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Semilinear elliptic problems on the half-space with a sign-changing nonlinearity

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Abstract: Semilinear elliptic problems of the form

$$\begin{cases} -\Delta w = g + \mu f(.,w), & \text{in } \mathbb{R}^n_+, \\ w > 0, & \text{in } \mathbb{R}^n_+, \\ \lim\limits_{y_n \to 0} w(y) = \phi(y'), & \\ \lim\limits_{|y| \to \infty} w(y) = 0, & \end{cases}$$

are considered, where $\mathbb{R}^n_+ = \{y = (y', y_n) \in \mathbb{R}^n : y_n > 0\}$ $(n \geq 2)$, $\mu > 0$, and ϕ is a nonnegative continuous function in $\mathbb{R}^{n-1}$ vanishing at infinity. The source term g is assumed to be a nontrivial nonnegative function belonging to a suitable Kato class, whereas the nonlinear term f may change sign. The existence and uniqueness of a positive continuous solution are established for a suitable range of μ under Lipschitz-type assumptions on f. The global behavior of this solution is also described. These results go beyond several earlier contributions by allowing a wider range of sign-changing nonlinearities, while preserving full qualitative information about the solution.

Keywords: semilinear elliptic problems; existence and uniqueness; qualitative behavior
Mathematics Subject Classification: 35J60, 35B40

1. Introduction

In this paper, we study the existence and uniqueness of positive continuous solutions to the semilinear elliptic problem

$$
\begin{cases}
-\Delta w = g + \mu f\,(.,w)\,, & \text{in } \mathbb{R}^n_+,\\
w > 0, & \text{in } \mathbb{R}^n_+,\\
\lim\limits_{y_n \to 0} w(y) = \phi(y'), & \\
\lim\limits_{|y| \to \infty} w(y) = 0. &
\end{cases}
\tag{1.1}
$$

Here, $n \geq 2$, $\mathbb{R}^n_+ = \{y = (y', y_n) \in \mathbb{R}^n : y_n > 0\}$ is the half space, $\mu > 0$, ϕ is a nonnegative continuous function in $\mathbb{R}^{n-1}$ vanishing at infinity, and $g : \mathbb{R}^n_+ \to [0, \infty)$ is a nontrivial function in $\mathcal{K}^\infty(\mathbb{R}^n_+)$ (see Definition 1.1). Solutions are understood in the distributional sense.

A main feature of problem (1.1) is that the nonlinear term $f(y, w)$ may to change sign. This considerably broadens the class of admissible models but introduces additional analytical difficulties. In particular, techniques that rely on monotonicity or positivity of the nonlinearity are no longer directly applicable. Our objective is to prove the existence and uniqueness of a positive solution in $C_0(\mathbb{R}^n_+)$ under appropriate assumptions on f and to describe its behavior near the boundary and at infinity.

Our approach relies mainly on the features of the class $\mathcal{K}^\infty\left(\mathbb{R}^n_+\right)$ introduced in [1] for $n \geq 3$ and in [2] for $n = 2$. We recall the definition below.

Definition 1.1. *We say that the function φ belongs to the class $\mathcal{K}^\infty(\mathbb{R}^n_+)$ if*

$$
\lim_{r \to 0} \left(\sup_{y \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+ \cap B(y,r)} \frac{z_n}{y_n} G(y,z) |\varphi(z)| dz \right) = 0, \tag{1.2}
$$

$$
\lim_{M \to \infty} \left(\sup_{y \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+ \cap (|z| \geq M)} \frac{z_n}{y_n} G(y,z) |\varphi(z)| dz \right) = 0. \tag{1.3}
$$

Here, $G(y,z)$ denotes the Green function for the Dirichlet Laplacian in $\mathbb{R}^n_+$.

For example, if $s > \frac{n}{2}$ and $a \in L^s\left(\mathbb{R}^n_+\right)$, then from [3, Proposition 2.11], functions of the form

$$
y \longrightarrow \frac{a(y)}{(|y|+1)^{\beta-\alpha} y_n^\alpha} \in \mathcal{K}^\infty(\mathbb{R}^n_+)
$$

belong to $\mathcal{K}^\infty\left(\mathbb{R}^n_+\right)$ provided that $\alpha < 2 - \frac{n}{s} < \beta$.

It has been proved (see [1, Proposition 9]) that $\mathcal{K}^\infty(\mathbb{R}^n_+)$ strictly contains the Kato class $K_n^\infty(\mathbb{R}^n_+)(n \geq 3)$, introduced in [4, 5].

Remark 1.2. *The generalized Kato class $\mathcal{K}^\infty\left(\mathbb{R}^n_+\right)$ is particularly suitable for the present problem because the nonlinear estimates are formulated in terms of Green potentials rather than solely in terms of Lebesgue norms. The two conditions in Definition 1.1 respectively, control the singularity of the Green function in the diagonal which allow us to obtain the continuity of potentials up to the boundary and appropriate decay at infinity. This is useful for source terms that may exhibit singular behavior near the boundary or decay too slowly at infinity to be conveniently handled by a single global L^p-condition. Thus, the generalized Kato framework provides the potential-theoretic control needed for the Green-potential estimates in the half-space.*

Recent progress concerns different classes of elliptic problems in half-space and unbounded domains. Ghergu and Rădulescu [6] presented a large collection of nonlinear partial differential equation models arising in biology, chemistry, and population dynamics, showing the importance of qualitative and analytical methods in the study of nonlinear elliptic equations.

In a related work, Ghergu and Rădulescu [7] proved the existence of ground state solutions for singular Lane-Emden-Fowler equations with sublinear convection terms. In [8], Jeanjean and Rădulescu studied nonhomogeneous quasilinear elliptic problems in the linear and sublinear cases, obtaining existence and multiplicity results by means of variational techniques. Recently, Montoro, Muglia and Sciunzi [9] have described the complete structure of solutions of a semilinear elliptic equation in the half-space and classified all its weak solutions. Also, Deng, Shi and Zhang [10] proved the existence results for critical Neumann problems with superlinear perturbation in the half space. Le [11] has proved symmetry properties of bounded solutions to quasilinear elliptic equations in a half-space. In the half-space, Li and Souplet [12] proved a Liouville theorem for the Lane–Emden system, which implies important nonexistence and classification results. Liang [13] studied the asymptotic behavior at infinity of solutions of the Monge-Ampère equations in the half space, and obtained higher order expansion formulas. These works show the great variety of analytical techniques used in the study of elliptic equations on unbounded domains and half-spaces, and emphasize the importance of the understanding of the existence, uniqueness, and qualitative behavior of solutions under general assumptions on the nonlinear terms.

On the other hand, it has been established in previous research (see, e.g., [1–3, 14, 15] and references therein) that $\mathcal{K}^{\infty}(\mathbb{R}^n_+)$ provides a useful framework for studying regularities of potentials functions. Specifically, it guarantees continuity up to the boundary and appropriate decay at infinity, but allows coefficients with singularities or slow decay which may be beyond the usual integrability assumptions. Thus, the class is very appropriate to solve existence and multiplicity questions for large families of semilinear elliptic boundary value problems.

Notations.

We use the following notation throughout the paper.

i. $\mathbb{R}^n_+ := \{y = (y_1, ..., y_n) = (y', y_n) \in \mathbb{R}^n : y_n > 0\}$, where $n \geq 2$.

ii. For $y \in \mathbb{R}^n_+$, write $\overline{y} = (y', -y_n)$.

iii. Denote by $\mathcal{B}(\mathbb{R}^n_+)$ the set of Borel measurable functions on $\mathbb{R}^n_+$, and by $\mathcal{B}^+(\mathbb{R}^n_+)$ its nonnegative elements.

iv. $\mathcal{K}^{\infty}_+(\mathbb{R}^n_+) := \{\varphi \in \mathcal{K}^{\infty}(\mathbb{R}^n_+) \text{ with } \varphi \geq 0\}$.

v. $C_0(\mathbb{R}^n_+)$ is the Banach space of continuous functions on $\mathbb{R}^n_+$ that vanish on the boundary $y_n \to 0$ and at infinity, equipped with the uniform norm:

$$\|\varphi\|_\infty = \sup_{y \in \mathbb{R}^n_+} |\varphi(y)|.$$

vi. $C_0(\mathbb{R}^{n-1})$ is the space of continuous functions in $\mathbb{R}^{n-1}$ that vanish at infinity.

vii. For $\varphi \in \mathcal{B}^+(\mathbb{R}^n_+)$, define $U\varphi$ by the Green potential

$$U\varphi(.) := \int_{\mathbb{R}^n_+} G(., z)\varphi(z)dz.$$

viii. For $\varphi \in \mathcal{B}(\mathbb{R}^n_+)$, define

$$\|\varphi\| := \sup_{y \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{z_n}{y_n} G(y,z)\, |\varphi(z)|\, dz.$$

ix. For φ and ψ nonnegative on $\mathbb{R}^n_+$, write $\varphi \asymp \psi$ if there exists $C > 0$ such that

$$\frac{1}{C}\psi(y) \leq \varphi(y) \leq C\psi(y), \text{ for all } y \in \mathbb{R}^n_+.$$

Throughout the paper, for a given nonnegative function ϕ in $C_0(\mathbb{R}^{n-1})$, we let $H\phi$ (see [16, Section 1.7]) be the unique nonnegative solution in $C_0(\mathbb{R}^n_+)$ of the Dirichlet problem

$$\begin{cases} \Delta w = 0, & \text{in } \mathbb{R}^n_+, \\ \lim\limits_{y_n \to 0} w(y) = \phi(y'), & \\ \lim\limits_{|y| \to \infty} w(y) = 0. & \end{cases} \tag{1.4}$$

We recall that $H\phi$ is given by

$$H\phi(y) := \frac{2}{\sigma_n} \int_{\mathbb{R}^{n-1}} \frac{y_n}{|y-z|^n} \phi(z) dz, \text{ for } y \in \mathbb{R}^n_+, \tag{1.5}$$

where $\sigma_n = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

We assume the following hypotheses:

$(H1)$ g is a nontrivial function in $\mathcal{K}^\infty_+(\mathbb{R}^n_+)$.

$(H2)$ f is a real measurable function on $\mathbb{R}^n_+ \times \mathbb{R}$ with $f(y,0) = 0$, for all $y \in \mathbb{R}^n_+$.

$(H3)$ There exists $q \in \mathcal{K}^\infty_+(\mathbb{R}^n_+)$, such that

$$|f(y,w_1) - f(y,w_2)| \leq q(y)\, |w_1 - w_2|, \text{ for all } y \in \mathbb{R}^n_+ \text{ and } w_1, w_2 \in \mathbb{R}.$$

Theorem 1.3. *Assume that g satisfies $(H1)$ and that f satisfies $(H2)$ and $(H3)$. Then, there exists a constant $\mu^* > 0$ such that, for every $\mu \in (0,\mu^*)$, problem (1.1) admits a unique continuous solution w satisfying on $\mathbb{R}^n_+$,*

$$\frac{2(\mu^* - \mu)}{2\mu^* - \mu}(H\phi(y) + Ug(y)) \leq w(y) \leq \frac{2\mu^*}{2\mu^* - \mu}(H\phi(y) + Ug(y)). \tag{1.6}$$

Hence, under small parameter perturbations, nonlinear effects do not alter the decay order of the linear solution $\theta = H\phi + Ug$ near the boundary and at infinity.

The present work is closely related to [14], but differs in an essential aspect of the nonlinear term and techniques used in the proof. While [14] considers a related semilinear problem under assumptions that do not cover the sign-changing framework considered here, our assumptions $(H2)$ and $(H3)$ permit $f(y,w)$ to change sign while maintaining a weighted Lipschitz control through $q \in \mathcal{K}^\infty_+(\mathbb{R}^n_+)$. This permits the Green-potential and 3G-inequality machinery to be combined with a contraction argument without imposing a fixed sign on the nonlinear perturbation. In addition, the present work provides the explicit two-sided estimates (1.6), which describe the global behavior of the nonlinear solution in terms of the corresponding linear solution.

Remark 1.4. *(see [17])*
For $\varphi \in \mathcal{B}^+(\mathbb{R}^n_+)$, if $U\varphi(z_0) < \infty$ for some $z_0 \in \mathbb{R}^n_+$, then $U\varphi \in L^1_{loc}(\mathbb{R}^n_+)$. Conversely, if $\varphi \in \mathrm{L}^1_{loc}(\mathbb{R}^n_+)$ and $U\varphi \in \mathrm{L}^1_{loc}(\mathbb{R}^n_+)$, then

$$-\Delta\,(U\varphi) = \varphi,\ \ in\ \mathbb{R}^n_+.$$

Remark 1.5. *Let $\phi \in C_0(\mathbb{R}^{n-1})$ with $\phi \geq 0$ and assume that g satisfies $(H1)$. Set*

$$\theta(y) := H\phi(y) + Ug(y),\ \ y \in \mathbb{R}^n_+. \tag{1.7}$$

Then, θ is the unique solution in $C_0(\mathbb{R}^n_+)$ to the associated linear problem (1.8)

$$\begin{cases} -\Delta w = g(y), & in\ \mathbb{R}^n_+, \\ \lim\limits_{y_n \to 0} w(y) = \phi(y'), & \\ \lim\limits_{|y| \to \infty} w(y) = 0. & \end{cases} \tag{1.8}$$

Indeed, from [1, Proposition 10] if $n \geq 3$ and [2, Proposition 3.5] if $n = 2$, we have $Ug \in C_0(\mathbb{R}^n_+)$, and by Remark 1.4 it is a solution of

$$\begin{cases} -\Delta w = g, & in\ \mathbb{R}^n_+, \\ \lim\limits_{y_n \to 0} w(y) = 0, & \\ \lim\limits_{|y| \to \infty} w(y) = 0. & \end{cases} \tag{1.9}$$

Combining this with (1.4), *we derive that $\theta \in C_0(\mathbb{R}^n_+)$, which solves problem* (1.8). *The uniqueness follows from the maximum principle (see [16, Theorem 3.1.5]).*

2. Basic properties of $G(y, z)$ and $\mathcal{K}^\infty(\mathbb{R}^n_+)$

Following [16, Theorem 4.1.6], the Green function $G(y, z)$, for $y, z \in \mathbb{R}^n_+$, is given by

$$G(y,z) = \begin{cases} \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}\left[\dfrac{1}{|y-z|^{n-2}} - \dfrac{1}{|y-\overline{z}|^{n-2}}\right], & \text{if } n \geq 3, \\ \dfrac{1}{4\pi}\log\left(1 + \dfrac{4y_n z_n}{|y-z|^2}\right), & \text{if } n = 2. \end{cases} \tag{2.1}$$

Next, we recall fundamental estimations for $G(y, z)$ and some features of functions from $\mathcal{K}^\infty(\mathbb{R}^n_+)$. The proofs may be found in [1] for $n \geq 3$ and in [2] for $n = 2$.

Proposition 2.1. *On $\mathbb{R}^n_+ \times \mathbb{R}^n_+$,*

$$G(y,z) \asymp \begin{cases} \dfrac{y_n z_n}{|\,y-z\,|^{n-2}\,|y-\overline{z}|^2}, & \textit{if } n \geq 3, \\ \dfrac{1}{4\pi}\log\left(1 + \dfrac{4y_n z_n}{|y-z|^2}\right), & \textit{if } n = 2. \end{cases} \tag{2.2}$$

Corollary 2.2. *For some $C > 0$,*

$$\frac{1}{C}\frac{y_n z_n}{(|y|+1)^n\,(|z|+1)^n} \leq G(y,z),\ \textit{for all } y, z \in \mathbb{R}^n_+. \tag{2.3}$$

Theorem 2.3. (3*G-inequality*) *For some* $C_0 > 0$,

$$\frac{G(x,z)G(z,y)}{G(x,y)} \leq C_0 \left[\frac{z_n}{x_n} G(x,z) + \frac{z_n}{y_n} G(y,z) \right], \text{ for all } x, y, z \in \mathbb{R}^n_+. \tag{2.4}$$

Remark 2.4. *In general, the 3G-inequality is used to establish the relation between the potential theory associated with the Laplace operator* Δ *and the Schrödinger operator* $\Delta - q$*. Here, it will play a central role in constructing the contraction mapping. In particular, the inequality produces the constant* Λ_q *(see (3.1)), which determines the contraction strength and hence the admissible smallness range for* μ*.*

Proposition 2.5. *For* $\varphi \in \mathcal{K}^{\infty}_{+}(\mathbb{R}^n_+)$, *we have*
i) $\|\varphi\| < \infty$.
ii) $\displaystyle\int_{\mathbb{R}^n_+} \frac{z_n}{(|z|+1)^n} |\varphi(z)|\, dz < \infty$.
iii) $z \longrightarrow U\varphi(z) \in C_0(\mathbb{R}^n_+)$.

Remark 2.6. *Combining Corollary 2.2 with Proposition 2.5 (ii), we obtain the lower bound*

$$\frac{1}{C} \frac{y_n}{(|y|+1)^n} \leq U\varphi(y). \tag{2.5}$$

From [1, Proposition 8] if $n \geq 3$ and [2, Proposition 3.7] if $n = 2$, we have the following remark.

Remark 2.7. *Let* $\alpha < 2 < \beta$ *and* $\varphi(y) = \dfrac{1}{(|y|+1)^{\beta-\alpha} y_n^{\alpha}}$. *Then,* $\varphi \in \mathcal{K}^{\infty}_{+}(\mathbb{R}^n_+)$ *and*

$$U\varphi(y) \leq C. \begin{cases} \dfrac{y_n^{2-\alpha}}{(|y|+1)^{n+2-2\alpha}}, & \text{if } 1 < \alpha < 2 \text{ and } \beta \geq n+2-\alpha, \\ \dfrac{y_n}{(|y|+1)^n} \log\left(\dfrac{2(|y|+1)^2}{y_n}\right), & \begin{array}{l} \text{if } \alpha = 1 \text{ and } \beta \geq n+1, \\ \text{or} \\ \alpha < 1 \text{ and } \beta = n+1, \end{array} \\ \dfrac{y_n}{(|y|+1)^n}, & \text{if } \alpha = 1 \text{ and } \beta > n+1, \\ \dfrac{y_n^{\beta-n}}{(|y|+1)^{2\beta-n-2}} & \text{if } n < \beta < \min(n+1, n+2-\alpha), \end{cases} \tag{2.6}$$

for some $C > 0$.

Remark 2.8. *(see [3, Example 2]) Let* $n \geq 3$ *and* $q \in L^s(\mathbb{R}^n_+) \cap L^1(\mathbb{R}^n_+)$ *with* $s > \frac{n}{2}$*. Then,* q *belongs to* $\mathcal{K}^{\infty}(\mathbb{R}^n_+)$.

3. Proof of the main result

For $\varphi \in \mathcal{K}^{\infty}_{+}(\mathbb{R}^n_+)$, define

$$\Lambda_\varphi := \sup_{x,y \in \mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{G(x,z)G(z,y)}{G(x,y)} \varphi(z) dz. \tag{3.1}$$

Lemma 3.1. *Let* $\varphi \in \mathcal{K}^{\infty}_{+}(\mathbb{R}^n_+)$. *Then,*

$$\|\varphi\| \leq \Lambda_\varphi \leq 2C_0 \|\varphi\|,$$

where C_0 *is the constant given in* (2.4). *Consequently,* Λ_φ *is finite.*

Proof. The upper bound follows from the 3G-inequality (2.4).

For the lower bound, by applying Fatou's lemma, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n_+} \frac{z_n}{x_n} G(x,z)\,|\varphi(z)|\,dz &= \int_{\mathbb{R}^n_+} \liminf_{|\zeta|\to\infty} \frac{z_n}{x_n}\frac{|x-\zeta|^n}{|z-\zeta|^n} G(x,z)\,|\varphi(z)|\,dz \\ &\le \liminf_{|\zeta|\to\infty} \int_{\mathbb{R}^n_+} \frac{z_n}{x_n}\frac{|x-\zeta|^n}{|z-\zeta|^n} G(x,z)\,|\varphi(z)|\,dz.\end{aligned}$$

Now, since from (2.1), we have

$$\lim_{y\to\zeta\in\partial\mathbb{R}^n_+} \frac{G(z,y)}{y_n} = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \frac{z_n}{|z-\zeta|^n}, \text{ if } n \ge 2, \tag{3.2}$$

then we obtain

$$\lim_{y\to\zeta\in\partial\mathbb{R}^n_+} \frac{G(z,y)}{G(x,y)} = \lim_{y\to\zeta\in\partial\mathbb{R}^n_+} \frac{\frac{G(z,y)}{y_n}}{\frac{G(x,y)}{y_n}} = \frac{z_n}{x_n}\frac{|x-\zeta|^n}{|z-\zeta|^n}.$$

Hence, again applying Fatou's lemma,

$$\begin{aligned}\int_{\mathbb{R}^n_+} \frac{z_n}{x_n}\frac{|x-\zeta|^n}{|z-\zeta|^n} G(x,z)\,|\varphi(z)|\,dz &\le \liminf_{y\to\zeta} \int_{\mathbb{R}^n_+} G(x,z)\frac{G(z,y)}{G(x,y)}\,|\varphi(z)|\,dz \\ &\le \Lambda_\varphi.\end{aligned}$$

This completes the proof of the lower bound. From the upper bound and Proposition 2.5 (i), we conclude that Λ_φ is finite. □

Lemma 3.2. *Let $\varphi \in \mathcal{K}^\infty_+(\mathbb{R}^n_+)$. Then, for every $y \in \mathbb{R}^n_+$,*

$$U(h\varphi)(y) \le \Lambda_\varphi h(y), \tag{3.3}$$

where h is a nonnegative superharmonic function in $\mathbb{R}^n_+$.

Proof. By [18, Theorem 2.1 p.164], there exists a sequence $(h_k)_k$ in $\mathcal{B}^+(\mathbb{R}^n_+)$ with

$$h(z) = \sup_k (Uh_k)(z), \text{ for all } z \in \mathbb{R}^n_+.$$

Using the Fubini-Tonelli theorem and the definition of the quantity (3.1), we obtain

$$\int_{\mathbb{R}^n_+} G(y,z)\varphi(z)Uh_k(z)dz \le \Lambda_\varphi Uh_k(y), \text{ for all } y \in \mathbb{R}^n_+.$$

By letting $k \to \infty$ and applying the monotone convergence theorem, we obtain the required result. □

Remark 3.3. *For $\varphi \in \mathcal{K}^\infty_+(\mathbb{R}^n_+)$, by taking $h \equiv 1$ in* (3.3), *we obtain*

$$\|U\varphi\|_\infty \le \Lambda_\varphi.$$

For $\varphi \in \mathcal{K}_+^\infty(\mathbb{R}_+^n)$, we set

$$\mathcal{M}_\varphi := \{\psi \in \mathcal{B}(\mathbb{R}_+^n),\ \ |\psi(y)| \leq \varphi(y),\ \text{for all } y \in \mathbb{R}_+^n\}.$$

We shall use the following lemma for the continuity argument.

Lemma 3.4. *For $\varphi \in \mathcal{K}_+^\infty(\mathbb{R}_+^n)$, let*

$$U\left(\mathcal{M}_\varphi\right) = \left\{U\psi :\ \psi \in \mathcal{M}_\varphi\right\}.$$

Then, $U\left(\mathcal{M}_\varphi\right)$ is relatively compact in $C_0(\mathbb{R}_+^n)$.

Proof. The proof is given in [1, Proposition 10] for $n \geq 3$ and in [2, Proposition 3.5] for $n = 2$. □

The next lemma gives the integral formulation of problem (1.1).

Lemma 3.5. *Let $w \in C_0(\mathbb{R}_+^n)$, and assume that $(H1)$–$(H3)$ hold. Then, w solves problem (1.1) if and only if*

$$w(y) = H\phi(y) + Ug(y) + \mu Uf(., w)(y),\ \textit{for } y \in \mathbb{R}_+^n. \tag{3.4}$$

Proof. Assume first that w satisfies the integral Eq (3.4). Using $(H2)$ and $(H3)$, we obtain

$$\begin{aligned} |f(z, w(z))| &= |f(z, w(z)) - f(z, 0)| \\ &\leq q(z)\,|w(z)| \\ &\leq \|w\|_\infty\, q(z). \end{aligned}$$

Since $q \in \mathcal{K}_+^\infty(\mathbb{R}_+^n)$, it follows that $f(., w) \in \mathcal{K}^\infty(\mathbb{R}_+^n)$. Then, Lemma 3.4 gives that Ug and $Uf(., w)$ are in $C_0(\mathbb{R}_+^n)$. Using Remarks 1.4 and 1.5, we conclude that w solves problem (1.1).

On the other hand, assume w solves problem (1.1). Define

$$w_0(y) : w(y) - H\phi(y) - Ug(y) - \mu Uf(., w)(y).$$

Then, w_0 is harmonic in $\mathbb{R}_+^n$ and vanishes on the boundary and at infinity. By the maximum principle (see [16, Theorem 3.1.5]), it follows that $w_0 \equiv 0$, and hence w satisfies (3.4). □

Proof of Theorem 1.3. By Remark 1.5, $\theta(y) := H\phi(y) + Ug(y)$ belongs to $C_0(\mathbb{R}_+^n)$, and from Lemma 3.2, we have

$$U(q\theta)(y) \leq \Lambda_q \theta(y),\ \text{for all } y \in \mathbb{R}_+^n. \tag{3.5}$$

Let

$$\mathcal{E} = \{w \in C_0(\mathbb{R}_+^n) :\ \sup_{y \in \mathbb{R}_+^n} \frac{|w(y)|}{\theta(y)} < \infty\}$$

equipped with the distance defined by

$$d(w_1, w_2) := \sup_{y \in \mathbb{R}_+^n} \frac{|w_1(y) - w_2(y)|}{\theta(y)}.$$

Then, $(\mathcal{E}, d)$ is a complete metric space. Set $\mu^* := \frac{1}{2\Lambda_q} > 0$. For $\mu \in (0, \mu^*)$, let

$$\alpha := \frac{2(\mu^* - \mu)}{2\mu^* - \mu} \text{ and } \beta := \frac{2\mu^*}{2\mu^* - \mu},$$

and define the closed subset $\mathcal{M}$ of $\mathcal{E}$ by

$$\mathcal{M} = \{w \in \mathcal{E},\ \alpha\theta(y) \le w(y) \le \beta\theta(y),\ \forall\ y \in \mathbb{R}^n_+\}.$$

Hence, $(\mathcal{M}, d)$ is a complete metric space. For $w \in \mathcal{M}$, define T by

$$Tw(y) = \theta(y) + \mu U(f(., w(.)))(y), \text{ for } y \in \mathbb{R}^n_+. \tag{3.6}$$

We claim that $T(\mathcal{M}) \subset \mathcal{M}$. Indeed, using $(H2)$ and $(H3)$, we obtain

$$\begin{aligned} |f(z, w(z))| &= |f(z, w(z)) - f(z, 0)| \\ &\le \beta\theta(z)q(z) \\ &\le \beta \|\theta\|_\infty q(z). \end{aligned} \tag{3.7}$$

Since $\theta \in C_0(\mathbb{R}^n_+)$ and $q \in \mathcal{K}^\infty_+(\mathbb{R}^n_+)$, Lemma 3.4 implies that $Tw \in C_0(\mathbb{R}^n_+)$. Combining this estimate with (3.5), we obtain

$$-\mu\beta\Lambda_q\theta(y) \le \mu U(f(., w(.)))(y) \le \mu\beta\Lambda_q\theta(y),$$

and therefore $T(\mathcal{M}) \subset \mathcal{M}$. Let $w_1, w_2 \in \mathcal{M}$. Using the Lipschitz condition $(H3)$ together with (3.5), we obtain

$$\begin{aligned} |Tw_1(y) - Tw_2(y)| &\le \mu \int_{\mathbb{R}^n_+} G(y, z)\, |f(z, w_1(z)) - f(z, w_2(z))|\, dz \\ &\le \mu \int_{\mathbb{R}^n_+} G(y, z)q(z)\, |w_1(z) - w_2(z)|\, dz \\ &\le \mu \int_{\mathbb{R}^n_+} G(y, z)q(z)\theta(z) \frac{|w_1(z) - w_2(z)|}{\theta(z)} dz \\ &\le \mu d(w_1, w_2) \int_{\mathbb{R}^n_+} G(y, z)q(z)\theta(z) dz \\ &\le \mu\Lambda_q d(w_1, w_2)\theta(y). \end{aligned}$$

So,

$$d(Tw_1, Tw_2) \le \mu\Lambda_q d(w_1, w_2).$$

Since $\mu\Lambda_q < \frac{1}{2}$, the mapping T is a contraction. Thus, by the Banach fixed point theorem, there is a unique fixed point $w \in \mathcal{M}$ such that

$$w(y) = \theta(y) + \mu \int_{\mathbb{R}^n_+} G(y, z) f(z, w(z)) dz, \text{ for } y \in \mathbb{R}^n_+. \tag{3.8}$$

According to Lemma 3.5, this fixed point represents the unique solution to problem (1.1). □

Remark 3.6. *The constant* $\mu^* := \frac{1}{2\Lambda_q}$ *is a sufficiently small threshold associated with the contraction mapping argument used in this paper. This threshold is not intended to represent a critical parameter for the existence of positive solutions. Indeed, consider the problem*

$$\begin{cases} -\Delta w = g + \mu q(y) w(y), & in\ \mathbb{R}^n_+, \\ w > 0, & in\ \mathbb{R}^n_+, \\ \lim\limits_{y_n \to 0} w(y) = \phi(y'), \\ \lim\limits_{|y| \to \infty} w(y) = 0, \end{cases} \tag{3.9}$$

where $\mu > 0$, ϕ is a nonnegative continuous function in $\mathbb{R}^{n-1}$ vanishing at infinity, and g, q are a nontrivial nonnegative functions in $\mathcal{K}^\infty(\mathbb{R}^n_+)$.

Let $w \in \mathcal{B}_b(\mathbb{R}^n_+)$, the set of Borel bounded measurable functions on $\mathbb{R}^n_+$. Then, w is a solution of (3.9) if and only if $w - \mu U(qw) = \theta$. Now, if

$$\mu \|U(q.)\| = \|U(q)\|_\infty < 1,$$

then $I - \mu U(q.)$ is invertible on $\mathcal{B}_b(\mathbb{R}^n_+)$ and $[I - \mu U(q.)]^{-1} = \sum_{n=0}^{\infty} \mu^n (U(q.))^n$.

Since $U(q) \leq \Lambda_q$, then $\|U(q)\|_\infty \leq \Lambda_q$. Thus, if $\mu\Lambda_q < 1$, then $w = \sum_{n=0}^{\infty} \mu^n (U(q.))^n(\theta)$.

From (3.5), it follows that

$$\theta \leq w \leq \frac{1}{1 - \mu\Lambda_q}\theta.$$

Consequently, the present result does not exclude the existence of positive solutions for $\mu \geq \mu^*$.

Example 3.7. *Let $\beta > n + 1$ and $q \in \mathcal{K}^\infty(\mathbb{R}^n_+)$. Then, by Theorem 1.3, there exists $\mu^* > 0$ such that for $\mu \in (0, \mu^*)$, the problem*

$$\begin{cases} -\Delta w = \dfrac{1}{(|y| + 1)^{\beta-1} y_n} + \mu q(y) \arctan(w(y)), & in\ \mathbb{R}^n_+, \\ w > 0, & in\ \mathbb{R}^n_+, \\ \lim\limits_{y_n \to 0} w(y) = 0, \\ \lim\limits_{|y| \to \infty} w(y) = 0, \end{cases}$$

admits a unique positive continuous solution w satisfying

$$w(y) \asymp \frac{y_n}{(|y| + 1)^n}. \tag{3.10}$$

Here, by Remark 2.7, the function $y \to g(y) := \dfrac{1}{(|y| + 1)^{\beta-1} y_n} \in \mathcal{K}^\infty(\mathbb{R}^n_+)$ and hypotheses $(H2)$ and $(H3)$ are satisfied with $f(y, w) = q(y) \arctan(w)$. The estimates in (3.10) follow from (1.6) with $\phi \equiv 0$ and Remarks 2.6 and 2.7.

This choice is particularly relevant to the present work because $\arctan(w)$ changes sign as it crosses zero, while remaining globally Lipschitz.

Example 3.8. *Let $n \geq 3$, ϕ be a nonnegative function in $C_0(\mathbb{R}^{n-1})$, and $g \in \mathcal{K}^\infty(\mathbb{R}^n_+)$. Let $q \in L^s(\mathbb{R}^n_+) \cap L^1(\mathbb{R}^n_+)$ with $s > \frac{n}{2}$. Then, from Remark 2.8, it follows that $q \in \mathcal{K}^\infty(\mathbb{R}^n_+)$.*
Therefore, by Theorem 1.3, for the admissible range of μ, the problem

$$\begin{cases} -\Delta w = g + \mu q(y) \sin(w(y)), & in\ \mathbb{R}^n_+, \\ w > 0, & in\ \mathbb{R}^n_+, \\ \lim\limits_{y_n \to 0} w(y) = \phi, \\ \lim\limits_{|y| \to \infty} w(y) = 0, \end{cases}$$

has a unique positive continuous solution w satisfying

$$w(y) \asymp H\phi(y) + Ug(y). \tag{3.11}$$

Here, $(H2)$ *and* $(H3)$ *are satisfied with* $f(y, w) = q(y)\sin(w)$ *.*

Similarly, the choice $\sin(w)$ provides an oscillatory globally Lipschitz nonlinearity that changes sign, illustrating that the present framework is not restricted to nonlinearities of fixed sign.

Remark 3.9. *Note that since the solution of problem (1.1) is bounded, then hypothesis* $(H3)$ *can be relaxed by considering the following form:*
$(H4)$ *For each* $c > 0$, *there exists* $q_c \in \mathcal{K}_+^\infty(\mathbb{R}_+^n)$, *such that*

$$|f(y, w_1) - f(y, w_2)| \leq q_c(y)\,|w_1 - w_2|\,,\ \textit{for all } y \in \mathbb{R}_+^n \textit{ and } w_1, w_2 \in [0, c].$$

In particular, nonlinearities of the form $f(y, w) = q(y)w^\gamma$, *with* $\gamma \geq 1$ *and* $q \in \mathcal{K}_+^\infty(\mathbb{R}_+^n)$, *can be considered.*

4. Conclusions

In this work we established the existence and uniqueness of positive solutions for a class of semilinear elliptic equations posed in the half-space involving sign-changing nonlinearities. By combining Green function estimates, generalized Kato classes, and a contraction mapping argument, we obtained explicit global estimates showing that the nonlinear perturbation preserves the asymptotic behavior of the associated linear problem. These results extend several earlier works where the nonlinearity was assumed to be nonnegative or monotone. An interesting direction for future work is the extension of the present framework to nonlocal operators, in particular the fractional Laplacian. While the general fixed-point strategy may remain useful, such an extension would require the development or use of suitable Green-function estimates, fractional 3G-type inequalities, and corresponding Kato-class potential estimates on the half-space . Thus, the fractional case is not a purely formal extension of the present argument but represents a natural problem requiring additional potential-theoretic tools.

Author contributions

Imed Bachar: Methodology, Validation, Formal analysis, Funding acquisition, Investigation, Writing–original draft, Writing–review and editing; Hassan Eltayeb: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing–review and editing, Visualization. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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