1)^\beta} \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \sum_{j=0}^{\infty} \frac{a^j}{j!} \int_0^1 x^{r+\alpha-1}(1-x)(2-x)^{\alpha-1} y^j dx.$$

On re-substituting $y = x^\alpha(2-x)^\alpha,\ a = \beta - k,$

$$= \frac{2\alpha\beta}{(e-1)^\beta} \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \sum_{j=0}^{\infty} \frac{(\beta-k)^j}{j!} \int_0^1 x^{r+\alpha-1}(1-x)(2-x)^{\alpha-1} x^{\alpha j}(2-x)^{\alpha j} dx,$$

$$= \frac{2\alpha\beta}{(e-1)^\beta} \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \sum_{j=0}^{\infty} \frac{(\beta-k)^j}{j!} \int_0^1 x^{r+\alpha+\alpha j-1}(1-x)(2-x)^{\alpha+\alpha j-1} dx,$$

$$= \frac{2\alpha\beta}{(e-1)^\beta} \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \sum_{j=0}^{\infty} \frac{(\beta-k)^j}{j!} \int_0^1 x^{r+\alpha+\alpha j-1}(1-x) 2^{\alpha+\alpha j-1} \left(1 - \frac{x}{2}\right)^{\alpha+\alpha j-1} dx,$$

$$= \frac{2\alpha\beta}{(e-1)^\beta} \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \sum_{j=0}^{\infty} \frac{(\beta-k)^j}{j!} \int_0^1 x^{r+\alpha+\alpha j-1}(1-x) 2^{\alpha+\alpha j-1} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+\alpha j-1}{i} \left(\frac{x}{2}\right)^i dx,$$

$$= \frac{2\alpha\beta}{(e-1)^\beta} \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} \sum_{j=0}^{\infty} \frac{(\beta-k)^j}{j!} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha+\alpha j-1}{i} 2^{\alpha+\alpha j-1-i} \int_0^1 x^{r+\alpha+\alpha j+i-1}(1-x)^{2-1} dx.$$

Using the beta integral and the relation between the beta and gamma function,

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1} dx = B(\mu,\nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}.$$

The above moment expression simplifies to

$$\mu_r' = \frac{2\alpha\beta}{(e-1)^\beta}\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^k\sum_{j=0}^{\infty}\frac{(\beta-k)^j}{j!}\sum_{i=0}^{\infty}\binom{\alpha+\alpha j-1}{i}\times 2^{\alpha+\alpha j-1-i}(-1)^i\frac{\Gamma(r+\alpha+\alpha j+i)}{\Gamma(r+\alpha+\alpha j+i+2)}. \quad (7)$$

Putting $r = 1, 2, 3,$ and 4 in Eq (7), we obtain first four moments about the origin,

$$\mu_1' = \frac{2\alpha\beta}{(e-1)^\beta}\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^k\sum_{j=0}^{\infty}\frac{(\beta-k)^j}{j!}\sum_{i=0}^{\infty}\binom{\alpha+\alpha j-1}{i}\times 2^{\alpha+\alpha j-1-i}(-1)^i\frac{\Gamma(1+\alpha+\alpha j+i)}{\Gamma(3+\alpha+\alpha j+i)}, \quad (8)$$

$$\mu_2' = \frac{2\alpha\beta}{(e-1)^\beta}\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^k\sum_{j=0}^{\infty}\frac{(\beta-k)^j}{j!}\sum_{i=0}^{\infty}\binom{\alpha+\alpha j-1}{i}\times 2^{\alpha+\alpha j-1-i}(-1)^i\frac{\Gamma(2+\alpha+\alpha j+i)}{\Gamma(4+\alpha+\alpha j+i)},$$

$$\mu_3' = \frac{2\alpha\beta}{(e-1)^\beta}\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^k\sum_{j=0}^{\infty}\frac{(\beta-k)^j}{j!}\sum_{i=0}^{\infty}\binom{\alpha+\alpha j-1}{i}\times 2^{\alpha+\alpha j-1-i}(-1)^i\frac{\Gamma(3+\alpha+\alpha j+i)}{\Gamma(5+\alpha+\alpha j+i)},$$

$$\mu_4' = \frac{2\alpha\beta}{(e-1)^\beta}\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^k\sum_{j=0}^{\infty}\frac{(\beta-k)^j}{j!}\sum_{i=0}^{\infty}\binom{\alpha+\alpha j-1}{i}\times 2^{\alpha+\alpha j-1-i}(-1)^i\frac{\Gamma(4+\alpha+\alpha j+i)}{\Gamma(6+\alpha+\alpha j+i)}.$$

So, the variance (μ_2) for the given model is given by

$$\begin{aligned}\mu_2 &= \left[\frac{2\alpha\beta}{(e-1)^\beta}\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^k\sum_{j=0}^{\infty}\frac{(\beta-k)^j}{j!}\sum_{i=0}^{\infty}\binom{\alpha+\alpha j-1}{i}2^{\alpha+\alpha j-1-i}(-1)^i\frac{\Gamma(2+\alpha+\alpha j+i)}{\Gamma(4+\alpha+\alpha j+i)}\right] \\ &- \left[\frac{2\alpha\beta}{(e-1)^\beta}\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^k\sum_{j=0}^{\infty}\frac{(\beta-k)^j}{j!}\sum_{i=0}^{\infty}\binom{\alpha+\alpha j-1}{i}2^{\alpha+\alpha j-1-i}(-1)^i\frac{\Gamma(1+\alpha+\alpha j+i)}{\Gamma(3+\alpha+\alpha j+i)}\right]^2.\end{aligned}$$

Moreover, the coefficient of skewness (γ_1) and the coefficient of kurtosis (γ_2) for the probability model are given by

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3}{(\mu_2)^{3/2}},$$

$$\gamma_2 = \frac{\mu_4}{(\mu_2)^2} - 3 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4}{(\mu_2)^2} - 3.$$

Behaviour of the mean, variance, coefficient of skewness, and kurtosis for the PGDUS-TL distribution for different values of the parameter involved in the distribution is presented in Table 1. Graphical behaviour of these measures is shown in Figures 3–6. It can be observed that as the values of α and β increase, the mean of the distribution increases, while the variance decreases. Simultaneously, the distribution becomes more negatively skewed and exhibits platykurtic behaviour, indicating a flatter peak and lighter tails. However, one unusual magnitude of skewness and kurtosis in Table 1. is due to the specific parameter configuration and the corresponding shape characteristics of the distribution rather than from computational instability.

Table 1. Summary measures for selected parameter values of the distribution.

Parameter α	β	Mean (μ_1')	μ_2'	μ_3'	μ_4'	Variance (σ^2)	Skewness (γ_1)	Kurtosis (γ_2)
1	2	0.5395	0.3358	0.2280	0.1643	0.0447	-0.1449	-0.7292
	3	0.6134	0.4108	0.2924	0.2177	0.0345	-0.3066	-0.5075
	4	0.6607	0.4643	0.3417	0.2606	0.0278	-0.3912	-0.3512
	5	0.6941	0.5050	0.3812	0.2963	0.0232	-0.4421	-0.2430
1	2	0.5395	0.3358	0.2280	0.1643	0.0447	-0.1449	-0.7292
1.5		0.6091	0.4071	0.2896	0.2156	0.0360	-0.3317	-0.4816
2		0.6550	0.4588	0.3373	0.2573	0.0297	-0.4244	-0.3336
2.5		0.7140	0.5243	0.4014	0.3176	0.0145	3.6176	-25.346

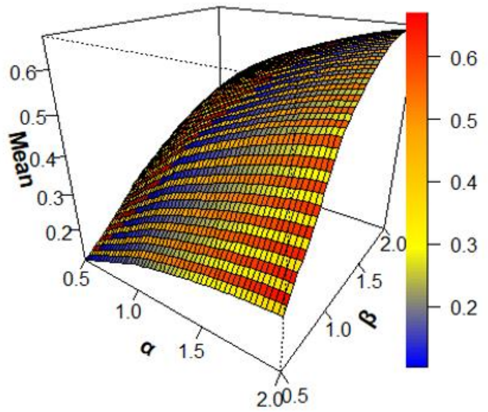

Figure 3. Mean plot of PGDUS-TL distribution.

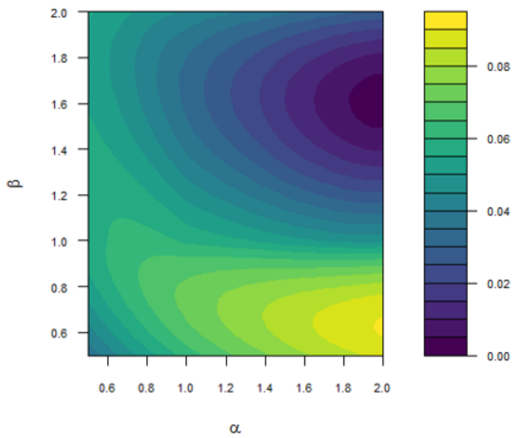

Figure 4. Variance plot of PGDUS-TL distribution.

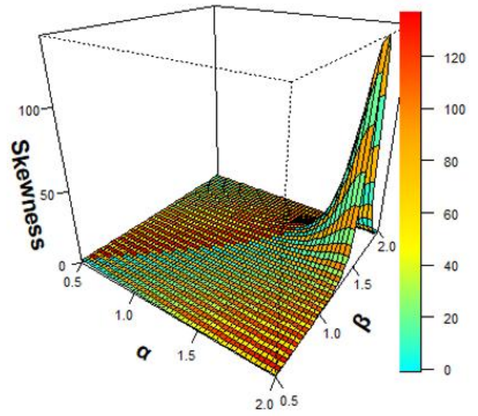

Figure 5. Skewness plot of PGDUS-TL distribution.

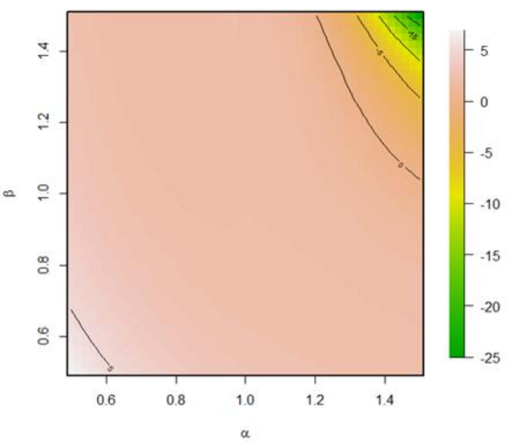

Figure 6. Kurtosis plot of PGDUS-TL distribution.

3.3. Incomplete moments

The r^{th} incomplete moment about the origin for the PGDUS-TL distribution is given by

$$\mathrm{m}_r' = \int_0^t x^r \psi(x)dx.$$

On using Eq (6) in the above expression,

$$\mathrm{m}_r' = \int_0^t x^r \frac{2\alpha\,\beta}{(e-1)^{\beta}}\left(e^{x^{\alpha}(2-x)^{\alpha}}-1\right)^{\beta-1} e^{x^{\alpha}(2-x)^{\alpha}} x^{\alpha-1}(1-\,x)(2\,-\,x)^{\alpha-1}dx.$$

Using generalized binomial expansion and the Taylor series expansion and simplifying the above expression similar to the moments expression, we get

$$= \frac{2\alpha\beta}{(e-1)^{\beta}} \sum_{k=0}^{\infty}(-1)^k\binom{\beta-1}{k}\sum_{j=0}^{\infty}\frac{(\beta-k)^j}{j!}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha+\alpha j-1}{i}2^{\alpha+\alpha j-1-i}\int_0^t x^{r+\alpha+\alpha j+i-1}(1-x)^{2-1}dx.$$

Substituting $p = r + \alpha + \alpha\mathrm{j} + i,$

$$= \frac{2\alpha\beta}{(e-1)^{\beta}} \sum_{k=0}^{\infty} (-1)^{k} \binom{\beta-1}{k} \sum_{j=0}^{\infty} \frac{(\beta-k)^{j}}{j!} \sum_{i=0}^{\infty} (-1)^{i} \binom{\alpha+\alpha j-1}{i} 2^{\alpha+\alpha j-1-i} \int_{0}^{t} x^{p-1} (1-x)^{2-1} dx,$$

$$\mathrm{m}_{r}' = \frac{2\alpha\beta}{(e-1)^{\beta}} \sum_{k=0}^{\infty} \binom{\beta-1}{k} (-1)^{k} \sum_{j=0}^{\infty} \frac{(\beta-k)^{j}}{j!} \sum_{i=0}^{\infty} \binom{\alpha+\alpha j-1}{i} 2^{\alpha+\alpha j-1-i} (-1)^{i} \mathrm{B}_{t}(p,2). \tag{9}$$

$\mathrm{B}_{t}(p,2)$ denotes the lower incomplete beta function defined as $\int_{0}^{t} x^{p-1}(1-x)dx, 0 < x < 1$.

3.4. Moment generating function

The moment generating function of the given model is computed as

$$M_X(t) = E(e^{tx}) = \int_0^1 e^{tx} \psi(x) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r).$$

Since, $\mu_r' = E(X^r)$, using Eq (7) in the above expression, we get moment generating function as

$$M_X(t) = \frac{2\alpha\beta}{(e-1)^{\beta}} \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k=0}^{\infty} \binom{\beta-1}{k} (-1)^k \sum_{j=0}^{\infty} \frac{(\beta-k)^j}{j!} \sum_{i=0}^{\infty} \binom{\alpha+\alpha j-1}{i} 2^{\alpha+\alpha j-1-i} (-1)^i \frac{\Gamma(r+\alpha+\alpha j+i)}{\Gamma(r+\alpha+\alpha j+i+2)}.$$

The characteristic function is given by

$$\varphi_X(t) = E\left(e^{itx}\right) = \int_0^1 e^{itx} \psi(x) dx = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} E(X^r).$$

Using Eq (7) for $E(X^r)$, the characteristic function simplifies to

$$\varphi_X(t) = \frac{2\alpha\beta}{(e-1)^{\beta}} \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \sum_{k=0}^{\infty} \binom{\beta-1}{k} (-1)^k \sum_{j=0}^{\infty} \frac{(\beta-k)^j}{j!} \sum_{i=0}^{\infty} \binom{\alpha+\alpha j-1}{i} 2^{\alpha+\alpha j-1-i} (-1)^i \frac{\Gamma(r+\alpha+\alpha j+i)}{\Gamma(r+\alpha+\alpha j+i+2)}.$$

The cumulant generating function for the proposed model is given by $K_X(t) = \log(M_X(t))$, where $M_X(t)$ is the moment generating function of a random variable X. Hence, the cumulant generating function of the PGDUS-TL model simplifies to

$$\mathrm{K}_{X}(t) = \log\left(\frac{2\alpha\beta}{(e-1)^{\beta}} \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k=0}^{\infty} \binom{\beta-1}{k} (-1)^k \sum_{j=0}^{\infty} \frac{(\beta-k)^j}{j!} \sum_{i=0}^{\infty} \binom{\alpha+\alpha j-1}{i} 2^{\alpha+\alpha j-1-i} (-1)^i \frac{\Gamma(r+\alpha+\alpha j+i)}{\Gamma(r+\alpha+\alpha j+i+2)}\right).$$

3.5. Reliability measures

The reliability behaviour of a probability distribution is characterized through the survival and hazard rate functions. The survival function quantifies the probability that a system or component remains operational beyond a certain time. The hazard rate function describes the instantaneous rate of failure at a particular time, conditional on survival up to that time.

The survival function for the PGDUS-TL model is $S_X(x) = 1 - \Psi_X(x)$. On using Eq (5),

$$S_X(x) = 1 - \left(\frac{e^{x^{\alpha}(2-x)^{\alpha}} - 1}{e-1}\right)^{\beta}.$$

The hazard rate is expressed as the ratio of the PDF and survival function. The hazard rate of the PGDUS-TL model is given as

$$H_X(x) = \frac{\psi_X(x)}{S_X(x)}.$$

On substituting Eq (6) with the expression expression of $S_X(x)$and simplifying, we get,

$$H_X(x) = \frac{2\alpha\beta\left(e^{x^{\alpha}(2-x)^{\alpha}}-1\right)^{\beta-1} e^{x^{\alpha}(2-x)^{\alpha}} (x)^{\alpha-1}(1-x)(2-x)^{\alpha-1}}{(e-1)^{\beta}-\left(e^{x^{\alpha}(2-x)^{\alpha}}-1\right)^{\beta}}.$$

The graphical behaviour of the survival function for different parameters is shown in Figures 7–9. From the graph, it can be observed that for lower values of α and β, the survival function declines sharply, indicating that failures occur early and survival probability decreases rapidly, which means that the system has short life span. As the values of α and β increase, the survival function shifts slightly to the right, indicating that failures are spread over a longer time and survival remains high for a longer time. Similar patterns are observed in the three-dimensional contour plots. In Figure 8, the parameter α is held constant while β varies, whereas in Figure 9, β is fixed, and α is allowed to vary. The contour shading represents the magnitude of the probability values, where lighter shades correspond to higher probabilities and darker shades indicate lower probabilities.

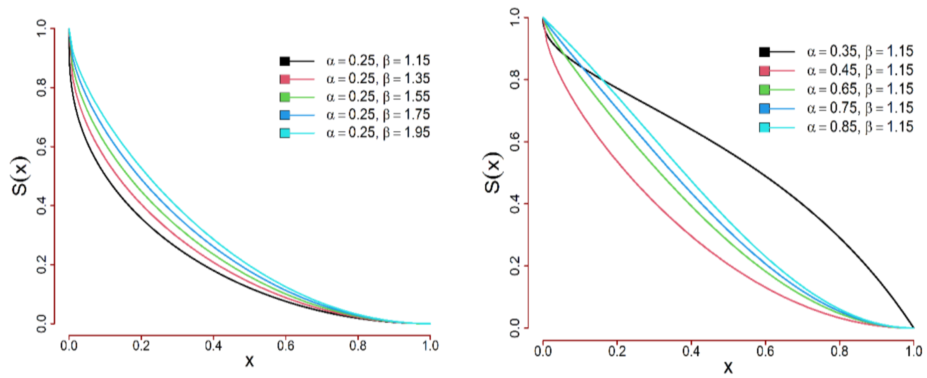

Figure 7. Survival function plot of PGDUS-TL distribution.

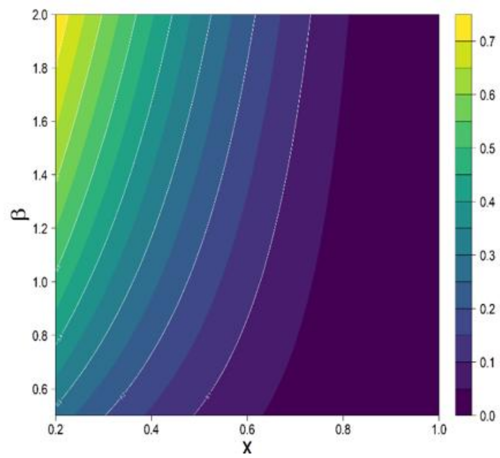

Figure 8. Survival surface plot of PGDUS-TL distribution (α=0.45).

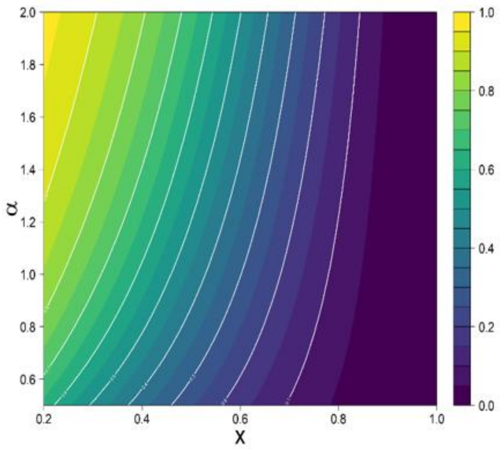

Figure 9. Survival surface plot of PGDUS-TL distribution (β = 1.35).

The graphical behaviour of the hazard rate for different parameter values shown in Figure 10 exhibits decreasing, constant, and increasing shapes. A decreasing hazard rate indicates early-life failures, a constant hazard rate reflects a memoryless failure mechanism, and an increasing hazard rate corresponds to aging or wear-out effects. This behaviour demonstrates the flexibility of the distribution in capturing different reliability patterns observed in practical lifetime data. The hazard contour plot in Figures 11 and 12 shows a clear increasing hazard rate as x approaches the upper bound, indicating strong ageing or wear-out behaviour. The closely spaced contours near $x = 1$ reflect rapid growth in risk, while the largely vertical contour pattern suggests that the hazard is primarily driven by the time variable, with the parameter mainly affecting its magnitude. This behaviour highlights the model's suitability for medical survival and engineering reliability data exhibiting risk over time.

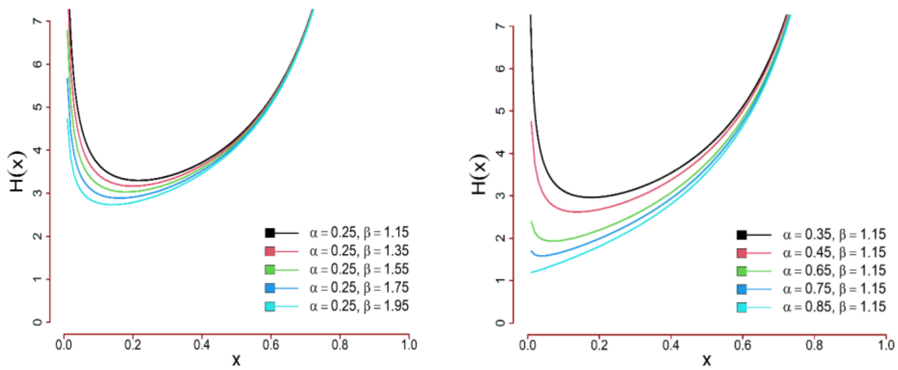

Figure 10. Hazard rate plot of PGDUS-TL distribution.

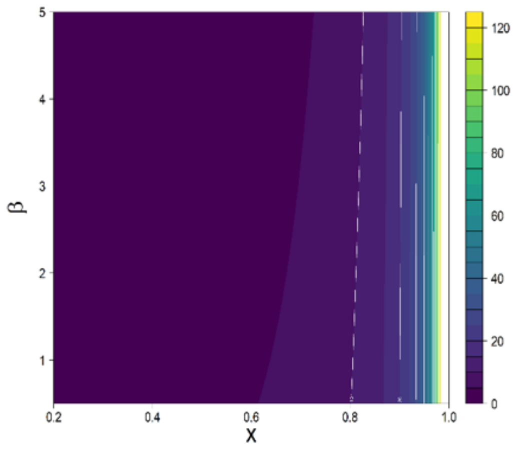

Figure 11. Hazard surface of PGDUS-TL distribution (α=1.25).

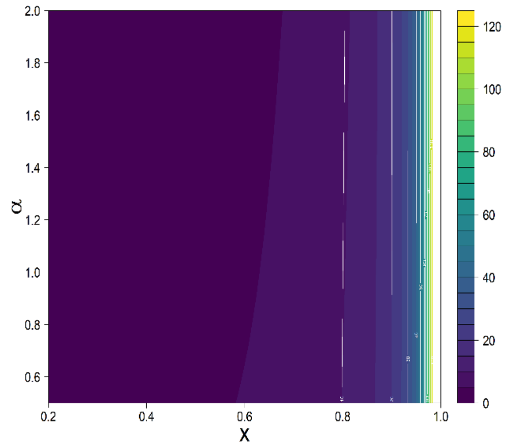

Figure 12. Hazard surface of PGDUS-TL distribution (β=0.45).

3.6. Order statistics

Let $X_1, X_2, \ldots \ldots, X_n$ be a random sample of size *n* drawn from the PGDUS-TL distribution, with corresponding values $x_1, x_2, \ldots \ldots, x_n$. Order statistics represent the sample observations arranged in ascending order. Accordingly, the ordered random variables are denoted by $X_{(1)} \leq X_{(2)} \leq \cdots \ldots \leq X_{(n)}$, where $X_{(1)} = min(X_1, X_2, \ldots \ldots, X_n)$ and $X_{(n)} = max(X_1, X_2, \ldots \ldots, X_n)$ correspond to the minimum and maximum order statistics, respectively.

The PDF of the r^{th} ordered statistics, $X_{(r)}$ for the PGDUS-TL distribution, is given by

$$\psi_{X_{(r)}}(x) = \frac{n!}{(n-r)!(r-1)!} \psi_X(x) \big(\Psi_X(x)\big)^{r-1} \big(1 - \Psi_X(x)\big)^{n-r}.$$

On using Eqs (5) and (6), the PDF expression of the r^{th} ordered statistics becomes

$$\psi_{X_{(r)}}(x) = \frac{n!}{(n-r)!\,(r-1)!} \left[\frac{2\alpha\beta}{(e-1)^\beta} \left(e^{x^\alpha (2-x)^\alpha} - 1 \right)^{\beta-1} e^{x^\alpha (2-x)^\alpha} x^{\alpha-1} (1-x)(2-x)^{\alpha-1} \right]$$
$$\times \left[\left(\frac{e^{x^\alpha (2-x)^\alpha} - 1}{e-1} \right)^{\beta} \right]^{r-1} \left[1 - \left(\frac{e^{x^\alpha (2-x)^\alpha} - 1}{e-1} \right)^{\beta} \right]^{n-r}. \tag{10}$$

On substituting $r = 1$ and n in Eq (10), we get the PDF of minimum and maximum ordered statistics respectively for the given model, which are expressed as

$$\psi_{X_{(1)}}(x) = \frac{2n\alpha\beta}{(e-1)^{\beta}}\left[\left(e^{x^{\alpha}(2-x)^{\alpha}} - 1\right)^{\beta-1} e^{x^{\alpha}(2-x)^{\alpha}} x^{\alpha-1}(1-x)(2-x)^{\alpha-1}\right]\left[1 - \left(\frac{e^{x^{\alpha}(2-x)^{\alpha}}-1}{e-1}\right)^{\beta}\right]^{n-1},$$

$$\psi_{X_{(n)}}(x) = \frac{2n\alpha\beta}{(e-1)^{\beta}}\left[\left(e^{x^{\alpha}(2-x)^{\alpha}} - 1\right)^{\beta-1} e^{x^{\alpha}(2-x)^{\alpha}} x^{\alpha-1}(1-x)(2-x)^{\alpha-1}\right]\left[\left(\frac{e^{x^{\alpha}(2-x)^{\alpha}}-1}{e-1}\right)^{\beta}\right]^{n-1}.$$

4. Entropy measures

Entropy is a fundamental measure used to quantify the uncertainty or randomness associated with a probability distribution. It reflects the amount of information contained in a random variable and provides insight into the variability and structural characteristics of statistical models. Entropy measures find wide utility in diverse fields such as reliability analysis, survival studies, economics, finance, signal processing and information theory, machine learning, engineering, and many more, making it essential for both theoretical exploration and practical implementation in various domains. In this section, Renyi's [37] and Tsalli's [38] measures of entropy are explored based on the PGDUS-TL distribution.

The Renyi entropy for the given model is

$$R(\theta) = \frac{1}{1-\theta} log\left(\int_0^1 \left(\psi(x)\right)^{\theta} dx\right) \quad \theta \neq 1,$$

$$R(\theta) = \frac{1}{1-\theta} log\left(\int_0^1 \left(\frac{2\alpha\beta}{(e-1)^{\beta}}\left(e^{x^{\alpha}(2-x)^{\alpha}} - 1\right)^{\beta-1} e^{x^{\alpha}(2-x)^{\alpha}} x^{\alpha-1}(1-x)(2-x)^{\alpha-1}\right)^{\theta} dx\right),$$

$$\mathrm{R}(\theta) = \frac{1}{1-\theta}\log\left(\int_0^1 \frac{2\alpha\beta\theta}{(e-1)^{\beta\theta}}\left(e^{x^{\alpha\theta}(2-x)^{\alpha\theta}} - 1\right)^{\beta\theta-\theta} e^{x^{\alpha\theta}(2-x)^{\alpha\theta}} x^{\alpha\theta-\theta}(1-x)^{\theta}(2-x)^{\alpha\theta-\theta} dx\right).$$

Using the generalized binomial expansion $(1-z)^{\alpha} = \sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}z^j \; |z| < 1$, and the Taylor series expansion $e^x = \sum_{j=0}^{\infty}\frac{x^j}{j!}$ and simplifying, we get the Renyi entropy for the proposed model as

$$\mathrm{R}(\theta) = \frac{1}{1-\theta}\log\left\{\frac{(2\alpha\beta)^{\theta}}{(e-1)^{\beta\theta}}\sum_{k=0}^{\infty}\binom{\beta\theta-\theta}{k}(-1)^k\sum_{j=0}^{\infty}\frac{(\theta\beta-k)^j}{j!}\sum_{i=0}^{\infty}\binom{\theta\alpha-\theta+\alpha j}{i}2^{\theta\alpha-\theta+\alpha j-i}(-1)^i\frac{\Gamma(\theta\alpha-\theta+\alpha j+i+1)\Gamma(\theta+1)}{\Gamma(\theta\alpha+\alpha j+i+2)}\right\}.$$

The Tsalli entropy for the given distribution is

$$T(\kappa) = \frac{1}{\kappa-1}\left(1 - \int_0^1\left(\psi(x)\right)^{\theta} dx\right) \; \kappa \neq 1,$$

$$T(\kappa) = \frac{1}{\kappa-1}\left(1 - \int_0^1\left(\frac{2\alpha\beta}{(e-1)^{\beta}}\left(e^{x^{\alpha}(2-x)^{\alpha}} - 1\right)^{\beta-1} e^{x^{\alpha}(2-x)^{\alpha}} x^{\alpha-1}(1-x)(2-x)^{\alpha-1}\right)^{\theta} dx\right).$$

Simplifying and using the Renyi entropy expression of the proposed model, we get Tsalli's entropy as

$$\mathrm{T}(\kappa)=\frac{1}{\kappa-1}\left[1-\left\{\frac{(2\alpha\beta)^{\theta}}{(e-1)^{\beta\theta}}\sum_{k=0}^{\infty}\binom{\beta\theta-\theta}{k}(-1)^{k}\sum_{j=0}^{\infty}\frac{(\theta\beta-k)^{j}}{j!}\times\right.\right.$$
$$\left.\left.\sum_{i=0}^{\infty}\binom{\theta\alpha-\theta+\alpha j}{i}2^{\theta\alpha-\theta+\alpha j-i}(-1)^{i}\frac{\Gamma(\theta\alpha-\theta+\alpha j+i+1)\Gamma(\theta+1)}{\Gamma(\theta\alpha+\alpha j+i+2)}\right\}\right].$$

5. Bonferroni and Lorenz curves

The Lorenz and the Bonferroni curves are functional representations used to quantify inequality in a non-negative random variable. The Lorenz curve is defined in terms of the cumulative distribution and mean of the variable, while the Bonferroni curve is derived from the Lorenz curve and reflects relative concentration, with particular sensitivity to the lower tail of the distribution. These curves are commonly utilized in economics, reliability analysis, actuarial science, and survival modelling to examine and compare distributional characteristics of probabilistic models.

The Bonferroni curve for the PGDUS-TL distribution is given by

$$B(p)=\frac{1}{p\mu_1'}\int_0^t x\psi(x)dx,\ \ 0<p\leq 1,\ \ 0<t<1.$$

On using Eqs (8) and (9) and simplifying, we get

$$\mathrm{B(p)}=\frac{\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{k}\sum_{j=0}^{\infty}\frac{(\beta-k)^{j}}{j!}\sum_{i=0}^{\infty}\binom{\alpha+\alpha j-1}{i}2^{\alpha+\alpha j-1-i}(-1)^{i}B_{t}(1+\alpha+\alpha j+i,2)}{p\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{k}\sum_{j=0}^{\infty}\frac{(\beta-k)^{j}}{j!}\sum_{i=0}^{\infty}\binom{\alpha+\alpha j-1}{i}2^{\alpha+\alpha j-1-i}(-1)^{i}\frac{\Gamma(\alpha+\alpha j+i+1)}{\Gamma(3+\alpha+\alpha j+i)}}.$$

$\mathrm{B}_{t}(1+\alpha+\alpha j+i,2)$ denotes the lower incomplete beta function defined as

$$\int_0^t x^{1+\alpha+\alpha j+i-1}(1-x)dx,\qquad 0<x<1.$$

The Lorenz curve for the given model is

$$L(p)=pB(p)=\frac{1}{p\mu_1'}\int_0^t x\psi(x)dx.$$

On simplification,

$$L(p)=\frac{\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{k}\sum_{j=0}^{\infty}\frac{(\beta-k)^{j}}{j!}\sum_{i=0}^{\infty}\binom{\alpha+\alpha j-1}{i}2^{\alpha+\alpha j-1-i}(-1)^{i}B_{t}(1+\alpha+\alpha j+i,2)}{\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{k}\sum_{j=0}^{\infty}\frac{(\beta-k)^{j}}{j!}\sum_{i=0}^{\infty}\binom{\alpha+\alpha j-1}{i}2^{\alpha+\alpha j-1-i}(-1)^{i}\frac{\Gamma(\alpha+\alpha j+i+1)}{\Gamma(3+\alpha+\alpha j+i)}}.$$

6. Estimation methods

In this section, several parameter estimation methods [12] for the proposed probability distribution are presented. These include maximum likelihood estimation (MLE), Anderson-Darling estimation (ADE), Cramer-von Mises estimation (CME), least squares estimation (LSE), and maximum product of spacing estimation (MPS). Each of these approaches yields parameter estimates by either maximizing or minimizing a specified objective function. The objective function is formulated in terms of the unknown parameters and the observed random samples drawn from the underlying population.

6.1. Maximum likelihood estimation

In this method, the likelihood function of a random sample is the required objective function that is to be maximized to estimate the unknown parameter. Let the random samples $x_1, x_2, x_3, \ldots\ldots, x_n$ be drawn from the PGDUS-TL distribution. Then the corresponding likelihood function of the probability model is given by

$$l = \prod_{i=1}^{n} \psi(x_i).$$

Using Eq (6), the above expression results in

$$l = \prod_{i=1}^{n} \left\{ \frac{2\alpha\beta}{(e-1)^{\beta}} \left(e^{{x_i}^{\alpha}(2-x_i)^{\alpha}} - 1 \right)^{\beta-1} e^{{x_i}^{\alpha}(2-x_i)^{\alpha}} {x_i}^{\alpha-1}(1-x_i)(2-x_i)^{\alpha-1} \right\}.$$

Taking the logarithm on both sides of the above expression, we get

$$\log l = n\log\left(\frac{2\alpha\beta}{(e-1)^{\beta}}\right) + (\beta-1)\sum_{i=1}^{n}\log(e^{z_i}-1) + \sum_{i=1}^{n} z_i$$
$$+(\alpha-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log(1-x_i) + (\alpha-1)\sum_{i=1}^{n}\log(2-x_i).$$

Partially differentiating the above expression with respect to α and β respectively and equating to zero, we get

$$\frac{n}{\alpha} + (\beta-1)\sum_{i=1}^{n}\frac{z_i e^{z_i}}{e^{z_i}-1}\log\big(x_i(2-x_i)\big) + \sum_{i=1}^{n} z_i \log\big(x_i(2-x_i)\big) + \sum_{i=1}^{n}\log\big(x_i(2-x_i)\big) = 0,$$

and

$$n\left(\frac{1}{\beta} - log(e-1)\right) + \sum_{i=1}^{n} log(e^{z_i}-1) = 0,$$

where $z_i = x_i^{\alpha}(2-x_i)^{\alpha}$.

The MLE's of the parameters α and β of the PGDUS-TL distribution were obtained by numerically maximizing the log-likelihood function.

6.2. Anderson-Darling estimation

The ADE method minimizes the Anderson-Darling statistic to find the value of the parameter that best fits the distribution. If $x_{(1)}, x_{(2)}, x_{(3)}, \ldots\ldots, x_{(n)}$ is the ordered random sample of size *n* drawn from the PGDUS-TL distribution, the Anderson -Darling statistic denoted by A is given by

$$A = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[log\,\Psi\big(x_{(i)}\big) + log\,S\big(x_{(n-i+1)}\big)\right],$$

$$A = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[log\left\{\left(\frac{e^{x_{(i)}^{\alpha}\left(2-x_{(i)}\right)^{\alpha}}-1}{e-1}\right)^{\beta}\right\} + log\left\{1-\left(\frac{e^{\left(x_{(n-i+1)}\right)^{\alpha}\left(2-x_{(n-i+1)}\right)^{\alpha}}-1}{e-1}\right)^{\beta}\right\}\right].$$

6.3. Cramer-von Mises estimation

For the CME, the goal is to obtain the parameters by minimizing the Cramer-von Mises statistic.

If $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(n)}$ is the ordered random sample drawn from the PGDUS-TL distribution, then the Cramer-von Mises statistic is given by

$$\mathrm{C} = \frac{1}{12n} + \sum_{i=1}^{n}\left[\Psi\left(x_{(i)}\right) - \frac{2i-1}{2n}\right]^2,\ \mathrm{C} = \frac{1}{12n} + \sum_{i=1}^{n}\left[\left(\frac{e^{x_{(i)}^{\alpha}\left(2-x_{(i)}\right)^{\alpha}}-1}{e-1}\right)^{\beta} - \frac{2i-1}{2n}\right]^2.$$

6.4. Least squares estimation

In the LSE method, parameters are estimated by minimizing the squared differences between the empirical and theoretical CDFs of the chosen distribution. The objective function that is to be minimized is expressed as

$$LS = \sum_{i=1}^{n}\left[\Psi\left(x_{(i)}\right) - \frac{i}{n+1}\right]^2,\ LS = \sum_{i=1}^{n}\left[\left(\frac{e^{{x_{(i)}}^{\alpha}\left(2-x_{(i)}\right)^{\alpha}}-1}{e-1}\right)^{\beta} - \frac{i}{n+1}\right]^2.$$

6.5. Maximum product of spacing estimation

This method aims to obtain the parameters by maximizing the product of spacing – the differences between successive values of the theoretical CDF evaluated at the ordered data points. Thus, the parameters are estimated by maximizing the following objective function:

$$MPS = \frac{1}{n+1}\sum_{i=1}^{n}\left[log\left\{\Psi_{X_{(i)}}(x) - \Psi_{X_{(i-1)}}(x)\right\}\right],$$

where $\Psi_{X_{(i)}}(x)$ is the CDF of i^{th} ordered statistics

7. Simulation study

A comprehensive simulation study is conducted to investigate the finite-sample behaviour of the parameter estimators associated with the PGDUS-TL model. The analysis considers multiple sample sizes (n = 75, 100, 150, 250, 400, and 600) under several combinations of different parameter values. Five sets of parameters are chosen as (i) (α = 2, β = 1), (ii) (α = 2.15, β = 1.95), (iii) (α = 0.5, β = 2), (iv) (α = 0.75, β = 2.8), and (v) (α = 0.65, β = 2.5). For each configuration, 1000 independent random samples are generated from the PGDUS-TL distribution using the inverse CDF technique, i.e., by using quantile function and by utilizing R software [40]. Then, we obtain estimators by solving maximum likelihood equations, and minimizing different statistics such as ADE, LSE, CME, and MPS estimator performance is quantified using three commonly employed accuracy measures: the average absolute bias (BIAS), the mean squared error (MSE), and the mean relative error (MRE). These measures are obtained using R software [39] and are computed as

$$\mathrm{BIAS} = \frac{1}{k}\sum_{i=1}^{k}|\hat{\eta} - \eta|,\ MSE = \frac{1}{k}\sum_{i=1}^{k}(\hat{\eta} - \eta)^2, \text{ and } MRE = \frac{1}{k}\sum_{i=1}^{k}\frac{|\hat{\eta}-\eta|}{\eta},$$

where $\eta = \alpha, \beta$ and k is the number of iterations in the simulation process.

Root mean square error (RMSE), given by

$$RMSE = \sqrt{\left(\frac{1}{k}\sum_{i=1}^{k}(\hat{\eta} - \eta)^2\right)},$$

is another estimator performance measure. Since RMSE is a square root of MSE, reporting both measures was considered redundant. Hence among the two, only MSE was computed.

In this simulation study, five estimation methods namely, MLE, ADE, CME, LSE, and MPS were employed to evaluate the performance of the model in terms of bias, MSE, and MRE of the parameter estimates. Since the estimating equations are nonlinear and complicated, numerical optimization techniques were employed for parameter estimation using R software. Specifically, the Limited-memory Broyden–Fletche–Goldfarb–Shanno with Bound Constraints (L-BFGS-B) optimization algorithm available in the optim() function was utilized to obtain the parameter estimates. As both the parameters are positive, box constraints were imposed with lower bounds of 0.01 and upper bounds of 20 for both α and β. The optimization procedure was initialized at $(\alpha,\ \beta)$=(2.15, 1.95), and a maximum of 1000 iterations was allowed. The optimization procedure was iterated until the default convergence criterion of optim(), based on the relative reduction in the objective function value, was met. All optimization runs terminated successfully with convergence code 0, indicating successful convergence.

To compare the performance of the different estimation methods, a ranking procedure based on bias, mean squared error (MSE), and MRE was adopted. For each fixed sample size, the estimation methods were ranked according to the corresponding values of bias, MSE, and MRE, where smaller values were assigned better ranks due to their higher estimation accuracy and efficiency.

Subsequently, the individual ranks obtained for bias, MSE, and MRE were summed for each estimation method to obtain partial rank corresponding to that sample size. This procedure was repeated for all considered sample sizes. Finally, the partial ranks were aggregated to determine the overall rank of each estimation method, where the method with the smallest total rank was regarded as the best performing estimator. From Tables 2–6, it is evident that the bias of the estimated parameter under all five estimation methods decreases as the sample size increases, indicating the estimators' converge toward the true parameter value. In addition, both the MSE and MRE exhibit a decreasing trend with increasing sample size, demonstrating improved accuracy and stability of the estimators. This behaviour confirms the consistency of the parameter estimates as the sample size grows. Also, based on the overall ranks, the MPS method attains the lowest (best) rank among other competing approaches, followed by ADE, MLE, CME, and LSE. Therefore, it can be concluded that MPS method is the most efficient method for estimating the parameters of the PGDUS-TL distribution, compared with the other four methods. The resulting partial and overall ranks are reported in Table 7.

Table 2. Bias, MSE, and MRE under different estimation methods for α = 2, β = 1.

n	Est.	Est. Par	MLE	ADE	CME	LSE	MPS
75	BIAS	α	0.916475{3}	0.921947{4}	0.789929{2}	0.787305{1}	0.945675{5}
		β	6.448655{3}	5.847387{2}	6.520425{4}	6.522346{5}	3.530938{1}
	MSE	α	22.23977{5}	21.12439{4}	20.35380{3}	20.26884{2}	19.47732{1}
		β	114.2487{5}	98.29254{2}	108.7200{3}	108.7765{4}	57.78152{1}
	MRE	α	1.372242{5}	1.370898{3}	1.371953{4}	1.370748{2}	1.185513{1}
		β	7.142356{3}	6.518212{2}	7.213119{4}	7.215102{5}	4.058810{1}
	∑ Ranks		24{5}	17{2}	20{4}	19{3}	10{1}

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n	Est.	Est. Par	MLE	ADE	CME	LSE	MPS
100	BIAS	α	0.912239$^{\{5\}}$	0.907023$^{\{4\}}$	0.746400$^{\{1\}}$	0.747888$^{\{2\}}$	0.885192$^{\{3\}}$
		β	6.144296$^{\{3\}}$	5.699963$^{\{2\}}$	6.346875$^{\{4\}}$	6.346955$^{\{5\}}$	3.413933$^{\{1\}}$
	MSE	α	20.91166$^{\{5\}}$	20.30721$^{\{4\}}$	18.81123$^{\{1\}}$	19.85355$^{\{3\}}$	19.27543$^{\{2\}}$
		β	107.0638$^{\{5\}}$	93.65570$^{\{2\}}$	102.5946$^{\{3\}}$	102.6406$^{\{4\}}$	53.47543$^{\{1\}}$
	MRE	α	1.428420$^{\{5\}}$	1.394959$^{\{2\}}$	1.414572$^{\{3\}}$	1.415233$^{\{4\}}$	1.199886$^{\{1\}}$
		β	6.824638$^{\{3\}}$	6.355381$^{\{2\}}$	7.037222$^{\{4\}}$	7.037331$^{\{5\}}$	3.920699$^{\{1\}}$
	$\sum$ Ranks		26$^{\{5\}}$	16$^{\{2.5\}}$	16$^{\{2.5\}}$	23$^{\{4\}}$	9$^{\{1\}}$
150	BIAS	α	0.215743$^{\{2\}}$	0.338168$^{\{5\}}$	0.309821$^{\{3\}}$	0.311274$^{\{4\}}$	0.154215$^{\{1\}}$
		β	5.578348$^{\{3\}}$	5.406092$^{\{2\}}$	6.026184$^{\{5\}}$	6.023938$^{\{4\}}$	3.246073$^{\{1\}}$
	MSE	α	13.67067$^{\{2\}}$	15.26237$^{\{3\}}$	15.97377$^{\{4\}}$	16.01139$^{\{5\}}$	10.71951$^{\{1\}}$
		β	90.80883$^{\{3\}}$	83.63109$^{\{2\}}$	91.80790$^{\{4\}}$	91.81310$^{\{5\}}$	46.76597$^{\{1\}}$
	MRE	α	1.109692$^{\{2\}}$	1.180682$^{\{3\}}$	1.210946$^{\{4\}}$	1.211560$^{\{5\}}$	0.925090$^{\{1\}}$
		β	6.133363$^{\{3\}}$	5.972422$^{\{2\}}$	6.625057$^{\{5\}}$	6.622723$^{\{4\}}$	3.648643$^{\{1\}}$
	$\sum$ Ranks		15$^{\{2\}}$	17$^{\{3\}}$	25$^{\{4\}}$	27$^{\{5\}}$	6$^{\{1\}}$
250	BIAS	α	-0.16885$^{\{1\}}$	-0.01807$^{\{3\}}$	0.081463$^{\{5\}}$	0.080287$^{\{4\}}$	-0.16108$^{\{2\}}$
		β	4.572506$^{\{3\}}$	4.387190$^{\{2\}}$	4.723663$^{\{5\}}$	4.722436$^{\{4\}}$	2.892835$^{\{1\}}$
	MSE	α	8.560965$^{\{2\}}$	10.60242$^{\{3\}}$	12.96932$^{\{5\}}$	12.92980$^{\{4\}}$	7.342764$^{\{1\}}$
		β	69.21981$^{\{5\}}$	63.78369$^{\{2\}}$	66.65763$^{\{4\}}$	66.56582$^{\{3\}}$	37.23507$^{\{1\}}$
	MRE	α	0.909310$^{\{2\}}$	0.972699$^{\{3\}}$	1.040911$^{\{5\}}$	1.040388$^{\{4\}}$	0.800028$^{\{1\}}$
		β	5.013052$^{\{3\}}$	4.829762$^{\{2\}}$	5.197379$^{\{5\}}$	5.196073$^{\{4\}}$	3.225455$^{\{1\}}$
	$\sum$ Ranks		16$^{\{3\}}$	15$^{\{2\}}$	29$^{\{5\}}$	23$^{\{4\}}$	7$^{\{1\}}$
400	BIAS	α	-0.29424$^{\{2\}}$	-0.29034$^{\{3\}}$	-0.16288$^{\{5\}}$	-0.16342$^{\{4\}}$	-0.33250$^{\{1\}}$
		β	3.776935$^{\{2\}}$	3.938023$^{\{3\}}$	4.306194$^{\{4\}}$	4.317518$^{\{5\}}$	2.488494$^{\{1\}}$
	MSE	α	7.962441$^{\{2\}}$	8.811623$^{\{3\}}$	11.97269$^{\{5\}}$	11.95663$^{\{4\}}$	6.581146$^{\{1\}}$
		β	49.05367$^{\{2\}}$	49.91769$^{\{3\}}$	53.79174$^{\{4\}}$	54.13602$^{\{5\}}$	26.21203$^{\{1\}}$
	MRE	α	0.879923$^{\{2\}}$	0.888849$^{\{3\}}$	0.968766$^{\{5\}}$	0.968606$^{\{4\}}$	0.772896$^{\{1\}}$
		β	4.142838$^{\{2\}}$	4.304050$^{\{3\}}$	4.704830$^{\{4\}}$	4.716745$^{\{5\}}$	2.762915$^{\{1\}}$
	$\sum$ Ranks		12$^{\{2\}}$	18$^{\{3\}}$	27$^{\{4.5\}}$	27$^{\{4.5\}}$	6$^{\{1\}}$
600	BIAS	α	-0.65367$^{\{5\}}$	-0.70623$^{\{1\}}$	-0.70322$^{\{3\}}$	-0.70325$^{\{2\}}$	-0.66142$^{\{4\}}$
		β	2.675420$^{\{2\}}$	2.765502$^{\{3\}}$	3.256144$^{\{4\}}$	3.258897$^{\{5\}}$	2.258897$^{\{1\}}$
	MSE	α	4.539015$^{\{3\}}$	4.481905$^{\{2\}}$	5.191922$^{\{5\}}$	5.191497$^{\{4\}}$	3.643935$^{\{1\}}$
		β	25.02432$^{\{3\}}$	24.87955$^{\{2\}}$	31.83747$^{\{4\}}$	31.95095$^{\{5\}}$	12.69327$^{\{1\}}$
	MRE	α	0.719334$^{\{3\}}$	0.692536$^{\{2\}}$	0.711955$^{\{5\}}$	0.711947$^{\{4\}}$	0.643562$^{\{1\}}$
		β	2.921886$^{\{2\}}$	3.000821$^{\{3\}}$	3.520026$^{\{4\}}$	3.522922$^{\{5\}}$	2.050296$^{\{1\}}$
	$\sum$ Ranks		18$^{\{3\}}$	13$^{\{2\}}$	25$^{\{4.5\}}$	25$^{\{4.5\}}$	9$^{\{1\}}$

Table 3. Bias, MSE, and MRE under different estimation methods for α = 2.15, β = 1.95.

n	Est.	Est. Par	MLE	ADE	CME	LSE	MPS
75	Bias	α	1.894115$^{\{1\}}$	3.165458$^{\{3\}}$	3.294834$^{\{5\}}$	3.294680$^{\{4\}}$	2.692530$^{\{2\}}$
		β	6.794762$^{\{5\}}$	5.713434$^{\{2\}}$	6.703899$^{\{3\}}$	6.709802$^{\{4\}}$	3.181158$^{\{1\}}$
	MSE	α	36.51816$^{\{1\}}$	55.98829$^{\{3\}}$	61.15727$^{\{5\}}$	61.15661$^{\{4\}}$	39.74830$^{\{2\}}$

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n	Est.	Est. Par	MLE	ADE	CME	LSE	MPS
		β	123.8087$^{\{5\}}$	105.5221$^{\{2\}}$	122.7090$^{\{3\}}$	122.8603$^{\{4\}}$	62.08878$^{\{1\}}$
	MRE	α	1.756473$^{\{1\}}$	2.254780$^{\{3\}}$	2.396865$^{\{4\}}$	2.396899$^{\{5\}}$	1.788169$^{\{2\}}$
		β	3.974120$^{\{3\}}$	3.523020$^{\{2\}}$	4.030643$^{\{4\}}$	4.033669$^{\{5\}}$	2.243299$^{\{1\}}$
	∑ Ranks		16$^{\{3\}}$	15$^{\{2\}}$	24$^{\{4\}}$	26$^{\{5\}}$	9$^{\{1\}}$
100	Bias	α	1.816899$^{\{1\}}$	2.882944$^{\{3\}}$	2.949937$^{\{5\}}$	2.948892$^{\{4\}}$	2.420846$^{\{2\}}$
		β	6.181973$^{\{5\}}$	4.858774$^{\{2\}}$	5.800971$^{\{3\}}$	5.801869$^{\{4\}}$	2.950420$^{\{1\}}$
	MSE	α	33.20870$^{\{1\}}$	49.32432$^{\{3\}}$	53.02502$^{\{5\}}$	52.99033$^{\{4\}}$	34.57770$^{\{2\}}$
		β	112.4558$^{\{5\}}$	88.54977$^{\{2\}}$	104.7552$^{\{3\}}$	104.7993$^{\{4\}}$	56.66504$^{\{1\}}$
	MRE	α	1.674118$^{\{2\}}$	2.069939$^{\{3\}}$	2.173278$^{\{5\}}$	2.172787$^{\{4\}}$	1.650523$^{\{1\}}$
		β	3.665671$^{\{5\}}$	3.075659$^{\{2\}}$	3.554241$^{\{3\}}$	3.554689$^{\{4\}}$	2.111165$^{\{1\}}$
	∑ Ranks		19$^{\{3\}}$	15$^{\{2\}}$	23$^{\{4\}}$	24$^{\{5\}}$	8$^{\{1\}}$
150	Bias	α	1.640276$^{\{1\}}$	2.348713$^{\{5\}}$	2.298089$^{\{4\}}$	2.297962$^{\{3\}}$	1.895422$^{\{2\}}$
		β	4.951426$^{\{3\}}$	4.672383$^{\{2\}}$	5.700402$^{\{4\}}$	5.707961$^{\{5\}}$	2.413992$^{\{1\}}$
	MSE	α	29.24190$^{\{2\}}$	40.79227$^{\{3\}}$	42.26140$^{\{4\}}$	42.26216$^{\{5\}}$	26.12352$^{\{1\}}$
		β	86.26833$^{\{3\}}$	82.00179$^{\{2\}}$	99.02318$^{\{4\}}$	99.22921$^{\{5\}}$	43.57296$^{\{1\}}$
	MRE	α	1.520929$^{\{2\}}$	1.823849$^{\{3\}}$	1.887893$^{\{4\}}$	1.887962$^{\{5\}}$	1.386065$^{\{1\}}$
		β	2.997702$^{\{2\}}$	2.914978$^{\{3\}}$	3.438695$^{\{4\}}$	3.442572$^{\{5\}}$	1.758361$^{\{1\}}$
	∑ Ranks		13$^{\{2\}}$	18$^{\{3\}}$	24$^{\{4\}}$	28$^{\{5\}}$	7$^{\{1\}}$
250	Bias	α	1.457050$^{\{1\}}$	2.009155$^{\{3\}}$	2.032646$^{\{5\}}$	2.031485$^{\{4\}}$	1.709592$^{\{2\}}$
		β	3.823932$^{\{3\}}$	3.442308$^{\{2\}}$	4.187191$^{\{5\}}$	4.181110$^{\{4\}}$	1.930248$^{\{1\}}$
	MSE	α	25.11888$^{\{2\}}$	33.61402$^{\{3\}}$	35.67559$^{\{5\}}$	35.63604$^{\{4\}}$	23.63076$^{\{1\}}$
		β	64.11809$^{\{3\}}$	57.32231$^{\{2\}}$	68.44957$^{\{5\}}$	68.28048$^{\{4\}}$	33.04622$^{\{1\}}$
	MRE	α	1.361409$^{\{2\}}$	1.577379$^{\{3\}}$	1.655816$^{\{5\}}$	1.655194$^{\{4\}}$	1.274189$^{\{1\}}$
		β	2.400602$^{\{3\}}$	2.263093$^{\{2\}}$	2.642388$^{\{5\}}$	2.639251$^{\{4\}}$	1.477982$^{\{1\}}$
	∑ Ranks		14$^{\{2\}}$	15$^{\{3\}}$	30$^{\{5\}}$	24$^{\{4\}}$	7$^{\{1\}}$
400	Bias	α	0.902496$^{\{1\}}$	1.191844$^{\{5\}}$	1.101326$^{\{4\}}$	1.101027$^{\{3\}}$	1.051720$^{\{2\}}$
		β	2.688253$^{\{2\}}$	2.995402$^{\{3\}}$	3.891699$^{\{5\}}$	3.889620$^{\{4\}}$	1.332533$^{\{1\}}$
	MSE	α	17.35410$^{\{2\}}$	21.67122$^{\{3\}}$	21.78247$^{\{5\}}$	21.77655$^{\{4\}}$	15.68133$^{\{1\}}$
		β	39.21807$^{\{2\}}$	45.20056$^{\{3\}}$	60.09648$^{\{5\}}$	60.05013$^{\{4\}}$	18.47996$^{\{1\}}$
	MRE	α	1.043617$^{\{2\}}$	1.188067$^{\{3\}}$	1.222316$^{\{5\}}$	1.222132$^{\{4\}}$	0.948371$^{\{1\}}$
		β	1.727097$^{\{2\}}$	1.929486$^{\{3\}}$	2.390635$^{\{5\}}$	2.389556$^{\{4\}}$	1.058471$^{\{1\}}$
	∑ Ranks		11$^{\{2\}}$	20$^{\{3\}}$	29$^{\{5\}}$	23$^{\{4\}}$	7$^{\{1\}}$
600	Bias	α	0.606748$^{\{1\}}$	0.875314$^{\{3\}}$	0.876557$^{\{4\}}$	0.877496$^{\{5\}}$	0.691297$^{\{2\}}$
		β	2.095183$^{\{3\}}$	1.917109$^{\{2\}}$	2.592429$^{\{5\}}$	2.592407$^{\{4\}}$	1.164367$^{\{1\}}$
	MSE	α	12.15299$^{\{2\}}$	15.92838$^{\{3\}}$	16.24745$^{\{4\}}$	16.28016$^{\{5\}}$	10.52553$^{\{1\}}$
		β	28.54152$^{\{3\}}$	25.97707$^{\{2\}}$	36.81370$^{\{4\}}$	36.81749$^{\{5\}}$	14.87921$^{\{1\}}$
	MRE	α	0.840357$^{\{2\}}$	0.942387$^{\{3\}}$	0.989389$^{\{4\}}$	0.989828$^{\{5\}}$	0.752715$^{\{1\}}$
		β	1.376385$^{\{3\}}$	1.324877$^{\{2\}}$	1.689182$^{\{4\}}$	1.689183$^{\{5\}}$	0.915880$^{\{1\}}$
	∑ Ranks		14$^{\{2\}}$	15$^{\{3\}}$	25$^{\{4\}}$	29$^{\{5\}}$	7$^{\{1\}}$

Table 4. Bias, MSE, and MRE under different estimation methods for α = 0.5, β = 2.0.

n	Est.	Est. Par	MLE	ADE	CME	LSE	MPS
75	Bias	α	$1.942098^{\{5\}}$	$1.755446^{\{3\}}$	$1.693640^{\{1\}}$	$1.696360^{\{2\}}$	$1.780218^{\{4\}}$
		β	$1.854846^{\{5\}}$	$0.972362^{\{2\}}$	$1.021375^{\{3\}}$	$1.023924^{\{4\}}$	$0.252889^{\{1\}}$
	MSE	α	$10.43427^{\{5\}}$	$8.429775^{\{3\}}$	$7.989963^{\{1\}}$	$8.025227^{\{2\}}$	$9.046134^{\{4\}}$
		β	$49.18624^{\{5\}}$	$27.04368^{\{4\}}$	$23.65966^{\{2\}}$	$23.76573^{\{3\}}$	$21.90255^{\{1\}}$
	MRE	α	$4.278563^{\{5\}}$	$3.887794^{\{4\}}$	$3.800621^{\{2\}}$	$3.805992^{\{3\}}$	$3.800342^{\{1\}}$
		β	$1.992510^{\{5\}}$	$1.515833^{\{4\}}$	$1.502754^{\{2\}}$	$1.504078^{\{3\}}$	$1.230145^{\{1\}}$
	∑ Ranks		$30^{\{5\}}$	$20^{\{4\}}$	$11^{\{1\}}$	$17^{\{3\}}$	$12^{\{2\}}$
100	Bias	α	$1.654854^{\{5\}}$	$1.534735^{\{2\}}$	$1.638052^{\{4\}}$	$1.636921^{\{3\}}$	$1.53434^{\{1\}}$
		β	$1.695408^{\{5\}}$	$0.894184^{\{4\}}$	$0.807417^{\{3\}}$	$0.802058^{\{2\}}$	$0.210286^{\{1\}}$
	MSE	α	$7.564239^{\{5\}}$	$6.422310^{\{1\}}$	$7.227046^{\{4\}}$	$7.201512^{\{3\}}$	$6.687975^{\{2\}}$
		β	$43.95833^{\{5\}}$	$25.09518^{\{4\}}$	$20.76741^{\{3\}}$	$20.60513^{\{2\}}$	$19.82840^{\{1\}}$
	MRE	α	$3.716116^{\{5\}}$	$3.436675^{\{2\}}$	$3.665531^{\{4\}}$	$3.663204^{\{3\}}$	$3.321704^{\{1\}}$
		β	$1.885430^{\{5\}}$	$1.458127^{\{4\}}$	$1.418494^{\{3\}}$	$1.415800^{\{2\}}$	$1.170962^{\{1\}}$
	∑ Ranks		$30^{\{5\}}$	$17^{\{3\}}$	$21^{\{4\}}$	$15^{\{2\}}$	$7^{\{1\}}$
150	Bias	α	$1.713455^{\{5\}}$	$1.686367^{\{2\}}$	$1.699907^{\{4\}}$	$1.698267^{\{3\}}$	$1.57225^{\{1\}}$
		β	$0.624171^{\{5\}}$	$-0.17638^{\{2\}}$	$-0.14128^{\{3\}}$	$-0.13581^{\{4\}}$	$-0.30740^{\{1\}}$
	MSE	α	$7.109276^{\{5\}}$	$6.809557^{\{2\}}$	$7.090605^{\{4\}}$	$7.070240^{\{3\}}$	$6.221798^{\{1\}}$
		β	$27.11381^{\{5\}}$	$10.50748^{\{3\}}$	$9.743990^{\{1\}}$	$9.851783^{\{2\}}$	$12.19618^{\{4\}}$
	MRE	α	$3.716201^{\{5\}}$	$3.619385^{\{2\}}$	$3.659653^{\{4\}}$	$3.656501^{\{3\}}$	$3.327428^{\{1\}}$
		β	$1.411495^{\{5\}}$	$0.998866^{\{2\}}$	$1.013272^{\{3\}}$	$1.015978^{\{4\}}$	$0.960675^{\{1\}}$
	∑ Ranks		$30^{\{5\}}$	$13^{\{2\}}$	$19^{\{3.5\}}$	$19^{\{3.5\}}$	$9^{\{1\}}$
250	Bias	α	$1.604562^{\{2\}}$	$1.609569^{\{3\}}$	$1.660654^{\{4\}}$	$1.660677^{\{5\}}$	$1.437534^{\{1\}}$
		β	$-0.18348^{\{5\}}$	$-0.71580^{\{3\}}$	$-0.72083^{\{2\}}$	$-0.72127^{\{1\}}$	$-0.62918^{\{4\}}$
	MSE	α	$5.667078^{\{3\}}$	$5.571991^{\{2\}}$	$6.128288^{\{4\}}$	$6.128585^{\{5\}}$	$4.789217^{\{1\}}$
		β	$14.18409^{\{5\}}$	$4.880502^{\{3\}}$	$4.412603^{\{2\}}$	$4.401410^{\{1\}}$	$7.618757^{\{4\}}$
	MRE	α	$3.419054^{\{3\}}$	$3.379458^{\{2\}}$	$3.487016^{\{4\}}$	$3.487075^{\{5\}}$	$3.020635^{\{1\}}$
		β	$1.058596^{\{5\}}$	$0.795251^{\{3\}}$	$0.788584^{\{2\}}$	$0.788365^{\{1\}}$	$0.821574^{\{4\}}$
	∑ Ranks		$21^{\{5\}}$	$16^{\{2\}}$	$18^{\{3.5\}}$	$18^{\{3.5\}}$	$15^{\{1\}}$
400	Bias	α	$1.637013^{\{3\}}$	$1.609650^{\{2\}}$	$1.686277^{\{5\}}$	$1.686087^{\{4\}}$	$1.526242^{\{1\}}$
		β	$-0.79124^{\{5\}}$	$-1.10089^{\{1\}}$	$-1.09358^{\{2\}}$	$-1.09278^{\{3\}}$	$-0.99493^{\{4\}}$
	MSE	α	$5.142664^{\{5\}}$	$4.925824^{\{2\}}$	$5.59504^{\{4\}}$	$5.59306^{\{3\}}$	$4.642095^{\{1\}}$
		β	$6.194038^{\{5\}}$	$2.450178^{\{3\}}$	$2.41462^{\{1\}}$	$2.427425^{\{2\}}$	$3.703227^{\{4\}}$
	MRE	α	$3.401711^{\{3\}}$	$3.292820^{\{2\}}$	$3.451594^{\{5\}}$	$3.451242^{\{4\}}$	$3.140788^{\{1\}}$
		β	$0.833413^{\{5\}}$	$0.674149^{\{1\}}$	$0.686143^{\{2\}}$	$0.686540^{\{3\}}$	$0.715855^{\{4\}}$
	∑ Ranks		$26^{\{4\}}$	$11^{\{1\}}$	$29^{\{5\}}$	$19^{\{3\}}$	$15^{\{2\}}$
600	Bias	α	$1.670496^{\{3\}}$	$1.634272^{\{2\}}$	$1.711541^{\{5\}}$	$1.709766^{\{4\}}$	$1.524599^{\{1\}}$
		β	$-1.11841^{\{5\}}$	$-1.24124^{\{3\}}$	$-1.24276^{\{1\}}$	$-1.24257^{\{2\}}$	$-1.17550^{\{4\}}$
	MSE	α	$4.737824^{\{3\}}$	$4.694118^{\{2\}}$	$5.411705^{\{5\}}$	$5.389020^{\{4\}}$	$4.134012^{\{1\}}$
		β	$3.178945^{\{5\}}$	$1.963769^{\{1\}}$	$2.017109^{\{2\}}$	$2.018299^{\{3\}}$	$2.209048^{\{4\}}$
	MRE	α	$3.418188^{\{3\}}$	$3.308661^{\{2\}}$	$3.463707^{\{5\}}$	$3.460169^{\{4\}}$	$3.108374^{\{1\}}$
		β	$0.737537^{\{5\}}$	$0.662985^{\{1\}}$	$0.672619^{\{2\}}$	$0.672686^{\{3\}}$	$0.678780^{\{4\}}$
	∑ Ranks		$24^{\{5\}}$	$11^{\{1\}}$	$20^{\{3.5\}}$	$20^{\{3.5\}}$	$15^{\{2\}}$

Table 5. Bias, MSE, and MRE under different estimation methods for α = 0.75, β = 2.8.

n	Est.	Est. Par	MLE	ADE	CME	LSE	MPS
75	Bias	α	$2.821860^{\{1\}}$	$3.383851^{\{3\}}$	$3.491924^{\{5\}}$	$3.488451^{\{4\}}$	$2.898747^{\{2\}}$
		β	$2.479120^{\{5\}}$	$1.293690^{\{2\}}$	$1.818245^{\{3\}}$	$1.822902^{\{4\}}$	$0.239170^{\{1\}}$
	MSE	α	$25.37815^{\{1\}}$	$31.61678^{\{3\}}$	$33.98676^{\{5\}}$	$33.89324^{\{4\}}$	$25.51067^{\{2\}}$
		β	$65.98355^{\{5\}}$	$46.24853^{\{2\}}$	$51.82488^{\{3\}}$	$51.94498^{\{4\}}$	$31.97942^{\{1\}}$
	MRE	α	$4.233887^{\{2\}}$	$4.888167^{\{3\}}$	$5.091112^{\{5\}}$	$5.086571^{\{4\}}$	$4.118406^{\{1\}}$
		β	$1.889993^{\{5\}}$	$1.530938^{\{4\}}$	$1.683222^{\{2\}}$	$1.684864^{\{3\}}$	$1.217102^{\{1\}}$
	∑ Ranks		$19^{\{3\}}$	$17^{\{2\}}$	$23^{\{4.5\}}$	$23^{\{4.5\}}$	$8^{\{1\}}$
100	Bias	α	$2.848621^{\{2\}}$	$3.244909^{\{3\}}$	$3.344468^{\{4\}}$	$3.346773^{\{5\}}$	$2.845128^{\{1\}}$
		β	$1.483660^{\{5\}}$	$0.386875^{\{2\}}$	$1.039202^{\{3\}}$	$1.040248^{\{4\}}$	$-0.25971^{\{1\}}$
	MSE	α	$24.89785^{\{2\}}$	$28.22552^{\{3\}}$	$30.95434^{\{4\}}$	$31.03761^{\{5\}}$	$23.50561^{\{1\}}$
		β	$50.22236^{\{5\}}$	$30.22286^{\{2\}}$	$38.62746^{\{3\}}$	$38.64123^{\{4\}}$	$24.04335^{\{1\}}$
	MRE	α	$4.176602^{\{2\}}$	$4.629796^{\{3\}}$	$4.832658^{\{4\}}$	$4.835786^{\{5\}}$	$3.996398^{\{1\}}$
		β	$1.570420^{\{5\}}$	$1.236515^{\{2\}}$	$1.431143^{\{3\}}$	$1.431528^{\{4\}}$	$1.060711^{\{1\}}$
	∑ Ranks		$213.5^{\{3.5\}}$	$15^{\{2\}}$	$21^{\{3.5\}}$	$27^{\{5\}}$	$6^{\{1\}}$
150	Bias	α	$2.901570^{\{2\}}$	$3.374607^{\{3\}}$	$3.510529^{\{5\}}$	$3.502455^{\{4\}}$	$2.742965^{\{1\}}$
		β	$0.791985^{\{5\}}$	$0.091163^{\{2\}}$	$0.536214^{\{3\}}$	$0.538712^{\{4\}}$	$-0.53110^{\{1\}}$
	MSE	α	$22.91269^{\{2\}}$	$27.98724^{\{3\}}$	$31.63962^{\{5\}}$	$31.37080^{\{4\}}$	$20.17740^{\{1\}}$
		β	$38.81968^{\{5\}}$	$26.97577^{\{2\}}$	$31.84480^{\{3\}}$	$31.91929^{\{4\}}$	$19.20600^{\{1\}}$
	MRE	α	$4.771087^{\{3\}}$	$4.771087^{\{2\}}$	$5.005453^{\{5\}}$	$4.994710^{\{4\}}$	$3.837589^{\{1\}}$
		β	$1.373748^{\{5\}}$	$1.177753^{\{2\}}$	$1.317487^{\{3\}}$	$1.317487^{\{4\}}$	$0.970675^{\{1\}}$
	∑ Ranks		$22^{\{3\}}$	$14^{\{2\}}$	$24^{\{4.5\}}$	$24^{\{4.5\}}$	$6^{\{1\}}$
250	Bias	α	$2.582758^{\{2\}}$	$2.87847^{\{3\}}$	$3.047029^{\{5\}}$	$3.045493^{\{4\}}$	$2.410132^{\{1\}}$
		β	$-0.31475^{\{5\}}$	$-0.93034^{\{2\}}$	$-0.64236^{\{3\}}$	$-0.64029^{\{4\}}$	$-1.06274^{\{1\}}$
	MSE	α	$16.51593^{\{2\}}$	$18.79726^{\{3\}}$	$22.65417^{\{5\}}$	$22.60441^{\{4\}}$	$13.67304^{\{1\}}$
		β	$20.38935^{\{5\}}$	$11.73094^{\{2\}}$	$14.87283^{\{3\}}$	$14.94472^{\{4\}}$	$11.20115^{\{1\}}$
	MRE	α	$3.666377^{\{2\}}$	$3.991764^{\{3\}}$	$4.257365^{\{5\}}$	$4.255326^{\{4\}}$	$3.330972^{\{1\}}$
		β	$1.017007^{\{5\}}$	$0.849184^{\{2\}}$	$0.936094^{\{3\}}$	$0.936825^{\{4\}}$	$0.797477^{\{1\}}$
	∑ Ranks		$21^{\{3\}}$	$15^{\{2\}}$	$24^{\{3.5\}}$	$24^{\{3.5\}}$	$6^{\{1\}}$
400	Bias	α	$2.614950^{\{2\}}$	$3.000087^{\{3\}}$	$3.129741^{\{4\}}$	$3.130132^{\{5\}}$	$2.427496^{\{1\}}$
		β	$-1.14150^{\{5\}}$	$-1.44581^{\{2\}}$	$-1.29012^{\{3\}}$	$-1.28988^{\{4\}}$	$-1.46310^{\{1\}}$
	MSE	α	$14.85169^{\{2\}}$	$19.27391^{\{3\}}$	$22.59817^{\{4\}}$	$22.61690^{\{5\}}$	$12.85717^{\{1\}}$
		β	$10.30263^{\{5\}}$	$7.449868^{\{2\}}$	$8.385744^{\{4\}}$	$8.378335^{\{3\}}$	$6.660574^{\{1\}}$
	MRE	α	$3.609099^{\{2\}}$	$4.078386^{\{3\}}$	$4.276676^{\{4\}}$	$4.277235^{\{5\}}$	$3.303533^{\{1\}}$
		β	$0.808282^{\{5\}}$	$0.748453^{\{2\}}$	$0.788118^{\{3\}}$	$0.788202^{\{4\}}$	$0.715979^{\{1\}}$
	∑ Ranks		$21^{\{3\}}$	$15^{\{2\}}$	$22^{\{4\}}$	$26^{\{5\}}$	$6^{\{1\}}$
600	Bias	α	$2.592063^{\{2\}}$	$2.928296^{\{3\}}$	$3.122474^{\{4\}}$	$3.125859^{\{5\}}$	$2.411390^{\{1\}}$
		β	$-1.59590^{\{5\}}$	$-1.79448^{\{1\}}$	$-1.74922^{\{3\}}$	$-1.75069^{\{2\}}$	$-1.70900^{\{4\}}$
	MSE	α	$12.99168^{\{2\}}$	$16.70779^{\{3\}}$	$20.83984^{\{4\}}$	$20.95400^{\{5\}}$	$11.28067^{\{1\}}$
		β	$5.852432^{\{5\}}$	$4.447881^{\{1\}}$	$4.676510^{\{3\}}$	$4.644083^{\{2\}}$	$4.684301^{\{4\}}$
	MRE	α	$3.513710^{\{2\}}$	$3.932207^{\{3\}}$	$4.199692^{\{4\}}$	$4.204166^{\{5\}}$	$3.248572^{\{1\}}$
		β	$0.719081^{\{5\}}$	$0.690182^{\{2\}}$	$0.697847^{\{4\}}$	$0.697338^{\{3\}}$	$0.684625^{\{1\}}$
	∑ Ranks		$21^{\{3\}}$	$13^{\{2\}}$	$22^{\{4.5\}}$	$22^{\{4.5\}}$	$12^{\{1\}}$

Table 6. Bias, MSE, and MRE under different estimation methods for α = 0.65, β = 2.5.

n	Est.	Est. Par	MLE	ADE	CME	LSE	MPS
75	Bias	α	2.457326$^{\{2\}}$	2.672946$^{\{5\}}$	2.654851$^{\{4\}}$	2.653279$^{\{3\}}$	2.431201$^{\{1\}}$
		β	2.462484$^{\{5\}}$	1.256500$^{\{2\}}$	1.762604$^{\{3\}}$	1.772553$^{\{4\}}$	0.295636$^{\{1\}}$
	MSE	α	18.78263$^{\{2\}}$	19.74652$^{\{3\}}$	19.95037$^{\{5\}}$	19.91525$^{\{4\}}$	17.39312$^{\{1\}}$
		β	63.79245$^{\{5\}}$	40.32641$^{\{2\}}$	45.12555$^{\{3\}}$	45.39177$^{\{4\}}$	28.79477$^{\{1\}}$
	MRE	α	4.236125$^{\{2\}}$	4.492459$^{\{3\}}$	4.530155$^{\{5\}}$	4.527879$^{\{4\}}$	3.989924$^{\{1\}}$
		β	1.999814$^{\{5\}}$	1.553041$^{\{2\}}$	1.715104$^{\{3\}}$	1.719064$^{\{4\}}$	1.235370$^{\{1\}}$
	∑ Ranks		21$^{\{3\}}$	17$^{\{2\}}$	23$^{\{4.5\}}$	23$^{\{4.5\}}$	6$^{\{1\}}$
100	Bias	α	2.185371$^{\{2\}}$	2.447090$^{\{3\}}$	2.447090$^{\{4\}}$	2.448740$^{\{5\}}$	2.077947$^{\{1\}}$
		β	1.973442$^{\{5\}}$	0.735138$^{\{2\}}$	1.143739$^{\{3\}}$	1.144324$^{\{4\}}$	0.128141$^{\{1\}}$
	MSE	α	14.96171$^{\{2\}}$	16.42282$^{\{3\}}$	17.06548$^{\{4\}}$	17.10761$^{\{5\}}$	12.77675$^{\{1\}}$
		β	54.39035$^{\{5\}}$	29.90059$^{\{2\}}$	34.00299$^{\{3\}}$	34.07252$^{\{4\}}$	24.75315$^{\{1\}}$
	MRE	α	3.776377$^{\{2\}}$	4.071654$^{\{3\}}$	4.165910$^{\{4\}}$	4.168414$^{\{5\}}$	3.431785$^{\{1\}}$
		β	1.796637$^{\{5\}}$	1.335696$^{\{2\}}$	1.461302$^{\{3\}}$	1.461545$^{\{4\}}$	1.142974$^{\{1\}}$
	∑ Ranks		21$^{\{3.5\}}$	15$^{\{2\}}$	21$^{\{3.5\}}$	27$^{\{5\}}$	6$^{\{1\}}$
150	Bias	α	2.291619$^{\{2\}}$	2.521278$^{\{3\}}$	2.618352$^{\{4\}}$	2.619699$^{\{5\}}$	2.122399$^{\{1\}}$
		β	0.950262$^{\{5\}}$	-0.14057$^{\{2\}}$	-0.02355$^{\{4\}}$	-0.02839$^{\{3\}}$	-0.148690$^{\{1\}}$
	MSE	α	13.78512$^{\{2\}}$	15.01984$^{\{3\}}$	16.93759$^{\{4\}}$	16.96552$^{\{5\}}$	11.84976$^{\{1\}}$
		β	40.05094$^{\{5\}}$	20.26527$^{\{3\}}$	19.62287$^{\{2\}}$	19.47664$^{\{1\}}$	21.69928$^{\{4\}}$
	MRE	α	3.836494$^{\{2\}}$	4.106999$^{\{3\}}$	4.278628$^{\{4\}}$	4.280606$^{\{5\}}$	3.468399$^{\{1\}}$
		β	1.483632$^{\{5\}}$	1.084892$^{\{1\}}$	1.110649$^{\{4\}}$	1.108724$^{\{2\}}$	1.089963$^{\{3\}}$
	∑ Ranks		21$^{\{3.5\}}$	15$^{\{2\}}$	22$^{\{5\}}$	21$^{\{3.5\}}$	11$^{\{1\}}$
250	Bias	α	2.231659$^{\{2\}}$	2.465664$^{\{3\}}$	2.583997$^{\{4\}}$	2.586135$^{\{5\}}$	2.072589$^{\{1\}}$
		β	-0.38538$^{\{5\}}$	-0.90880$^{\{2\}}$	-0.76400$^{\{3\}}$	-0.76192$^{\{4\}}$	-0.94675$^{\{1\}}$
	MSE	α	11.24365$^{\{2\}}$	13.03789$^{\{3\}}$	15.38480$^{\{4\}}$	15.44269$^{\{5\}}$	9.728149$^{\{1\}}$
		β	17.88656$^{\{5\}}$	9.784375$^{\{1\}}$	10.16574$^{\{3\}}$	10.24414$^{\{4\}}$	10.08138$^{\{2\}}$
	MRE	α	3.626821$^{\{2\}}$	3.930130$^{\{3\}}$	4.143781$^{\{4\}}$	4.147059$^{\{5\}}$	3.301171$^{\{1\}}$
		β	1.013145$^{\{5\}}$	0.838214$^{\{2\}}$	0.881978$^{\{3\}}$	0.882827$^{\{4\}}$	0.813041$^{\{1\}}$
	∑ Ranks		21$^{\{3\}}$	14$^{\{2\}}$	22$^{\{4\}}$	27$^{\{5\}}$	7$^{\{1\}}$
400	Bias	α	2.135412$^{\{2\}}$	2.390995$^{\{3\}}$	2.506785$^{\{5\}}$	2.505279$^{\{4\}}$	1.951929$^{\{1\}}$
		β	-0.82903$^{\{5\}}$	-1.34921$^{\{1\}}$	-1.22884$^{\{3\}}$	-1.22931$^{\{2\}}$	-1.19632$^{\{4\}}$
	MSE	α	9.504957$^{\{2\}}$	11.62150$^{\{3\}}$	13.39402$^{\{5\}}$	13.35874$^{\{4\}}$	8.086142$^{\{1\}}$
		β	11.63483$^{\{5\}}$	4.535450$^{\{1\}}$	5.525895$^{\{3\}}$	5.508352$^{\{2\}}$	6.047363$^{\{4\}}$
	MRE	α	3.416373$^{\{2\}}$	3.756748$^{\{3\}}$	3.953702$^{\{4\}}$	3.953702$^{\{5\}}$	3.088674$^{\{1\}}$
		β	0.874089$^{\{5\}}$	0.714739$^{\{1\}}$	0.759298$^{\{4\}}$	0.759093$^{\{3\}}$	0.732491$^{\{2\}}$
	∑ Ranks		21$^{\{4\}}$	12$^{\{1\}}$	24$^{\{5\}}$	20$^{\{3\}}$	13$^{\{2\}}$
600	Bias	α	2.182044$^{\{2\}}$	2.247993$^{\{3\}}$	2.410164$^{\{4\}}$	2.410374$^{\{5\}}$	2.006760$^{\{1\}}$
		β	-1.43681$^{\{5\}}$	-1.56168$^{\{1\}}$	-1.50405$^{\{3\}}$	-1.50344$^{\{4\}}$	-1.51919$^{\{2\}}$
	MSE	α	8.810349$^{\{2\}}$	9.211089$^{\{3\}}$	11.36896$^{\{4\}}$	11.37717$^{\{5\}}$	7.579276$^{\{1\}}$
		β	4.287185$^{\{5\}}$	3.33766$^{\{2\}}$	3.836334$^{\{3\}}$	3.847589$^{\{4\}}$	3.332461$^{\{1\}}$
	MRE	α	3.414142$^{\{2\}}$	3.491927$^{\{3\}}$	3.756819$^{\{5\}}$	3.756819$^{\{4\}}$	3.121597$^{\{1\}}$
		β	0.699238$^{\{3\}}$	0.677223$^{\{2\}}$	0.706317$^{\{4\}}$	0.706567$^{\{5\}}$	0.662666$^{\{1\}}$
	∑ Ranks		19$^{\{3\}}$	14$^{\{2\}}$	23$^{\{4\}}$	27$^{\{5\}}$	7$^{\{1\}}$

Table 7. Partial and overall ranks of all the methods of estimation of the proposed distribution by various values of model parameters.

Parameter	Sample size	MLE	ADE	CME	LSE	MPS
α=2.0, β=1.0	75	5.0	2.0	4.0	3	1.0
	100	5.0	2.5	2.5	4	1.0
	150	2.0	3.0	4.0	5	1.0
	250	3.0	2.0	5.0	4	1.0
	400	2.0	3.0	4.5	4.5	1.0
	600	3.0	2.0	4.5	4.5	1.0
α=2.15, β=1.95	75	3.0	2.0	4.0	5.0	1.0
	100	3.0	2.0	4.0	5.0	1.0
	150	2.0	3.0	4.0	5.0	1.0
	250	2.0	3.0	5.0	4.0	1.0
	400	2.0	3.0	5.0	4.0	1.0
	600	2.0	3.0	4.0	4.0	1.0
α=0.5 β=2.0	75	5.0	4.0	1.0	3.0	2.0
	100	5.0	3.0	4.0	2.0	1.0
	150	5.0	2.0	3.5	3.5	1.0
	250	5.0	2.0	3.5	3.5	1.0
	400	4.0	1.0	5.0	3.0	2.0
	600	5.0	1.0	3.5	3.5	2.0
α=0.75, β=2.8	75	3.0	2.0	4.5	4.5	1.0
	100	3.5	2.0	3.5	5.0	1.0
	150	3.0	2.0	4.5	4.5	1.0
	250	3.0	2.0	3.5	3.5	1.0
	400	3.0	2.0	4.0	5.0	1.0
	600	3.0	2.0	4.5	4.5	1.0
α=0.65, β=2.5	75	3.0	2.0	4.5	4.5	1.0
	100	3.5	2.0	3.5	5.0	1.0
	150	3.5	2.0	5.0	3.5	1.0
	250	3.0	2.0	4.0	5.0	1.0
	400	4.0	1.0	5.0	3.0	2.0
	600	3.0	2.0	4.0	5.0	1.0
∑ Ranks		101.5	66.5	121.5	123.5	34
Overall rank		3	2	4	5	1

8. Real data application

This section examines the flexibility and practical applicability of the PGDUS-TL distribution on a real-world data set. Its adaptability is assessed by comparing its goodness-of-fit with that of other related distributions present in the literature, whose PDF and CDF denoted by $g(x)$ and $G(x)$ are given as follows.

- Power generalized DUS Lindley distribution (PGDUSL):

$$g(x)=\frac{\beta\theta^2(1+x)e^{-\theta x}e^{1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}}}{(\theta+1)(e-1)^\beta}\left(e^{1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}}-1\right)^{\beta-1},\ x>0\ \ \theta,\beta>0,$$

$$G(x)=\left(\frac{e^{1-\left(1+\frac{\theta x}{\theta+1}\right)e^{-\theta x}}-1}{e-1}\right)^\beta.$$

- Power generalized DUS Rayleigh distribution (PGDUSRL):

$$g(x)=\frac{\beta x e^{\left(\frac{-x^2}{2\theta^2}\right)}}{\theta^2(e-1)^\beta}\left(e^{1-e^{\left(\frac{-x^2}{2\theta^2}\right)}}-1\right)^{\beta-1}e^{1-e^{\left(\frac{-x^2}{2\theta^2}\right)}},x>0\ \ \theta,\beta>0,\ G(x)=\left(\frac{e^{1-e^{\left(\frac{-x^2}{2\theta^2}\right)}}-1}{e-1}\right)^\beta.$$

- Power generalized DUS exponential distribution (PGDUSEX) [34]:

$$g(x)=\frac{\beta\lambda e^{1-\lambda x-e^{-\lambda x}}\left(e^{1-e^{-\lambda x}}-1\right)^{\beta-1}}{(e-1)^\beta},x>0,\lambda,\beta>0,\ G(x)=\left(\frac{e^{1-e^{-\lambda x}}-1}{e-1}\right)^\beta.$$

- Topp-Leone distribution (TL) [27]:

$$g(x)=2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1},0<x<1,\alpha>0,\ G(x)=x^\alpha(2-x)^\alpha.$$

- Unit Lindley distribution (UL) [40]:

$$g(x)=\frac{\theta^2}{1+\theta}(1-x)^{-3}e^{\left(\frac{-\theta x}{1-x}\right)},0<x<1,\theta>0,$$

$$G(x)=1-\left(1-\frac{\theta x}{(1+\theta)(1-x)}\right)e^{\left(\frac{-\theta x}{1-x}\right)}.$$

- Bounded Rayleigh distribution (BRL):

$$g(x)=\frac{2\lambda x}{(1-x)^3}e^{-\lambda\left(\frac{x}{1-x}\right)^2},0<x<1,\lambda>0,G(x)=1-e^{\left[-\lambda\left(\frac{x}{1-x}\right)^2\right]}.$$

- Lindley distribution (LD) [40]:

$$g(x)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x},x>0,\theta>0,\ G(x)=1-\frac{e^{-\theta x}(1+\theta+\theta x)}{\theta+1}.$$

- Rayleigh distribution (RL) [41]:

$$g(x)=\frac{x}{\theta^2}e^{\left(\frac{-x^2}{2\theta^2}\right)},x>0,\theta>0,\ G(x)=1-e^{\left(\frac{-x^2}{2\theta^2}\right)}.$$

- Exponential distribution (ED):

$$g(x)=\lambda e^{-\lambda x},x>0,\lambda>0,\ G(x)=1-e^{-\lambda x}.$$

- Unit Weibull distribution (UWL) [42]:

$$g(x)=\frac{1}{x}\alpha\beta(-\log x)^{\beta-1}e^{-\alpha(-\log x)^\beta},0<x<1,\ \alpha,\beta>0,\ G(x)=e^{-\alpha(-\log x)^\beta}.$$

- Kumaraswamy distribution (KUM) [43]:

$$g(x) = abx^{a-1}(1-x^a)^{b-1}, 0 < x < 1, a, b > 0,\ G(x) = 1-(1-x^a)^b.$$

- Beta distribution (B) [44]:

$$g(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 < x < 1, \alpha, \beta > 0.$$

$B(\alpha,\beta)$ is the beta function.

$$G(x) = \frac{B(x;\alpha,\beta)}{B(\alpha,\beta)}.$$

$B(x;\alpha,\beta)$ is the lower incomplete beta function $\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt$.

- Topp-Leone Weibull distribution (TLWL) [30]:

$$g(x) = 2\theta\beta\alpha x^{\beta-1}e^{-2\alpha x^\beta}\left(1-e^{-2\alpha x^\beta}\right)^{\theta-1} 0 < x < 1, \theta, \alpha, \beta > 0,\ G(x) = \left(1-e^{-2\alpha x^\beta}\right)^{\theta}.$$

- Exponentiated Topp-Leone distribution (ExTL) [28]:

$$g(x) = \frac{2\nu\beta}{b}\left(1-\frac{x}{b}\right)\left[\frac{x}{b}\left(2-\frac{x}{b}\right)\right]^{\nu-1}\left[1-\left(\frac{x}{b}\right)^{\nu}\left(2-\frac{x}{b}\right)^{\nu}\right]^{\beta-1} 0 < x < b \quad \nu, \beta > 0,$$

$$G(x) = 1-\left[1-\left(\frac{x}{b}\right)^{\nu}\left(2-\frac{x}{b}\right)^{\nu}\right]^{\beta}.$$

- Bounded exponential distribution (BEX):

$$g(x) = \frac{\lambda}{(1-x)^2}e^{\left(\frac{-\lambda x}{1-x}\right)}, 0 < x < 1, \lambda > 0,\ G(x) = 1-e^{\left(\frac{-\lambda x}{1-x}\right)}.$$

To evaluate model performance and identify the most suitable distribution for modelling real-life data, several information criteria and goodness-of fit measures are employed. These include the log-likelihood (-2logL), Akaike information criterion (AIC), corrected Akaike information criterion (AICC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Cramer-von Mises statistic (CVMS), Anderson-Darling statistic (ADS), the Kolmogorov-Smirnov (K-S) test, and the corresponding K-S p-value.

These criteria are defined as follows:

$$AIC = -2ln(l) + 2\omega,\ AICC = AIC + \frac{2\omega^2+2\omega}{n-\omega-1},$$

$$BIC = -2ln(l) + \omega ln(n),\ HQIC = -2ln(l) + 2\omega\ ln\{ln(n)\},$$

where ω represents number of parameters involved in the model and *n* denotes the number of observations in the given data set.

The distribution that yields the smallest values of these criteria is considered the most appropriate model for fitting the data.

Ethical consideration in data collection:

The data analysed in this study are secondary data obtained from previously published and publicly available sources, all of which are appropriately cited in the manuscript. No new human participants were involved, and therefore ethical approval and informed consent were not required.

Data-set: This data pertains to the ordered life time (in years) of patients suffering from blood cancer (leukaemia) from the Ministry of Health hospital in Saudi Arabia.

The dataset was normalized to the interval (0,1) using the transformation $x_i = [y_i - min(y_i)]/[max(y_i) - min(y_i)]$ as described in [14] for $i = 1, 2, \ldots, 40,$ where y_i is the value on the original dataset given in [45], which has 40 observations. Though the original dataset had 40 observations, two boundary values were excluded after normalization, as domain values of the model are between 0 and 1, and the final analysis was based on 38 observations as given in Table 8.

Table 8. Life time of leukaemia patients.

0.0363	0.0604	0.1665	0.1791	0.1901	0.2204	0.3430	0.3452
0.3704	0.3803	0.4122	0.4446	0.4769	0.4994	0.5205	0.5209
0.5770	0.5913	0.6083	0.6083	0.6241	0.6386	0.6456	0.6891
0.6923	0.7106	0.7363	0.7489	0.7880	0.8039	0.8155	0.8177
0.8187	0.8688	0.8901	0.9254	0.9342	0.9545		

The summary statistics for the data set are presented in Table 9. In Table 10, we have summarized the MLE of the parameter(s) of the distributions mentioned above for the underlying data set.

Table 9. Summary statistics of life time of leukaemia patients.

Min.	Q_1	Q_2	Q_3	Max.	Mean	Variance	C.V	Skewness	Kurtosis
0.0363	0.3883	0.6083	0.7782	0.9545	0.5698	0.0641	0.444	-0.4308	2.2675

Table 10. Distribution parameter estimation for life time of leukaemia patients.

Model	Parameter	MLE
PGDUS-TL	α	3.81474
	β	0.56342
PGDUSL	θ	3.98499
	β	2.28106
PGDUSRL	θ	0.39912
	β	0.93260
PGDUSEX	λ	3.41673
	β	2.37986
TL	α	2.54581
UL	θ	0.53914
BRL	λ	0.03608
LD	θ	2.28856
RL	θ	0.43997
ED	λ	1.75492
UWL	α	1.34106
	β	1.11757
KUM	a	1.63964
	b	1.34659

Continued on next page

Model	Parameter	MLE
B	α	1.65966
	β	1.31232
TLWL	α	1.64518
	β	10.79268
	θ	0.14058
ExTL	b	0.95481
	ν	1.51956
	β	0.46750
BEX	λ	0.32681

The summary statistics for the data set is presented in Table 9. In Table 10, we have summarized the MLE of the parameter(s) of the distributions mentioned above for the underlying data set.

For the data set, the modelling performance of the proposed model was similarly evaluated using model selection criteria and goodness-of-fit measures. As shown in Table 11, the PGDUS-TL distribution yields the smallest statistic values and the highest p-value. The graphical analyses in Figures 13–19 further confirm that the model provides a good fit to the above dataset.

Table 11. Distribution performance and information criterion values based on life time of leukaemia patients.

Model	-2logL	AIC	BIC	AICC	HQIC	K-S statistic	CVMS	ADS	p-value
PGDUS-TL	-1.444	2.5556	5.8308	2.8985	3.7209	0.0879	0.0438	0.3072	0.930
PGDUSL	12.992	16.992	20.267	17.335	18.157	0.1534	0.2336	1.4512	0.332
PGDUSRL	4.2359	8.2359	11.511	8.5787	9.4011	0.1330	0.1142	0.7628	0.512
PGDUSEX	13.868	17.867	21.143	18.211	19.033	0.1571	0.2466	1.5230	0.305
TL	3.5387	5.5387	7.1762	5.6498	6.1213	0.1843	0.0430	0.2964	0.151
ULD	8.784	10.784	12.421	10.895	11.3668	0.2187	0.1177	0.7504	0.052
BRL	88.427	90.427	92.065	90.538	91.010	0.4924	0.2073	1.2693	1.9e-08
LD	30.506	32.506	34.144	32.617	33.089	0.3791	0.4535	2.5617	0.003
RL	7.1091	9.1091	10.746	9.2202	9.6918	0.2652	0.3442	1.9979	0.090
ED	33.255	35.255	36.893	35.366	35.838	0.3963	0.4760	2.6761	0.001
UWL	-4.033	-0.0336	3.2415	0.3092	1.1316	0.0818	0.0455	0.3125	0.961
KUM	-4.874	-0.8747	2.4004	-0.531	0.2905	0.0829	0.0348	0.2450	0.956
B	-4.765	-0.7653	2.5098	-0.422	0.3999	0.0840	0.0365	0.2557	0.951
TLWL	-4.661	1.33859	6.2513	2.0443	3.0865	0.0724	0.0233	0.1809	0.988
ExTL	-8.352	-2.3527	2.5599	-1.646	-0.604	0.0868	0.0419	0.2857	0.936
BEX	-1.182	0.8174	2.4550	0.9285	1.400	0.15663	0.0750	0.4877	0.308

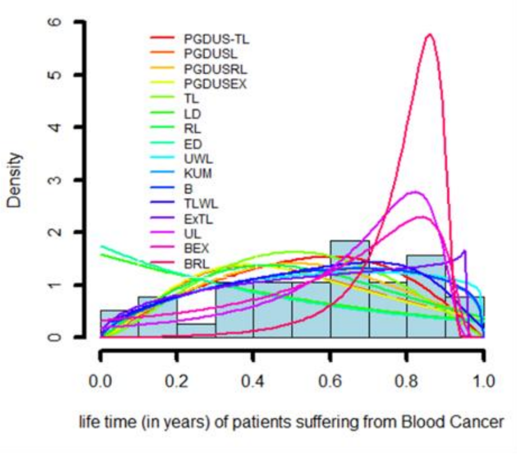

Figure 13. Histogram and density plot of data.

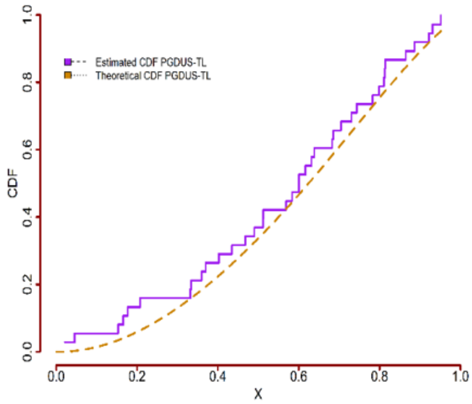

Figure 14. Estimated and theoretical CDF plot of data

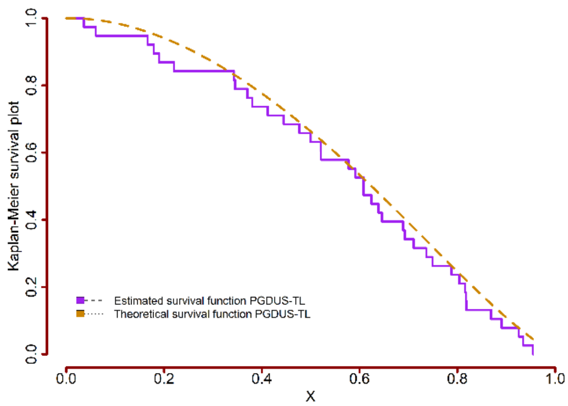

Figure 15. Theoretical and estimated survival function of data.

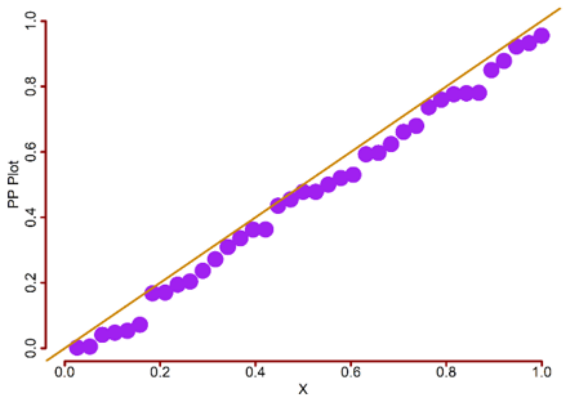

Figure 16. P-P plot of PGDUS-TL distribution for data.

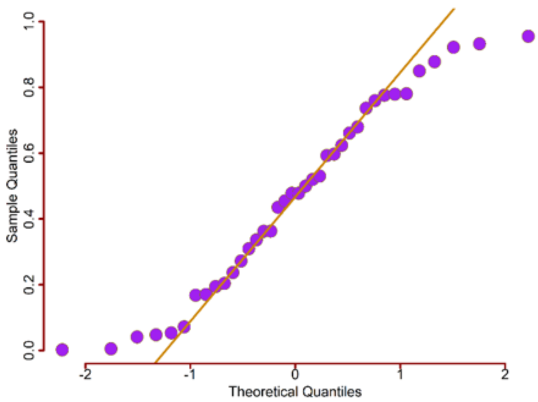

Figure 17. Q-Q plot of PGDUS-TL distribution of data.

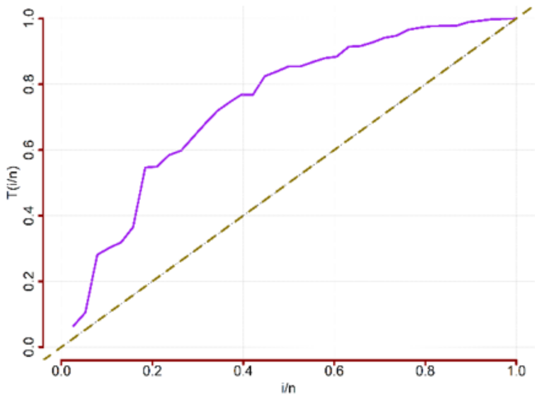

Figure 18. TTT plot of PGDUS-TL distribution of data.

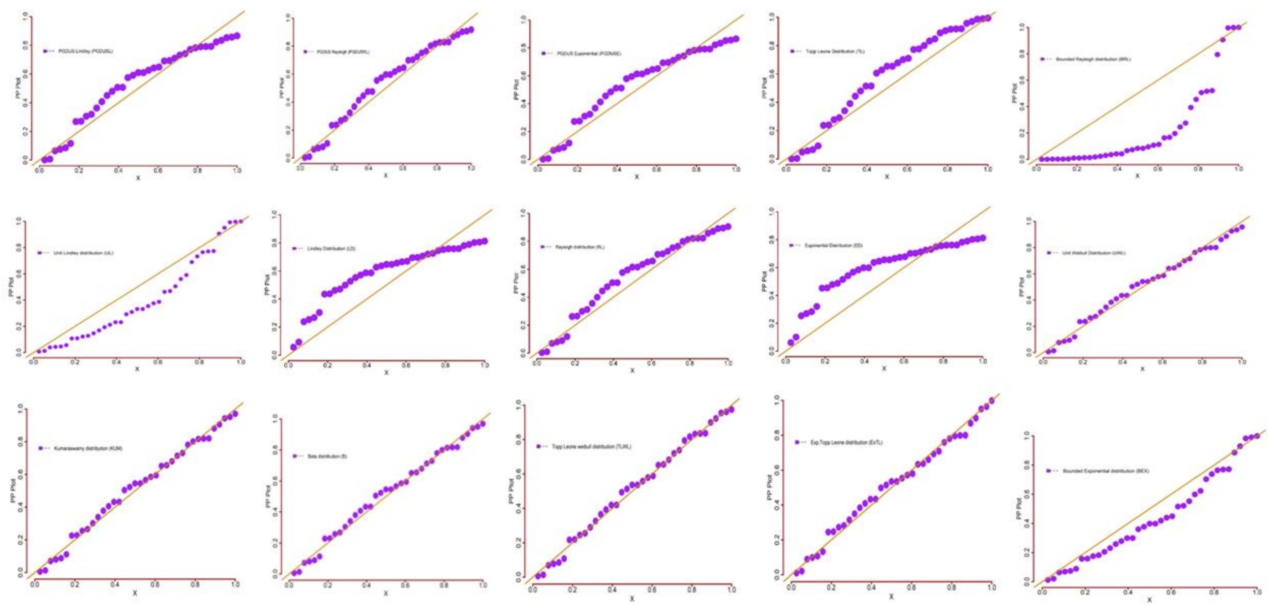

Figure 19. PP plots of competitive models for data.

9. Conclusions

This paper introduces the PGDUS-TL distribution as a flexible extension of the classical Topp-Leone distribution, constructed via the PGDUS transformation approach. The fundamental statistical properties of the proposed model are thoroughly investigated. To obtain accurate and reliable parameter estimates, multiple estimation techniques are considered and comparatively evaluated. An extensive Monte Carlo simulation study is conducted to assess the efficiency and stability of these estimators, with the MPS method emerging as the most reliable due to its lower bias, smaller MSE, and reduced MRE. The practical applicability of the PGDUS-TL distribution is demonstrated through real data analysis in the context of medical domain.

A comparative study was made with established competing models, including the PGDUS Rayleigh distribution, PGDUS Lindley distribution, PGDUS exponential distribution, bounded Rayleigh distribution, unit Lindley distribution, Topp-Leone distribution, unit Weibull distribution, Kumaraswamy distribution, beta distribution, Topp-Leone Weibull distribution, exponentiated Topp-Leone distribution, bounded exponential distribution, Lindley distribution, Rayleigh distribution, and exponential distribution. Among the above-mentioned competitors and for the analysed dataset, the proposed model proved a better fit over the baseline Topp-Leone distribution, some of PGDUS transformation distributions such as the PGDUS Rayleigh, PGDUS Lindley, and PGDUS exponential, bounded distributions such as the bounded Rayleigh distribution, and unit Lindley distribution; and base distributions like the Lindley distribution, Rayleigh distribution, and exponential distribution. However, for the analysed dataset, a few of the considered competing models outperformed the proposed model in terms of the selected goodness-of-fit measures. Nevertheless, the proposed model yet effectively captures the underlying structural characteristics of the data. Hence the PGDUS-TL can be considered as a good modelling tool with applications in applied disciplines.

Future research may focus on the further theoretical development of the PGDUS-TL distribution by exploring potential extensions, generalizations, and alternative formulations. This includes a comprehensive investigation of Bayesian frameworks for parameter estimation, as well as stress strength reliability properties. Additionally, extending the model to reliability, and risk-based decision-making contexts would broaden its applicability. Such advancements would deepen the theoretical understanding of the distribution while significantly enhancing its practical relevance across diverse real-world applications.

Author contributions

Vidya Yerneni: Conceptualization, Methodology, Formal analysis, Software, Investigation, Data curation, Writing-original draft preparation; Aafaq A. Rather: Conceptualization, Supervision, Validation, Writing- review & editing, Visualization, Project administration; Sadiah M. Aljeddani: Investigation, Data curation, Visualization; M. I. Khan: Methodology, Software; Alaa A. Elnazer: Formal analysis, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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