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*Research article*

# Maximal-time criteria for a generalized derivative nonlinear Schrödinger equation

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**Abstract:** We study a generalized derivative nonlinear Schrödinger equation on a bounded interval under homogeneous Dirichlet boundary conditions. Starting from the local $H^2$ well-posedness theory, we derive a quantitative $H^2$ energy inequality in physical space. The estimate yields a genuine continuation criterion at a finite endpoint, divergence of the controlling $W^{1,\infty}$ time integral whenever the maximal time is finite, the $H^2$ blow-up alternative, a quantitative lower blow-up rate, and an explicit lower bound for the lifespan in terms of the initial $H^2$ norm. The key mechanism is a derivative-loss cancellation in the highest-order transport term. Thus, the local existence theory is supplemented by explicit maximal-time information and computable growth scales.

**Keywords:** generalized derivative nonlinear Schrödinger equation; maximal-time criteria; blow-up alternative; $H^2$ energy estimates
**Mathematics Subject Classification:** 35B30, 35C07, 35Q55

## 1. Introduction and main results

This paper studies the homogeneous Dirichlet problem on a bounded interval $\Omega = (a, b) \subset \mathbb{R}$:

$$\begin{cases} iu_t + u_{xx} + i\,|u|^{2\sigma}\, u_x = 0, & (t,x) \in [0,T) \times \Omega, \\ u(t,a) = u(t,b) = 0, & t \in [0,T), \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Throughout the paper, we impose

$$\sigma \geq 1. \tag{1.2}$$

The local $H^2$ theory used below is available already for $\sigma \geq \frac{1}{2}$; see [1, Theorem 1.1]. The stronger restriction (1.2) is imposed only by the direct twice-in-space energy method developed here. Indeed, for

$$F(y) = |y|^{2\sigma}, \qquad y \in \mathbb{R}^2,$$

one has $D^2F(y) \sim |y|^{2\sigma-2}$ near $y = 0$. Hence, $F \in C^2(\mathbb{R}^2)$ when $\sigma \geq 1$, whereas for $\frac{1}{2} \leq \sigma < 1$, the second derivative is singular at zeros of $u$. The lower range can be treated in the local theory by differentiating in time and using Yosida regularization, but it is not covered by the present explicit estimate for $\partial_x^2 |u|^{2\sigma}$. Thus, Theorems 1.1–1.3 are not asserted below for $\frac{1}{2} \leq \sigma < 1$.

Kaup and Newell introduced an integrable derivative nonlinear Schrödinger model and obtained an exact solution [2]. Related derivative Schrödinger equations arise in models for Alfvén-wave propagation in magnetized plasma [3, 4] and for ultrashort optical pulses with self-steepening effects [5]. On a finite spatial window, the homogeneous Dirichlet condition idealizes a wave envelope that vanishes at the two endpoints. From an applied and numerical viewpoint, a continuation criterion identifies a quantity whose finiteness rules out loss of $H^2$ regularity, while an explicit lifespan bound supplies an a priori time interval on which a strong solution must exist.

For the classical derivative nonlinear Schrödinger equation, Tsutsumi and Fukuda established an early existence and uniqueness theory [6], while Hayashi and Ozawa developed the Cauchy and finite-energy theories [7, 8]; further energy-space results were obtained in [9, 10]. Low-regularity results on the line include well-posedness and ill-posedness analyses [11–13], while the periodic Cauchy problem was studied in [14, 15]. Global well-posedness results at critical or energy regularity were obtained in [16, 17]; for long-time behavior, see [18]. For the generalized model with nonlinearity $|u|^{2\sigma} u_x$, local existence for noninteger powers was studied in [19], and related generalized derivative Schrödinger equations were analyzed in [20].

The local result that is relevant here is [1, Theorem 1.1]: It treats the Dirichlet problem on an arbitrary open interval $\Omega \subset \mathbb{R}$ and therefore includes the bounded interval $\Omega = (a, b)$ considered in this paper. It establishes local $H^2(\Omega) \cap H_0^1(\Omega)$ well-posedness for $\sigma \geq \frac{1}{2}$ by a Yosida-regularization argument. The periodic lifespan estimate in [21] concerns the torus rather than the Dirichlet interval; we cite it only as a neighboring quantitative result showing that lifespan questions depend strongly on the underlying boundary geometry. Earlier bounded-domain initial-boundary value and numerical questions were considered in [22]. Recent whole-line results include [23, 24].

The present paper does not reconstruct the local solution. Its purpose is to extract quantitative information that is not contained in the bare local existence statement. More precisely, the paper supplies the following chain:

exact differentiated-energy identities and derivative-loss cancellation
$\Longrightarrow$ time-integrable continuation criterion;
$\Longrightarrow$ finite-time divergence and the $H^2$ blow-up alternative;
$\Longrightarrow$ quantitative lower blow-up rate and explicit lifespan bound.

The new structural point is the identity

$$\operatorname{Re} \int_\Omega |u|^{2\sigma}\, u_{xxx}\overline{u_{xx}}\, dx = -\frac{1}{2} \int_\Omega \partial_x(|u|^{2\sigma})\, |u_{xx}|^2\, dx,$$

which removes the apparent loss of one derivative. The remaining terms involve only

$$\partial_x |u|^{2\sigma}, \qquad \partial_x^2 |u|^{2\sigma}$$

and are controlled by one-dimensional Sobolev embedding. This produces an explicit differential inequality whose constant depends only on $\sigma$ and $\Omega$, not on the solution or on time.

The distinction from the preceding local theory is therefore quantitative rather than existential. The local theorem gives a solution on some data-dependent interval. The results below identify a concrete spacetime quantity that decides continuation at a finite endpoint, prove that this quantity must diverge at every finite maximal time, and quantify both the minimum rate at which the $H^2$ norm can become unbounded and the minimum lifespan forced by the initial $H^2$ size. These statements are useful in neighboring problems involving quasilinear dispersive transport, where the principal issue is likewise to prevent a derivative-carrying coefficient from destroying top-order energy control.

The first theorem is the main a priori estimate.

**Theorem 1.1.** *(Energy inequality) Assume* (1.2). *Let $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, and let*

$$u \in C([0, T_{\max}); H^2(\Omega) \cap H_0^1(\Omega))$$

*be the maximal strong solution of* (1.1). *There exist constants*

$$C_{\rm en} = C_{\rm en}(\sigma) > 0, \qquad C_{\rm ode} = C_{\rm ode}(\sigma, \Omega) > 0$$

*such that*

$$\frac{d}{dt} \|u(t)\|^2_{H^2(\Omega)} \le C_{\rm en}\Big(1 + \|u(t)\|^{2\sigma+1}_{W^{1,\infty}(\Omega)}\Big) \|u(t)\|^2_{H^2(\Omega)} \tag{1.3}$$

*in the sense of distributions on $(0, T_{\max})$ and hence at almost every point of differentiability. Equivalently, for every $0 \le s \le t < T_{\max}$,*

$$\|u(t)\|^2_{H^2(\Omega)} \le \|u(s)\|^2_{H^2(\Omega)} \exp\!\left(C_{\rm en} \int_s^t \Big(1 + \|u(\tau)\|^{2\sigma+1}_{W^{1,\infty}(\Omega)}\Big)\, d\tau\right). \tag{1.4}$$

*Moreover, with $X(t) := \|u(t)\|_{H^2(\Omega)}$,*

$$\frac{d}{dt} X(t) \le C_{\rm ode}(1 + X(t))^{2\sigma+2} \tag{1.5}$$

*in the sense of distributions.*

The next theorem gives the continuation principle at a genuine finite endpoint and its maximal-time consequences.

**Theorem 1.2.** *(Continuation and blow-up alternative) Assume* (1.2).

*(i) Let $0 < T < \infty$, and let $u \in C([0, T); H^2(\Omega) \cap H_0^1(\Omega))$ be a strong solution of* (1.1). *If*

$$\int_0^T \Big(1 + \|u(t)\|^{2\sigma+1}_{W^{1,\infty}(\Omega)}\Big)\, dt < \infty, \tag{1.6}$$

*then there exists $\varepsilon > 0$ and a strong solution on $[0, T + \varepsilon)$ that agrees with $u$ on $[0, T)$.*

*(ii) Let $u \in C([0,T_{\max}); H^2(\Omega) \cap H_0^1(\Omega))$ be the maximal strong solution. If $T_{\max} < \infty$, then*

$$\int_0^{T_{\max}} \left(1 + \|u(t)\|_{W^{1,\infty}(\Omega)}^{2\sigma+1}\right) dt = \infty, \tag{1.7}$$

*and*

$$\limsup_{t \to T_{\max}} \|u(t)\|_{H^2(\Omega)} = \infty. \tag{1.8}$$

*Moreover, there exists $c_1 = c_1(\sigma, \Omega) > 0$ such that*

$$1 + \|u(t)\|_{H^2(\Omega)} \geq c_1 \left(T_{\max} - t\right)^{-1/(2\sigma+1)}, \qquad 0 \leq t < T_{\max}, \tag{1.9}$$

*where one may take*

$$c_1 = \left((2\sigma + 1)C_{\mathrm{ode}}\right)^{-1/(2\sigma+1)}.$$

The final theorem is the explicit lower bound for the maximal time.

**Theorem 1.3.** *(Lifespan) Assume* (1.2). *Let $u \in C([0,T_{\max}); H^2(\Omega) \cap H_0^1(\Omega))$ be the maximal strong solution of* (1.1). *Then*

$$T_{\max} \geq c_0 (1 + \|u_0\|_{H^2(\Omega)})^{-(2\sigma+1)}, \qquad c_0 := \frac{1}{(2\sigma + 1)C_{\mathrm{ode}}}. \tag{1.10}$$

The paper is organized as follows. Section 1 states the problem and the main results. Section 2 records the local $H^2$ input, clarifies the regularity restriction on $\sigma$, and proves the coefficient bounds for $|u|^{2\sigma}$. Section 3 derives the exact differentiated-energy identities and passes rigorously from smooth approximations to strong $H^2$ solutions. Section 4 proves the continuation criterion, the finite-time divergence and blow-up alternative, the lower blow-up rate, and the explicit lifespan bound. Section 5 discusses the role of the boundary condition and the limits of the method.

## 2. Preliminaries and coefficient estimates

Throughout Sections 2–4, $C$ denotes a positive constant that may change from line to line. Unless an additional dependence is displayed, it depends only on $\sigma$ and on the fixed interval $\Omega$. Constants are independent of the solution, the approximation index, and time. The constants in Theorem 1.1 are fixed at the end of Proposition 3.1: $C_{\mathrm{en}}$ is determined by the coefficient constants in Proposition 2.3, and $C_{\mathrm{ode}}$ is then determined by $C_{\mathrm{en}}$ and the embedding constant in Lemma 2.1.

We use the following specialization of [1, Theorems 1.1 and 1.3, Section 2]. That paper formulates the homogeneous Dirichlet problem on an arbitrary open interval $\Omega \subset \mathbb{R}$; hence, its result applies to the bounded interval considered here.

**Proposition 2.1.** *(Local $H^2$ theory) Assume $\sigma \geq \frac{1}{2}$. For every $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, problem* (1.1) *has a unique maximal strong solution*

$$u \in C([0,T_{\max}); H^2(\Omega) \cap H_0^1(\Omega)), \qquad 0 < T_{\max} \leq \infty.$$

*Moreover, the following properties hold.*

(a) *For every $M > 0$, there exists $\tau = \tau(M, \sigma, \Omega) > 0$ such that every datum $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying $\|v_0\|_{H^2(\Omega)} \leq M$ generates a strong solution on $[0, \tau]$.*

(b) *If $v_0^{(n)} \to v_0$ in $H^2(\Omega)$ and $v^{(n)}, v$ are the corresponding solutions, then, on a common sufficiently short interval,*

$$v^{(n)} \longrightarrow v \quad \textit{in } C([0,\tau]; H^s(\Omega)) \quad \textit{for every } 0 \leq s < 2.$$

(c) *If $t_0 \in [0, T_{\max})$, then $v(t) := u(t + t_0)$ solves* (1.1) *on its interval of existence with initial value $v(0) = u(t_0)$.*

**Lemma 2.1.** *(Embedding) There exists a constant $C_{\mathrm{emb}} = C_{\mathrm{emb}}(\Omega) > 0$ such that*

$$\|v\|_{W^{1,\infty}(\Omega)} \leq C_{\mathrm{emb}} \|v\|_{H^2(\Omega)} \tag{2.1}$$

*for every $v \in H^2(\Omega)$.*

*Proof.* Since $\Omega$ is a bounded interval, the one-dimensional Sobolev theorem gives

$$H^2(\Omega) \hookrightarrow C^1(\overline{\Omega}).$$

Therefore, there exists $C_{\mathrm{emb}}(\Omega) > 0$ such that

$$\|v\|_{L^\infty(\Omega)} + \|v_x\|_{L^\infty(\Omega)} \leq C_{\mathrm{emb}} \|v\|_{H^2(\Omega)}.$$

This is exactly (2.1).

**Remark 2.1.** *(Why the direct argument stops at $\sigma = 1$) For $\frac{1}{2} \leq \sigma < 1$, Proposition 2.1 still supplies a local $H^2$ solution. However, the pointwise estimate used below would contain*

$$|u|^{2\sigma-2} |u_x|^2,$$

*whose coefficient is singular at zeros of u. Consequently, neither* (2.8) *nor the closed estimate* (2.10) *follows from a direct $C^2$ chain rule. The time-differentiation method in [1] bypasses this singularity for local existence, but it does not automatically yield the explicit physical-space inequality* (1.3)*. The continuation, blow-up-rate, and lifespan conclusions would extend to this lower range once an analogue of* (1.3) *is proved; establishing such an analogue requires a separate argument and is outside the scope of this paper.*

For later brevity, we set

$$X(t) := \|u(t)\|_{H^2(\Omega)}. \tag{2.2}$$

The shorthand (2.2) will be used repeatedly in Section 4.

We now turn to the coefficient $|u|^{2\sigma}$. The proof of Theorem 1.1 uses only the first two spatial derivatives of this quantity. Proposition 2.2 gives the exact identities (2.5) and (2.6), and Lemma 2.3 turns them into the pointwise bounds (2.7) and (2.8).

We begin the proof with the only coefficient bounds needed later. The nonlinear coefficient is

$$a(u) := |u|^{2\sigma}.$$

We need only its first two spatial derivatives. Since $u$ is complex-valued, it is convenient to rewrite the calculation in real coordinates.

Write

$$u = u_1 + iu_2,$$

and define

$$F(y_1, y_2) := (y_1^2 + y_2^2)^{\sigma}, \qquad (y_1, y_2) \in \mathbb{R}^2.$$

Then $a(u) = F(u_1, u_2)$. Since $\sigma \geq 1$, the function $F$ belongs to $C^2(\mathbb{R}^2)$.

**Lemma 2.2.** *Assume* (1.2). *There exists* $C = C(\sigma) > 0$ *such that for every* $y \in \mathbb{R}^2$,

$$|DF(y)| \leq C\, |y|^{2\sigma-1}\,, \tag{2.3}$$

*and*

$$\left\| D^2 F(y) \right\| \leq C\, |y|^{2\sigma-2}\,. \tag{2.4}$$

*Proof.* A direct computation gives

$$DF(y) = 2\sigma\, |y|^{2\sigma-2}\, y.$$

Hence, (2.3) is immediate. For the Hessian,

$$D^2 F(y) = 2\sigma\, |y|^{2\sigma-2}\, I + 4\sigma(\sigma - 1)\, |y|^{2\sigma-4}\, y \otimes y$$

for $y \neq 0$. Since $2\sigma - 2 \geq 0$, the same expression extends continuously to $y = 0$. Therefore,

$$\left\| D^2 F(y) \right\| \leq 2\sigma\, |y|^{2\sigma-2} + 4\sigma(\sigma - 1)\, |y|^{2\sigma-2} \leq C(\sigma)\, |y|^{2\sigma-2}\,,$$

which proves (2.4).

**Proposition 2.2.** *(Exact identities) Assume* (1.2)*, and let* $u \in H^2(\Omega)$*. Then*

$$\partial_x\, |u|^{2\sigma} = 2\sigma\, |u|^{2\sigma-2}\, \mathrm{Re}(\overline{u} u_x), \tag{2.5}$$

*and*

$$\partial_x^2\, |u|^{2\sigma} = 2\sigma\, |u|^{2\sigma-2}\, \mathrm{Re}(\overline{u} u_{xx}) + 2\sigma\, |u|^{2\sigma-2}\, |u_x|^2 + 4\sigma(\sigma - 1)\, |u|^{2\sigma-4}\, \mathrm{Re}(\overline{u} u_x)^2, \tag{2.6}$$

*where the last term is interpreted by its continuous extension at* $u = 0$*; when* $\sigma = 1$*, that term is absent.*

*Proof.* Since

$$|u|^2 = u\overline{u},$$

we have

$$\partial_x\, |u|^2 = u_x \overline{u} + u \overline{u_x} = 2\,\mathrm{Re}(\overline{u} u_x).$$

Therefore,

$$\partial_x\, |u|^{2\sigma} = \partial_x((|u|^2)^{\sigma}) = \sigma(|u|^2)^{\sigma-1} \partial_x\, |u|^2 = 2\sigma\, |u|^{2\sigma-2}\, \mathrm{Re}(\overline{u} u_x),$$

which is (2.5).

Differentiate once more:

$$\partial_x^2 |u|^{2\sigma} = 2\sigma\, \partial_x\Big(|u|^{2\sigma-2}\, \mathrm{Re}(\overline{u} u_x)\Big).$$

Using the product rule,

$$\partial_x^2 |u|^{2\sigma} = 2\sigma\, \partial_x(|u|^{2\sigma-2})\, \mathrm{Re}(\overline{u} u_x) + 2\sigma\, |u|^{2\sigma-2}\, \partial_x\, \mathrm{Re}(\overline{u} u_x).$$

Again,

$$\partial_x(|u|^{2\sigma-2}) = (\sigma-1)(|u|^2)^{\sigma-2}\partial_x\, |u|^2 = 2(\sigma-1)\, |u|^{2\sigma-4}\, \mathrm{Re}(\overline{u} u_x),$$

and

$$\partial_x\, \mathrm{Re}(\overline{u} u_x) = \mathrm{Re}(\overline{u_x} u_x + \overline{u} u_{xx}) = |u_x|^2 + \mathrm{Re}(\overline{u} u_{xx}).$$

Substituting these identities gives

$$\partial_x^2 |u|^{2\sigma} = 4\sigma(\sigma-1)\, |u|^{2\sigma-4}\, \mathrm{Re}(\overline{u} u_x)^2 + 2\sigma\, |u|^{2\sigma-2}\, \Big(|u_x|^2 + \mathrm{Re}(\overline{u} u_{xx})\Big),$$

which is (2.6).

**Lemma 2.3.** *Assume* (1.2)*, and let $u \in H^2(\Omega)$. Then*

$$\left|\partial_x\, |u|^{2\sigma}\right| \le C\, |u|^{2\sigma-1}\, |u_x|\,, \tag{2.7}$$

*and*

$$\left|\partial_x^2\, |u|^{2\sigma}\right| \le C\, |u|^{2\sigma-1}\, |u_{xx}| + C\, |u|^{2\sigma-2}\, |u_x|^2 \tag{2.8}$$

*almost everywhere on $\Omega$, where $C = C(\sigma) > 0$.*

*Proof.* Let $U = (u_1, u_2)$. Since $a(u) = F(U)$, the chain rule gives

$$\partial_x a(u) = DF(U)U_x,$$

and

$$\partial_x^2 a(u) = DF(U)U_{xx} + D^2F(U)[U_x, U_x].$$

Using Lemma 2.2, we obtain

$$|\partial_x a(u)| \le |DF(U)|\, |U_x| \le C\, |U|^{2\sigma-1}\, |U_x|\,,$$

which is exactly (2.7). Likewise,

$$\left|\partial_x^2 a(u)\right| \le |DF(U)|\, |U_{xx}| + \left\|D^2F(U)\right\|\, |U_x|^2 \le C\, |U|^{2\sigma-1}\, |U_{xx}| + C\, |U|^{2\sigma-2}\, |U_x|^2\,.$$

Since $|U| = |u|$, $|U_x| = |u_x|$, and $|U_{xx}| = |u_{xx}|$, this is (2.8).

**Proposition 2.3.** *(Coefficient bounds) Assume* (1.2)*, and let $u \in H^2(\Omega)$. Then*

$$\left\|\partial_x\, |u|^{2\sigma}\right\|_{L^\infty(\Omega)} \le C\, \|u\|_{W^{1,\infty}(\Omega)}^{2\sigma}\,, \tag{2.9}$$

*and*

$$\left\|\partial_x^2\, |u|^{2\sigma}\right\|_{L^2(\Omega)} \le C\Big(1 + \|u\|_{W^{1,\infty}(\Omega)}^{2\sigma}\Big)\, \|u\|_{H^2(\Omega)}\,. \tag{2.10}$$

*Here, $C = C(\sigma) > 0$.*

*Proof.* From (2.7),

$$\left|\partial_x |u|^{2\sigma}\right| \leq C\,|u|^{2\sigma-1}\,|u_x|\,.$$

Taking the $L^\infty$ norm gives

$$\left\|\partial_x |u|^{2\sigma}\right\|_{L^\infty} \leq C\,\|u\|_{L^\infty}^{2\sigma-1}\,\|u_x\|_{L^\infty} \leq C\,\|u\|_{W^{1,\infty}}^{2\sigma}\,,$$

which proves (2.9).

For the second estimate, (2.8) yields

$$\left\|\partial_x^2 |u|^{2\sigma}\right\|_{L^2} \leq C\left\||u|^{2\sigma-1}\,u_{xx}\right\|_{L^2} + C\left\||u|^{2\sigma-2}\,|u_x|^2\right\|_{L^2}.$$

For the first term,

$$\left\||u|^{2\sigma-1}\,u_{xx}\right\|_{L^2} \leq \|u\|_{L^\infty}^{2\sigma-1}\,\|u_{xx}\|_{L^2}\,.$$

For the second term,

$$\left\||u|^{2\sigma-2}\,|u_x|^2\right\|_{L^2} \leq \|u\|_{L^\infty}^{2\sigma-2}\,\|u_x\|_{L^\infty}\,\|u_x\|_{L^2}\,.$$

Therefore,

$$\left\|\partial_x^2 |u|^{2\sigma}\right\|_{L^2} \leq C\,\|u\|_{L^\infty}^{2\sigma-1}\,\|u_{xx}\|_{L^2} + C\,\|u\|_{L^\infty}^{2\sigma-2}\,\|u_x\|_{L^\infty}\,\|u_x\|_{L^2}\,. \tag{2.11}$$

If $\|u\|_{W^{1,\infty}} \geq 1$, then (2.11) gives

$$\left\|\partial_x^2 |u|^{2\sigma}\right\|_{L^2} \leq C\,\|u\|_{W^{1,\infty}}^{2\sigma}\,(\|u_{xx}\|_{L^2} + \|u_x\|_{L^2}) \leq C\,\|u\|_{W^{1,\infty}}^{2\sigma}\,\|u\|_{H^2}\,.$$

If $\|u\|_{W^{1,\infty}} \leq 1$, then

$$\|u\|_{L^\infty}^{2\sigma-1} \leq 1, \qquad \|u\|_{L^\infty}^{2\sigma-2}\,\|u_x\|_{L^\infty} \leq 1,$$

and (2.11) reduces to

$$\left\|\partial_x^2 |u|^{2\sigma}\right\|_{L^2} \leq C(\|u_{xx}\|_{L^2} + \|u_x\|_{L^2}) \leq C\,\|u\|_{H^2}\,.$$

Combining the two cases proves (2.10).

## 3. The $H^2$ energy estimate

With the coefficient bounds from Section 2 in hand, we now turn to the core estimate of the paper. This section proves Theorem 1.1. We first work with smooth solutions and then pass to strong $H^2$ solutions by approximation.

### *3.1. Mass, first derivative, and second derivative*

For a smooth solution, the equation can be rewritten as

$$u_t = iu_{xx} - |u|^{2\sigma}\,u_x. \tag{3.1}$$

The linear Schrödinger part is skew-adjoint, while the nonlinear term behaves like a transport term. We estimate the $L^2$, $H^1$, and $H^2$ pieces separately.

**Lemma 3.1.** *(Boundary identities) Let $u$ be a smooth solution of* (1.1). *Then for every time in the interval of existence,*

$$u(t,a) = u(t,b) = u_t(t,a) = u_t(t,b) = u_{xx}(t,a) = u_{xx}(t,b) = 0. \tag{3.2}$$

*Consequently,*

$$|u(t,a)|^{2\sigma} = |u(t,b)|^{2\sigma} = 0. \tag{3.3}$$

*Proof.* The boundary condition in (1.1) gives

$$u(t,a) = u(t,b) = 0.$$

Differentiating these identities with respect to $t$ yields

$$u_t(t,a) = u_t(t,b) = 0.$$

Next, evaluate the equation

$$iu_t + u_{xx} + i\,|u|^{2\sigma}\,u_x = 0$$

at $x = a$ and $x = b$. Since $u = 0$ on $\partial\Omega$, one has $|u|^{2\sigma} = 0$ on $\partial\Omega$. Hence,

$$u_{xx}(t,a) = -iu_t(t,a) - i\,|u(t,a)|^{2\sigma}\,u_x(t,a) = 0,$$

and likewise

$$u_{xx}(t,b) = -iu_t(t,b) - i\,|u(t,b)|^{2\sigma}\,u_x(t,b) = 0.$$

This proves (3.2), and (3.3) is immediate from $u = 0$ on $\partial\Omega$.

**Lemma 3.2.** *(Mass) Let $u$ be a smooth solution of* (1.1). *Then*

$$\frac{d}{dt}\,\|u(t)\|^2_{L^2(\Omega)} = 0. \tag{3.4}$$

*Proof.* Using (3.1),

$$\frac{d}{dt}\,\|u\|^2_{L^2} = 2\,\mathrm{Re}\int_\Omega u_t\overline{u}\,dx = 2\,\mathrm{Re}\int_\Omega (iu_{xx} - |u|^{2\sigma}\,u_x)\overline{u}\,dx.$$

For the linear term,

$$2\,\mathrm{Re}\int_\Omega iu_{xx}\overline{u}\,dx = -2\,\mathrm{Re}\int_\Omega i\,|u_x|^2\;dx = 0$$

because $u = 0$ on $\partial\Omega$. For the nonlinear term,

$$-2\,\mathrm{Re}\int_\Omega |u|^{2\sigma}\,u_x\overline{u}\,dx = -\int_\Omega |u|^{2\sigma}\,\partial_x\,|u|^2\;dx = -\frac{1}{\sigma+1}\int_\Omega \partial_x\,|u|^{2\sigma+2}\;dx = 0$$

again because $u = 0$ on $\partial\Omega$.

**Lemma 3.3.** *($H^1$ bound) Let $u$ be a smooth solution of* (1.1). *Then*

$$\frac{d}{dt}\,\|u_x(t)\|^2_{L^2(\Omega)} \le C\,\|u(t)\|^{2\sigma}_{W^{1,\infty}(\Omega)}\,\|u_x(t)\|^2_{L^2(\Omega)} \tag{3.5}$$

*for all times on which the solution exists.*

*Proof.* Differentiate (3.1) once in $x$:

$$(u_x)_t = iu_{xxx} - \partial_x(|u|^{2\sigma}\, u_x).$$

Multiply by $\overline{u_x}$, integrate over $\Omega$, and take the real part:

$$\frac{1}{2}\frac{d}{dt}\,\|u_x\|_{L^2}^2 = \mathrm{Re}\int_\Omega iu_{xxx}\overline{u_x}\,dx - \mathrm{Re}\int_\Omega \partial_x(|u|^{2\sigma}\, u_x)\overline{u_x}\,dx. \tag{3.6}$$

For the linear term, integrate by parts once:

$$\int_\Omega iu_{xxx}\overline{u_x}\,dx = [iu_{xx}\overline{u_x}]_{x=a}^{x=b} - \int_\Omega iu_{xx}\overline{u_{xx}}\,dx.$$

By Lemma 3.1,

$$u_{xx}(t,a) = u_{xx}(t,b) = 0,$$

hence

$$\mathrm{Re}\int_\Omega iu_{xxx}\overline{u_x}\,dx = \mathrm{Re}\Big(-i\int_\Omega |u_{xx}|^2\;dx\Big) = 0.$$

For the nonlinear term, write

$$a := |u|^{2\sigma}.$$

Then

$$\partial_x(au_x) = a_x u_x + au_{xx},$$

so

$$\begin{aligned}-\mathrm{Re}\int_\Omega \partial_x(au_x)\overline{u_x}\,dx &= -\mathrm{Re}\int_\Omega a_x u_x\overline{u_x}\,dx - \mathrm{Re}\int_\Omega au_{xx}\overline{u_x}\,dx\\ &= -\int_\Omega a_x\,|u_x|^2\;dx - \mathrm{Re}\int_\Omega au_{xx}\overline{u_x}\,dx.\end{aligned}$$

Since

$$2\,\mathrm{Re}(u_{xx}\overline{u_x}) = \partial_x\,|u_x|^2,$$

we have

$$\mathrm{Re}\int_\Omega au_{xx}\overline{u_x}\,dx = \frac{1}{2}\int_\Omega a\,\partial_x\,|u_x|^2\;dx.$$

Integrating by parts and using Lemma 3.1, which gives $a(t,a) = a(t,b) = 0$, we obtain

$$\int_\Omega a\,\partial_x\,|u_x|^2\;dx = [a\,|u_x|^2]_{x=a}^{x=b} - \int_\Omega a_x\,|u_x|^2\;dx = -\int_\Omega a_x\,|u_x|^2\;dx.$$

Hence,

$$-\mathrm{Re}\int_\Omega \partial_x(au_x)\overline{u_x}\,dx = -\frac{1}{2}\int_\Omega a_x\,|u_x|^2\;dx.$$

Returning to (3.6), we conclude that

$$\frac{d}{dt}\,\|u_x\|_{L^2}^2 \le \|a_x\|_{L^\infty}\,\|u_x\|_{L^2}^2.$$

Now apply Proposition 2.3, namely (2.9), to deduce (3.5).

**Lemma 3.4.** *(Cancellation) Let u be a smooth solution of* (1.1), *and set* $a = |u|^{2\sigma}$. *Then*

$$\operatorname{Re}\int_\Omega a u_{xxx}\overline{u_{xx}}\,dx = -\frac{1}{2}\int_\Omega a_x\,|u_{xx}|^2\,dx. \tag{3.7}$$

*Proof.* Start from

$$2\operatorname{Re}(u_{xxx}\overline{u_{xx}}) = \partial_x\,|u_{xx}|^2\,.$$

Therefore,

$$\operatorname{Re}\int_\Omega a u_{xxx}\overline{u_{xx}}\,dx = \frac{1}{2}\int_\Omega a\,\partial_x\,|u_{xx}|^2\,dx.$$

Integrating by parts and using Lemma 3.1, which yields $u_{xx}(t,a) = u_{xx}(t,b) = 0$, we get

$$\operatorname{Re}\int_\Omega a u_{xxx}\overline{u_{xx}}\,dx = \frac{1}{2}\Big[a\,|u_{xx}|^2\Big]_{x=a}^{x=b} - \frac{1}{2}\int_\Omega a_x\,|u_{xx}|^2\,dx = -\frac{1}{2}\int_\Omega a_x\,|u_{xx}|^2\,dx.$$

This is (3.7).

**Lemma 3.5.** *($H^2$ bound) Let u be a smooth solution of* (1.1). *Then*

$$\frac{d}{dt}\,\|u_{xx}(t)\|^2_{L^2(\Omega)} \le C\Big(1 + \|u(t)\|^{2\sigma+1}_{W^{1,\infty}(\Omega)}\Big)\,\|u(t)\|^2_{H^2(\Omega)} \tag{3.8}$$

*for all times on which the solution exists.*

*Proof.* Differentiate (3.1) twice in $x$:

$$(u_{xx})_t = iu_{xxxx} - \partial_x^2(|u|^{2\sigma}\,u_x).$$

Multiply by $\overline{u_{xx}}$, integrate over $\Omega$, and take the real part:

$$\frac{1}{2}\frac{d}{dt}\,\|u_{xx}\|^2_{L^2} = \operatorname{Re}\int_\Omega iu_{xxxx}\overline{u_{xx}}\,dx - \operatorname{Re}\int_\Omega \partial_x^2(|u|^{2\sigma}\,u_x)\overline{u_{xx}}\,dx. \tag{3.9}$$

Integration by parts gives

$$\int_\Omega iu_{xxxx}\overline{u_{xx}}\,dx = [iu_{xxx}\overline{u_{xx}}]_{x=a}^{x=b} - \int_\Omega iu_{xxx}\overline{u_{xxx}}\,dx.$$

By Lemma 3.1,

$$u_{xx}(t,a) = u_{xx}(t,b) = 0,$$

and therefore

$$\operatorname{Re}\int_\Omega iu_{xxxx}\overline{u_{xx}}\,dx = \operatorname{Re}\Big(-i\int_\Omega |u_{xxx}|^2\,dx\Big) = 0.$$

For the nonlinear term, write

$$a := |u|^{2\sigma}\,.$$

A direct differentiation gives

$$\partial_x^2(au_x) = au_{xxx} + 2a_xu_{xx} + a_{xx}u_x. \tag{3.10}$$

Substituting (3.10) into (3.9), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|_{L^2}^2 = -\operatorname{Re}\int_\Omega a u_{xxx}\overline{u_{xx}}\,dx - 2\operatorname{Re}\int_\Omega a_x u_{xx}\overline{u_{xx}}\,dx - \operatorname{Re}\int_\Omega a_{xx}u_x\overline{u_{xx}}\,dx. \tag{3.11}$$

Since $a_x$ is real-valued,

$$\operatorname{Re}\int_\Omega a_x u_{xx}\overline{u_{xx}}\,dx = \int_\Omega a_x |u_{xx}|^2\,dx.$$

Applying Lemma 3.4 to the first term in (3.11) gives

$$-\operatorname{Re}\int_\Omega a u_{xxx}\overline{u_{xx}}\,dx = \frac{1}{2}\int_\Omega a_x |u_{xx}|^2\,dx.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|_{L^2}^2 = -\frac{3}{2}\int_\Omega a_x |u_{xx}|^2\,dx - \operatorname{Re}\int_\Omega a_{xx}u_x\overline{u_{xx}}\,dx. \tag{3.12}$$

Now estimate the two terms on the right-hand side. First,

$$\left|\int_\Omega a_x |u_{xx}|^2\,dx\right| \le \|a_x\|_{L^\infty}\|u_{xx}\|_{L^2}^2.$$

Second,

$$\left|\operatorname{Re}\int_\Omega a_{xx}u_x\overline{u_{xx}}\,dx\right| \le \|a_{xx}\|_{L^2}\|u_x\|_{L^\infty}\|u_{xx}\|_{L^2}.$$

Therefore, (3.12) implies

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|_{L^2}^2 \le \frac{3}{2}\|a_x\|_{L^\infty}\|u_{xx}\|_{L^2}^2 + \|a_{xx}\|_{L^2}\|u_x\|_{L^\infty}\|u_{xx}\|_{L^2}. \tag{3.13}$$

Using Proposition 2.3, namely (2.9) and (2.10), in (3.13), we get

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|_{L^2}^2 \le C\|u\|_{W^{1,\infty}}^{2\sigma}\|u_{xx}\|_{L^2}^2 + C\Big(1+\|u\|_{W^{1,\infty}}^{2\sigma}\Big)\|u\|_{H^2}\|u_x\|_{L^\infty}\|u_{xx}\|_{L^2}.$$

Since

$$\|u_x\|_{L^\infty} \le \|u\|_{W^{1,\infty}}, \qquad \|u_{xx}\|_{L^2} \le \|u\|_{H^2},$$

we conclude that

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|_{L^2}^2 \le C\|u\|_{W^{1,\infty}}^{2\sigma}\|u\|_{H^2}^2 + C\Big(1+\|u\|_{W^{1,\infty}}^{2\sigma}\Big)\|u\|_{W^{1,\infty}}\|u\|_{H^2}^2.$$

Finally,

$$\|u\|_{W^{1,\infty}}^{2\sigma} + \Big(1+\|u\|_{W^{1,\infty}}^{2\sigma}\Big)\|u\|_{W^{1,\infty}} \le C\Big(1+\|u\|_{W^{1,\infty}}^{2\sigma+1}\Big),$$

and this proves (3.8).

### 3.2. The full $H^2$ estimate

The preceding lemmas have separated the calculation into its natural pieces. We now combine them and then pass from smooth solutions to strong $H^2$ solutions.

**Proposition 3.1.** *(Smooth energy bound) Let $u$ be a smooth solution of* (1.1). *Then*

$$\frac{d}{dt}\|u(t)\|_{H^2(\Omega)}^2 \leq C\Big(1+\|u(t)\|_{W^{1,\infty}(\Omega)}^{2\sigma+1}\Big)\|u(t)\|_{H^2(\Omega)}^2 \tag{3.14}$$

*for all times on which the solution exists.*

*Proof.* By Lemma 3.2, that is, by the conserved identity (3.4),

$$\frac{d}{dt}\|u\|_{L^2}^2 = 0.$$

By Lemma 3.3,

$$\frac{d}{dt}\|u_x\|_{L^2}^2 \leq C\,\|u\|_{W^{1,\infty}}^{2\sigma}\,\|u_x\|_{L^2}^2\,.$$

By Lemma 3.5,

$$\frac{d}{dt}\|u_{xx}\|_{L^2}^2 \leq C\Big(1+\|u\|_{W^{1,\infty}}^{2\sigma+1}\Big)\|u\|_{H^2}^2\,.$$

Adding the three identities and using

$$\|u_x\|_{L^2}^2 \leq \|u\|_{H^2}^2\,, \qquad \|u\|_{W^{1,\infty}}^{2\sigma} \leq 1+\|u\|_{W^{1,\infty}}^{2\sigma+1}\,,$$

we obtain

$$\begin{aligned}\frac{d}{dt}\|u\|_{H^2}^2 &= \frac{d}{dt}\|u\|_{L^2}^2+\frac{d}{dt}\|u_x\|_{L^2}^2+\frac{d}{dt}\|u_{xx}\|_{L^2}^2\\ &\leq C\Big(1+\|u\|_{W^{1,\infty}}^{2\sigma+1}\Big)\|u\|_{H^2}^2\,.\end{aligned}$$

This is (3.14). From now on, fix one admissible constant in (3.14) and denote it by $C_{\mathrm{en}}$.

**Lemma 3.6.** *(Dirichlet Galerkin approximation) Let $A=-\partial_x^2$ with domain $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$, let $\{e_k\}_{k\geq 1}$ be its $L^2$-orthonormal eigenbasis, and let $P_N$ denote the orthogonal projection onto* $\mathrm{span}\{e_1,\dots,e_N\}$. *For $v_0\in H^2(\Omega)\cap H_0^1(\Omega)$, consider*

$$\begin{cases} i\partial_t v_N+\partial_x^2 v_N+iP_N(|v_N|^{2\sigma}\,\partial_x v_N)=0,\\ v_N(0)=P_N v_0.\end{cases} \tag{3.15}$$

*For every $M>0$, there exists $\delta=\delta(M,\sigma,\Omega)>0$ such that, whenever $\|v_0\|_{H^2}\leq M$, the solutions of* (3.15) *exist on* $[0,\delta]$ *and satisfy*

$$\sup_N \sup_{0\leq r\leq\delta}\|v_N(r)\|_{H^2}<\infty. \tag{3.16}$$

*Moreover, if $v$ is the strong solution of* (1.1) *with $v(0)=v_0$, then*

$$v_N \longrightarrow v \quad \text{in } C([0,\delta];H^s(\Omega)) \quad \text{for every } 0\leq s<2. \tag{3.17}$$

*In particular, for every fixed $r\in[0,\delta]$,*

$$\|v(r)\|_{H^2} \leq \liminf_{N\to\infty}\|v_N(r)\|_{H^2}\,. \tag{3.18}$$

*Proof.* Because (3.15) is a finite-dimensional ordinary differential equation with a locally Lipschitz vector field, it has a unique maximal solution. The projection $P_N$ is self-adjoint, commutes with $A$, and satisfies $P_N v_N = v_N$. Consequently, in the $L^2$, $H^1$, and $H^2$ energy pairings, one has

$$\langle P_N f, v_N\rangle = \langle f, v_N\rangle, \qquad \langle P_N f, Av_N\rangle = \langle f, Av_N\rangle, \qquad \langle P_N f, A^2 v_N\rangle = \langle f, A^2 v_N\rangle.$$

Here,

$$\|\partial_x v_N\|_{L^2}^2 = \langle Av_N, v_N\rangle, \qquad \left\|\partial_x^2 v_N\right\|_{L^2}^2 = \|Av_N\|_{L^2}^2\,,$$

so differentiating the three quadratic energies produces exactly these pairings. Self-adjointness therefore removes $P_N$ before the spatial integrations by parts; no commutation of $P_N$ with $\partial_x$ is being used. Each $v_N$ is a finite linear combination of Dirichlet eigenfunctions. Hence, $v_N = \partial_x^2 v_N = 0$ at $x = a, b$, and the integrations by parts in Lemmas 3.2–3.5 apply without an additional boundary term. Repeating those calculations gives

$$\frac{d}{dr}\|v_N(r)\|_{H^2}^2 \le C_{\mathrm{en}}\Big(1 + \|v_N(r)\|_{W^{1,\infty}}^{2\sigma+1}\Big)\|v_N(r)\|_{H^2}^2\,. \tag{3.19}$$

By Lemma 2.1, with $Y_N(r) := \|v_N(r)\|_{H^2}$, there is $C_{\mathrm{G}} = C_{\mathrm{G}}(\sigma,\Omega) > 0$ such that

$$Y_N'(r) \le C_{\mathrm{G}}(1 + Y_N(r))^{2\sigma+2}. \tag{3.20}$$

Multiplication by $(1 + Y_N)^{-(2\sigma+2)}$ and integration give

$$(1 + Y_N(r))^{-(2\sigma+1)} \ge (1 + Y_N(0))^{-(2\sigma+1)} - (2\sigma + 1)C_{\mathrm{G}} r.$$

Thus, one may choose, for example,

$$\delta := \frac{1}{2(2\sigma + 1)C_{\mathrm{G}}}(1 + M)^{-(2\sigma+1)}.$$

The preceding inequality bounds $Y_N$ uniformly on $[0, \delta]$. The finite-dimensional continuation theorem therefore gives existence on that common interval and proves (3.16).

The Galerkin equation (3.15) and the bound (3.16) also give

$$\sup_N \|\partial_t v_N\|_{L^\infty(0,\delta;L^2)} < \infty \tag{3.21}$$

because

$$\left\|P_N(|v_N|^{2\sigma}\,\partial_x v_N)\right\|_{L^2} \le \|v_N\|_{L^\infty}^{2\sigma}\,\|\partial_x v_N\|_{L^2} \le C(M, \sigma, \Omega).$$

The compact embedding $H^2(\Omega) \Subset H^s(\Omega)$, the equicontinuity in $L^2$ furnished by (3.21), and interpolation imply relative compactness in $C([0, \delta]; H^s(\Omega))$ for every $s < 2$. Choose $s_0 \in (3/2, 2)$. Along any convergent subsequence,

$$v_N \to \widetilde{v} \quad \text{in } C([0, \delta]; H^{s_0}(\Omega)) \hookrightarrow C([0, \delta]; W^{1,\infty}(\Omega)).$$

It follows that

$$|v_N|^{2\sigma}\,\partial_x v_N \longrightarrow |\widetilde{v}|^{2\sigma}\,\partial_x \widetilde{v} \quad \text{in } C([0, \delta]; L^2(\Omega)).$$

Since $P_N \to I$ strongly on $L^2$, passing to the limit in (3.15) shows that $\widetilde{v}$ solves (1.1) with initial value $v_0$. The uniqueness theorem in [1, Theorem 1.3] applies in $H^{3/2}(\Omega) \cap H_0^1(\Omega)$, and therefore $\widetilde{v} = v$. Every convergent subsequence has the same limit, so the full sequence satisfies (3.17). Finally, (3.16) gives weak $H^2$ compactness at each fixed time; the strong $H^{s_0}$ limit identifies every weak limit with $v(r)$, and weak lower semicontinuity yields (3.18).

**Proposition 3.2.** *(Passage to strong solutions) Assume* (1.2)*, and let*

$$u \in C([0, T_{\max}); H^2(\Omega) \cap H_0^1(\Omega))$$

*be the maximal strong solution of* (1.1)*. Then* (1.4) *holds for every* $0 \le s \le t < T_{\max}$*. Consequently,* (1.3) *holds in the sense of distributions and at almost every point of differentiability.*

*Proof.* Fix $0 \le s < t < T_{\max}$, and set

$$M := 1 + \sup_{\tau \in [s,t]} \|u(\tau)\|_{H^2(\Omega)} < \infty.$$

Let $\delta = \delta(M, \sigma, \Omega)$ be supplied by Lemma 3.6, and choose a partition

$$s = t_0 < t_1 < \cdots < t_J = t, \qquad t_{j+1} - t_j < \delta.$$

Fix $j$, put $\delta_j = t_{j+1} - t_j$, and apply Lemma 3.6 with initial value $u(t_j)$. Denote the Galerkin solution by $v_{j,N}$ and the corresponding strong solution by $v_j(r) = u(t_j + r)$. Choosing $s_0 \in (3/2, 2)$, we have

$$v_{j,N} \longrightarrow v_j \quad \text{in } C([0, \delta_j]; W^{1,\infty}(\Omega)), \tag{3.22}$$

and

$$\left\|u(t_{j+1})\right\|_{H^2} \le \liminf_{N \to \infty} \left\|v_{j,N}(\delta_j)\right\|_{H^2}. \tag{3.23}$$

For each $N$, (3.19) and Gronwall's inequality give

$$\left\|v_{j,N}(\delta_j)\right\|_{H^2}^2 \le \left\|P_N u(t_j)\right\|_{H^2}^2 \exp\!\left(C_{\mathrm{en}} \int_0^{\delta_j} \left(1 + \left\|v_{j,N}(r)\right\|_{W^{1,\infty}}^{2\sigma+1}\right) dr\right). \tag{3.24}$$

The spectral convergence $P_N u(t_j) \to u(t_j)$ in $H^2$, (3.22), and (3.23) yield

$$\left\|u(t_{j+1})\right\|_{H^2}^2 \le \left\|u(t_j)\right\|_{H^2}^2 \exp\!\left(C_{\mathrm{en}} \int_{t_j}^{t_{j+1}} \left(1 + \|u(r)\|_{W^{1,\infty}}^{2\sigma+1}\right) dr\right).$$

Multiplication over $j = 0, \dots, J-1$ proves (1.4).

Finally, define

$$F(t) := \|u(t)\|_{H^2}^2, \qquad A(t) := C_{\mathrm{en}}\left(1 + \|u(t)\|_{W^{1,\infty}}^{2\sigma+1}\right).$$

Inequality (1.4) says that

$$t \longmapsto F(t) \exp\!\left(-\int_0^t A(r)\, dr\right)$$

is nonincreasing. Hence, $F$ is locally of bounded variation and $DF \le AF\, dt$ as measures, which is precisely (1.3) in the distributional sense. The almost-everywhere form follows from the differentiability of functions of bounded variation.

*Proof of Theorem 1.1.* Proposition 3.2 proves (1.3) and (1.4).

To identify the dependence of the constants, let $C_1 = C_1(\sigma)$ and $C_2 = C_2(\sigma)$ be fixed constants for (2.9) and (2.10), respectively. After fixing the numerical constants in (3.5) and (3.13), define

$$C_{\mathrm{en}} := C_*(\sigma)(1 + C_1 + C_2),$$

where $C_*(\sigma)$ is the constant in the elementary inequality $r^{2\sigma} + (1 + r^{2\sigma})r \le C_*(\sigma)(1 + r^{2\sigma+1})$ for $r \ge 0$. Thus, $C_{\mathrm{en}}$ depends only on $\sigma$. Let $C_{\mathrm{emb}} = C_{\mathrm{emb}}(\Omega)$ be the constant in (2.1), and put

$$C_{\mathrm{ode}} := \frac{C_{\mathrm{en}}}{2} \max\{1, C_{\mathrm{emb}}^{2\sigma+1}\}.$$

To justify the square-root chain rule at possible zeros of $X$, set $X_\varepsilon = (X^2 + \varepsilon)^{1/2}$. From $D(X^2) \le AX^2\,dt$ and the bounded-variation chain rule,

$$DX_\varepsilon \le \frac{AX^2}{2X_\varepsilon}\,dt \le \frac{A}{2} X_\varepsilon\,dt.$$

Apply (2.1), and let $\varepsilon \downarrow 0$ in distributions. This gives

$$DX \le C_{\mathrm{ode}}(1 + X^{2\sigma+1})X\,dt \le C_{\mathrm{ode}}(1 + X)^{2\sigma+2}\,dt,$$

which is (1.5). In particular, both constants depend only on the parameters stated in Theorem 1.1.

## 4. Quantitative consequences

We now turn the energy inequality into endpoint continuation, maximal-time divergence, a lower blow-up rate, and a lifespan estimate.

### *4.1. Continuation and maximal-time divergence*

*Proof of Theorem 1.2(i).* By (1.4) with $s = 0$ and assumption (1.6),

$$M_0 := \sup_{0 \le t < T} \|u(t)\|_{H^2(\Omega)} < \infty. \tag{4.1}$$

Set $M := M_0 + 1$. Proposition 2.1(a) provides a time $\tau = \tau(M, \sigma, \Omega) > 0$ for all data whose $H^2$ norm is at most $M$. Choose one time

$$t_0 \in (T - \tau/2, T).$$

The local theory gives a strong solution $v$ on $[0, \tau]$ with $v(0) = u(t_0)$. The shifted function $w(r) := u(t_0 + r)$ is a strong solution with the same initial datum on $[0, T - t_0)$. By uniqueness in Proposition 2.1,

$$v(r) = w(r), \qquad 0 \le r < T - t_0.$$

Therefore, the function obtained by using $u$ on $[0, t_0]$ and $v(\cdot - t_0)$ on $[t_0, t_0 + \tau]$ is a well-defined strong solution. Since $t_0 + \tau > T$, it extends $u$ beyond $T$.

*Proof of Theorem 1.2(ii).* If $T_{\max} < \infty$ and the integral in (1.7) were finite, part (i), applied with $T = T_{\max}$, would extend the maximal solution beyond $T_{\max}$. This contradiction proves (1.7).

If instead

$$\sup_{0 \le t < T_{\max}} \|u(t)\|_{H^2(\Omega)} < \infty,$$

then Lemma 2.1 would imply a uniform $W^{1,\infty}$ bound. Since $T_{\max} < \infty$, the integral in (1.7) would then be finite, again a contradiction. Hence, (1.8) holds.

*4.2. Scalar comparison, blow-up rate, and lifespan*

**Lemma 4.1.** *(Ordinary differential equation comparison for distributional inequalities) Let $y : [0,T) \to [0,\infty)$ be continuous and locally of bounded variation, and assume*

$$Dy \le C(1+y)^p\,dt \tag{4.2}$$

*in the sense of measures, where $C > 0$ and $p > 1$.*

*(i) For every $s \in [0,T)$ and every*

$$s \le t < s + \frac{1}{(p-1)C}(1+y(s))^{-(p-1)},$$

*one has*

$$y(t) \le \frac{1+y(s)}{\left(1-(p-1)C(t-s)(1+y(s))^{p-1}\right)^{1/(p-1)}} - 1. \tag{4.3}$$

*The estimate is asserted only where*

$$1-(p-1)C(t-s)(1+y(s))^{p-1} > 0. \tag{4.4}$$

*(ii) If $T < \infty$ and $\limsup_{t\uparrow T} y(t) = \infty$, then*

$$1+y(t) \ge \left((p-1)C(T-t)\right)^{-1/(p-1)}, \qquad 0 \le t < T. \tag{4.5}$$

*Proof.* Set $z := 1+y \ge 1$. The bounded-variation chain rule applied to $r \mapsto r^{-(p-1)}$ and (4.2) gives

$$D(z^{-(p-1)}) \ge -(p-1)C\,dt.$$

Therefore, for $0 \le s \le t < T$,

$$z(t)^{-(p-1)} \ge z(s)^{-(p-1)} - (p-1)C(t-s). \tag{4.6}$$

Under the positivity condition (4.4), inversion of (4.6) gives (4.3).

For part (ii), choose $t_n \uparrow T$ such that $y(t_n) \to \infty$. Apply (4.6) with $s = t$ and $t = t_n$, and let $n \to \infty$. Since $z(t_n)^{-(p-1)} \to 0$, one obtains

$$z(t)^{-(p-1)} \le (p-1)C(T-t),$$

which is equivalent to (4.5).

*Proof of the lower rate in Theorem 1.2(ii).* Apply Lemma 4.1(ii) to $y = X$, $p = 2\sigma+2$, and $C = C_{\mathrm{ode}}$ using (1.5) and (1.8). This gives (1.9) with the stated value of $c_1$.

*Proof of Theorem 1.3.* Apply Lemma 4.1(i) to $y = X$, $s = 0$, $p = 2\sigma+2$, and $C = C_{\mathrm{ode}}$. Then

$$X(t) \le \frac{1+X(0)}{\left(1-(2\sigma+1)C_{\mathrm{ode}}t(1+X(0))^{2\sigma+1}\right)^{1/(2\sigma+1)}} - 1 \tag{4.7}$$

whenever

$$0 \le t < T_* := \frac{1}{(2\sigma+1)C_{\mathrm{ode}}}(1+X(0))^{-(2\sigma+1)}. \tag{4.8}$$

The denominator in (4.7) is positive precisely on the interval (4.8). If $T_{\max} < T_*$, then (4.7) would keep $X$ bounded on $[0,T_{\max})$, contradicting Theorem 1.2(ii). Therefore, $T_{\max} \ge T_*$, which is (1.10).

## 5. Conclusions

Starting from the local Dirichlet $H^2$ theory, we derived an explicit maximal-time mechanism for (1.1). The top-order transport term does not cause derivative loss because of the identity in Lemma 3.4. Together with the coefficient estimates in Proposition 2.3, this yields the distributional energy inequality (1.3) and its integrated form (1.4). The resulting consequences are the endpoint continuation criterion (1.6), divergence at every finite maximal time (1.7), the $H^2$ blow-up alternative (1.8), the quantitative lower rate (1.9), and the lifespan lower bound (1.10).

The restriction $\sigma \geq 1$ belongs to the direct second-spatial-derivative estimate rather than to the local $H^2$ existence theory. When $\frac{1}{2} \leq \sigma < 1$, $D^2(|z|^{2\sigma})$ is singular at $z = 0$, so the coefficient estimate (2.10) is unavailable without a different argument. This explains why the present quantitative chain is not claimed in that range.

The boundary condition also enters the cancellation mechanism. Under periodic boundary conditions, all endpoint terms cancel pairwise, so the same physical-space calculation extends once the corresponding periodic local theory is inserted. Under homogeneous Neumann conditions, however, $|u|^{2\sigma}$ need not vanish at the endpoints, and the $H^2$ calculation leaves boundary flux terms involving $|u|^{2\sigma}|u_{xx}|^2$. Thus, the Neumann problem requires additional compatibility conditions, a boundary correction to the energy, or separate trace estimates; the Dirichlet proof does not extend verbatim.

### Author contributions

Senyue Luo: Conceptualization, methodology, formal analysis, writing–original draft; Meilan Qiu: Validation, investigation, funding acquisition and editing; Zhiming Tan: Supervision, project administration and writing–review. All authors have read and agreed to the published version of the manuscript.

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The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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### Conflict of interest

The authors declare no conflicts of interest.

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