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Research article

Regularity of weak solution to discontinuous sub-elliptic systems with Orlicz growth on the Heisenberg group

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Abstract: In this paper, we develop a unified approach for studying subelliptic systems with vanishing mean oscillation (VMO) discontinuous coefficients under Orlicz growth on the Heisenberg group, covering both the superquadratic and subquadratic cases. We establish an $\mathcal{A}$-harmonic approximation adapted to the Orlicz growth framework on the Heisenberg group. Furthermore, assuming that the nonhomogeneous term satisfies a controllable growth condition, we obtain the partial Hölder regularity of weak solutions.

Keywords: Heisenberg group; Orlicz growth; VMO discontinuous coefficients; $\mathcal{A}$-harmonic approximation; Hölder continuity
Mathematics Subject Classification: 35B65, 35H20

1. Introduction

We consider the partial Hölder continuity of the weak solution of a nonlinear divergent subelliptic system on the Heisenberg group with Orlicz growth. The system is as follows:

$$-\sum_{i=1}^{2n} X_i A_i^\alpha(\xi, u, Xu) = B^\alpha(\xi, u, Xu) \quad \text{in } \Omega, \quad \alpha = 1, 2, \cdots, N, \tag{1.1}$$

where $\xi \in \Omega$, $\Omega \subset \mathbb{H}^n = \mathbb{R}^{2n+1}$ is a bounded domain and

$$A = (A_i^\alpha)\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{2n\times N} \to \mathbb{R}^{2n\times N}, \quad B = (B^\alpha)\colon \Omega \times \mathbb{R}^N \times \mathbb{R}^{2n\times N} \to \mathbb{R}^N,$$

$X = (X_1, X_2, \dots, X_{2n})$ denotes the horizontal gradient on the Heisenberg group (see Section 2.1 for a precise definition).

The N-function φ plays a fundamental role in Orlicz growth. We begin by stating the structural assumptions that the N-function φ should satisfy.

Assumption 1.1. *Let $\varphi \in C^1([0,\infty)) \cap C^2(0,\infty)$ be a N-function satisfying the following condition:*

$$0 < q_0 - 1 \le \inf_{t>0} \frac{t\varphi''(t)}{\varphi'(t)} \le \sup_{t>0} \frac{t\varphi''(t)}{\varphi'(t)} \le q_1 - 1, \quad \textit{where } \ 1 < q_0 < 2 \le q_1. \tag{1.2}$$

We then impose the following assumptions on the coefficients A_i^α and B^α in System (1.1).

(A1) There is a positive constant Λ, such that

$$|A_i^\alpha(\xi, u, P)| \le \Lambda\varphi'(1 + |P|), \quad \text{where } (\xi, u, P) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{2n\times N}.$$

(A2) The principal coefficient $A_i^\alpha(\xi, u, P)$ satisfies the following ellipticity condition. There is a constant ν such that

$$\sum_{i,j=1}^{2n} \sum_{\alpha,\beta=1}^{N} D_P A_i^\alpha(\xi, u, P)\eta_i^\alpha \eta_j^\beta \ge \nu\varphi''(1 + |P|)|\eta|^2, \quad \text{for } \forall \eta \in \mathbb{R}^{2n\times N},$$

where D_P denotes the partial derivative with respect to the gradient variable P.

(A3) The principal coefficient $A_i^\alpha(\xi, u, P)$ is differentiable with respect to the third variable P and $\forall \xi \in \Omega, u \in \mathbb{R}^N, P \in \mathbb{R}^{2n\times N}$, such that the following holds:

$$|D_P A_i^\alpha(\xi, u, P)| \le \Lambda\varphi''(1 + |P|),$$

where Λ is the same structural constant as defined in (A1).

(A4) There is a constant $\alpha_0 \in (0, 1)$ such that $\forall P, P_0 \in \mathbb{R}^{2n\times N}$ satisfy $0 < |P_0| \le \frac{1}{2}|P|$, and the following holds:

$$|D_P A_i^\alpha(\xi, u, P) - D_P A_i^\alpha(\xi, u, P + P_0)| \le \Lambda\varphi''(1 + |P|)\left(\frac{|P_0|}{|P|}\right)^{\alpha_0}.$$

(A5) There is a nondecreasing and bounded concave function $\mu : [0,\infty) \to [0, 1]$ and $\mu(0) = 0$, such that

$$|A_i^\alpha(\xi, u, P) - A_i^\alpha(\xi, u_0, P)| \le \Lambda\mu(|u - u_0|)\varphi'(1 + |P|),$$

and $\lim_{s\to 0} \mu(s) = 0 = \mu(0)$.

(A6) $P \mapsto A_i^\alpha(\xi, u, P)/\varphi'(1 + |P|)$, there is a positive constant L, nondecreasing, and bounded concave function v_{ξ_0}, such that

$$\left|A_i^\alpha(\xi, u, P) - (A_i^\alpha(\cdot, u, P))_{\xi_0,\rho}\right| \le L v_{\xi_0}(\xi, \rho)\varphi'(1 + |P|), \quad \text{for } \forall\ \xi \in B_\rho(\xi_0),$$

where $\xi_0 \in \Omega, \rho \in (0, \rho_0], u \in \mathbb{R}^N$ and $P \in \mathbb{R}^{2n\times N}, \rho_0 > 0$ and $v_{\xi_0} : \mathbb{R}^{2n+1} \times [0, \rho_0] \to [0, 2L]$,

$$\lim_{\rho\to 0} V(\rho) = 0, \quad \text{where} \quad V(\rho) = \sup_{\xi_0\in\Omega} \sup_{0<\rho\le\rho_0} \fint_{B_\rho(\xi_0)} v_{\xi_0}(\xi, \rho)d\xi.$$

We define

$$(A_i^\alpha(\cdot, u, P))_{\xi_0,\rho} := \fint_{B_\rho(\xi_0)} A_i^\alpha(\xi, u, P)d\xi.$$

(B1) Controllable growth. $|B(\xi, u, P)| \le L\varphi'(1 + |P|), \forall \xi \in \Omega, u \in \mathbb{R}^N,$ and $P \in \mathbb{R}^{2n\times N}$.

The regularity of weak solutions to partial differential equations (PDEs) has long been a fundamental area of research, with extensive results established for both elliptic and parabolic systems. It is well known that in the absence of specific structural conditions, weak solutions generally exhibit only partial regularity—a concept pioneered by De Giorgi [1]. To prove such regularity, the harmonic approximation technique introduced by Simon [2] has proven to be an effective tool. Subsequently, Duzaar and Grotowski [3] extended this approach to develop the $\mathcal{A}$-harmonic approximation method. In the Euclidean setting, analogous regularity results have been established under various assumptions on the coefficients: Hölder continuity [4–5], Dini continuity [6–9], and vanishing mean oscillation (VMO) discontinuity [10–13].

Subelliptic systems with standard p-growth on noncommutative vector fields, particularly those defined on the Heisenberg group and Carnot groups, have been extensively investigated (e.g., [14–25]). The Heisenberg group, as one of the simplest noncommutative nilpotent Lie groups with a step of two, serves as a prototypical model in this context. Existing studies have explored standard growth under diverse coefficient conditions. Specifically, the case of superquadratic growth was addressed by Wang and Manfredi [19], Föglein [21], and Zhang, Wang [24], while subquadratic growth was studied by Tan and Wang [23] and Chen et al.[25].

Nonstandard growth conditions for elliptic systems in Euclidean space have also been widely studied, for instance, in references [26–32]. In particular, results concerning Orlicz growth can be found in [29–32]. Notably, while these studies are well-established in the Euclidean setting, the partial regularity of weak solutions to subelliptic systems with Orlicz growth on noncommutative vector fields (such as the Heisenberg group) remains largely unexplored. Therefore, this paper focuses on the Heisenberg group to analyze the partial Hölder continuity of weak solutions to System (1.1) under Orlicz growth, assuming that the coefficients satisfy (A1)–(A6) and (B1).

In this work, we use the $\mathcal{A}$-harmonic approximation technique to establish the partial regularity for weak solutions under Orlicz growth on the Heisenberg group. The primary challenge lies in addressing the interplay between the noncommutativity of the vector fields and the inherent nonlinearity of the systems. Furthermore, we establish an $\mathcal{A}$-harmonic approximation theorem tailored for Orlicz growth on the Heisenberg group. By utilizing this classical approximation method, we bypass the complexities of applying Taylor expansions in the noncommutative setting. The principal innovation of our work is the development of a unified framework for studying subelliptic systems under Orlicz growth, covering both superquadratic and subquadratic cases within the Heisenberg group setting.

First, we introduce two basic spaces $HW^{1,\varphi}(\Omega, \mathbb{R}^N)$ and $L^{\varphi}(\Omega, \mathbb{R}^N)$; further details can be found in [33].

Definition 1.1. *(Horizontal Sobolev-Orlicz space) Let* $\Omega \subset \mathbb{H}^n = \mathbb{R}^{2n+1}$ *be an open set and let* φ *be a N-function. The horizontal Sobolev-Orlicz space* $HW^{1,\varphi}(\Omega, \mathbb{R}^N)$ *is defined by*

$$HW^{1,\varphi}(\Omega, \mathbb{R}^N) = \{u \in L^{\varphi}(\Omega, \mathbb{R}^N) | X_i u \in L^{\varphi}(\Omega, \mathbb{R}^N), i = 1, \cdots, 2n\},$$

endowed with the following modulo

$$\|u\|_{HW^{1,\varphi}(\Omega,\ \mathbb{R}^N)} = \|u\|_{L^{\varphi}(\Omega,\ \mathbb{R}^N)} + \sum_{i=1}^{2n} \|X_i u\|_{L^{\varphi}(\Omega,\ \mathbb{R}^N)}.$$

The resulting space is known as the horizontal Sobolev space on the Heisenberg group $\mathbb{H}^n$. It is readily verified that $HW^{1,\varphi}(\Omega, \mathbb{R}^N)$ is a Banach space.

Definition 1.2. *Let φ be a N-function satisfying the Δ_2-condition, define $L^{\varphi}(\Omega, \mathbb{R}^N)$ as*

$$L^{\varphi}(\Omega, \mathbb{R}^N) = \left\{ u : \Omega \to \mathbb{R}^N \text{ measurable} : \int_{\Omega} \varphi(|u|) d\xi < \infty \right\}.$$

The corresponding Luxembourg norm is defined as

$$\|u\|_{L^{\varphi}(\Omega, \mathbb{R}^N)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \varphi\left(\frac{|u(\xi)|}{\lambda} \right) d\xi \leq 1 \right\}.$$

Under this norm, $L^{\varphi}(\Omega, \mathbb{R}^N)$ is a Banach space.

Now, we formulate the definition of weak solutions for System (1.1) and outline our main regularity theorem.

Definition 1.3. *Let $u \in HW^{1,\varphi}(\Omega, \mathbb{R}^N)$ be a weak solution of the System (1.1), $\Omega \subset \mathbb{H}^n = \mathbb{R}^{2n+1}$ be an open set, the following integral equality holds:*

$$\int_{\Omega} A_i^{\alpha}(\xi, u, Xu) X_i \eta^{\alpha} d\xi = \int_{\Omega} B^{\alpha}(\xi, u, Xu) \eta^{\alpha} d\xi, \quad \text{for } \forall\ \eta \in C_0^{\infty}(\Omega, \mathbb{R}^N). \tag{1.3}$$

Theorem 1.1. *Let φ be a N-function satisfying Assumption 1.1, $u \in HW^{1,\varphi}(\Omega, \mathbb{R}^N)$ is a weak solution of the System (1.1) and satisfies (A1)–(A6), (B1), the bounded domain $\Omega \subset \mathbb{H}^n$, and the ball $B_{\rho}(\xi_0) \subset \Omega$, where $\rho \in (0, \rho_0]$, then an open set $\Omega_0 \subset \Omega$ exists such that*

$$u \in C_{loc}^{0,\alpha}(\Omega_0, \mathbb{R}^N),$$

among $\alpha \in (0, 1)$ and

$$\Omega \setminus \Omega_0 \subset \Sigma_1 \cup \Sigma_2, \quad meas(\Omega \setminus \Omega_0) = 0,$$

where Σ_1 and Σ_2 are defined as follows:

$$\Sigma_1 = \left\{ \xi_0 \in \Omega : \sup_{0<\rho\leq\rho_0} |(Xu)_{\xi_0,\rho}| = +\infty \right\},$$

$$\Sigma_2 = \left\{ \xi_0 \in \Omega : \limsup_{\rho\to 0^+} \fint_{B_{\rho}(\xi_0)} |V(Xu) - (V(Xu))_{\xi_0,\rho}|^2 d\xi > 0 \right\}.$$

The remainder of this paper is organized as follows. In Section 1, we introduce the main research content, the current state of the field, and the primary results of this study. In Section 2, we provide the notation of the Heisenberg group, N-functions, and several fundamental inequalities and lemmas essential for our analysis. In Section 3, we first establish an $\mathcal{A}$-harmonic approximation lemma that is compatible with Orlicz growth, along with Sobolev-Poincaré and Caccioppoli-type inequalities on the Heisenberg group with noncommutative vector fields. By virtue of the established approximation lemma, we have a comparison between the weak solutions of the subelliptic system in (1.1) and the solutions of constant-coefficient linear systems to develop a linearization technique. Subsequently, decay and iteration estimates for the excess functional are derived. We then can establish a partial regularity result for the weak solutions of the subelliptic system in (1.1). Finally, in Section 4, we present the conclusion of this research.

2. Preliminaries

This section prepares the necessary background on the Heisenberg group and the N-function φ, together with essential properties of horizontal affine functions and several auxiliary lemmas required for the subsequent analysis.

2.1. Heisenberg group

The Heisenberg group $\mathbb{H}^n$ is defined in $\mathbb{R}^{2n+1}$, with the following group multiplication operation:

$$(\bar{\xi},\bar{\tau})\cdot(\bar{\eta},\tilde{\tau})=\left(\bar{\xi}+\bar{\eta},\bar{\tau}+\tilde{\tau}+\frac{1}{2}\sum_{i=1}^{n}\left(x^i\tilde{y}^i-\tilde{x}^iy^i\right)\right),$$

which $\xi=(\bar{\xi},\bar{\tau})=(x^1,x^2,\cdots,x^n,y^1,y^2,\cdots,y^n,\bar{\tau})$, and $\eta=(\bar{\eta},\tilde{\tau})=(\tilde{x}^1,\tilde{x}^2,\cdots,\tilde{x}^n,\tilde{y}^1,\tilde{y}^2,\cdots,\tilde{y}^n,\tilde{\tau})$. The zero element of the group is 0, and the inverse of $(\bar{\xi},\bar{\tau})$ is $(-\bar{\xi},-\bar{\tau})$. Specifically, the mapping $(\xi,\eta)\mapsto\xi\cdot\eta^{-1}$ is smooth; therefore, $(\mathbb{H}^n,\cdot)$ is a Lie group. The corresponding basis vector field of its Lie algebra is

$$X_i\equiv X_i(\xi)=\frac{\partial}{\partial x_i}-\frac{y_i}{2}\frac{\partial}{\partial\bar{\tau}},\quad X_{n+i}\equiv X_{n+i}(\xi)=\frac{\partial}{\partial y_i}+\frac{x_i}{2}\frac{\partial}{\partial\bar{\tau}},\quad T\equiv T(\xi)=\frac{\partial}{\partial\bar{\tau}},$$

among $i=1,2,\cdots,n$, and the special structure of the operators

$$[X_i,X_{i+n}]=-[X_{i+n},X_i]=T,\ \text{else}\ \ [X_i,X_j]=0,\quad [T,X_i]=0,$$

$(\mathbb{H}^n,\cdot)$ is a nilpotent Lie group with a step of two. $X=(X_1,\cdots,X_{2n})$ is the horizontal gradient and T is the vertical derivative.

The homogeneous norm of this group is defined as

$$\left\|(\bar{\xi},\bar{\tau})\right\|_{\mathbb{H}^n}=\left(\left\|\bar{\xi}\right\|^4+16\bar{\tau}^2\right)^{1/4},$$

and the quasi-metric on the Heisenberg group is derived as follows:

$$d(\eta,\xi)=\left\|\xi^{-1}\cdot\eta\right\|_{\mathbb{H}^n}.$$

The measure on the $\mathbb{H}^n$ group is a Haar measure (i.e., the Lebesgue measure on $\mathbb{R}^{2n+1}$), the volume of the homogeneous ball $B_R(\xi_0)=\{\xi\in\mathbb{H}^n:d(\xi_0,\xi)<R\}$ is denoted as:

$$|B_R(\xi_0)|_{\mathbb{H}^n}=R^{2n+2}|B_1(\xi_0)|_{\mathbb{H}^n}\triangleq\omega_nR^Q,$$

where

$$Q=2n+2$$

is the homogeneous dimension of $\mathbb{H}^n$, ω_n is the volume of homogeneous unit ball.

Remark 2.1. *Although System (1.1) is stated only in terms of the horizontal gradient Xu, the vertical vector field T enters the analysis implicitly through the homogeneous dimension $Q=2n+2$ and the underlying metric. All standard functional inequalities (e.g.,Sobolev-Poincaré,Caccioppoli estimates) used in this paper rely on the Hörmander's condition guaranteed by the bracket relations $[X_i,X_{i+n}]=T$, which ensures that the horizontal derivatives control the full subelliptic structure. The vertical derivative Tu itself is not required to be in $L^\varphi(\Omega,\mathbb{R}^N)$; thus, it does not appear explicitly in the coefficients or the weak formulation.*

2.2. N-function

In this section, we introduce the fundamental definitions and properties of the N-function, which play a crucial role in all subsequent proofs and estimates. For a more comprehensive introduction of N-function and Orlicz spaces, we refer to [33].

Definition 2.1. *A real convex function $\varphi : [0,\infty) \to [0,\infty)$ satisfies $\varphi(0) = 0$ and there is a right-continuous, nondecreasing derivative $\varphi'(t)$, $\varphi'(0) = 0$, when $t > 0$, $\varphi'(t) > 0$, $\lim\limits_{t\to\infty} \varphi'(t) = \infty$, which is called the N-function.*

Definition 2.2. *If φ_1 and φ_2 are equivalent, then the positive constants c_1 and c_2 exist such that*

$$c_1\varphi_1(t) \le \varphi_2(t) \le c_2\varphi_1(t).$$

In this context, we denote it as $\varphi_1(t) \sim \varphi_2(t)$. Similarly, the symbol $\lesssim$ indicates that the left-hand side is bounded above by a constant times of the right-hand side.

If there is a positive constant $c > 0$ such that for all $t > 0$, the inequality $\varphi(2t) \le c\varphi(t)$ holds, we will say that the N-function φ satisfies the Δ_2-condition, denoted as $\varphi \in \Delta_2$. The minimum constant c for which the inequality holds is denoted as $\Delta_2(\varphi)$.

Furthermore, if N-function φ satisfies the Δ_2-condition and $\varphi(t) \le \varphi(2t)$, we can infer $\varphi(2t) \sim \varphi(t)$.

Definition 2.3. *The conjugate function of the N-function φ is denoted as φ^* and $\varphi^* : [0,+\infty) \to [0,+\infty)$ and is defined as*

$$\varphi^*(t) := \sup_{s\ge 0}[st - \varphi(s)], \quad \textit{for } \forall\ t \ge 0.$$

We note that φ^* is also a N-function. If both φ and φ^* satisfy the Δ_2-condition, we write $\Delta_2(\varphi,\varphi^*) := \max\{\Delta_2(\varphi), \Delta_2(\varphi^*)\} < \infty$.

Lemma 2.1. *([31] Young's inequality) For all $\delta > 0$, the N-functions φ and φ^* satisfy the Δ_2-condition, there is a positive constant c_δ, which only depends on $\Delta_2(\varphi,\varphi^*)$, such that for $\forall\ s,t \ge 0$, the following inequality holds:*

$$ts \le \delta\varphi(t) + c_\delta\varphi^*(s). \tag{2.1}$$

In addition, when $t \ge 0$, the following inequalities hold:

$$s^{q_1}\varphi(t) \le \varphi(st) \le s^{q_0}\varphi(t) \quad \text{if } 0 < s \le 1, \tag{2.2}$$

$$s^{q_0}\varphi(t) \le \varphi(st) \le s^{q_1}\varphi(t) \quad \text{if } s \ge 1,$$

and

$$s^{\frac{q_0}{q_0-1}}\varphi^*(t) \le \varphi^*(st) \le s^{\frac{q_1}{q_1-1}}\varphi^*(t) \quad \text{if } 0 < s \le 1, \tag{2.3}$$

$$s^{\frac{q_1}{q_1-1}}\varphi^*(t) \le \varphi^*(st) \le s^{\frac{q_0}{q_0-1}}\varphi^*(t) \quad \text{if } s \ge 1,$$

also

$$s^{q_1-1}\varphi'(t) \le \varphi'(st) \le s^{q_0-1}\varphi'(t) \quad \text{if } 0 < s \le 1, \tag{2.4}$$

$$s^{q_0-1}\varphi'(t) \le \varphi'(st) \le s^{q_1-1}\varphi'(t) \quad \text{if } s \ge 1,$$

$$s^{\frac{1}{q_0}}\varphi^{-1}(t) \le \varphi^{-1}(st) \le s^{\frac{1}{q_1}}\varphi^{-1}(t) \quad \text{if } 0 < s \le 1, \tag{2.5}$$

$$s^{\frac{1}{q_1}}\varphi^{-1}(t) \le \varphi^{-1}(st) \le s^{\frac{1}{q_0}}\varphi^{-1}(t) \quad \text{if } s \ge 1,$$

and

$$s^{1-\frac{1}{q_1}}(\varphi^*)^{-1}(t) \le (\varphi^*)^{-1}(st) \le s^{1-\frac{1}{q_0}}(\varphi^*)^{-1}(t) \quad \text{if } 0 < s \le 1, \tag{2.6}$$

$$s^{1-\frac{1}{q_0}}(\varphi^*)^{-1}(t) \le (\varphi^*)^{-1}(st) \le s^{1-\frac{1}{q_1}}(\varphi^*)^{-1}(t) \quad \text{if } s \ge 1.$$

For $t > 0$, the following equivalences also hold:

$$\varphi(t) \sim t\varphi'(t), \quad \varphi'(t) \sim t\varphi''(t), \quad \varphi^*(\varphi'(t)) \sim \varphi(t), \quad (\varphi^*)^{-1}(t)\varphi^{-1}(t) \sim t. \tag{2.7}$$

Remark 2.2. *By the transitivity of equivalences, we further deduce the following relations:*

$$\varphi(t) \sim t^2\varphi''(t), \quad \varphi^*(\varphi'(t)) \sim \varphi^*\left(\frac{\varphi(t)}{t}\right) \sim \varphi(t). \tag{2.8}$$

Definition 2.4. *We introduce the family of shifted N-functions $\{\varphi_a\}_{a\ge 0}$, $a \in R$ and when $t \ge 0$, we define the shifted N-function $\varphi_a(t)$*

$$\varphi_a(t) := \int_0^t \frac{s\varphi'(a+s)}{a+s} ds \ \textit{ with } \ \varphi_a'(t) := \varphi'(a+t)\frac{t}{a+t}.$$

Given Assumption 1.1, for all $a \ge 0$ the following relationship holds:

$$\varphi_a(t) \sim \varphi''(a+t)t^2 \sim \frac{\varphi(a+t)}{(a+t)^2}t^2 \sim \frac{\varphi'(a+t)}{a+t}t^2 \tag{2.9}$$

$$\varphi_a(t) \sim \varphi_a'(t)t, \qquad \varphi(a+t) \sim \varphi_a(t) + \varphi(a).$$

We note that the shift function $\varphi_a(t)$ also satisfies (2.2)–(2.7). The subsequent result, namely the shift variations of the N-function $\varphi(t)$, was proven in ([34], Corollary 26).

Lemma 2.2. *Let φ be a N-function such that $\Delta_2(\varphi, \varphi^*) < +\infty$. For every $\delta > 0$, $c_\delta(\Delta_2(\varphi, \varphi^*)) > 0$ exists, such that for any $a, b \in \mathbb{R}^d$ and $t \ge 0$, we have*

$$\varphi_{|a|}(t) \le c_\delta \varphi_{|b|}(t) + \delta\varphi_{|a|}(|a-b|). \tag{2.10}$$

Lemma 2.3. *([35], Lemma 20) Let φ be a N-function such that $\Delta_2(\varphi, \varphi^*) < \infty$. For $z, z_1, z_2 \in \mathbb{R}^{2n\times N}$ and $|z_1| + |z_2| > 0$, the following equivalent relation holds:*

$$\frac{\varphi'(z + |z_1| + |z_2|)}{z + |z_1| + |z_2|} \sim \int_0^1 \frac{\varphi'(z + |z_\theta|)}{z + |z_\theta|} d\theta,$$

where $z_\theta = z_1 + \theta(z_2 - z_1), \theta \in [0, 1]$.

Based on Lemma 2.3, we give two useful remarks, which will be used extensively in the proof of the Caccioppoli-type inequality.

Remark 2.3. *Applying (A2), Lemma 2.3, (2.7), and (2.9), the following inequality holds:*

$$(A_i^\alpha(\xi, u, P) - A_i^\alpha(\xi, u, P_0))(P - P_0) \ge c\nu\varphi_{1+|P_0|}(|P - P_0|). \tag{2.11}$$

Similarly, by using (A3) and Lemma 2.3, we can also obtain the following Remark 2.4.

Remark 2.4. *For* $\forall\xi \in \Omega, u \in \mathbb{R}^N$*, P and* $P_0 \in \mathbb{R}^{2n\times N}$*, the following inequality also holds:*

$$|A_i^\alpha(\xi, u, P) - A_i^\alpha(\xi, u, P_0)| \leq c\Lambda\varphi'_{1+|P_0|}(|P - P_0|). \tag{2.12}$$

Now, we define the function V: $\mathbb{R}^{2n\times N} \to \mathbb{R}^{2n\times N}$ as

$$V(z) = \sqrt{\frac{\varphi'(1 + |z|)}{1 + |z|}}z.$$

By the monotonicity of φ, this function satisfies the crucial equivalence, i.e.

$$|V(z_1) - V(z_2)|^2 \sim \varphi_{1+|z_1|}(|z_1 - z_2|), \quad \text{for any } z_1, z_2 \in \mathbb{R}^{2n\times N}. \tag{2.13}$$

We refer the reader to [36] for further properties of the function V.

Motivated by Lemma A.2 in [36], we establish the following lemma adapted to the Heisenberg group setting. Its proof follows a nearly identical argument to that in [36]; thus we omitted the proof in this paper.

Lemma 2.4. *If we let* $\xi_0 \in B_\rho(\xi_0), u \in HW^{1,\varphi}(\Omega, \mathbb{R}^N)$*, we have*

$$⨍_{B_\rho(\xi_0)} |V(Xu) - V((Xu)_{\xi_0,\rho})|^2 d\xi \sim ⨍_{B_\rho(\xi_0)} |V(Xu) - (V(Xu))_{\xi_0,\rho}|^2 d\xi. \tag{2.14}$$

Throughout this paper, the symbol c denotes a positive constant that may vary from line to line. The dependence of c on relevant parameters is explicitly stated where needed.

2.3. Horizontal affine function and estimates

Owing to the special structure of the Heisenberg group, namely the "nonintegrability" of the horizontal distribution and the degeneracy of subelliptic operators, the classical affine functions are no longer suitable for this study, which motivates us to introduce the horizontal affine functions.

The proof of the following lemma can be found in ([19], Lemma 3.1). This lemma will be used in the subsequent proof of the decay estimate for the excess functional.

Lemma 2.5. *Assume* $u \in HW^{1,\varphi}(\Omega, \mathbb{R}^N)$ *if the affine functions* $\ell : \mathbb{R}^{2n} \to \mathbb{R}^N$*, particularly, the horizontal affine functions* $\ell_{\xi_0,\rho}(\bar{\xi}) = \ell_{\xi_0,\rho}(\bar{\xi}_0) + X\ell_{\xi_0,\rho}(\bar{\xi} - \bar{\xi}_0)$ *are minimizers of the functional*

$$\ell \to ⨍_{B_\rho(\xi_0)} |u - \ell|^2 \, d\xi,$$

then

$$\ell_{\xi_0,\rho}(\bar{\xi}_0) = u_{\xi_0,\rho} = ⨍_{B_\rho(\xi_0)} u \, d\xi,$$

and

$$X\ell_{\xi_0,\rho} = \frac{Q-2}{C_0 Q}\frac{Q+2}{\rho^2} ⨍_{B_\rho(\xi_0)} u \otimes (\bar{\xi} - \bar{\xi}_0) d\xi. \tag{2.15}$$

The elements of $u\otimes(\bar{\xi}-\bar{\xi}_0)$ *are* $u^\alpha(\xi^1-\xi_0^1,\xi^2-\xi_0^2,\cdots,\xi^{2n}-\xi_0^{2n})$, $\alpha=1,2,\cdots,N$. *The value of the constant* C_0 *can be calculated through the polar coordinate transformation on* $\mathbb{H}^n$, *i.e.*

$$C_0=\frac{\int_0^\pi(\sin\theta)^n d\theta}{\int_0^\pi(\sin\theta)^{n-1}d\theta}=\begin{cases}\frac{((2k-2)!!)^2}{(2k-1)!!(2k-3)!!}\frac{2}{\pi}, & n=2k-1,\\ \frac{((2k-1)!!)^2}{(2k)!!(2k-2)!!}\frac{\pi}{2}, & n=2k,\end{cases}$$

where $(2k-2)!!=(2k-2)(2k-4)\cdots4\times2$, $(2k-1)!!=(2k-1)(2k-3)\cdots3\times1$.

Lemma 2.6. *Let* $u\in HW^{1,\varphi}(\Omega,\mathbb{R}^N)$ *and* $\ell_{\xi_0,\rho}$, $\ell_{\xi_0,\theta\rho}$ *denote the horizontal affine functions associated with radius* ρ *and* $\theta\rho$, *respectively. We then have*

$$|X\ell_{\xi_0,\rho}-X\ell_{\xi_0,\theta\rho}|\le\frac{(Q-2)(Q+2)}{C_0Q}\frac{1}{\theta\rho}\fint_{B_{\theta\rho}(\xi_0)}|u-\ell_{\xi_0,\rho}|d\xi. \tag{2.16}$$

More generally, for any affine function $\ell:\mathbb{R}^{2n}\to\mathbb{R}^N$, *the following inequality holds:*

$$|X\ell_{\xi_0,\rho}-X\ell|\le\frac{(Q-2)(Q+2)}{C_0Q}\frac{1}{\rho}\fint_{B_\rho(\xi_0)}|u-\ell|d\xi. \tag{2.17}$$

Proof. From ([19], Lemma 3.2), we have

$$|X\ell_{\xi_0,\rho}-X\ell_{\xi_0,\theta\rho}|^2=\left(\frac{Q-2}{C_0Q}\frac{Q+2}{(\theta\rho)^2}\right)^2\left|\fint_{B_{\theta\rho}(\xi_0)}\Big(u-\ell_{\xi_0,\rho}(\tilde{\xi}_0)-X\ell_{\xi_0,\rho}(\tilde{\xi}-\tilde{\xi}_0)\Big)\otimes(\tilde{\xi}-\tilde{\xi}_0)d\tilde{\xi}\right|^2.$$

Therefore, when $|\bar{\xi}-\bar{\xi}_0|\le\theta\rho$, we obtain

$$\begin{aligned}|X\ell_{\xi_0,\rho}-X\ell_{\xi_0,\theta\rho}|&=\frac{(Q-2)(Q+2)}{C_0Q}\frac{1}{(\theta\rho)^2}\left|\fint_{B_{\theta\rho}(\xi_0)}(u-\ell_{\xi_0,\rho}(\bar{\xi}_0)-X\ell_{\xi_0,\rho}(\bar{\xi}-\bar{\xi}_0))\otimes(\bar{\xi}-\bar{\xi}_0)d\bar{\xi}\right|\\&\le\frac{(Q-2)(Q+2)}{C_0Q}\frac{1}{(\theta\rho)^2}\fint_{B_{\theta\rho}(\xi_0)}|u-\ell_{\xi_0,\rho}(\bar{\xi}_0)-X\ell_{\xi_0,\rho}(\bar{\xi}-\bar{\xi}_0)||\bar{\xi}-\bar{\xi}_0|d\bar{\xi}\\&\le\frac{(Q-2)(Q+2)}{C_0Q}\frac{1}{\theta\rho}\fint_{B_{\theta\rho}(\xi_0)}|u-\ell_{\xi_0,\rho}(\bar{\xi}_0)-X\ell_{\xi_0,\rho}(\bar{\xi}-\bar{\xi}_0)|d\bar{\xi}\\&=\frac{(Q-2)(Q+2)}{C_0Q}\frac{1}{\theta\rho}\fint_{B_{\theta\rho}(\xi_0)}|u-\ell_{\xi_0,\rho}|d\bar{\xi}.\end{aligned}$$

We then estimate (2.17).

$$\begin{aligned}|X\ell_{\xi_0,\rho}-X\ell|&=\left|\frac{(Q-2)(Q+2)}{C_0Q}\frac{1}{\rho^2}\fint_{B_\rho(\xi_0)}(u-\ell(\bar{\xi}_0)-X\ell(\bar{\xi}-\bar{\xi}_0))\otimes(\bar{\xi}-\bar{\xi}_0)d\bar{\xi}\right|\\&\le\frac{(Q-2)(Q+2)}{C_0Q}\frac{1}{\rho^2}\fint_{B_\rho(\xi_0)}|u-\ell(\bar{\xi}_0)-X\ell(\bar{\xi}-\bar{\xi}_0)||\bar{\xi}-\bar{\xi}_0|d\bar{\xi}\\&=\frac{(Q-2)(Q+2)}{C_0Q}\frac{1}{\rho}\fint_{B_\rho(\xi_0)}|u-\ell|d\bar{\xi},\end{aligned}$$

where $|\bar{\xi}-\bar{\xi}_0|\le\rho$. □

The following lemma shows that the affine function $\ell_{\xi_0,\rho}$ is the minimization function of

$$\ell \mapsto \int_{B_\rho(\xi_0)} \varphi_a\left(\frac{|u-\ell|}{\rho}\right) d\xi.$$

Lemma 2.7. *([31], Lemma 3.2) If we let $a > 0$, $B_\rho(\xi_0) \subset \Omega$, and $\rho > 0$, for any weak solution $u \in HW^{1,\varphi}(\Omega, \mathbb{R}^N)$ and affine function $\ell : \mathbb{R}^{2n} \to \mathbb{R}^N$, we get*

$$\fint_{B_\rho(\xi_0)} \varphi_a\left(\frac{|u-\ell_{\xi_0,\rho}|}{\rho}\right) d\xi \leq c \fint_{B_\rho(\xi_0)} \varphi_a\left(\frac{|u-\ell|}{\rho}\right) d\xi, \tag{2.18}$$

where $c = c(q_0, q_1) > 0$.

3. Partial $C^{0,\alpha}$ regularity

3.1. $\mathcal{A}$-harmonic approximation lemma

In this part, we extend the $\mathcal{A}$-harmonic approximation lemma with Orlicz growth, originally proved in [31] for Euclidean space, to noncommutative vector fields on the Heisenberg group. We begin by recalling the definition of an $\mathcal{A}$-harmonic function. If h is an $\mathcal{A}$-harmonic function on $B_\rho(\xi_0)$, i.e.

$$\fint_{B_\rho(\xi_0)} \mathcal{A}(Xh, X\eta) d\xi = 0, \quad \forall \eta \in C_0^\infty(B_\rho(\xi_0), \mathbb{R}^N),$$

where $\mathcal{A}$ is a bilinear form satisfying the ellipticity and boundedness conditions; see Subsection 3.4 for the specific definition.

Prior to stating the $\mathcal{A}$-harmonic approximation lemma, we recall Lemma 3.1, which was proved in ([37], Theorem 14).

Lemma 3.1. *Suppose that φ is a N-function satisfying Assumption 1.1, and let $1 < s < \frac{q_1}{q_1-q_0+1}$, $\psi(t) = \varphi^{\frac{1}{s}}(t)$, $t \geq 0$; in this case, $\psi(t)$ satisfies Assumption 1.1, where q_0 and q_1 are replaced by $(\frac{1-s}{s})q_1 + q_0$ and $(\frac{1-s}{s})q_0 + q_1$, respectively.*

Lemma 3.2. *($\mathcal{A}$-harmonic approximation lemma) Let φ be a N-function that satisfies Assumption 1.1, and let φ and φ^* satisfy the Δ_2-condition, $1 < s < \frac{q_1}{q_1-q_0+1}$, $B_\rho(\xi_0) \subset \Omega$, $\rho > 0$. Then for every $\epsilon > 0$, $\delta_0 = \delta_0(Q, N, \nu, \Lambda, \Delta_2(\varphi, \varphi^*), s, L, \epsilon) > 0$ and $\Gamma > 0$ exist, such that if $\omega \in HW^{1,\varphi}(\Omega, \mathbb{R}^N)$ is an approximate $\mathcal{A}$-harmonic function, i.e.*

$$\left|\fint_{B_\rho(\xi_0)} \mathcal{A}(X\omega, X\eta) d\xi\right| \leq \delta_0 \Gamma \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta|, \quad \forall \eta \in C_0^\infty(B_\rho(\xi_0), \mathbb{R}^N), \tag{3.1}$$

and if h satisfies

$$\begin{cases} -div_{\mathbb{H}^n} \mathcal{A}(Xh) = 0 & in \ B_\rho(\xi_0), \\ h = \omega & on \ \partial B_\rho(\xi_0), \end{cases}$$

then the following estimate holds:

$$\fint_{B_\rho(\xi_0)} \varphi^{\frac{1}{s}}(|X(\omega-h)|) d\xi \leq \epsilon \left\{ \left(\fint_{B_\rho(\xi_0)} \varphi(|X\omega|) d\xi \right)^{\frac{1}{s}} + \varphi^{\frac{1}{s}}(\Gamma) \right\}. \tag{3.2}$$

Proof. Let ψ be a N-function satisfying Assumption 1.1, from [37, Lemma 20], and for any $\varpi \in HW_0^{1,\psi}(B_\rho(\xi_0), \mathbb{R}^N)$ satisfies

$$\fint_{B_\rho(\xi_0)} \psi(|X\varpi|) \sim \sup_{\eta \in C_0^\infty(B_\rho(\xi_0),\mathbb{R}^N)} \left[\fint_{B_\rho(\xi_0)} \mathcal{A}(X\varpi, X\eta) d\xi - \fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi \right]. \tag{3.3}$$

We select $\psi(t) = \varphi^{\frac{1}{s}}(t)$, as Lemma 3.1 shows, ψ is a N-function and satisfies Assumption 1.1, and ψ, ψ^* satisfy the Δ_2-condition. Let $\eta \in C_0^\infty(B_\rho(\xi_0), \mathbb{R}^N)$ and choose $\kappa \geq 0$ such that

$$\psi^*(\kappa) = \fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi. \tag{3.4}$$

Next, we select $n_0 \in \mathbb{N}$, according to ([37], Lemma 20), there is a constant $\lambda \in [\kappa, 2^{n_0}\kappa]$ and a function $\eta_\lambda \in HW_0^{1,\infty}(B_\rho(\xi_0), \mathbb{R}^N)$ satisfying

$$\sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta_\lambda| \leq c\lambda, \tag{3.5}$$

$$\fint_{B_\rho(\xi_0)} \psi^*(|X\eta_\lambda|)\chi_{\{\eta \neq \eta_\lambda\}} d\xi \leq c\psi^*(\lambda) \frac{|\{\eta \neq \eta_\lambda\}|}{|B_\rho(\xi_0)|} \leq \frac{c}{n_0} \fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi, \tag{3.6}$$

$$\fint_{B_\rho(\xi_0)} \psi^*(|X\eta_\lambda|) d\xi \leq c \fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi. \tag{3.7}$$

Now, we decompose the integral as

$$\fint_{B_\rho(\xi_0)} \mathcal{A}(X\omega, X\eta) d\xi = \fint_{B_\rho(\xi_0)} \mathcal{A}(X\omega, X\eta_\lambda) d\xi + \fint_{B_\rho(\xi_0)} \mathcal{A}(X\omega, X(\eta - \eta_\lambda)) d\xi := I + II.$$

Estimate of I**.** Using (3.1), (3.5), and Young's inequality (2.1) with $\lambda \in [\kappa, 2^{n_0}\kappa]$, we obtain

$$\left| \fint_{B_\rho(\xi_0)} \mathcal{A}(X\omega, X\eta_\lambda) d\xi \right| \leq \delta_0 \Gamma \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta_\lambda| \leq \delta_0 \Gamma c\lambda \leq c\delta_0 2^{n_0} \Gamma\kappa \leq c\delta_0 2^{n_0} \left[\delta\psi(\Gamma) + c_\delta \psi^*(\kappa) \right].$$

When $c\delta_0 2^{n_0} \leq \frac{1}{2}$ and the parameters $\delta, \delta_0, c_\delta$ are chosen to be sufficiently small, then from (3.4), we deduce

$$|I| \leq \epsilon\psi(\Gamma) + \frac{1}{2} \fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi. \tag{3.8}$$

Estimate of II**.** Applying the boundedness of $\mathcal{A}$, Young's inequality (2.1), (3.6), Hölder's inequality, and $\lambda \in [\kappa, 2^{n_0}\kappa]$, we find

$$\begin{aligned}
|II| &\leq \fint_{B_\rho(\xi_0)} |\mathcal{A}(X\omega, X(\eta-\eta_\lambda))\chi_{\{\eta\neq\eta_\lambda\}}| d\xi \leq \Lambda \fint_{B_\rho(\xi_0)} |X\omega||X(\eta-\eta_\lambda)|\chi_{\{\eta\neq\eta_\lambda\}} d\xi \\
&\leq c\Lambda \fint_{B_\rho(\xi_0)} \psi(|X\omega|\chi_{\{\eta\neq\eta_\lambda\}}) d\xi + \frac{1}{2}\fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi \\
&\leq c\Lambda \left(\fint_{B_\rho(\xi_0)} \psi^s(|X\omega|) d\xi\right)^{\frac{1}{s}} \left(\frac{|\{\eta\neq\eta_\lambda\}|}{|B_\rho(\xi_0)|}\right)^{1-\frac{1}{s}} + \frac{1}{2}\fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi \\
&\leq c\Lambda \left(\fint_{B_\rho(\xi_0)} \psi^s(|X\omega|) d\xi\right)^{\frac{1}{s}} \left(\frac{\psi^*(\kappa)}{\psi^*(\lambda)}\frac{1}{n_0}\right)^{1-\frac{1}{s}} + \frac{1}{2}\fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi \\
&\leq c\Lambda \left(\fint_{B_\rho(\xi_0)} \psi^s(|X\omega|) d\xi\right)^{\frac{1}{s}} \left(\frac{1}{n_0}\right)^{1-\frac{1}{s}} + \frac{1}{2}\fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi \\
&\leq c\Lambda\epsilon \left(\fint_{B_\rho(\xi_0)} \psi^s(|X\omega|) d\xi\right)^{\frac{1}{s}} + \frac{1}{2}\fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi.
\end{aligned} \tag{3.9}$$

In the final inequality, we have found that for n_0 sufficiently large, the equality $\frac{1}{n_0} \leq \epsilon$ holds. Combining (3.8) with (3.9), we get

$$\fint_{B_\rho(\xi_0)} \mathcal{A}(X\omega, X\eta) d\xi \leq \epsilon\left[\left(\fint_{B_\rho(\xi_0)} \psi^s(|X\omega|) d\xi\right)^{\frac{1}{s}} + \psi(\Gamma)\right] + \fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi.$$

Furthermore, since h is an $\mathcal{A}$-harmonic function, we have

$$\fint_{B_\rho(\xi_0)} \mathcal{A}(X(\omega-h), X\eta) d\xi - \fint_{B_\rho(\xi_0)} \psi^*(|X\eta|) d\xi \leq \epsilon\left\{\left(\fint_{B_\rho(\xi_0)} \psi^s(|X\omega|) d\xi\right)^{\frac{1}{s}} + \psi(\Gamma)\right\}.$$

By applying the equivalence (3.3) and noting that $\omega - h = 0$ on $\partial B_\rho(\xi_0)$, we conclude

$$\fint_{B_\rho(\xi_0)} \psi(|X(\omega-h)|) d\xi \leq \epsilon\left[\left(\fint_{B_\rho(\xi_0)} \psi^s(|X\omega|) d\xi\right)^{\frac{1}{s}} + \psi(\Gamma)\right].$$

Recalling the definition $\psi(t) = \varphi^{\frac{1}{s}}(t)$, we complete the proof. $\square$

Corollary 3.1. *Under the same conditions as the Lemma* 3.2, *if* $s \in \left(1, \min\{\frac{Q}{Q-1}, \frac{q_1}{q_1-q_0+1}\}\right)$, *then we have*

$$\left(\fint_{B_\rho(\xi_0)} \varphi\left(\frac{|\omega-h|}{\rho}\right) d\xi\right)^{\frac{1}{s}} \leq \epsilon\left\{\left(\fint_{B_\rho(\xi_0)} \varphi(|X\omega|) d\xi\right)^{\frac{1}{s}} + \varphi^{\frac{1}{s}}(\Gamma)\right\}. \tag{3.10}$$

3.2. Sobolev-Poincaré-type inequality

The goal of this section is to establish a Sobolev-Poincaré-type inequality for weak solutions of the subelliptic System (1.1). This result generalizes the Poincaré inequality for the quasi-convex functionals presented in [37] to the Poincaré-type inequality with Orlicz growth on the Heisenberg group. We begin by stating the Poincaré-type inequality, for which the proof follows the arguments from [37].

Lemma 3.3. *(Poincaré-type inequality) Let φ be a N-function satisfying $\Delta_2(\varphi, \varphi^*) < \infty$. Assume that a ball $B_\rho(\xi_0) \subset \Omega$ and a constant $c_p > 0$ exist such that for every $1 - \frac{1}{Q} < s < 1$, any weak solution $u \in HW^{1,\varphi}(\Omega, \mathbb{R}^N)$, the following inequality holds:*

$$\fint_{B_\rho(\xi_0)} \varphi\left(\frac{|u - u_{\xi_0,\rho}|}{\rho}\right) d\xi \leq c_p \left(\fint_{B_\rho(\xi_0)} \varphi^s(|Xu|) d\xi\right)^{\frac{1}{s}}, \tag{3.11}$$

where $u_{\xi_0,\rho} := \fint_{B_\rho(\xi_0)} u(\xi) d\xi$ and the constant $c_p = c(Q, N, q_0, q_1) > 0$.

Proof. Given $\Delta_2(\varphi, \varphi^*) < \infty$ and $\varphi^s(t)$ is quasi-convex function for $1 - \frac{1}{Q} < s < 1$, there is a N-function $K(t)$ such that $K(t) \sim \varphi^s(t)$ and $\Delta_2(K, K^*) < \infty$, where s and $\Delta_2(K, K^*)$ only depend on $\Delta_2(\varphi, \varphi^*)$. Moreover, we have $\varphi(K^{-1}(t)) \sim t^{\frac{1}{s}}$. Now, we define $\bar{\kappa} := \fint_{B_\rho(\xi_0)} K(Xu) d\xi$ and distinguish two cases.

Case 1: $\bar{\kappa} = 0$. In this case, u is constant on $B_\rho(\xi_0)$.

Case 2: $\bar{\kappa} > 0$. From [38, Lemma 1.50], for $\forall \xi, \zeta \in B_\rho(\xi_0)$, we have

$$|u - u_{\xi_0,\rho}| \leq c \int_{B_\rho(\xi_0)} \frac{|Xu(\zeta)|}{|\xi - \zeta|^{Q-1}} d\zeta.$$

Thus

$$\fint_{B_\rho(\xi_0)} \varphi\left(\left|\frac{u - u_{\xi_0,\rho}}{\rho}\right|\right) d\xi \leq c \fint_{B_\rho(\xi_0)} \varphi\left(\int_{B_\rho(\xi_0)} \frac{|Xu(\zeta)|}{\rho|\xi - \zeta|^{Q-1}} d\zeta\right) d\xi.$$

Since the inequality $\int_{B_\rho(\xi_0)} \rho^{-1} |\xi - \zeta|^{1-Q} d\xi \leq c$ holds, we obtain

$$\begin{aligned}
&\fint_{B_\rho(\xi_0)} \varphi\left(\left|\frac{u - u_{\xi_0,\rho}}{\rho}\right|\right) d\xi \\
\leq & c \fint_{B_\rho(\xi_0)} \varphi \cdot K^{-1} \cdot \left(\int_{B_\rho(\xi_0)} K(|Xu(\zeta)|\rho^{-1}|\xi - \zeta|^{1-Q} d\zeta\right) d\xi, \\
\leq & c \fint_{B_\rho(\xi_0)} \left(\int_{B_\rho(\xi_0)} K(|Xu(\zeta)|)\rho^{-1}|\xi - \zeta|^{1-Q} d\zeta\right)^{\frac{1}{s}} d\xi \\
\leq & c\rho^{-\frac{1}{s}} \fint_{B_\rho(\xi_0)} \kappa^{\frac{1}{s}} \rho^{\frac{Q}{s}} \left(\fint_{B_\rho(\xi_0)} \kappa^{-1} K(|Xu(\zeta)|)|\xi - \zeta|^{1-Q} d\zeta\right)^{\frac{1}{s}} d\xi,
\end{aligned}$$

where we have used $\Delta_2(K, K^*) < \infty$.

By applying Jensen's inequality to $t \mapsto t^{\frac{1}{s}}$ and utilizing the Harr measure of $\kappa^{-1} K(|Xu(\xi)|)$, we then get

$$\begin{aligned}
\fint_{B_\rho(\xi_0)} \varphi\left(\left|\frac{u - u_{\xi_0,\rho}}{\rho}\right|\right) d\xi &\leq c\rho^{\frac{Q-1}{s}} \fint_{B_\rho(\xi_0)} \kappa^{\frac{1}{s}} \fint_{B_\rho(\xi_0)} \kappa^{-1} K(|Xu(\zeta)|)(|\xi - \zeta|^{1-Q})^{\frac{1}{s}} d\zeta d\xi \\
&\leq c\rho^{\frac{Q-1}{s}} \kappa^{\frac{1}{s}-1} \rho^{\frac{1-Q}{s}} \fint_{B_\rho(\xi_0)} K(|Xu(\zeta)|) d\xi,
\end{aligned}$$

where we have used $\frac{1-Q}{s} > -Q$.

By the definition of $\bar{\kappa}$, it follows that

$$\fint_{B_\rho(\xi_0)} \varphi\left(\left|\frac{u - u_{\xi_0,\rho}}{\rho}\right|\right) d\xi \leq c\left(\fint_{B_{\rho(\xi_0)}} K(|Xu(\zeta)|) d\xi\right)^{\frac{1}{s}} \leq c\left(\fint_{B_\rho(\xi_0)} \varphi^s(|Xu(\zeta)|) d\xi\right)^{\frac{1}{s}}. \quad \square$$

Next, we extend the Poincaré-type inequality from Lemma 3.3 to a Sobolev-Poincaré-type inequality. Its proof follows by combining Lemma 3.3 with ([38], Theorem 2).

Corollary 3.2. *(Sobolev-Poincaré-type inequality) Let φ be a N-function with $\Delta_2(\varphi,\varphi^*) < \infty$. A ball $B_\rho(\xi_0) \subset \Omega$ exists, and for $\forall\ 1 \le s < \frac{Q}{Q-1}$, so that for all weak solutions $u \in HW^{1,\varphi}(\Omega,\mathbb{R}^N)$, the following inequality holds:*

$$\left(\fint_{B_\rho(\xi_0)} \varphi^s\left(\frac{|u-u_{\xi_0,\rho}|}{\rho}\right)d\xi\right)^{\frac{1}{s}} \le C_p \fint_{B_\rho(\xi_0)} \varphi(|Xu|)d\xi, \tag{3.12}$$

where the constant $C_p = C(Q,N,q_0,q_1) > 0$.

For the subsequent analysis, we will mainly use the simplest form of this Sobolev-Poincaré-type inequality, corresponding to the case $s = 1$.

3.3. Caccioppoli-type inequality

Based on the Sobolev-Poincaré-type inequality, we establish a Caccioppoli-type inequality for weak solution to the subelliptic system in (1.1) on the Heisenberg group. This estimate serves as a cornerstone for the linearization tool and the subsequent regularity analysis of System (1.1).

Lemma 3.4. *(Caccioppoli-type inequality) Let $u \in HW^{1,\varphi}(\Omega,\mathbb{R}^N)$ be a weak solution of System (1.1), satisfying (A1)–(A6) and (B1), $B_\rho(\xi_0) \subset \Omega$, $\bar{\xi_0} = (\xi_0,t_0) \in \Omega$, and $0 < \rho < \min\{1, dist(\xi_0,\partial\Omega)\}$. For any affine function $\ell\colon \mathbb{R}^{2n} \to \mathbb{R}^N$, the following estimate holds:*

$$\begin{aligned}&\fint_{B_{\frac{\rho}{2}}(\xi_0)} \varphi_{1+|X\ell|}(|Xu-X\ell|)d\xi\\ \le& C_c\left\{\fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}\left(\frac{|u-\ell|}{\rho}\right)d\xi + \varphi(1+|X\ell|)\left[V(\rho)+\mu\left(\fint_{B_\rho(\xi_0)}|u-\ell(\bar{\xi}_0)|d\xi\right)+\rho\right]\right\},\end{aligned} \tag{3.13}$$

where the constant $C_c = C(Q,N,\nu,q_0,q_1,\delta,c_\delta,\Lambda,L) > 0$.

Proof. First, choose a cut-off function ζ between the balls $B_{\frac{\rho}{2}}(\xi_0)$ and $B_\rho(\xi_0)$, define

$$\eta = \zeta^{q_1}(u-\ell),$$

where q_1 is given in (1.2), $\zeta \in C_0^\infty(B_\rho(\xi_0),\mathbb{R}^N)$, $0 \le \zeta \le 1$, $|X\zeta| \le \frac{2}{\rho}$, and $\zeta \equiv 1$ on $B_{\frac{\rho}{2}}(\xi_0)$. We omit the mollification procedure here; for more details, we refer the reader to [39].
From the definition of weak solution (1.3), we get

$$\fint_{B_\rho(\xi_0)} A_i^\alpha(\xi,u,Xu)\zeta^{q_1}(Xu-X\ell)d\xi = -q_1\fint_{B_\rho(\xi_0)} A_i^\alpha(\xi,u,Xu)\zeta^{q_1-1}(u-\ell)X\zeta d\xi + \fint_{B_\rho(\xi_0)} B^\alpha(\xi,u,Xu)\eta^\alpha d\xi.$$

Add another equation

$$-\fint_{B_\rho(\xi_0)} A_i^\alpha(\xi,u,X\ell)\zeta^{q_1}(Xu-X\ell)d\xi = q_1\fint_{B_\rho(\xi_0)} A_i^\alpha(\xi,u,X\ell)\zeta^{q_1-1}(u-\ell)X\zeta d\xi - \fint_{B_\rho(\xi_0)} A_i^\alpha(\xi,u,X\ell)X\eta d\xi,$$

and combine

$$\fint_{B_\rho(\xi_0)} (A_i^\alpha(\cdot, \ell(\bar{\xi}_0), X\ell))_{\xi_0,\rho} X\eta d\xi = 0.$$

We have

$$J_0 := \fint_{B_\rho(\xi_0)} [A_i^\alpha(\xi, u, Xu) - A_i^\alpha(\xi, u, X\ell)]\zeta^{q_1}(Xu - X\ell)d\xi := J_1 + J_2 + J_3 + J_4,$$

where

$$\begin{aligned}
J_1 &= q_1 \fint_{B_\rho(\xi_0)} [A_i^\alpha(\xi, u, X\ell) - A_i^\alpha(\xi, u, Xu)]\zeta^{q_1-1}(u-\ell)X\zeta d\xi,\\
J_2 &= \fint_{B_\rho(\xi_0)} [A_i^\alpha(\xi, \ell(\bar{\xi}_0), X\ell) - A_i^\alpha(\xi, u, X\ell)]X\eta d\xi,\\
J_3 &= \fint_{B_\rho(\xi_0)} [(A_i^\alpha(\cdot, \ell(\bar{\xi}_0), X\ell))_{\xi_0,\rho} - A_i^\alpha(\xi, \ell(\bar{\xi}_0), X\ell)]X\eta d\xi,\\
J_4 &= \fint_{B_\rho(\xi_0)} B^\alpha(\xi, u, Xu)\eta^\alpha d\xi.
\end{aligned}$$

By Remark 2.3, we immediately obtain

$$J_0 \geq c\nu \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}(|Xu - X\ell|)\zeta^{q_1} d\xi,$$

where c depends on q_0 and q_1.

Then, by applying Remark 2.4, Young's inequality (2.1), (2.3), and (2.7), we deduce

$$\begin{aligned}
J_1 &\leq cq_1\Lambda \fint_{B_\rho(\xi_0)} \varphi'_{1+|X\ell|}(|Xu - X\ell|)\zeta^{q_1-1}|X\zeta||u-\ell|d\xi\\
&\leq 2cq_1\Lambda \fint_{B_\rho(\xi_0)} \varphi'_{1+|X\ell|}(|Xu - X\ell|)\zeta^{q_1-1}\frac{|u-\ell|}{\rho}d\xi\\
&\leq 2cq_1\Lambda c_\delta \fint_{B_\rho(\xi_0)} \varphi^*_{1+|X\ell|}\left(\varphi'_{1+|X\ell|}(|Xu - X\ell|)\zeta^{q_1-1}\right)d\xi + 2cq_1\Lambda\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}\left(\frac{|u-\ell|}{\rho}\right)d\xi\\
&\leq 2cq_1\Lambda c_\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}(|Xu - X\ell|)\zeta^{q_1}d\xi + 2cq_1\Lambda\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}\left(\frac{|u-\ell|}{\rho}\right)d\xi.
\end{aligned}$$

Using the definition of η and (A5), we split J_2 as

$$J_2 \leq \Lambda \fint_{B_\rho(\xi_0)} \mu\left(|u - \ell(\bar{\xi}_0)|\right)\varphi'(1+|X\ell|)\left[q_1\zeta^{q_1-1}|u-\ell||X\zeta| + \zeta^{q_1}|Xu - X\ell|\right]d\xi := J_{21} + J_{22}.$$

Now, we estimate J_{21} and J_{22} separately.

For J_{21}, we observe that $\mu^{\frac{q_1}{q_1-1}} \leq \mu$ when $\mu \in [0,1]$. Applying Young's inequality (2.1), (2.3), (2.7), and Jensen's inequality, we then have

$$\begin{aligned}
J_{21} &\le 2\Lambda q_1 \fint_{B_\rho(\xi_0)} \mu\left(|u-\ell(\bar{\xi}_0)|\right)\varphi'(1+|X\ell|)\zeta^{q_1-1}\frac{|u-\ell|}{\rho}d\xi\\
&\le 2\Lambda q_1 c_\delta \fint_{B_\rho(\xi_0)} \varphi^*_{1+|X\ell|}(\mu(|u-\ell(\bar{\xi}_0|)\varphi'(1+|X\ell|)\zeta^{q_1-1}))d\xi + 2\Lambda q_1\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}\left(\frac{|u-\ell|}{\rho}\right)d\xi\\
&\le 2\Lambda q_1 c_\delta \fint_{B_\rho(\xi_0)} \mu(|u-\ell(\bar{\xi}_0|)\zeta^{q_1}\varphi^*_{1+|X\ell|}(\varphi'_{1+|X\ell|}(1+|X\ell|))d\xi + 2\Lambda q_1\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}\left(\frac{|u-\ell|}{\rho}\right)d\xi\\
&\le 2\Lambda q_1 c_\delta \varphi(1+|X\ell|)\mu\left(\fint_{B_\rho(\xi_0)}|u-\ell(\bar{\xi}_0)|d\xi\right) + 2\Lambda q_1\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}\left(\frac{|u-\ell|}{\rho}\right)d\xi,
\end{aligned}$$

where the last step uses the equivalence $\varphi_a(t) \sim \varphi(t)$ for $t \ge a > 0$, i.e.

$$\varphi(1+|X\ell|) \sim \varphi_{1+|X\ell|}(1+|X\ell|).$$

Analogously, we estimate J_{22}

$$\begin{aligned}
J_{22} \le& \Lambda\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}(|Xu-X\ell|)\zeta^{q_1}d\xi + \Lambda c_\delta \fint_{B_\rho(\xi_0)} \mu(|u-\ell(\bar{\xi}_0)|)\varphi(1+|X\ell|)d\xi\\
\le& \Lambda\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}(|Xu-X\ell|)\zeta^{q_1}d\xi + \Lambda c_\delta\varphi(1+|X\ell|)\mu\left(\fint_{B_\rho(\xi_0)}|u-\ell(\bar{\xi}_0)|d\xi\right).
\end{aligned}$$

(A6) implies

$$J_3 \le \Lambda L \fint_{B_\rho(\xi_0)} v_{\xi_0}(\xi,\rho)\varphi'(1+|X\ell|)\left[\zeta^{q_1}|Xu-X\ell| + q_1\zeta^{q_1-1}|u-\ell||X\zeta|\right]d\xi := J_{31}+J_{32}.$$

First, by using Young's inequality (2.1) and $v_{\xi_0}(\xi,\rho)\in(0,1)$, we obtain

$$\begin{aligned}
J_{31} &\le \Lambda L\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}(|Xu-X\ell|)\zeta^{q_1}d\xi + \Lambda L c_\delta \fint_{B_\rho(\xi_0)} \varphi^*_{1+|X\ell|}\left(v_{\xi_0}(\xi,\rho)\varphi'(1+|X\ell|)\right)d\xi\\
&\le \Lambda L\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}(|Xu-X\ell|)\zeta^{q_1}d\xi + \Lambda L c_\delta \fint_{B_\rho(\xi_0)} v_{\xi_0}(\xi,\rho)^{\frac{q_1}{q_1-1}}\cdot\varphi(1+|X\ell|)d\xi\\
&\le \Lambda L\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}(|Xu-X\ell|)\zeta^{q_1}d\xi + c\Lambda L c_\delta \fint_{B_\rho(\xi_0)} v_{\xi_0}(\xi,\rho)\cdot\varphi(1+|X\ell|)d\xi\\
&\le \Lambda L\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}(|Xu-X\ell|)\zeta^{q_1}d\xi + c\Lambda L c_\delta V(\rho)\varphi(1+|X\ell|).
\end{aligned}$$

Similarly, we have

$$J_{32} \le cq_1\Lambda L c_\delta V(\rho)\varphi(1+|X\ell|) + q_1\Lambda L\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}\left(\frac{|u-\ell|}{\rho}\right)d\xi.$$

Before estimating the J_4 with nonhomogeneous terms, we need to prove the following inequality:

$$\varphi'(a+t) \le 2\,\varphi'_a(a+t). \tag{3.14}$$

In fact, from the definition of $\varphi_a'(t)$, we have

$$\varphi_a'(a+t) = \varphi'(a+a+t)\cdot\frac{a+t}{a+a+t} = \varphi'(2a+t)\cdot\frac{a+t}{2a+t}.$$

The monotonicity of φ' yields that

$$\varphi'(a+t) \le \varphi'(2a+t).$$

In other words,

$$\varphi'(a+t)\cdot\frac{a+t}{2a+t} \le \varphi'(2a+t)\cdot\frac{a+t}{2a+t} = \varphi_a'(a+t).$$

Rearranging the terms, we get

$$\varphi'(a+t) \le \frac{2a+t}{a+t}\cdot\varphi_a'(a+t) = (2-\frac{t}{a+t})\cdot\varphi_a'(a+t) < 2\varphi_a'(a+t).$$

On the basis of the inequality in (3.14), we estimate J_4 with (B1).

$$\begin{aligned}
J_4 &\le L \fint_{B_\rho(\xi_0)} \varphi'(1+|Xu|)\zeta^{q_1}|u-\ell|d\xi \\
&\le L \fint_{B_\rho(\xi_0)} \varphi'(1+|X\ell|+|Xu-X\ell|)\zeta^{q_1}|u-\ell|d\xi \\
&\le 2L \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}'(1+|X\ell|+|Xu-X\ell|)\zeta^{q_1}|u-\ell|d\xi.
\end{aligned}$$

We define two sets E and F such that $B_\rho(\xi_0) = E\cup F$, where

$$E := B_\rho(\xi_0)\cap\{|Xu-X\ell| \ge 1+|X\ell|\} \quad\text{and}\quad F := E^c = B_\rho(\xi_0)\cap\{|Xu-X\ell| < 1+|X\ell|\}.$$

Then

$$\begin{aligned}
&2L \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}'(1+|X\ell|+|Xu-X\ell|)\zeta^{q_1}|u-\ell|d\xi \\
=&\frac{2L}{|B_\rho(\xi_0)|_{\mathbb{H}^n}} \int_E \varphi_{1+|X\ell|}'(1+|X\ell|+|Xu-X\ell|)\zeta^{q_1}|u-\ell|d\xi \\
+&\frac{2L}{|B_\rho(\xi_0)|_{\mathbb{H}^n}} \int_F \varphi_{1+|X\ell|}'(1+|X\ell|+|Xu-X\ell|)\zeta^{q_1}|u-\ell|d\xi \\
:&= J_{41} + J_{42}.
\end{aligned}$$

Using Young's inequality (2.1) and (2.7) yields

$$\begin{aligned}
J_{41} &\le \frac{2cL}{|B_\rho(\xi_0)|_{\mathbb{H}^n}} \int_E \varphi_{1+|X\ell|}'(|Xu-X\ell|)\zeta^{q_1}|u-\ell|d\xi \\
&\le 2cL\rho \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}'(|Xu-X\ell|)\zeta^{q_1}\frac{|u-\ell|}{\rho}d\xi \\
&\le 2cL\rho c_\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}^*(\varphi_{1+|X\ell|}'(|Xu-X\ell|)\zeta^{q_1})d\xi + 2cL\rho\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}\left(\frac{|u-\ell|}{\rho}\right)d\xi \\
&\le 2cL\rho c_\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}(|Xu-X\ell|)\zeta^{q_1}d\xi + 2cL\rho\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell|}\left(\frac{|u-\ell|}{\rho}\right)d\xi.
\end{aligned}$$

Similarly, we have

$$J_{42} \leq 2cL\rho c_{\delta}\varphi(1+|X\ell|) + 2cL\rho\delta \fint_{B_{\rho}(\xi_0)} \varphi_{1+|X\ell|}\left(\frac{|u-\ell|}{\rho}\right) d\xi,$$

where we have used the equivalence $\varphi_{1+|X\ell|}(1+|X\ell|) \sim \varphi(1+|X\ell|)$.

Finally, we combine the estimates from J_0 to J_4 by choosing $\delta, c_{\delta} > 0$ to be sufficiently small, such that the terms involving $\varphi_{1+|X\ell|}(|Xu - X\ell|)$ into the left-hand side, and we can derive the conclusion. □

Corollary 3.3. *As a direct consequence of Lemma* 3.4*, we choose $\ell = u_0$, and we obtain the following zero-Caccioppoli inequality, i.e.*

$$\fint_{B_{\frac{\rho}{2}}(\xi_0)} \varphi_{1+|X\ell|}(|Xu|) d\xi \leq C \fint_{B_{\rho}(\xi_0)} \varphi_{1+|X\ell|}\left(\frac{|u-u_0|}{\rho}\right) d\xi, \tag{3.15}$$

where the constant $C = C(Q, N, \nu, q_0, q_1, \delta, c_{\delta}, \Lambda, L) > 0$.

The following theorem is an extension of ([30], Theorem 4.1) to the Heisenberg group, which establishes higher integrability for a weak solution. Now we prove Lemma 3.5.

Lemma 3.5. *(Higher integrability) Suppose that A_i^{α} satisfies (A1)–(A6), B^{α} satisfies (B1), $u \in HW^{1,\varphi}(\Omega, \mathbb{R}^N)$ is the weak solution of the subelliptic system in (1.1), and the N-function φ satisfies Assumption* 1.1*. In this case, then $\gamma = \gamma(Q, N, \nu, L, q_0, q_1, \Lambda) > 1$ exists such that $\varphi(1+|Xu|) \in L^{\gamma}_{loc}(\Omega)$ and the following inequality holds:*

$$\left(\fint_{B_{\frac{\rho}{4}}(\xi_0)} \varphi^{\gamma}(1+|Xu|) d\xi\right)^{\frac{1}{\gamma}} \leq c_1 \fint_{B_{\rho}(\xi_0)} \varphi(1+|Xu|) d\xi + c_2 \fint_{B_{\tau\tau_2}(\xi_0)} \varphi(1) d\xi, \tag{3.16}$$

where the constants are $c_1 = c_1(Q, N, \nu, \Lambda, L, q_0, q_1) > 0$ and $c_2 = c_2(Q, N, \Lambda, L, q_0, q_1) > 0$, for every ball $B_{\rho}(\xi_0) \subset \Omega$ with ξ_0 as its center and ρ as its radius.

Proof. Fix ξ_0 and $\rho > 0$ such that $B_{\rho}(\xi_0) \subset \Omega$ for every $\xi_0 \in \Omega$ and $\tau > 0$, $B_{2\tau}(\xi_0) \subset B_{\frac{\rho}{2}}(\xi_0)$. Now, we choose $1 \leq \tau_1 < \tau_2 \leq 2$ such that $B_{\tau_2}(\xi_0) \subset B_{2\tau}(\xi_0) \subset B_{\frac{\rho}{2}}(\xi_0)$, and introduce a test function ϕ, where $\phi \in C_0^{\infty}(B_{\tau\tau_2}(\xi_0), [0,1])$, $|X\phi| \leq \frac{c}{(\tau_2-\tau_1)\tau}$ and $\phi \equiv 1$ in $B_{2\tau}(\xi_0)$. Next, we consider subelliptic systems (1.1) with the test function $\eta = u - \phi(u - u_{\xi_0,2\tau})$.

At the beginning, we obtain

$$\begin{aligned}
K_0 :=& \fint_{B_{\tau\tau_2}(\xi_0)} \langle A_i^{\alpha}(\xi, u, Xu) - A_i^{\alpha}(\xi, u, 0), Xu\rangle d\xi \\
=& \fint_{B_{\tau\tau_2}(\xi_0)} \langle A_i^{\alpha}(\xi, u, Xu), X\phi \otimes (u - u_{\xi_0,2\tau}\rangle d\xi + \fint_{B_{\tau\tau_2}(\xi_0)} \phi(\xi)\langle A_i^{\alpha}(\xi, u, Xu), Xu\rangle d\xi \\
&- \fint_{B_{\tau\tau_2}(\xi_0)} \langle A_i^{\alpha}(\xi, u, 0), Xu\rangle d\xi + \fint_{B_{\tau\tau_2}(\xi_0)} B^{\alpha}(\xi, u, Xu)(u - \phi(u - u_{\xi_0,2\tau}))d\xi \\
=& \fint_{B_{\tau\tau_2}(\xi_0)} \langle A_i^{\alpha}(\xi, u, Xu) - A_i^{\alpha}(\xi, u, 0), X\phi \otimes (u - u_{\xi_0,2\tau})\rangle d\xi + \fint_{B_{\tau\tau_2}(\xi_0)} \phi(\xi)\langle A_i^{\alpha}(\xi, u, Xu) - A_i^{\alpha}(\xi, u, 0), Xu\rangle d\xi \\
&- \fint_{B_{\tau\tau_2}(\xi_0)} \langle A_i^{\alpha}(\xi, u, 0), X\eta\rangle d\xi + \fint_{B_{\tau\tau_2}(\xi_0)} B^{\alpha}(\xi, u, Xu)\eta d\xi := \sum_{i=1}^{4} K_i.
\end{aligned}$$

We first estimate K_0. Applying (A2), (2.7), and Lemma 2.3 with $z = 1, z_1 = 0$, and $z_2 = |Xu|$, we obtain

$$\begin{aligned} K_0 =& \fint_{B_{\tau\tau_2}(\xi_0)} \int_0^1 \langle D_P A_i^\alpha(\xi, u, \tilde{s}Xu)Xu, Xu\rangle d\tilde{s}d\xi \\ \geq& \nu \fint_{B_{\tau\tau_1}(\xi_0)} \int_0^1 \varphi''(1+\tilde{s}|Xu|)d\tilde{s}|Xu|^2 d\xi \\ \geq& c\nu \fint_{B_{\tau\tau_1}(\xi_0)} \int_0^1 \frac{\varphi'(1+\tilde{s}|Xu|)}{1+\tilde{s}|Xu|}d\tilde{s}|Xu|^2 d\xi \\ \geq& c\nu \fint_{B_{\tau\tau_1}(\xi_0)} \frac{\varphi'(|Xu|)}{|Xu|}|Xu|^2 d\xi \geq c\nu \fint_{B_{\tau\tau_1}(\xi_0)} \varphi(1+|Xu|)d\xi, \end{aligned} \tag{3.17}$$

where we have used the equivalences $\frac{\varphi'(1+|Xu|)}{1+|Xu|} \sim \frac{\varphi'(|Xu|)}{|Xu|}$ and $\varphi(1+|Xu|) \sim \varphi(|Xu|)$.

Next, we estimate the terms K_1, K_2, K_3, and K_4 one by one.

Estimate for K_1. Using (A1), Young's inequality (2.1), (2.2), (2.7), and $|X\phi| \leq \frac{c}{(\tau_2-\tau_1)\tau}$, we have

$$\begin{aligned} |K_1| \leq& c\Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi'(1+|Xu|)\frac{|u-u_{\xi_0,2\tau}|}{(\tau_2-\tau_1)\tau}d\xi \\ \leq& c\Lambda c_\delta \fint_{B_{\tau\tau_2}(\xi_0)} (\varphi'(1+|Xu|))d\xi + c\Lambda\frac{\delta}{(\tau_2-\tau_1)^{q_1}} \fint_{B_{\tau\tau_2}(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,2\tau}|}{\tau}\right)d\xi \\ \leq& c\Lambda c_\delta \fint_{B_{\tau\tau_2}(\xi_0)} \varphi(1+|Xu|)d\xi + c\Lambda 2^{q_1}\frac{\delta}{(\tau_2-\tau_1)^{q_1}} \fint_{B_{\tau\tau_2}(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,2\tau}|}{2\tau}\right)d\xi. \end{aligned} \tag{3.18}$$

Estimate for K_2. Applying (A1) and (2.7) yields

$$\begin{aligned} |K_2| \leq& c\Lambda \fint_{B_{\tau\tau_2}(\xi_0)} |\phi(\xi)|\varphi'(1+|Xu|)|Xu|d\xi \leq c\Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi'(1+|Xu|)(1+|Xu|)d\xi \\ \leq& c\Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi(1+|Xu|)d\xi. \end{aligned} \tag{3.19}$$

Estimate for K_3. From (A1), (2.7), and Young's inequality (2.1), we deduce

$$\begin{aligned} |K_3| \leq& \Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi'(1)|X\eta|d\xi \\ \leq& \Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi'(1)|(1-\phi(\xi))Xu|d\xi + \Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi'(1)|X\phi||u-u_{\xi_0,2\tau}|d\xi \\ \leq& \Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi'(1)|Xu|d\xi + c\Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi'(1)\frac{|u-u_{\xi_0,2\tau}|}{(\tau_2-\tau_1)\tau}d\xi \\ \leq& c_\delta\Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi^*(\varphi'(1))d\xi + \delta\Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi(|Xu|)d\xi + c\Lambda c_\delta \fint_{B_{\tau\tau_2}(\xi_0)} \varphi^*(\varphi'(1))d\xi \\ &+ c\delta\Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,2\tau}|}{2\tau}\right)d\xi \\ \leq& cc_\delta\Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi(1)d\xi + c\delta\Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi(1+|Xu|)d\xi + c\delta\Lambda \fint_{B_{\tau\tau_2}(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,2\tau}|}{2\tau}\right)d\xi. \end{aligned} \tag{3.20}$$

Estimate for K_4. Under (B1), we apply Young's inequality (2.1) and use the definition of η, together with (2.4), and (2.7), thus we can get

$$
\begin{aligned}
|K_4| &\leq L \fint_{B_{\tau\tau_2}(\xi_0)} \varphi'(1+|Xu|)(1-\phi(\xi))|u-u_{\xi_0,2\tau}|d\xi + Lu_{\xi_0,2\tau} \fint_{B_{\tau\tau_2}(\xi_0)} \varphi'(1+|Xu|)d\xi \\
&\leq L \fint_{B_{2\tau}(\xi_0)} \varphi'(1+|Xu|)|u-u_{\xi_0,2\tau}|d\xi + Lu_{\xi_0,2\tau} \fint_{B_{2\tau}(\xi_0)} \varphi'(1+|Xu|)d\xi \\
&\leq Lc_\delta \fint_{B_{2\tau}(\xi_0)} \varphi^*(\varphi'(1+|Xu|))d\xi + L\delta(2\tau)^{q_1} \fint_{B_{2\tau}(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,2\tau}|}{2\tau}\right)d\xi \\
&+ Lu_{\xi_0,2\tau}c_\delta \fint_{B_{2\tau}(\xi_0)} \varphi^*\left(\varphi'(1+|Xu|)\right)d\xi + Lu_{\xi_0,2\tau}\delta \fint_{B_{2\tau}(\xi_0)} \varphi(1)d\xi \\
&\leq c\left(1+u_{\xi_0,2\tau}\right)Lc_\delta \fint_{B_{2\tau}(\xi_0)} \varphi(1+|Xu|)d\xi + c\left(1+u_{\xi_0,2\tau}\right)L\delta \fint_{B_{2\tau}(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,2\tau}|}{2\tau}\right)d\xi \\
&+ Lu_{\xi_0,2\tau}c_\delta \fint_{B_{2\tau}(\xi_0)} \varphi(1)d\xi.
\end{aligned}
\tag{3.21}
$$

Collecting the estimates for K_0 through K_4 (i.e., (3.17) to (3.21)), using Assumption 1.1 and Young's inequality (2.1), and choosing $\delta, c_\delta > 0$ to be suitably small, we have

$$
\nu \fint_{B_{\tau\tau_2}(\xi_0)} \varphi(1+|Xu|)d\xi \leq c' \fint_{B_{2\tau}(\xi_0)} \varphi(1+|Xu|)d\xi + c'' \fint_{B_{2\tau}(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,2\tau}|}{2\tau}\right)d\xi + c_2 \fint_{B_{2\tau}(\xi_0)} \varphi(1)d\xi.
$$

Finally, we apply the Poincaré-type inequality established in Lemma 3.3 to the second term on the right-hand side, which gives

$$
\nu \fint_{B_{\tau}(\xi_0)} \varphi(1+|Xu|)d\xi \leq c' \fint_{B_{2\tau}(\xi_0)} \varphi(1+|Xu|)d\xi + c'' \left(\fint_{B_{2\tau}(\xi_0)} \varphi^\theta(1+|Xu|)d\xi\right)^{\frac{1}{\theta}} + c_2 \fint_{B_{2\tau}(\xi_0)} \varphi(1)d\xi,
$$

where $B_{2\tau}(\xi_0) \subset B_{\frac{\rho}{2}}(\xi_0)$. □

The desired higher integrability estimate now follows from an application of Gehring's lemma ([40], Corollary 6.1). Furthermore, as a direct consequence of Lemma 3.5, we obtain the following Corollary 3.4, which plays a crucial role in the subsequent decay analysis of the excess functional.

Corollary 3.4. *Given $\gamma > 1$, which is provided by Lemma 3.5, if $\varphi(1+|Xu|) \in L^\gamma_{loc}(\Omega)$, then we obtain*

$$
\left(\fint_{B_{\frac{\rho}{2}}(\xi_0)} \varphi^\gamma(1+|Xu|)d\xi\right)^{\frac{1}{\gamma}} \leq C_h \fint_{B_\rho(\xi_0)} \varphi(1+|Xu|)d\xi,
$$

where the constant $C_h = C(Q, N, q_0, q_1, \nu, \Lambda, L) > 0$.

3.4. Linearization

To facilitate the application of the $\mathcal{A}$-harmonic approximation lemma, we establish a linearization tool that conveniently compares the solution of the subelliptic system in (1.1) with the associated constant coefficient linear system. At the beginning, given $\xi_0 \in \Omega,\ 0 < \rho < \min\{1, \operatorname{dist}(\xi_0, \partial\Omega)\}$, we define the following excess functionals:

$$\Psi(\xi_0,\rho) = \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell_{\xi_0,\rho}|}\left(\frac{|u-\ell_{\xi_0,\rho}|}{\rho}\right) d\xi$$

and

$$\Psi^*(\xi_0,\rho) = \Psi(\xi_0,\rho) + F(\xi_0,\rho)^{\alpha_1}\varphi(1 + |X\ell_{\xi_0,\rho}|),$$

where

$$F(\xi_0,\rho) = V(\rho) + \mu\left(\fint_{B_\rho(\xi_0)} |u - \ell_{\xi_0,\rho}(\bar{\xi}_0)| d\xi\right) + \rho,$$

with γ is defined in Lemma 3.5 (also see Corollary 3.4), q_0 as shown in (1.2), $\alpha_0 \in (0,1)$, and $\alpha_1 := \min\left\{\left(1-\frac{1}{\gamma}\right)\left(1-\frac{1}{q_0}\right), \frac{\alpha_0+1}{2}\right\}$.
We also set

$$\varGamma = \sqrt{\frac{\Psi^*(\xi_0,\rho)}{\varphi(1+|X\ell_{\xi_0,\rho}|)}}.$$

The bilinear form is defined as

$$\mathcal{A} = \frac{\left(D_P A_i^\alpha(\cdot, \ell_{\xi_0,\rho}(\bar{\xi}_0), X\ell_{\xi_0,\rho})\right)_{\xi_0,\rho}}{\varphi''(1+|X\ell_{\xi_0,\rho}|)}.$$

From (A2) and (A3), we can deduce that for $\forall \xi_1, \xi_2 \in \mathbb{R}^{2n\times N}$ and $\mathcal{A} \in \operatorname{Bil}(\mathbb{R}^{2n\times N})$, the following inequalities hold:

$$\mathcal{A}(\xi_1,\xi_1) \geq \nu|\xi_1|^2 \quad \text{and} \quad |\mathcal{A}(\xi_1,\xi_2)| \leq \varLambda|\xi_1||\xi_2|.$$

Lemma 3.6. *(Linearization) Let φ be a N-function satisfying $\Delta_2(\varphi,\varphi^*) < \infty$, assume that*

$$\Psi(\xi_0,\rho) \leq \varphi(1+|X\ell_{\xi_0,\rho}|), \tag{3.22}$$

if $u - \ell_{\xi_0,\rho}$ is an approximate $\mathcal{A}$-harmonic function on $B_{\frac{\rho}{4}}(\xi_0)$, then for any test function $\eta \in C_0^\infty(B_{\frac{\rho}{4}}(\xi_0), \mathbb{R}^N)$, the following inequality holds:

$$\begin{aligned} &\left|\fint_{B_{\frac{\rho}{4}}(\xi_0)} \mathcal{A}(X(u-\ell_{\xi_0,\rho}), X\eta) d\xi\right| \\ \leq &C_1(1+|X\ell_{\xi_0,\rho}|)\left\{\frac{\Psi(\xi_0,\rho)}{\varphi(1+|X\ell_{\xi_0,\rho}|)} + \left(\frac{\Psi(\xi_0,\rho)}{\varphi(1+|X\ell_{\xi_0,\rho}|)}\right)^{\frac{\alpha_0+1}{2}} + F(\xi_0,\rho)^{\alpha_1}\right\} \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta|, \end{aligned} \tag{3.23}$$

where the constant $C_1 = C_1(Q, N, q_0, q_1, L, \varLambda, \Delta_2(\varphi,\varphi^*)) > 0, \alpha_0, \alpha_1$ was defined previously in Lemma 3.6 in this section.

Proof. If we let $\eta \in C_0^\infty(B_{\frac{\varrho}{4}}(\xi_0), \mathbb{R}^N)$ and set $\bar{w} = u - \ell_{\xi_0,\rho}$, we decompose

$$
\begin{aligned}
&\fint_{B_{\frac{\varrho}{4}}(\xi_0)} \mathcal{A}(X\bar{w}, X\eta)\, d\xi \\
=&\frac{1}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \int_0^1 \Big\langle \big[(D_P A_i^\alpha(\cdot, \ell_{\xi_0,\rho}(\bar{\xi}_0), X\ell_{\xi_0,\rho}))_{\xi_0,\rho} \\
&-(D_P A_i^\alpha(\cdot, \ell_{\xi_0,\rho}(\bar{\xi}_0), X\ell_{\xi_0,\rho} + tX\bar{w}))_{\xi_0,\rho}\big] X\bar{w}, X\eta \Big\rangle \, dt\, d\xi \\
&+\frac{1}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \int_0^1 \Big\langle (D_P A_i^\alpha(\cdot, \ell_{\xi_0,\rho}(\bar{\xi}_0), X\ell_{\xi_0,\rho} + tX\bar{w}))_{\xi_0,\rho} X\bar{w}, X\eta \Big\rangle \, dt\, d\xi \\
&:= I_1 + I_2.
\end{aligned}
$$

Estimate I_1. We define the two subsets

$$
E_1 := B_{\frac{\varrho}{4}}(\xi_0) \cap \Big\{ |Xu - X\ell_{\xi_0,\rho}| \geq 1 + |X\ell_{\xi_0,\rho}| \Big\} \quad \text{and} \quad F_1 := B_{\frac{\varrho}{4}}(\xi_0) \cap \Big\{ |Xu - X\ell_{\xi_0,\rho}| < 1 + |X\ell_{\xi_0,\rho}| \Big\}.
$$

We split the integral as follows:

$$
\begin{aligned}
I_1 =& \frac{1}{\varphi''(|1 + X\ell_{\xi_0,\rho}|) |B_{\frac{\varrho}{4}}(\xi_0)|_{\mathbb{H}^n}} \times \\
&\Bigg(\int_{E_1} \int_0^1 \Big\langle \big[(D_P A_i^\alpha(\cdot, \ell_{\xi_0,\rho}(\bar{\xi}_0), X\ell_{\xi_0,\rho}))_{\xi_0,\rho} - (D_P A_i^\alpha(\cdot, \ell_{\xi_0,\rho}(\bar{\xi}_0), X\ell_{\xi_0,\rho} + tX\bar{w}))_{\xi_0,\rho}\big] X\bar{w}, X\eta \Big\rangle \, dt d\xi \\
&+ \int_{F_1} \int_0^1 \Big\langle \big[(D_P A_i^\alpha(\cdot, \ell_{\xi_0,\rho}(\bar{\xi}_0), X\ell_{\xi_0,\rho}))_{\xi_0,\rho} - (D_P A_i^\alpha(\cdot, \ell_{\xi_0,\rho}(\bar{\xi}_0), X\ell_{\xi_0,\rho} + tX\bar{w}))_{\xi_0,\rho}\big] X\bar{w}, X\eta \Big\rangle dt d\xi \Bigg) \\
:=& I_{11} + I_{12}.
\end{aligned}
$$

Applying (A3) to I_{11}, we obtain

$$
\begin{aligned}
I_{11} \leq& \frac{1}{\varphi''(|1 + X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \int_0^1 \Big\langle \big[(D_P A_i^\alpha(\cdot, \ell_{\xi_0,\rho}(\bar{\xi}_0), X\ell_{\xi_0,\rho}))_{\xi_0,\rho} \\
&- (D_P A_i^\alpha(\cdot, \ell_{\xi_0,\rho}(\bar{\xi}_0), X\ell_{\xi_0,\rho} + tX\bar{w}))_{\xi_0,\rho}\big] X\bar{w},\ X\eta \Big\rangle \, dt d\xi \\
\leq& \frac{2\Lambda}{\varphi''(1 + |X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \int_0^1 \Big(\varphi''(1 + |X\ell_{\xi_0,\rho}|) + \varphi''(1 + |X\ell_{\xi_0,\rho} + tX\bar{w}|) \Big) |X\bar{w}|\, |X\eta| \, dt d\xi.
\end{aligned}
$$

Using (2.7) and Lemma 2.3, we can also get

$$
\begin{aligned}
&\int_0^1 \Big[\varphi''(1 + |X\ell_{\xi_0,\rho}|) + \varphi''(1 + |X\ell_{\xi_0,\rho} + tX\bar{w}|) \Big] dt \\
\sim& \int_0^1 \left[\frac{\varphi'(1 + |X\ell_{\xi_0,\rho}|)}{1 + |X\ell_{\xi_0,\rho}|} + \frac{\varphi'(1 + |X\ell_{\xi_0,\rho} + t(Xu - X\ell_{\xi_0,\rho})|)}{1 + |X\ell_{\xi_0,\rho} + t(Xu - X\ell_{\xi_0,\rho})|} \right] dt \\
\sim& \frac{\varphi'(1 + |X\ell_{\xi_0,\rho}|)}{1 + |X\ell_{\xi_0,\rho}|} + \frac{\varphi'(1 + |X\ell_{\xi_0,\rho}| + |Xu|)}{1 + |X\ell_{\xi_0,\rho}| + |Xu|} \\
\leq& \frac{\varphi'(1 + |X\ell_{\xi_0,\rho}|) + \varphi'(1 + |X\ell_{\xi_0,\rho}| + |Xu|)}{1 + |X\ell_{\xi_0,\rho}|} \leq \frac{2\varphi'(1 + |X\ell_{\xi_0,\rho}| + |Xu|)}{1 + |X\ell_{\xi_0,\rho}|}.
\end{aligned}
$$

On E_1, we have $|Xu - X\ell_{\xi_0,\rho}| \geq 1 + |X\ell_{\xi_0,\rho}|$, i.e.

$$\varphi'(1 + |X\ell_{\xi_0,\rho}| + |Xu|) \sim \varphi'(|Xu - X\ell_{\xi_0,\rho}|).$$

For $t \geq a > 0$, $\varphi_a(t) \sim \varphi(t)$ holds. Together with (2.7) and Lemma 3.4, this yields

$$\begin{aligned}
I_{11} &\leq \frac{4\Lambda}{\varphi''(1 + |X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \frac{\varphi'(1 + |X\ell_{\xi_0,\rho}| + |Xu|)}{1 + |X\ell_{\xi_0,\rho}|} |Xu - X\ell_{\xi_0,\rho}| d\xi \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \\
&\leq 4c\Lambda \frac{1 + |X\ell_{\xi_0,\rho}|}{\varphi(1 + |X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,\rho}|}(|Xu - X\ell_{\xi_0,\rho}|) d\xi \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \\
&\leq 4c\Lambda \frac{1 + |X\ell_{\xi_0,\rho}|}{\varphi(1 + |X\ell_{\xi_0,\rho}|)} \Big(\Psi(\xi_0, \rho) + \varphi(1 + |X\ell_{\xi_0,\rho}|) F(\xi_0, \rho) \Big) \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \\
&= 4c\Lambda (1 + |X\ell_{\xi_0,\rho}|) \left(\frac{\Psi(\xi_0, \rho)}{\varphi(1 + |X\ell_{\xi_0,\rho}|)} + F(\xi_0, \rho) \right) \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta|.
\end{aligned}$$

For I_{12}, we use (A5), Hölder's inequality, (2.9), and Lemma 3.4,

$$\begin{aligned}
I_{12} &\leq \frac{2\Lambda}{|B_{\frac{\varrho}{4}}(\xi_0)|_{\mathbb{H}^n}} \int_0^1 \int_{F_1} \left(\frac{t|Xu - X\ell_{\xi_0,\rho}|}{|X\ell_{\xi_0,\rho}|} \right)^{\alpha_0} |Xu - X\ell_{\xi_0,\rho}| d\xi dt \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \\
&\leq 2c\Lambda (1 + |X\ell_{\xi_0,\rho}|) \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \left(\frac{|Xu - X\ell_{\xi_0,\rho}|}{|X\ell_{\xi_0,\rho}|} \right)^{\alpha_0+1} d\xi \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \\
&\leq 2c\Lambda (1 + |X\ell_{\xi_0,\rho}|) \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \left(\frac{\varphi'(1 + |X\ell_{\xi_0,\rho}|)|Xu - X\ell_{\xi_0,\rho}|^2}{\varphi'(1 + |X\ell_{\xi_0,\rho}|)||X\ell_{\xi_0,\rho}|^2} \right)^{\frac{\alpha_0+1}{2}} d\xi \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \\
&\leq 2c\Lambda (1 + |X\ell_{\xi_0,\rho}|) \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \left[\frac{\varphi'(1 + |X\ell_{\xi_0,\rho}| + |Xu - X\ell_{\xi_0,\rho}|)|Xu - X\ell_{\xi_0,\rho}|^2}{\varphi(1 + |X\ell_{\xi_0,\rho}|)(1 + |X\ell_{\xi_0,\rho}| + |Xu - X\ell_{\xi_0,\rho}|)} \right]^{\frac{\alpha_0+1}{2}} d\xi \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \\
&\leq 2c\Lambda (1 + |X\ell_{\xi_0,\rho}|) \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \left(\frac{\varphi_{1+|X\ell_{\xi_0,\rho}|}(|Xu - X\ell_{\xi_0,\rho}|)}{\varphi(1 + |X\ell_{\xi_0,\rho}|)} \right)^{\frac{\alpha_0+1}{2}} d\xi \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \\
&\leq 2c\Lambda (1 + |X\ell_{\xi_0,\rho}|) \left(\frac{1}{\varphi(1 + |X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,\rho}|}(|Xu - X\ell_{\xi_0,\rho}|) d\xi \right)^{\frac{\alpha_0+1}{2}} \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \\
&\leq 2c\Lambda (1 + |X\ell_{\xi_0,\rho}|) \left(\frac{\Psi(\xi_0, \rho)}{\varphi(1 + |X\ell_{\xi_0,\rho}|)} + F(\xi_0, \rho) \right)^{\frac{\alpha_0+1}{2}} \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta|.
\end{aligned}$$

In the fourth to last inequality, we use the equivalence

$$\frac{\varphi'(1 + |X\ell_{\xi_0,\rho}|)}{1 + |X\ell_{\xi_0,\rho}|} \sim \frac{\varphi'(1 + |X\ell_{\xi_0,\rho}| + |Xu - X\ell_{\xi_0,\rho}|)}{1 + |X\ell_{\xi_0,\rho}| + |Xu - X\ell_{\xi_0,\rho}|}.$$

Estimate of I_2. From the definition of the weak solution (1.3)

$$\fint_{B_{\frac{\varrho}{4}}(\xi_0)} A_i^\alpha(\xi,u,Xu)X\eta d\xi = \fint_{B_{\frac{\varrho}{4}}(\xi_0)} B^\alpha(\xi,u,Xu)\eta d\xi \quad \text{and} \quad \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \Big\langle (A_i^\alpha(\cdot,\ell_{\xi_0,\rho}(\bar{\xi}_0), X\ell_{\xi_0,\rho})_{\xi_0,\rho}, X\eta\Big\rangle d\xi = 0,$$

we have

$$\begin{aligned}
I_2 &= \frac{1}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \Big\langle (A_i^\alpha(\cdot,\ell_{\xi_0,\rho}(\bar{\xi}_0),Xu))_{\xi_0,\rho} - (A_i^\alpha(\cdot,\ell_{\xi_0,\rho}(\bar{\xi}_0),X\ell_{\xi_0,\rho}))_{\xi_0,\rho}, X\eta\Big\rangle d\xi \\
&= \frac{1}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \Big\langle (A_i^\alpha(\cdot,\ell_{\xi_0,\rho}(\bar{\xi}_0),Xu))_{\xi_0,\rho} - A_i^\alpha(\xi,\ell_{\xi_0,\rho}(\bar{\xi}_0),Xu), X\eta\Big\rangle d\xi \\
&+ \frac{1}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} \Big\langle A_i^\alpha(\xi,\ell_{\xi_0,\rho}(\bar{\xi}_0),Xu) - A_i^\alpha(\xi,u,Xu), X\eta\Big\rangle d\xi \\
&+ \frac{1}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} B^\alpha(\xi,u,Xu)\eta d\xi \\
&:= I_{21} + I_{22} + I_{23}.
\end{aligned}$$

By using (A6), $v_{\xi_0}(\xi,\rho) \le 2L$, Jensen's inequality, Corollary 3.4, (2.3), (2.5), (2.7), and Hölder's inequality, we obtain

$$\begin{aligned}
I_{21} &\le \frac{L}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\varrho}{4}}(\xi_0)} v_{\xi_0}(\xi,\rho)\varphi'(1+|Xu|)|X\eta| d\xi \\
&\le \frac{L}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} (\varphi^*)^{-1}\left(\fint_{B_{\frac{\varrho}{4}}(\xi_0)} \varphi^*(v_{\xi_0}(\xi,\rho)\varphi'(1+|Xu|))d\xi\right) \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \\
&\le \frac{cL}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} (\varphi^*)^{-1}\left[\left(\fint_{B_{\frac{\varrho}{4}}(\xi_0)} v_{\xi_0}^{\frac{\gamma}{\gamma-1}}(\xi,\rho)d\xi\right)^{1-\frac{1}{\gamma}} \left(\fint_{B_{\frac{\varrho}{4}}(\xi_0)} \varphi^\gamma(1+|Xu|)d\xi\right)^{\frac{1}{\gamma}}\right] \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \qquad (3.24) \\
&\le \frac{cL}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} (\varphi^*)^{-1}\left[V(\rho)^{1-\frac{1}{\gamma}} \fint_{B_{\frac{\varrho}{2}}(\xi_0)} \varphi(|Xu|)d\xi\right] \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta| \\
&\le \frac{cL}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} \left(V(\rho)^{1-\frac{1}{\gamma}}\right)^{1-\frac{1}{q_0}} (\varphi^*)^{-1}\left[\fint_{B_{\frac{\varrho}{2}}(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,\rho}|}{\rho}\right)d\xi\right] \sup_{B_{\frac{\varrho}{4}}(\xi_0)} |X\eta|.
\end{aligned}$$

In the third to last inequality, we used $\varphi(1+|Xu|) \sim \varphi(|Xu|)$. The final step relies on the fact that $V(\rho) \in [0,1]$ for a sufficiently small ρ, which follows from (A6).

From the hypotheses of Lemma 3.6, (2.9), and (3.22) and for all $s,t > 0$, $\varphi(s+t) \sim \varphi(s) + \varphi(t)$

holds. Together with the Sobolev-Poincaré-type inequality (3.12) with $s = 1$, we further deduce

$$\begin{aligned}
\fint_{B_\rho(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,\rho}|}{\rho}\right) d\xi &\le \fint_{B_\rho(\xi_0)} \varphi\left(\frac{|u-\ell_{\xi_0,\rho}|}{\rho} + \frac{|\ell_{\xi_0,\rho}-u_{\xi_0,\rho}|}{\rho}\right) d\xi \\
&\le c \fint_{B_\rho(\xi_0)} \varphi\left(\frac{|u-\ell_{\xi_0,\rho}|}{\rho}\right) d\xi + c \fint_{B_\rho(\xi_0)} \varphi\left(\frac{|\ell_{\xi_0,\rho}-u_{\xi_0,\rho}|}{\rho}\right) d\xi \\
&\le c \fint_{B_\rho(\xi_0)} \varphi\left(\frac{|u-\ell_{\xi_0,\rho}|}{\rho}\right) d\xi + C_p \fint_{B_\rho(\xi_0)} \varphi(|X\ell_{\xi_0,\rho}|) d\xi \\
&\le c \fint_{B_\rho(\xi_0)} \varphi\left(\frac{|u-\ell_{\xi_0,\rho}|}{\rho} + |X\ell_{\xi_0,\rho}|\right) d\xi \\
&\le c \fint_{B_\rho(\xi_0)} \varphi\left(\frac{|u-\ell_{\xi_0,\rho}|}{\rho} + 1 + |X\ell_{\xi_0,\rho}|\right) d\xi \\
&\le c \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell_{\xi_0,\rho}|}\left(\frac{|u-\ell_{\xi_0,\rho}|}{\rho}\right) d\xi + c\varphi(1+|X\ell_{\xi_0,\rho}|) \\
&= c[\Psi(\xi_0,\rho) + \varphi(1+|X\ell_{\xi_0,\rho}|)] \le 2c\varphi(1+|X\ell_{\xi_0,\rho}|).
\end{aligned} \tag{3.25}$$

Applying (2.8) together with $\varphi^*\left(\frac{\varphi(t)}{t}\right) \sim \varphi(t)$, and combining (3.24) with (3.25), we get

$$I_{21} \le \frac{cL}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} V(\rho)^{\bar{\alpha}} (\varphi^*)^{-1}(\varphi(1+|X\ell_{\xi_0,\rho}|)) \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta| \le cL(1+|X\ell_{\xi_0,\rho}|) V(\rho)^{\bar{\alpha}} \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta|,$$

where $\bar{\alpha} := \left(1-\frac{1}{\gamma}\right)\left(1-\frac{1}{q_0}\right)$.

Analogously, by using (A5), we estimate I_{22}.

$$\begin{aligned}
I_{22} &\le \frac{\Lambda}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} \fint_{B_{\frac{\rho}{4}}(\xi_0)} \mu(|u-\ell_{\xi_0,\rho}(\bar{\xi}_0)|)\varphi'(1+|Xu|) d\xi \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta| \\
&\le \frac{c\Lambda}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} (\varphi^*)^{-1}\left(\fint_{B_{\frac{\rho}{4}}(\xi_0)} \varphi^*\left(\mu(|u-\ell_{\xi_0,\rho}(\bar{\xi}_0)|)\varphi'(1+|Xu|)\right) d\xi\right) \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta| \\
&\le \frac{c\Lambda}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} (\varphi^*)^{-1}\left[\left(\fint_{B_{\frac{\rho}{4}}(\xi_0)} \mu^{\frac{\gamma}{\gamma-1}}(|u-\ell_{\xi_0,\rho}(\bar{\xi}_0)|) d\xi\right)^{1-\frac{1}{\gamma}} \left(\fint_{B_{\frac{\rho}{4}}(\xi_0)} \varphi^\gamma(1+|Xu|) d\xi\right)^{\frac{1}{\gamma}}\right] \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta| \\
&\le \frac{c\Lambda}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} \left[\mu\left(\fint_{B_{\frac{\rho}{4}}(\xi_0)} |u-\ell_{\xi_0,\rho}(\bar{\xi}_0)| d\xi\right)\right]^{\bar{\alpha}} (\varphi^*)^{-1}(\varphi(1+|X\ell_{\xi_0,\rho}|)) \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta| \\
&\le c\Lambda(1+|X\ell_{\xi_0,\rho}|)\mu\left(\fint_{B_{\frac{\rho}{4}}(\xi_0)} |u-\ell_{\xi_0,\rho}(\bar{\xi}_0)| d\xi\right)^{\bar{\alpha}} \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta|.
\end{aligned}$$

Finally, applying (B1), (2.7), (2.9), and Lemma 3.4, we estimate I_{23}.

$$
\begin{aligned}
I_{23} &\leq \frac{L}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} ⨍_{B_{\frac{\rho}{4}}(\xi_0)} \varphi'(1+|Xu|)|\eta| d\xi \\
&\leq \frac{cL\rho}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} ⨍_{B_{\frac{\rho}{4}}(\xi_0)} \varphi'(1+|Xu|) d\xi \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta| \\
&\leq \frac{cL\rho}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} ⨍_{B_{\frac{\rho}{4}}(\xi_0)} \varphi'_{1+|X\ell_{\xi_0,\rho}|}(|Xu - X\ell_{\xi_0,\rho}|) \frac{|Xu - X\ell_{\xi_0,\rho}|}{1+|X\ell_{\xi_0,\rho}|} d\xi \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta| \\
&+ \frac{cL\rho}{\varphi''(1+|X\ell_{\xi_0,\rho}|)} ⨍_{B_{\frac{\rho}{4}}(\xi_0)} \varphi'(1+|X\ell_{\xi_0,\rho}|) d\xi \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta| \\
&\leq cL\rho \frac{1+|X\ell_{\xi_0,\rho}|}{\varphi(1+|X\ell_{\xi_0,\rho}|)} ⨍_{B_{\frac{\rho}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,\rho}|}(|Xu - X\ell_{\xi_0,\rho}|) d\xi \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\zeta| + cL\rho(1+|X\ell_{\xi_0,\rho}|) \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta| \\
&\leq cL(1+|X\ell_{\xi_0,\rho}|) \left\{ \left[\frac{\Psi(\xi_0,\rho)}{\varphi(1+|X\ell_{\xi_0,\rho}|)} + F(\xi_0,\rho) \right] + \rho \right\} \sup_{B_{\frac{\rho}{4}}(\xi_0)} |X\eta|.
\end{aligned}
$$

Collecting all estimates, we obtain the estimate of Lemma 3.6. □

3.5. Decay estimation and iteration

In this section, we use linearization tools, together with the $\mathcal{A}$-harmonic approximation lemma and the Caccioppoli-type inequality, to establish a decay estimate for the excess functional under some smallness assumptions. First of all, we extend Lemma 5 from [41] to noncommubtative vector fields on the Heisenberg group, then give the following useful lemma.

Lemma 3.7. *(A priori estimate) If we let $h \in HW^{1,\varphi}(B_\rho(\xi_0), \mathbb{R}^N)$ be an $\mathcal{A}$-harmonic function, i.e.*

$$
⨍_{B_\rho(\xi_0)} \mathcal{A}(Xh, X\eta) d\xi = 0, \quad \forall \eta \in C_0^\infty(B_\rho(\xi_0), \mathbb{R}^N),
$$

then the following inequality holds:

$$
r \sup_{B_{\frac{r}{8}}(\xi_0)} |X^2 h| + \sup_{B_{\frac{r}{8}}(\xi_0)} |Xh| \leq c ⨍_{B_{\frac{r}{4}}(\xi_0)} |Xh| d\xi, \tag{3.26}
$$

where the constant c depends on Q, N, ν, and Λ.

Proof. Let $v \in HW^{1,\varphi}\left(B_\rho(\xi_0), \mathbb{R}^N\right)$ be a weak solution of the system in (1.1). From Proposition 2.10 of [42], we can deduce

$$
\sup_{B_{\frac{r}{8}}(\xi_0)} |v| \leq \frac{c}{r^Q} \int_{B_{\frac{r}{4}}(\xi_0)} |v| d\xi,
$$

where the constant c depends only on Q, N, ν, and Λ.

If we set $0 < \rho < r$ and $\tau = \frac{\rho}{r}$, then for $\forall \varepsilon > 0$, we can find $\xi_* \in B_\rho(\xi_0)$ such that

$$\sup_{B_\rho(\xi_0)} |v| = \sup_{B_{\tau r}(\xi_0)} |v| \le \varepsilon + |v(\xi_*)| \le \varepsilon + \sup_{B_{\frac{(1-\tau)\rho}{4}}(\xi_*)} |v| \le \varepsilon + c \fint_{B_{\frac{(1-\tau)\rho}{2}}(\xi_*)} |v| d\xi$$
$$\le \varepsilon + c\frac{2^Q}{(1-\tau)^Q} \fint_{B_\rho(\xi_0)} |v| d\xi = \varepsilon + 2^Q c \left(\frac{r}{r-\rho}\right)^Q \fint_{B_\rho(\xi_0)} |v| d\xi.$$

Since $\varepsilon > 0$ is arbitrary, when $\varepsilon \to 0$, for Xv and X^2v, it yields

$$\sup_{B_\rho(\xi_0)} |X^2 v| \le c \left(\frac{r}{r-\rho}\right)^Q \fint_{B_r(\xi_0)} |X^2 v| d\xi, \quad \sup_{B_\rho(\xi_0)} |Xv| \le c \left(\frac{r}{r-\rho}\right)^Q \fint_{B_r(\xi_0)} |Xv| d\xi. \tag{3.27}$$

Now, by combining (3.27), Hölder's inequality, and the Caccioppoli inequality, we derive

$$\begin{aligned}
\sup_{B_{\frac{r}{8}}(\xi_0)} |X^2 v| &\le c \fint_{B_{\frac{r}{5}}(\xi_0)} |X^2 v| d\xi \le c \left(\fint_{B_{\frac{r}{5}}(\xi_0)} |X^2 v|^2 d\xi \right)^{\frac{1}{2}} \\
&\le \frac{c}{r} \left(\fint_{B_{\frac{r}{5}}(\xi_0)} |Xv|^2 d\xi \right)^{\frac{1}{2}} \le \frac{c}{r} \left(\fint_{B_{\frac{r}{4}}(\xi_0)} |Xv| d\xi \right)^{\frac{1}{2}} \left(\sup_{B_{\frac{r}{4}}(\xi_0)} |Xv| \right)^{\frac{1}{2}} \\
&\le \frac{c}{r} \fint_{B_{\frac{r}{4}}(\xi_0)} |Xv| d\xi.
\end{aligned} \tag{3.28}$$

Combining with (3.27) and (3.28), for $v \in HW^{1,\varphi}\left(B_\rho(\xi_0), \mathbb{R}^N\right)$ and a standard approximation argument ([42], the proof of Proposition 2.10, Step 2), we can get the desired a priori estimate for a general $v \in HW^{1,\varphi}\left(B_\rho(\xi_0), \mathbb{R}^N\right)$.

Remark 3.1. *Although Proposition 2.1 from [42] is stated in the Euclidean setting, its proof relies only on the doubling property of the underlying measure and the Sobolev-Poincaré-type inequality. Both properties are available in the Heisenberg group setting, and the Sobolev-Poincaré-type inequality for the horizontal gradient X was established in Lemma 3.3 and Corollary 3.2. Hence the same argument applies verbatim by replacing the Euclidean dimension n with the homogeneous dimension* $Q = 2n + 2$ *and the standard gradient with the horizontal gradient X.*

On the basis of the abovementioned estimates, we can now establish the estimation of the excess functional.

Lemma 3.8. *(Decay estimation) Assume* $\theta \in (0, 1]$ *and that for any ball* $B_r(\xi_0) \subset \Omega$ *with* $r \le \rho_0$*, the following smallness conditions hold:*

$$\Psi(\xi_0, r) \le \delta_1 \varphi(1 + |X\ell_{\xi_0,r}|) \quad \textit{and} \quad F(\xi_0, r)^{\alpha_1} \le \delta_2, \tag{3.29}$$

where

$$\delta_1 = \min\left\{ \left(\frac{C_0 Q \theta^{Q+1}}{8(Q+2)(Q-2)} \right)^{q_1}, \left(\frac{\delta_0}{4}\right)^{\frac{2}{\alpha_0}} \right\},$$
$$\delta_2 = \left(\frac{\delta_0}{2}\right)^2,$$

As δ_0 is given in Lemma 3.1 and if $\alpha_0 \in (0,1)$, α_1 is provided in Lemma 3.6, then the following inequality holds:

$$\Psi(\xi_0, \theta r) \leq C_2 \theta^2 \Psi^*(\xi_0, r),$$

where the constant C_2 only depends on $Q, N, \nu, \Lambda, L, q_0$, and q_1.

Proof. For any ball $B_r(\xi_0) \subset \Omega$, the definition of $\Psi^*(\xi_0, r)$ together with (3.29) yields

$$\varGamma^2 = \frac{\Psi^*(\xi_0, r)}{\varphi(1 + |X\ell_{\xi_0,r}|)} = \frac{\Psi(\xi_0, r)}{\varphi(1 + |X\ell_{\xi_0,r}|)} + F(\xi_0, r)^{\alpha_1} \leq \delta_1 + \delta_2 \leq 1. \tag{3.30}$$

We set

$$\tilde{w} := \frac{u - \ell_{\xi_0,r}}{C_1(1 + |X\ell_{\xi_0,r}|)}.$$

We then deduce the estimate

$$\fint_{B_{\frac{r}{4}}(\xi_0)} \mathcal{A}(X\tilde{w}, X\eta) d\xi \leq \varGamma \left[\left(\frac{\Psi(\xi_0, r)}{\varphi(1 + |X\ell_{\xi_0,r}|)} \right)^{\frac{1}{2}} + \left(\frac{\Psi(\xi_0, r)}{\varphi(1 + |X\ell_{\xi_0,r}|)} \right)^{\frac{\alpha_0}{2}} + F(\xi_0, r)^{\frac{\alpha_1}{2}} \right] \sup_{B_{\frac{r}{4}}(\xi_0)} |X\eta|.$$

Given the assumptions in (3.29) and the selection of δ_1, δ_2, we conclude that $\tilde{w}$ is an approximate $\mathcal{A}$-harmonic approximation function and have the following estimates:

$$\fint_{B_{\frac{r}{4}}(\xi_0)} \mathcal{A}(X\tilde{w}, X\eta) d\xi \leq \varGamma \left[\delta_1^{\frac{1}{2}} + \delta_1^{\frac{\alpha_0}{2}} + \delta_2^{\frac{1}{2}} \right] \sup_{B_{\frac{r}{4}}(\xi_0)} |X\eta| \leq \varGamma \delta_0 \sup_{B_{\frac{r}{4}}(\xi_0)} |X\eta|.$$

Applying Lemma 3.2 and Corollary 3.1 to the mapping $t \to \varphi_{1+|X\ell_{\xi_0,r}|}((1 + |X\ell_{\xi_0,r}|)t)$, we find that there is a real $\mathcal{A}$-harmonic function h, such that

$$\begin{aligned} &\fint_{B_{\frac{r}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|} \left((1 + |X\ell_{\xi_0,r}|) \left| \frac{w - h}{\frac{r}{4}} \right| \right) d\xi \\ \leq & c\epsilon^s \left[\fint_{B_{\frac{r}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}((1 + |X\ell_{\xi_0,r}|)|X\tilde{w}|) d\xi + \varphi_{1+|X\ell_{\xi_0,r}|}((1 + |X\ell_{\xi_0,r}|)\varGamma) \right]. \end{aligned} \tag{3.31}$$

Using the inequality $\varphi_a(sa) \leq cs^2 \varphi_a(a)$ for all $s \in [0,1]$ and (3.30), we obtain

$$\varphi_{1+|X\ell_{\xi_0,r}|}((1 + |X\ell_{\xi_0,r}|)\varGamma) \leq c\varGamma^2 \varphi_{1+|X\ell_{\xi_0,r}|}(1 + |X\ell_{\xi_0,r}|) \leq c\varGamma^2 \varphi(1 + |X\ell_{\xi_0,r}|), \tag{3.32}$$

where the final step relies on the equivalence $\varphi_a(t) \sim \varphi(a)$ for $t \geq a > 0$.

Next, by Lemma 3.4, (2.2) and the definition of $\Psi^*(\xi_0, r)$, we estimate

$$\begin{aligned}
&\fint_{B_{\frac{r}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}(1+|X\ell_{\xi_0,r})||X\tilde{w}|)d\xi \\
=&\fint_{B_{\frac{r}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left(\frac{|Xu - X\ell_{\xi_0,r}|}{C_1}\right)d\xi \\
\le& c\fint_{B_{\frac{r}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left(|Xu - X\ell_{\xi_0,r}|\right)d\xi \\
\le& C_c\left\{\fint_{B_{\frac{r}{2}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left(\frac{|u-\ell_{\xi_0,r}|}{\frac{r}{2}}\right)d\xi + \varphi(1+|X\ell_{\xi_0,r}|)\left[V(r) + \mu\left(\fint_{B_{\frac{r}{2}}(\xi_0)}|u-u_{\xi_0,r}|d\xi\right) + r\right]\right\} \\
\le& c\left\{\fint_{B_r(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left(\frac{|u-\ell_{\xi_0,r}|}{r}\right)d\xi + \varphi(1+|X\ell_{\xi_0,r}|)F(\xi_0,r)\right\} \\
\le& c\left[\Psi(\xi_0,r) + \varphi(1+|X\ell_{\xi_0,r}|)F(\xi_0,r)^{\alpha_1}\right] = c\frac{\Psi^*(\xi_0,r)}{\varphi(1+|X\ell_{\xi_0,r}|)}\varphi(1+|X\ell_{\xi_0,r}|) = c\Gamma^2\varphi(1+|X\ell_{\xi_0,r}|).
\end{aligned} \tag{3.33}$$

Therefore, we substitute (3.32) and (3.33) into (3.31) and yield

$$\fint_{B_{\frac{r}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left((1+|X\ell_{\xi_0,r}|)\left|\frac{w-h}{\frac{r}{4}}\right|\right)d\xi \le c\epsilon^s\Gamma^2\varphi(1+|X\ell_{\xi_0,r}|). \tag{3.34}$$

Subsequently, by applying Theorem 18 from [37] to the $t \to \varphi_{1+|X\ell_{\xi_0,r}|}((1+|X\ell_{\xi_0,r}|)t)$ and combining it with (3.33), we have

$$\fint_{B_{\frac{r}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\Big((1+|X\ell_{\xi_0,r}|)|Xh|\Big)d\xi \le c\fint_{B_{\frac{r}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\Big(|Xu - X\ell_{\xi_0,r}|\Big)d\xi \le c\Gamma^2\varphi(1+|X\ell_{\xi_0,r}|).$$

For $t \le 1$, the following inequality and equivalence holds:

$$\frac{\varphi_{1+|X\ell_{\xi_0,r}|}((1+|X\ell_{\xi_0,r}|)t)}{\varphi(1+|X\ell_{\xi_0,r}|)} \sim \frac{t^2}{(1+t)}\frac{\varphi((1+|X\ell_{\xi_0,r}|)(1+t)^2)}{\varphi(1+|X\ell_{\xi_0,r}|)} \ge \frac{t^2}{(1+t)^2} \ge \frac{t^2}{4}.$$

Therefore, we set $\psi(t) = \varphi_{1+|X\ell_{\xi_0,r}|}((1+|X\ell_{\xi_0,r}|)t)$ and by applying Jensen's inequality, (3.26), and (3.34) to get the estimate

$$\begin{aligned}
\fint_{B_{\frac{r}{4}}(\xi_0)} |Xh|d\xi &= \psi^{-1}\cdot\psi\left(\fint_{B_{\frac{r}{4}}(\xi_0)}|Xh|d\xi\right) \le \psi^{-1}\left(\fint_{B_{\frac{r}{4}}(\xi_0)}\psi(|Xh|)d\xi\right) \\
&= \psi^{-1}\left(\fint_{B_{\frac{r}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}((1+|X\ell_{\xi_0,r}|)|Xh|)d\xi\right) \\
&\le c\psi^{-1}(\Gamma^2\varphi(1+|X\ell_{\xi_0,r}|)) \le c\psi^{-1}(\varphi_{1+|X\ell_{\xi_0,r}|}(1+|X\ell_{\xi_0,r}|)\Gamma) \\
&= c\psi^{-1}(\psi(\Gamma)) = c\Gamma.
\end{aligned} \tag{3.35}$$

Fix $\theta \in \left(0, \frac{1}{8}\right]$ and choose the special affine function

$$\ell^{(h)}(\bar{\xi}) := h_{\xi_0,\theta r} + (Xh)_{\xi_0,\theta r}(\bar{\xi} - \bar{\xi}_0),$$

Using (3.26), (3.35), and Poincaré inequality, we find

$$\sup_{B_{\theta r}(\xi_0)} |h(\bar{\xi}) - \ell^{(h)}(\bar{\xi})| = \sup_{B_{\theta r}(\xi_0)} |h(\bar{\xi}) - h_{\xi_0,\theta r}(\bar{\xi}) - (Xh)_{\xi_0,\theta r}(\bar{\xi} - \bar{\xi}_0)|$$
$$\leq {c_p}^2\theta^2 r^2 ⨍_{B_{\frac{r}{8}}(\xi_0)} |X^2 h| d\xi \leq c\theta^2 r ⨍_{B_{\frac{r}{4}}(\xi_0)} |Xh| d\xi \leq c\theta^2 r\Gamma.$$

Applying $\varphi_a(sa) \leq cs^2\varphi_a(a)$ for $s \in [0,1]$ again, we deduce

$$⨍_{B_{\theta r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left((1+|X\ell_{\xi_0,r}|)\left|\frac{h-\ell^{(h)}}{\theta r}\right|\right) d\xi \leq c\Gamma^2\theta^2\varphi(1+|X\ell_{\xi_0,r}|). \tag{3.36}$$

On the one hand, combining (3.33) with (2.2) and (3.36), it yields

$$\begin{aligned}
&⨍_{B_{\theta r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left((1+|X\ell_{\xi_0,r}|)\left|\frac{\tilde{w}-\ell^{(h)}}{\theta r}\right|\right) d\xi \\
\leq& c⨍_{B_{\theta r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left((1+|X\ell_{\xi_0,r}|)\left|\frac{\tilde{w}-h}{\theta r}\right|\right) d\xi + c⨍_{B_{\theta r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left((1+|X\ell_{\xi_0,r}|)\left|\frac{h-\ell^{(h)}}{\theta r}\right|\right) d\xi \\
\leq& \frac{c}{(4\theta)^Q}\frac{1}{(4\theta)^{q_1}} ⨍_{B_{\frac{r}{4}}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left((1+|X\ell_{\xi_0,r}|)\left|\frac{\tilde{w}-h}{\frac{r}{4}}\right|\right) d\xi + c\Gamma^2\theta^2\varphi(1+|X\ell_{\xi_0,r}|) \\
\leq& c\theta^{-Q-q_1}\epsilon^s\Gamma^2\varphi(1+|X\ell_{\xi_0,r}|) + c\Gamma^2\theta^2\varphi(1+|X\ell_{\xi_0,r}|).
\end{aligned}$$

Furthermore, we choose $\epsilon^s \leq \theta^{Q+q_1+2}$ and combine the definition of $\Psi^*(\xi_0, r)$, and we obtain

$$⨍_{B_{\theta r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left((1+|X\ell_{\xi_0,r}|)\left|\frac{\tilde{w}-\ell^{(h)}}{\theta r}\right|\right) d\xi \leq c\Gamma^2\theta^2\varphi(1+|X\ell_{\xi_0,r}|) = c\theta^2\Psi^*(\xi_0, r). \tag{3.37}$$

On the other hand, we apply (2.2) and Lemma 2.7

$$\begin{aligned}
&⨍_{B_{\theta r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left((1+|X\ell_{\xi_0,r}|)\left|\frac{\tilde{w}-\ell^{(h)}}{\theta r}\right|\right) d\xi \\
=& ⨍_{B_{\theta r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left((1+|X\ell_{\xi_0,r}|)\left|\frac{u-\ell_{\xi_0,r}}{C_1(1+|X\ell_{\xi_0,r}|)} - \ell^{(h)}\right|\frac{1}{\theta r}\right) d\xi \\
\geq& c⨍_{B_{\theta r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left(\frac{|u-\ell_{\xi_0,r} - \ell^{(h)}C_1(1+|X\ell_{\xi_0,r}|)|}{\theta r}\right) d\xi \\
\geq& c⨍_{B_{\theta r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left(\frac{|u-\ell_{\xi_0,\theta r}|}{\theta r}\right) d\xi.
\end{aligned} \tag{3.38}$$

Therefore, from (3.37) and (3.38), we get

$$⨍_{B_{\theta r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left(\frac{|u-\ell_{\xi_0,\theta r}|}{\theta r}\right) d\xi \leq c\theta^2\Psi^*(\xi_0, r). \tag{3.39}$$

Next, we prove that $1 + |X\ell_{\xi_0,r}| \sim 1 + |X\ell_{\xi_0,\theta r}|$. From (2.16), Jensen's inequality, (3.29), and (2.5), the following inequality holds:

$$
\begin{aligned}
|X\ell_{\xi_0,r} - X\ell_{\xi_0,\theta r}| &\le \frac{(Q-2)(Q+2)}{C_0 Q} \fint_{B_{\theta r}(\xi_0)} \left|\frac{u-\ell_{\xi_0,r}}{\theta r}\right| d\xi \\
&\le \frac{(Q-2)(Q+2)}{C_0 Q} \frac{1}{\theta^{Q+1}} \fint_{B_r(\xi_0)} \left|\frac{u-\ell_{\xi_0,r}}{r}\right| d\xi \\
&\le \frac{(Q-2)(Q+2)}{C_0 Q} \frac{1}{\theta^{Q+1}} \varphi^{-1}_{1+|X\ell_{\xi_0,r}|}\left(\fint_{B_r(\xi_0)} \varphi_{1+|X\ell_{\xi_0,r}|}\left(\left|\frac{u-\ell_{\xi_0,r}}{r}\right|\right) d\xi\right) \\
&\le \frac{(Q-2)(Q+2)}{C_0 Q} \frac{1}{\theta^{Q+1}} \varphi^{-1}_{1+|X\ell_{\xi_0,r}|}(\delta_1 \varphi(1+|X\ell_{\xi_0,r}|)) \\
&\le \frac{(Q-2)(Q+2)}{C_0 Q} \frac{1}{\theta^{Q+1}} \delta_1^{\frac{1}{q_1}} \varphi^{-1}_{1+|X\ell_{\xi_0,r}|}(16\, \varphi_{1+|X\ell_{\xi_0,r}|}(1+|X\ell_{\xi_0,r}|)) \\
&\le 4\frac{(Q-2)(Q+2)}{C_0 Q} \frac{1}{\theta^{Q+1}} \delta_1^{\frac{1}{q_1}} (1+|X\ell_{\xi_0,r}|) \le \frac{1}{2}(1+|X\ell_{\xi_0,r}|),
\end{aligned}
$$

where we have used $\delta_1 \le \left(\frac{C_0 Q \theta^{Q+1}}{8(Q+2)(Q-2)}\right)^{q_1}$.

Consequently

$$
1 + |X\ell_{\xi_0,\theta r}| \le |X\ell_{\xi_0,\theta r} - X\ell_{\xi_0,r}| + |X\ell_{\xi_0,r}| + 1 \le \frac{3}{2}(1 + |X\ell_{\xi_0,r}|),
$$

and

$$
1 + |X\ell_{\xi_0,r}| \le |X\ell_{\xi_0,r} - X\ell_{\xi_0,\theta r}| + |X\ell_{\xi_0,\theta r}| + 1 \le \frac{1}{2}(1 + |X\ell_{\xi_0,r}|) + |X\ell_{\xi_0,\theta r}| + 1.
$$

We rearrange the terms, i.e.

$$
\frac{1}{2}(1 + X\ell_{\xi_0,r}|) \le 1 + |X\ell_{\xi_0,\theta r}|,
$$

which implies

$$
1 + |X\ell_{\xi_0,r}| \sim 1 + |X\ell_{\xi_0,\theta r}|.
$$

Furthurmore, we have $\varphi_{1+|X\ell_{\xi_0,\theta r}|}(\cdot) \sim \varphi_{1+|X\ell_{\xi_0,r}|}(\cdot)$.

Combining this with the estimate (3.39), we finally obtain

$$
\fint_{B_{\theta r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,\theta r}|}\left(\frac{|u-\ell_{\xi_0,\theta r}|}{\theta r}\right) d\xi \le C_2 \theta^2 \Psi^*(\xi_0, r).
$$

That is

$$
\Psi(\xi_0, \theta r) \le C_2 \theta^2 \Psi^*(\xi_0, r). \qquad \square
$$

We now introduce an additional excess functional which will be used in the subsequent iteration of the decay estimate from Lemma 3.7, i.e.,

$$
\Psi_\alpha(\xi_0, r) := r^{q_0(1-\alpha)} \fint_{B_r(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,r}|}{r}\right) d\xi, \quad \text{for } \alpha \in (0,1). \tag{3.40}
$$

Lemma 3.9. *(Iteration) Assuming the conditions of Theorem* 1.1*, for any* $\alpha \in (0,1)$*, there are the constant* $\varepsilon_*, \sigma_*, \rho_* > 0$*, and* $\theta \in (0, \frac{1}{8}]$*, where all constants only depend on* $Q, N, \nu, \Lambda, L, q_0$*, and* q_1*, such that for any* $r \in (0, \rho_*)$*,* $B_r(\xi_0) \subset \Omega$ *satisfies*

$$\Psi(\xi_0, r, l_{\xi_0,r}) < \varepsilon_* \varphi(1 + |X\ell_{\xi_0,r}|) \quad \textit{and} \quad \Psi_\alpha(\xi_0, r) < \sigma_*, \tag{3.41}$$

and

$$\Psi(\xi_0, r, l_{\xi_0,\theta^j r}) < \varepsilon_* \varphi(1 + |X\ell_{\xi_0,\theta^j r}|) \quad \textit{and} \quad \Psi_\alpha(\xi_0, \theta^j r) < \sigma_*. \tag{3.42}$$

Proof. At the beginning, we suppose

$$\theta \le \min\left\{ \frac{1}{8}, (2^{q_1+1}C_2)^{-\frac{1}{2}}, \left(2c\sqrt{\frac{C_3(Q-2)(Q+2)}{C_0 Q}}_{q_1} \right)^{-\frac{1}{q_0(1-\alpha)}} \right\},$$

and

$$\varepsilon_* \le \min\left\{ \left(\frac{C_0 Q \theta^{Q+1}}{8(Q+2)(Q-2)} \right)^{q_1}, \left(\frac{\delta_0}{4} \right)^{\frac{2}{\alpha_0}}, \left(\frac{1}{2c} \right)^{\frac{q_1+Q}{q_0(1-\alpha)} \left(\frac{C_3(Q-2)(Q+2)}{C_0 Q} \right)^{-q_1 - \frac{1}{q_1}\left(\frac{q_1+Q}{q_0(1-\alpha)} - 1 \right)}} \right\},$$

and when σ_* and r_* are sufficiently small, we have

$$\left[V(r_*) + r_* + \mu\left(r_* + \bar{c} r_*^\alpha \sigma_*^{\frac{1}{q_0}} \right) \right]^{\alpha_1} \le \min\left\{ \left(\frac{\delta_0}{4} \right)^2, \varepsilon_* \right\},$$

where $\bar{c} := \frac{1}{\varphi(1)^{\frac{1}{q_0}}}$, C_2 is given in Lemma 3.7, and C_3 is defined as follows.

We now apply mathematical induction on the index j to obtain the desired conclusion. For $j = 0$, (3.42) reduces to (3.41), and then the conclusion follows. Assuming that the conclusion holds for some $j \in \mathbb{N}_0$, we proceed to prove it for $j + 1$.

First, we use Hölder's inequality and (2.2) to obtain

$$\begin{aligned}
&\fint_{B_r(\xi_0)} |u - u_{\xi_0,r}| d\xi \\
=& \frac{1}{|B_r(\xi_0)|_{\mathbb{H}^n}} \int_{B_r(\xi_0) \cap \left\{ \frac{|u-u_{\xi_0,r}|}{r} \le 1 \right\}} |u - u_{\xi_0,r}| d\xi + \frac{1}{|B_r(\xi_0)|_{\mathbb{H}^n}} \int_{B_r(\xi_0) \cap \left\{ \frac{|u-u_{\xi_0,r}|}{r} > 1 \right\}} |u - u_{\xi_0,r}| d\xi \\
\le& r + \frac{r}{|B_r(\xi_0)|_{\mathbb{H}^n}} \int_{B_r(\xi_0) \cap \left\{ \frac{|u-u_{\xi_0,r}|}{r} > 1 \right\}} \frac{|u - u_{\xi_0,r}|}{r} d\xi \\
\le& r + \frac{r}{|B_r(\xi_0)|_{\mathbb{H}^n}} \left(\int_{B_r(\xi_0) \cap \left\{ \frac{|u-u_{\xi_0,r}|}{r} > 1 \right\}} \left| \frac{u - u_{\xi_0,r}}{r} \right|^{q_0} d\xi \right)^{\frac{1}{q_0}} |B_r(\xi_0)|_{\mathbb{H}^n}^{1-\frac{1}{q_0}} \\
\le& r + \frac{r}{\varphi(1)^{\frac{1}{q_0}}} \left(\fint_{B_r(\xi_0)} \varphi\left(\frac{|u - u_{\xi_0,r}|}{r} \right) d\xi \right)^{\frac{1}{q_0}}.
\end{aligned}$$

From (3.40), we can obtain

$$F(\xi_0, r) \le V(r) + r + \mu(r + \bar{c} r^\alpha \Psi_\alpha(\xi_0, r)^{\frac{1}{q_0}}).$$

We note that $r \le r_*$ and $\theta \le 1$, we can derive the following inequalities from (3.41):

$$\begin{aligned} F(\xi_0, \theta^j r) &\le V(\theta^j r) + \theta^j r + \mu\left(\theta^j r + \bar{c}(\theta^j r)^\alpha \Psi_\alpha(\theta^j r)^{\frac{1}{q_0}}\right) \\ &\le V(r_*) + r_* + \mu\left(r_* + \bar{c} r_*^\alpha \sigma_*^{\frac{1}{q_0}}\right). \end{aligned}$$

By selecting an appropriate parameter ε_* and applying Lemma 3.7, we can deduce that

$$\Psi(\xi_0, \theta^{j+1} r) \le C_2\theta^2 \Psi^*(\xi_0, \theta^j r) \le 2C_2\theta^2\varepsilon_*\varphi(1 + |X\ell_{\xi_0,\theta^j r}|). \tag{3.43}$$

Next, we use (2.16) and Jensen's inequality to calculate

$$\begin{aligned} |X\ell_{\xi_0,\theta^j r} - X\ell_{\xi_0,\theta^{j+1} r}| &\le \frac{(Q-2)(Q+2)}{C_0 Q}\frac{1}{\theta^{j+1} r}\fint_{B_{\theta^{j+1} r}(\xi_0)} |u - \ell_{\xi_0,\theta^j r}| d\xi \\ &\le \frac{(Q-2)(Q+2)}{C_0 Q}\frac{1}{\theta^{Q+1}}\fint_{B_{\theta^j r}(\xi_0)} \frac{|u - \ell_{\xi_0,\theta^j r}|}{\theta^j r} d\xi \\ &\le \frac{(Q-2)(Q+2)}{C_0 Q}\frac{1}{\theta^{Q+1}}\varphi^{-1}_{1+|X\ell_{\xi_0,\theta^j r}|}\left(\fint_{B_{\theta^j r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,\theta^j r}|}\left(\frac{|u - \ell_{\xi_0,\theta^j r}|}{\theta^j r}\right) d\xi\right). \end{aligned}$$

From the definition of the excess functional $\Psi(\xi_0, \theta^j r)$, (2.5), and (3.42), we can also obtain

$$\begin{aligned} |X\ell_{\xi_0,\theta^j r} - X\ell_{\xi_0,\theta^{j+1} r}| &\le \frac{(Q-2)(Q+2)}{C_0 Q}\frac{1}{\theta^{Q+1}}\varphi^{-1}_{1+|X\ell_{\xi_0,\theta^j r}|}\left(\Psi(\xi_0, \theta^j r)\right) \\ &\le \frac{(Q-2)(Q+2)}{C_0 Q}\frac{1}{\theta^{Q+1}}\varphi^{-1}_{1+|X\ell_{\xi_0,\theta^j r}|}\left(\varepsilon_*\varphi(1 + |X\ell_{\xi_0,\theta^j r}|)\right) \\ &\le \frac{(Q-2)(Q+2)}{C_0 Q}\frac{\varepsilon_*^{\frac{1}{q_1}}}{\theta^{Q+1}}\varphi^{-1}_{1+|X\ell_{\xi_0,\theta^j r}|}\varphi(1 + |X\ell_{\xi_0,\theta^j r}|) \\ &\le \frac{(Q-2)(Q+2)}{C_0 Q}\frac{\varepsilon_*^{\frac{1}{q_1}}}{\theta^{Q+1}}\varphi^{-1}_{1+|X\ell_{\xi_0,\theta^j r}|}\left(16\varphi_{1+|X\ell_{\xi_0,\theta^j r}|}(1 + |X\ell_{\xi_0,\theta^j r}|)\right) \\ &\le 4\frac{(Q-2)(Q+2)}{C_0 Q}\frac{\varepsilon_*^{\frac{1}{q_1}}}{\theta^{Q+1}}(1 + |X\ell_{\xi_0,\theta^j r}|) \le \frac{1}{2}(1 + |X\ell_{\xi_0,\theta^j r}|). \end{aligned} \tag{3.44}$$

Here, we utilize (3.44)

$$1 + |X\ell_{\xi_0,\theta^j r}| \le 1 + |X\ell_{\xi_0,\theta^j r} - X\ell_{\xi_0,\theta^{j+1} r}| + |X\ell_{\xi_0,\theta^{j+1} r}| \le \frac{1}{2}(1 + |X\ell_{\xi_0,\theta^j r}|) + (1 + |X\ell_{\xi_0,\theta^{j+1} r}|).$$

By rearranging the terms, we have

$$1 + |X\ell_{\xi_0,\theta^j r}| \le 2(1 + |X\ell_{\xi_0,\theta^{j+1} r}|). \tag{3.45}$$

Therefore, combining (3.43), (3.45), and (2.2) with a suitable θ, we can obtain

$$\begin{aligned} \Psi(\xi_0, \theta^{j+1} r) &\le 2C_2\theta^2\varepsilon_*\varphi(1 + |X\ell_{\xi_0,\theta^j r}|) \le 2C_2\theta^2\varepsilon_*\varphi(2(1 + |X\ell_{\xi_0,\theta^{j+1} r}|)) \\ &\le C_2\theta^2\varepsilon_* 2^{q_1+1}\varphi(1 + |X\ell_{\xi_0,\theta^{j+1} r}|) \le \varepsilon_*\varphi(1 + |X\ell_{\xi_0,\theta^{j+1} r}|). \end{aligned}$$

Next, we prove the second conclusion. Select a special affine function

$$\ell_{\xi_0,\theta^j r}(\bar{\xi}) := u_{\xi_0,\theta^j r} + X\ell_{\xi_0,\theta^j r}(\bar{\xi} - \bar{\xi}_0).$$

By using (2.17), (2.9), and the Sobolev-Poincaré-type inequality (3.12) with $s = 1$ and a suitable θ, we obtain

$$\begin{aligned}
\Psi_\alpha(\xi_0, \theta^{j+1} r) &= (\theta^{j+1} r)^{q_0(1-\alpha)} \fint_{B_{\theta^{j+1} r}(\xi_0)} \varphi\left(\frac{|u - u_{\xi_0,\theta^{j+1} r}|}{\theta^{j+1} r}\right) d\xi \\
&\le c(\theta^{j+1} r)^{q_0(1-\alpha)} \fint_{B_{\theta^{j+1} r}(\xi_0)} \varphi\left(\frac{|u - u_{\xi_0,\theta^j r}|}{\theta^{j+1} r}\right) d\xi \\
&\le c(\theta^{j+1} r)^{q_0(1-\alpha)} \fint_{B_{\theta^{j+1} r}(\xi_0)} \varphi\left(\frac{|u - \ell_{\xi_0,\theta^j r}|}{\theta^{j+1} r} + \frac{|\ell_{\xi_0,\theta^j r} - u_{\xi_0,\theta^j r}|}{\theta^{j+1} r}\right) d\xi \\
&\le c(\theta^{j+1} r)^{q_0(1-\alpha)} \left[\fint_{B_{\theta^{j+1} r}(\xi_0)} \varphi\left(\frac{|u - \ell_{\xi_0,\theta^j r}|}{\theta^{j+1} r}\right) d\xi + \fint_{B_{\theta^{j+1} r}(\xi_0)} \varphi\left(\frac{|\ell_{\xi_0,\theta^j r} - u_{\xi_0,\theta^j r}|}{\theta^{j+1} r}\right) d\xi\right] \\
&\le c(\theta^{j+1} r)^{q_0(1-\alpha)} \left[\fint_{B_{\theta^{j+1} r}(\xi_0)} \varphi\left(\frac{|u - \ell_{\xi_0,\theta^j r}|}{\theta^{j+1} r}\right) d\xi + C_p \fint_{B_{\theta^{j+1} r}(\xi_0)} \varphi\left(|X\ell_{\xi_0,\theta^j r}|\right) d\xi\right] \\
&\le c(\theta^{j+1} r)^{q_0(1-\alpha)} \fint_{B_{\theta^{j+1} r}(\xi_0)} \varphi\left(\frac{|u - \ell_{\xi_0,\theta^j r}|}{\theta^{j+1} r} + |X\ell_{\xi_0,\theta^j r}|\right) d\xi \\
&\le c(\theta^{j+1} r)^{q_0(1-\alpha)} \fint_{B_{\theta^{j+1} r}(\xi_0)} \varphi_{1+|X\ell_{\xi_0,\theta^j r}|}\left(\frac{|u - \ell_{\xi_0,\theta^j r}|}{\theta^{j+1} r}\right) d\xi + c(\theta^{j+1} r)^{q_0(1-\alpha)} \varphi(|X\ell_{\xi_0,\theta^j r}|) \\
&\le \frac{c(\theta^{j+1} r)^{q_0(1-\alpha)}}{\theta^{q_1+Q}} \Psi(\xi_0, \theta^j r) + c(\theta^{j+1} r)^{q_0(1-\alpha)} \varphi(1 + |X\ell_{\xi_0,\theta^j r}|) \\
&\le \frac{c(\theta^{j+1} r)^{q_0(1-\alpha)}}{\theta^{q_1+Q}} \varepsilon_* \varphi(1 + |X\ell_{\xi_0,\theta^j r}|) + c(\theta^{j+1} r)^{q_0(1-\alpha)} \varphi(1 + |X\ell_{\xi_0,\theta^j r}|).
\end{aligned}$$

From (2.16), we obtain

$$|X\ell_{\xi_0,\theta^j r}| \le \frac{(Q-2)(Q+2)}{C_0 Q} \frac{1}{\theta^j r} \fint_{B_{\theta^j r}(\xi_0)} \left|u - u_{\xi_0,\theta^j r}\right| d\xi = \frac{(Q-2)(Q+2)}{C_0 Q} \fint_{B_{\theta^j r}(\xi_0)} \left|\frac{u - u_{\xi_0,\theta^j r}}{\theta^j r}\right| d\xi.$$

We then choose the constant c such that $C_0 Q \le c(Q+2)(Q-2) \fint_{B_{\theta^j r}(\xi_0)} \left|\frac{u - u_{\xi_0,\theta^j r}}{\theta^j r}\right| d\xi$ and we define $C_3 := c + 1$. In this way, the following inequalitiy (3.46) holds, i.e.,

$$1 + |X\ell_{\xi_0,\theta^j r}| \le \frac{C_3(Q-2)(Q+2)}{C_0 Q} \fint_{B_{\theta^j r}(\xi_0)} \left|\frac{u - u_{\xi_0,\theta^j r}}{\theta^j r}\right| d\xi. \tag{3.46}$$

Since the N-function φ is increasing, using Jensen's inequality, we have

$$\begin{aligned}\varphi(1+|X\ell_{\xi_0,\theta^j r}|) &\le \varphi\left(\frac{C_3(Q-2)(Q+2)}{C_0 Q}\fint_{B_{\theta^j r}(\xi_0)}\left|\frac{u-u_{\xi_0,\theta^j r}}{\theta^j r}\right|d\xi\right)\\
&\le \left(\frac{C_3(Q-2)(Q+2)}{C_0 Q}\right)^{q_1}\fint_{B_{\theta^j r}(\xi_0)}\varphi\left(\frac{|u-u_{\xi_0,\theta^j r}|}{\theta^j r}\right)d\xi\\
&= \left(\frac{C_3(Q-2)(Q+2)}{C_0 Q}\right)^{q_1}\frac{1}{(\theta^j r)^{q_0(1-\alpha)}}\Psi_\alpha(\xi_0,\theta^j r)\\
&\le \left(\frac{C_3(Q-2)(Q+2)}{C_0 Q}\right)^{q_1}\frac{1}{(\theta^j r)^{q_0(1-\alpha)}}\sigma_*.\end{aligned}$$

By selecting an appropriate θ, we can estimate

$$\begin{aligned}\Psi_\alpha(\xi_0,\theta^{j+1}r) \le{}& \frac{c(\theta^{j+1}r)^{q_0(1-\alpha)}}{\theta^{q_1+Q}}\varepsilon_*\left(\frac{C_3(Q-2)(Q+2)}{C_0 Q}\right)^{q_1}\frac{1}{(\theta^j r)^{q_0(1-\alpha)}}\sigma_*\\
&+(\theta^{j+1}r)^{q_0(1-\alpha)}c\left(\frac{C_3(Q-2)(Q+2)}{C_0 Q}\right)^{q_1}\frac{1}{(\theta^j r)^{q_0(1-\alpha)}}\sigma_*\\
\le{}& \frac{1}{2}\sigma_*+\frac{1}{2}\sigma_*=\sigma_*.\end{aligned}$$

Thus, we have completed the proof. □

3.6. The proof of Theorem 1.1

On the basis of the preceding series of estimates, we now prove Theorem 1.1 of the present paper to obtain the partial regularity result for weak solutions.

Suppose that

$$\liminf_{\rho\to 0}\fint_{B_\rho(\xi_0)}|V(Xu)-V(Xu)_{\xi_0,\rho})|^2 d\xi = 0,$$

and

$$\limsup_{\rho\to 0}\left|\fint_{B_\rho(\xi_0)}Xud\xi\right|<\infty.$$

From (2.13) and (2.14) in Lemma 2.4, we have

$$\fint_{B_\rho(\xi_0)}|V(Xu)-(V(Xu))_{\xi_0,\rho}|^2 d\xi \sim \fint_{B_\rho(\xi_0)}\varphi_{1+|Xu_{\xi_0,\rho}|}(|Xu-(Xu)_{\xi_0,\rho}|)d\xi.$$

We then get

$$\liminf_{\rho\to 0}\fint_{B_\rho(\xi_0)}\varphi_{1+|(Xu)_{\xi_0,\rho}|}(|Xu-(Xu)_{\xi_0,\rho}|)d\xi = 0. \tag{3.47}$$

Next, by applying Sobolev-Poincaré-type inequality (3.12) with $s = 1$, (2.9), and (3.47), we obtain

$$\begin{aligned}\Psi_\alpha(\xi_0,\rho) &= \rho^{q_0(1-\alpha)} \fint_{B_\rho(\xi_0)} \varphi\left(\left|\frac{u-u_{\xi_0,\rho}}{\rho}\right|\right)d\xi \\ &\le C_p\rho^{q_0(1-\alpha)} \fint_{B_\rho(\xi_0)} \varphi(|Xu|)d\xi \\ &\le c\rho^{q_0(1-\alpha)} \fint_{B_\rho(\xi_0)} \varphi(1+|Xu|)d\xi \\ &\le c\rho^{q_0(1-\alpha)} \fint_{B_\rho(\xi_0)} \varphi(1+|(Xu)_{\xi_0,\rho}|+|Xu-(Xu)_{\xi_0,\rho}|)d\xi \\ &\le c\rho^{q_0(1-\alpha)} \fint_{B_\rho(\xi_0)} \varphi_{1+|(Xu)_{\xi_0,\rho}|}(|Xu-(Xu)_{\xi_0,\rho}|)d\xi + c\rho^{q_0(1-\alpha)}\varphi(1+|(Xu)_{\xi_0,\rho}|).\end{aligned}$$

Furthermore, by applying Lemma 2.2, we have

$$\begin{aligned}\Psi(\xi_0,\rho) &= \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell_{\xi_0,\rho}|}\left(\frac{|u-\ell_{\xi_0,\rho}|}{\rho}\right)d\xi \\ &\le C_p \fint_{B_\rho(\xi_0)} \varphi_{1+|X\ell_{\xi_0,\rho}|}\left(|Xu-(Xu)_{\xi_0,\rho}|\right)d\xi \\ &\le c\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|(Xu)_{\xi_0,\rho}|}\left(|Xu-(Xu)_{\xi_0,\rho}|\right)d\xi + c_\delta\varphi_{1+|X\ell_{\xi_0,\rho}|}\left(|(Xu)_{\xi_0,\rho}-X\ell_{\xi_0,\rho}|\right) \\ &\le c\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|(Xu)_{\xi_0,\rho}|}\left(|Xu-(Xu)_{\xi_0,\rho}|\right)d\xi + c_\delta\varphi_{1+|(Xu)_{\xi_0,\rho}|}\left(|(Xu)_{\xi_0,\rho}-X\ell_{\xi_0,\rho}|\right) \\ &\le c\delta \fint_{B_\rho(\xi_0)} \varphi_{1+|(Xu)_{\xi_0,\rho}|}\left(|Xu-(Xu)_{\xi_0,\rho}|\right)d\xi.\end{aligned}$$

The selection of ξ_0 is determined by the aforementioned estimation, where $0<\rho<r$ exists such that

$$\Psi(\xi_0,\rho)<\delta_* \quad \text{and} \quad \Psi_\alpha(\xi_0,\rho)<\sigma_*. \tag{3.48}$$

By the absolute continuity of the integral, we can find that in a neighborhood U of ξ_0, we have

$$\Psi(\xi,\rho)<\delta_* \quad \text{and} \quad \Psi_\alpha(\xi,\rho)<\sigma_* \quad \forall\xi\in U.$$

Next, if we let $\rho\in(0,r]$, $k\in N_0$ exists such that $\sigma^{k+1}r<\rho\le\sigma^k r$. From Remark 2.3, (2.3), and (3.48), we can obtain

$$\begin{aligned}\Psi_\alpha(\xi_0,\rho) &= \rho^{q_0(1-\alpha)} \fint_{B_\rho(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,\rho}|}{\rho}\right)d\xi \\ &\le \frac{c(\sigma^k r)^{q_0(1-\alpha)}}{|B_\rho(\xi_0)|_{\mathbb{H}^n}} \int_{B_\rho(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,\sigma^k r}|}{\rho}\right)d\xi \\ &\le \frac{c(\sigma^k r)^{q_0(1-\alpha)}|B_{\sigma^k r(\xi_0)}|_{\mathbb{H}^n}}{|B_\rho(\xi_0)|_{\mathbb{H}^n}|B_{\sigma^k r(\xi_0)}|_{\mathbb{H}^n}} \int_{B_\rho(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,\sigma^k r}|}{\sigma^{k+1}r}\right)d\xi \\ &\le \frac{c(\sigma^k r)^{q_0(1-\alpha)}}{\sigma^{Q+q_1}} \fint_{B_{\sigma^k r}(\xi_0)} \varphi\left(\frac{|u-u_{\xi_0,\sigma^k r}|}{\sigma^k r}\right)d\xi \\ &= \frac{c}{\sigma^{Q+q_1}}\Psi_\alpha(\xi_0,\sigma^k r) \le \frac{c}{\sigma^{Q+q_1}}\sigma_* < \infty.\end{aligned}$$

Therefore

$$\sup_{\xi\in U,\sigma\in(0,\rho]} \Psi_\alpha(\xi,\sigma) < \infty.$$

Finally, applying (2.2), we get

$$\begin{aligned}
&\sigma^{q_0(1-\alpha)} \fint_{B_\sigma(\xi)} \left|\frac{u-u_{\xi,\sigma}}{\sigma}\right|^{q_0} d\xi \\
=&\frac{\sigma^{q_0(1-\alpha)}}{|B_\sigma(\xi)|_{\mathbb{H}^n}} \int_{B_\sigma(\xi)\cap\left\{\left|\frac{u-u_{\xi,\sigma}}{\sigma}\right|\le 1\right\}} \left|\frac{u-u_{\xi,\sigma}}{\sigma}\right|^{q_0} d\xi \\
+&\frac{\sigma^{q_0(1-\alpha)}}{|B_\sigma(\xi)|_{\mathbb{H}^n}\varphi(1)} \int_{B_\sigma(\xi)\cap\left\{\left|\frac{u-u_{\xi,\sigma}}{\sigma}\right|> 1\right\}} \left|\frac{u-u_{\xi,\sigma}}{\sigma}\right|^{q_0} \varphi(1) d\xi \\
\le&\sigma^{q_0(1-\alpha)} + \frac{\sigma^{q_0(1-\alpha)}}{|B_\sigma(\xi)|_{\mathbb{H}^n}\varphi(1)} \int_{B_\sigma(\xi)} \varphi\left(\left|\frac{u-u_{\xi,\sigma}}{\sigma}\right|\right) d\xi \\
=&\sigma^{q_0(1-\alpha)} + \frac{\sigma^{q_0(1-\alpha)}}{\varphi(1)} \fint_{B_\sigma(\xi)} \varphi\left(\left|\frac{u-u_{\xi,\sigma}}{\sigma}\right|\right) d\xi
\end{aligned}$$

and

$$\sup_{\xi\in U,\sigma\in(0,\rho)} \frac{1}{\sigma^{Q+q_0\alpha}} \fint_{B_\sigma(\xi)} |u-u_{\xi,\sigma}|^{q_0} d\xi < \infty.$$

From Campanato's characterization of Hölder's continuity, we get the weak solution $u \in C^{0,\alpha}(U,\mathbb{R}^N)$.

4. Conclusions

In this paper, we have studied the partial regularity for weak solutions of subelliptic systems (1.1) under Orlicz growth on the Heisenberg group. The proposed approach is based on the $\mathcal{A}$-harmonic approximation technique, specifically adapted to handle the noncommutativity of vector fields and the nonlinearity of the systems. This framework provides a unified method for studying both superquadratic and subquadratic cases, effectively avoiding the complexities typically associated with applying Taylor expansions on the Heisenberg group.

We established several key theoretical components, including an $\mathcal{A}$-harmonic approximation lemma and a Sobolev-Poincaré-type inequality compatible with Orlicz growth. By developing linearization tools that compare weak solutions with homogeneous linear systems, we derived decay estimates for the excess functional. These results ultimately led to the proof of Theorem 1.1, characterizing the singular set and demonstrating that weak solutions achieve Hölder's continuity $C^{0,\alpha}$ within the framework of Campanato spaces.

From a theoretical perspective, this work extends the regularity theory for variational problems with nonstandard growth to the setting of noncommutative geometry. The simulation of the iteration process and the decay analysis confirm that the $\mathcal{A}$-harmonic approximation method is highly effective for addressing subelliptic systems. Our findings provide new insights into the regularity of solutions on Carnot groups and offer a robust foundation for future research into more general degenerate elliptic equations.

Author contributions

Shengling Tu: Formal analysis, methodology, writing—original draft; Jialin Wang: Formal analysis, methodology, writing—original draft, funding acquisition, supervision, writing—review and editing; Shuijin Zhang: Formal analysis, methodology, funding acquisition, supervision, writing—review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of generative AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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