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**Research article****On the region  $\pi$  chromatic number of snake graph families****Sulochana Aradhe<sup>1</sup>, Archana A. Bhang<sup>2,\*</sup> and Haribhau R. Bhapkar<sup>3</sup>**

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**Abstract:** A region  $\pi$  coloring of a plane graph assigns colors to its faces so that two faces sharing an edge receive different colors, and so that two faces with different closed edge-neighborhoods receive different sets of colors on those neighborhoods. The least number of colors that permits such an assignment is the region  $\pi$  chromatic number  $R\pi\chi(G)$ . This article determines  $R\pi\chi$  for the linear, alternate, cyclic, and double families of planar snake graphs. The values are obtained from two general theorems rather than from separate arguments for each family. The first treats plane graphs in which every interior face meets the outer face, and the interior faces split into  $n$  mutually nonadjacent groups of  $d$  pairwise adjacent faces, and gives  $R\pi\chi = 1 + \min\{k : \binom{k}{d} \geq n\}$ . The second treats annular plane graphs with  $p$  petals and gives  $R\pi\chi = p + 2$ . The linear and alternate families are the case  $d = 1$ , where  $R\pi\chi$  equals the number of faces, and hence,  $|E| - |V| + 2$  by Euler's formula. The cyclic families are annular, and the double families are the case  $d = 2$ , where the value is  $2 + \lceil (\sqrt{8n+1} - 1)/2 \rceil$  and rises by one as  $n$  passes a triangular number. The article also shows that  $R\pi\chi$  is an invariant of a plane graph and not of the underlying planar graph: A nested drawing of the triangular snake  $TS_7$  needs four colors, whereas the standard drawing needs eight. For the linear and alternate families, the standard drawing is shown to maximize  $R\pi\chi$  over all embeddings. A linear-time algorithm recognizes both patterns and returns an optimal coloring, the associated decision problem is shown to lie in NP, and  $R\pi\chi$  is bounded below by the chromatic number of the dual graph and above by the number of faces.

**Keywords:** region  $\pi$  coloring; region  $\pi$  chromatic number; snake graphs; face coloring; plane graphs; planar embedding; triangular numbers

**Mathematics Subject Classification:** 05C10, 05C15

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## 1. Introduction

A planar graph carries structure beyond its vertices and its edges. Every planar embedding cuts the plane into bounded regions and one unbounded region, and the pattern in which these regions meet is an object of study in its own right. Face coloring assigns colors to regions rather than to points, and it grows out of the coloring of maps. The classical version of the problem was settled by the four color theorem, for which Robertson and coauthors gave a simplified proof [1]. Proper face coloring asks only that two regions sharing an edge receive different colors. The region  $\pi$  coloring studied here asks for more. Each region must also be identified by the set of colors appearing on the regions that surround it.

Colorings that identify vertices by their neighborhoods have a substantial literature. Burris and Schelp introduced vertex-distinguishing proper edge coloring, in which every vertex is determined by the colors on its incident edges [2]. Bazgan and coauthors sharpened the resulting bounds [3]. Balister and coauthors settled the case of graphs with maximum degree two [4]. Zhang and coauthors studied adjacent strong edge coloring, in which adjacent vertices must see different sets of colors on their incident edges [5]. Albertson and Collins studied symmetry breaking by coloring, which applies the same idea to automorphisms [6]. Thakare and Bhapkar introduced incident vertex  $\pi$ -coloring, in which every vertex must display a color set different from the color sets of its neighbors [7]. Region  $\pi$  coloring carries that requirement from vertex incidence to face adjacency in a plane graph [8]. Bhange and Bhapkar studied perfect coloring and several of its structural variants [9]. Bhapkar and Salunke analyzed planar regions and their adjacent region sets in connection with the four color problem [10].

Work on coloring parameters of planar graphs is broad. Brooks bounded the vertex chromatic number of a connected graph by its maximum degree, with two exceptions [11]. Vizing's theorem placed the edge chromatic number of a simple graph within one of its maximum degree, and his later survey collected many of the problems that shaped the field [12]. Thomassen proved that every planar graph is five-choosable [13]. He also obtained the exact counterparts of Grötzsch's theorem for the torus and the projective plane [14]. Borodin and coauthors proved that planar graphs with no cycle of length four to seven are three-colorable [15]. Sanders and Zhao bounded the cyclic chromatic number, which counts the colors needed so that vertices on a common face boundary differ [16]. Jendrol' and Voss surveyed light subgraphs of embedded graphs [17].

Snake graphs are planar graphs built by chaining copies of a fixed cycle along a base path. Their recursive structure has made them a standard test case for labeling and coloring problems. Gallian's survey records a large body of such results [18]. Graham and Sloane began the study of harmonious labelings [19]. Acharya and Hegde studied arithmetic graphs [20]. Bača and coauthors established irregular total labelings for several families [21]. Bača and coauthors also studied strong labelings of linear forests, which include snake-type chains [22]. Chartrand and coauthors introduced rainbow connectivity [23]. Geetha and Somasundaram computed total colorings for product graphs related to the families treated here [24].

This article determines the region  $\pi$  chromatic number for the planar snake graph families it studies. The five linear families, namely the triangular snake  $TS_n$ , the quadrilateral snake  $Ql_n$ , the square snake  $Sqs_n$ , the quintet snake  $Qt_n$ , and the heptagonal snake  $Hts_n$ , all satisfy  $R\pi\chi = n + 1$ , where  $n$  counts the cycle attachments. The alternate families  $ATS_n$  and  $AQS_n$  obey a parity-sensitive formula, namely  $\lfloor n/2 \rfloor + 1$  for closed-end graphs with  $n$  even,  $n/2$  for open-end graphs with  $n$  even, and  $\lceil n/2 \rceil$  for one-

end-open graphs with  $n$  odd. The cyclic quadrilateral snake  $Cqs_n$  and the line of sunlet graph  $Ls_n$  both satisfy  $R\pi\chi = n + 2$ , as do the alternate triangular and quadrilateral cycle snakes ( $ATCS_n$  and  $AQCS_n$ ) up to a parity shift. For the double snake families ( $DT_n, Dqls_n, ADT_n, ADQ_n$ ), the chromatic number follows a stepped formula  $R\pi\chi = 2 + \lceil (\sqrt{8n+1} - 1)/2 \rceil$ , whose value is  $m + 2$  on the range  $T_{m-1} < n \leq T_m$ , where  $T_m = m(m+1)/2$ , so that it increases by one as  $n$  passes a triangular number.

Three further contributions run alongside these values. The first is that  $R\pi\chi$  depends on the planar embedding and not only on the abstract graph, and that the drawings used here are the ones for which the value is largest. The second is that all of the family results are instances of two general theorems about plane graphs, so the polygon size plays no role. The third is that the parameter can be computed in linear time on the classes that arise here, and that the general decision problem lies in nondeterministic polynomial time (NP).

The article is organized as follows. Section 2 recalls the definition of region  $\pi$  coloring, illustrates it on the Jellyfish graph, and records two bounds used throughout. Section 3 analyzes the effect of the planar embedding. Section 4 proves the two general theorems and records which instance each family is. Section 5 treats the five linear snake families and establishes the uniform formula  $n + 1$ . Section 6 handles the parity-sensitive alternate families. Section 7 covers cyclic variants producing chromatic numbers  $n + 2$  and  $\lfloor n/2 \rfloor + 2$ . Section 8 addresses double snake families and the triangular-number step structure. Section 9 gives the algorithm and the complexity analysis. Section 10 relates  $R\pi\chi$  to the chromatic number of the dual graph. Section 11 summarizes all results and states six open problems.

## 2. Preliminaries

Throughout, every graph  $G$  is finite, simple, and undirected. Standard coloring terminology follows Chartrand and Zhang [25]. A planar graph is one that can be drawn in the plane without edge crossings. Such a drawing is called a planar embedding, and it partitions the plane into connected regions. A planar graph together with a fixed planar embedding is a plane graph. A bounded region is called an interior face, and the unique unbounded region is the exterior face, denoted  $R_0$ . Adjacency between two regions is defined strictly through shared edges: Two regions are adjacent if and only if they share at least one common edge. Sharing only a vertex does not constitute adjacency.

The following definition introduces the coloring concept studied throughout this article.

**Definition 2.1** (Region  $\pi$  coloring adapted from [7]). Let  $G$  be a plane graph with region set  $\{R_0, R_1, \dots, R_k\}$ . For each region  $R_i$ , define its *adjacent region set*

$$P_i = \{R_i\} \cup \{R_j \mid R_j \text{ shares an edge with } R_i\}.$$

Let  $\mathcal{K} = \{1, 2, \dots, h\}$  be a color set and  $\mathcal{P}(\mathcal{K})$  its power set. An assignment  $c : \{R_0, R_1, \dots, R_k\} \rightarrow \mathcal{K}$  induces a *delta function*

$$\delta(P_i) = \{c(R_j) \mid R_j \in P_i\},$$

which collects the colors of all regions in  $P_i$  as a set (repetitions collapsed). The assignment  $c$  is a *region  $\pi$  coloring* of  $G$  if both of the following conditions hold:

$$c(R_i) \neq c(R_j) \quad \text{whenever } R_i \text{ and } R_j \text{ share an edge,} \quad (2.1)$$

$$\delta(P_i) \neq \delta(P_j) \quad \forall i \neq j \text{ with } P_i \neq P_j. \quad (2.2)$$

The *region  $\pi$  chromatic number*  $R\pi\chi(G)$  is the minimum value of  $h$  over all region  $\pi$  colorings of  $G$ .

Condition (2.2) requires distinctness only between structurally different adjacent region sets. If  $P_i = P_j$  as sets of regions, the corresponding delta values are automatically equal, and no conflict arises. Relabelling the colors if necessary, the exterior face  $R_0$  may be assumed to carry the color 1.

**Remark 2.2.** Conditions (2.1) and (2.2) are independent, and neither one implies the other. To see that (2.1) does not imply (2.2), consider the triangular snake  $TS_3$  with  $R_0 \rightarrow 1$  and  $R_1, R_2, R_3 \rightarrow 2$ . No two adjacent regions share a color, so (2.1) holds. Here,  $\delta(P_0) = \delta(P_1) = \{1, 2\}$  when  $P_0 \neq P_1$ , so (2.2) fails. To see that (2.2) does not imply (2.1), assign  $R_0 \rightarrow 1$ ,  $R_1 \rightarrow 1$ ,  $R_2 \rightarrow 2$ , and  $R_3 \rightarrow 3$  in the same graph. The delta values  $\{1, 2, 3\}$ ,  $\{1\}$ ,  $\{1, 2\}$ , and  $\{1, 3\}$  are pairwise distinct, so (2.2) holds. The regions  $R_0$  and  $R_1$  share an edge and carry the same color, so (2.1) fails. Both conditions are therefore needed in Definition 2.1.

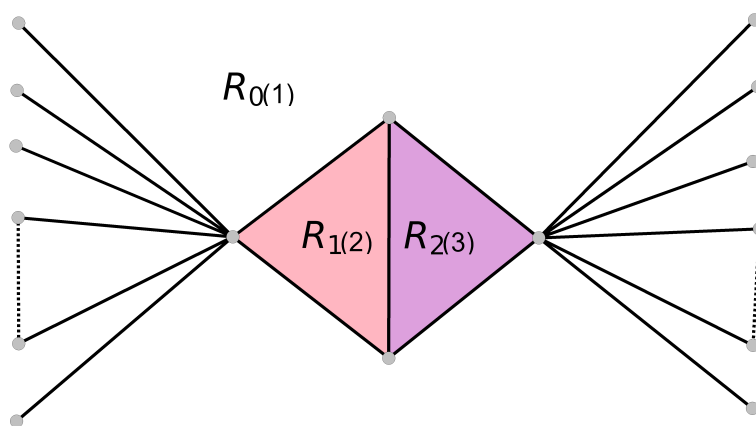
The following example illustrates the definition on a small graph and establishes the base case referenced in later sections.

**Example 2.3.** The region  $\pi$  chromatic number of the jellyfish graph  $Jl_f$  is 3.

The graph  $Jl_f$  is a planar graph containing three regions: the exterior face  $R_0$  and two interior faces,  $R_1$  and  $R_2$ . By observing Figure 1, every pair of regions in  $Jl_f$  shares a common edge, so  $P_i = \{R_0, R_1, R_2\}$  for  $i = 0, 1, 2$ . Because all three adjacent region sets coincide, the region  $\pi$  coloring condition requires only that the three regions receive distinct colors. Assigning

$$R_0 \rightarrow 1, \quad R_1 \rightarrow 2, \quad R_2 \rightarrow 3,$$

gives  $\delta(P_i) = \{1, 2, 3\}$  for  $i = 0, 1, 2$ . The sets are vacuously distinct (there is exactly one distinct adjacent region set), and three colors suffice. Therefore,  $R\pi\chi(Jl_f) = 3$ .



**Figure 1.** Jellyfish graph  $Jl_f$  with region  $\pi$  coloring;  $R\pi\chi(Jl_f) = 3$ .

The next two bounds hold for every plane graph and are used repeatedly below.

**Lemma 2.4.** For every plane graph  $G$ ,  $R\pi\chi(G) \leq |F(G)|$ , where  $F(G)$  denotes the set of regions of  $G$ .

*Proof.* Assign a different color to every region. Two adjacent regions are distinct regions, so (2.1) holds. The assignment is injective, so  $\delta(P_i)$  determines  $P_i$ . Regions with different adjacent region sets therefore receive different delta values, and (2.2) holds. The coloring uses  $|F(G)|$  colors.  $\square$

**Lemma 2.5.** *Let  $G$  be a plane graph, let  $t$  be the number of distinct sets among  $P_0, P_1, \dots, P_k$ , and let  $s = \max_i |P_i|$ . Every region  $\pi$  coloring of  $G$  with  $h$  colors satisfies*

$$\sum_{r=1}^s \binom{h}{r} \geq t. \quad (2.3)$$

*Proof.* Each region  $R_i$  lies in  $P_i$ , so  $\delta(P_i)$  is a nonempty subset of  $\mathcal{K}$ . Repetitions are collapsed, so  $|\delta(P_i)| \leq |P_i| \leq s$ . Condition (2.2) makes the map  $P_i \mapsto \delta(P_i)$  injective on the  $t$  distinct adjacent region sets. The number of nonempty subsets of  $\mathcal{K}$  of size at most  $s$  is the left-hand side of (2.3), which must therefore be at least  $t$ .  $\square$

### 3. Region $\pi$ coloring and the planar embedding

Definition 2.1 refers to the regions of a graph already drawn in the plane. A planar graph with cut vertices admits more than one embedding, and different embeddings may produce different region adjacency patterns. This section determines what happens to  $R\pi\chi$  under a change of drawing.

Write  $H(G, \Phi)$  for the *region adjacency graph* of a plane graph  $(G, \Phi)$ . Its vertices are the regions of  $(G, \Phi)$ , and two vertices are joined when the corresponding regions share an edge. The adjacent region set  $P_i$  is exactly the closed neighborhood of  $R_i$  in  $H(G, \Phi)$ . Both conditions of Definition 2.1, therefore refer only to  $H(G, \Phi)$ , so  $R\pi\chi(G, \Phi)$  is determined by the region adjacency graph alone. In particular, it does not depend on which region is taken to be unbounded.

A connected graph is considered a cactus when every edge lies in at most one cycle, or equivalently when every block is a cycle or a single edge. The five linear families  $TS_n$ ,  $Ql_n$ ,  $Sqs_n$ ,  $Qt_n$ , and  $Hts_n$  are cacti, as are the alternate families  $ATS_n$  and  $AQS_n$ .

**Lemma 3.1.** *Let  $G$  be a connected cactus with  $n$  cycles, and let  $\Phi$  be any planar embedding of  $G$ . Then,  $(G, \Phi)$  has exactly  $n + 1$  regions, and  $H(G, \Phi)$  is a tree on  $n + 1$  vertices.*

*Proof.* A bridge of  $G$  lies in no cycle, so the cycle rank of  $G$  is  $|E| - |V| + 1 = n$ . Euler's formula for a connected plane graph gives  $|F| = |E| - |V| + 2 = n + 1$ , and this count does not depend on  $\Phi$ .

Fix  $\Phi$ . Each of the  $n$  cycles is drawn as a simple closed curve. Two distinct cycles of a cactus lie in different blocks, so they share at most one vertex and never cross. For two such curves, the open interiors are therefore either disjoint or nested. The  $n$  interiors form a laminar family. Say that a cycle  $D$  is *directly nested* in a cycle  $C$  when the interior of  $D$  is contained in the interior of  $C$ , and no cycle lies strictly between them. The region carried by  $C$  is the interior of  $C$  with the closed interiors of all cycles directly nested in  $C$  removed, and the unbounded region is the complement of the closed interiors of the outermost cycles. This accounts for all  $n + 1$  regions.

The boundary of the region carried by  $C$  consists of the edges of  $C$ , the edges of the cycles directly nested in  $C$ , and possibly some bridges. A bridge has the same region on both of its sides, so it is never shared by two distinct regions. Two cycles nested in a common cycle share at most a vertex and hence no edge. Two regions therefore share an edge exactly when one of the two cycles is directly nested

in the other. The graph  $H(G, \Phi)$  is the containment forest of the laminar family with the unbounded region as its root, which is a tree on  $n + 1$  vertices.  $\square$

**Theorem 3.2.** *Let  $G$  be a connected cactus with  $n \geq 1$  cycles. Then,  $R\pi\chi(G, \Phi) \leq n + 1$  for every planar embedding  $\Phi$ . The standard embedding  $\Phi_0$ , in which no cycle is drawn inside the region bounded by another cycle, satisfies  $R\pi\chi(G, \Phi_0) = n + 1$ . Consequently,*

$$\max_{\Phi} R\pi\chi(G, \Phi) = n + 1,$$

and the maximum is attained by the standard embedding.

*Proof.* Lemma 3.1 gives  $|F| = n + 1$  for every  $\Phi$ , so Lemma 2.4 gives  $R\pi\chi(G, \Phi) \leq n + 1$ .

Under  $\Phi_0$ , no cycle is nested in another, so the containment tree is the star with center  $R_0$  and leaves  $F_1, F_2, \dots, F_n$ . If  $n = 1$ , then  $P_0 = P_1$ , no instance of (2.2) applies, and (2.1) alone forces two colors, which equals  $n + 1$ . Let  $n \geq 2$ . Here,  $P_0 = \{R_0, F_1, \dots, F_n\}$ , and  $P_j = \{R_0, F_j\}$  for  $1 \leq j \leq n$ . Every leaf is adjacent to  $R_0$ , so (2.1) gives  $c(F_j) \neq c(R_0)$ , and hence,  $\delta(P_j) = \{c(R_0), c(F_j)\}$  has exactly two elements. For  $j \neq l$ , the sets  $P_j$  and  $P_l$  differ, so (2.2) forces  $\delta(P_j) \neq \delta(P_l)$ , which holds only if  $c(F_j) \neq c(F_l)$ . The  $n$  leaf colors are therefore pairwise distinct, and all differ from  $c(R_0)$ , so at least  $n + 1$  colors are needed. Lemma 2.4 shows that  $n + 1$  colors suffice.  $\square$

**Theorem 3.3.** *Let  $n \geq 2$ , and let  $\Phi_1$  be the embedding of the triangular snake  $TS_n$  in which the triangle  $T_{i+1}$  is drawn inside the region bounded by  $T_i$  for  $i = 1, \dots, n - 1$ . Then,  $H(TS_n, \Phi_1)$  is the path on  $n + 1$  vertices, and every region  $\pi$  coloring of  $(TS_n, \Phi_1)$  with  $h$  colors satisfies*

$$\binom{h}{2} + \binom{h}{3} \geq n + 1. \quad (3.1)$$

In particular,  $R\pi\chi(TS_7, \Phi_1) = 4$ , whereas  $R\pi\chi(TS_7, \Phi_0) = 8$ .

*Proof.* Write  $TS_n$  as a path  $v_0v_1 \cdots v_n$  with a triangle  $T_i$  on the edge  $v_{i-1}v_i$  and apex  $u_i$ . The triangles  $T_i$  and  $T_{i+1}$  share the vertex  $v_i$ , which lies on the boundary of the region carried by  $T_i$ . The triangle  $T_{i+1}$ , together with everything attached beyond it, can therefore be drawn inside that region without crossings. Repeating the construction for  $i = 1, \dots, n - 1$  gives a planar embedding  $\Phi_1$  in which  $T_{i+1}$  is directly nested in  $T_i$  for every  $i$ . By Lemma 3.1, the region adjacency graph is the containment tree, which here is the path  $R_0, F_1, F_2, \dots, F_n$ .

On a path, every closed neighborhood has at most three elements, so  $s = 3$  in Lemma 2.5. For  $n \geq 2$ , the  $n + 1$  closed neighborhoods of a path are pairwise distinct, so  $t = n + 1$ , and Lemma 2.5 applies. The bound can be sharpened here. Every closed neighborhood on a path contains two regions that share an edge, so by (2.1), no delta value consists of a single color. Each of the  $n + 1$  delta values is therefore a subset of  $\mathcal{K}$  of size two or three, and these are pairwise distinct by (2.2). Counting the available subsets of those two sizes gives (3.1).

Figure 2 shows the two embeddings for  $n = 5$ , together with the star and the path that arise as their region adjacency graphs.

Let  $n = 7$ , so that  $t = 8$ . Taking  $h = 3$  in (3.1) gives  $3 + 1 = 4 < 8$ , so three colors do not suffice. Four colors do. Assign along the path

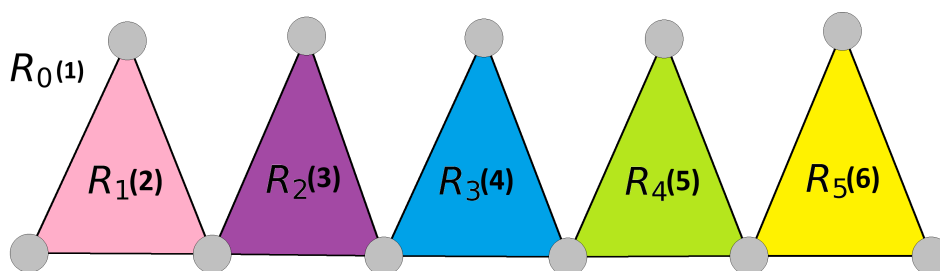
$$R_0 \rightarrow 1, \quad F_1 \rightarrow 2, \quad F_2 \rightarrow 3, \quad F_3 \rightarrow 2, \quad F_4 \rightarrow 4, \quad F_5 \rightarrow 1, \quad F_6 \rightarrow 4, \quad F_7 \rightarrow 3.$$

Consecutive regions carry different colors, so (2.1) holds. The eight delta values are

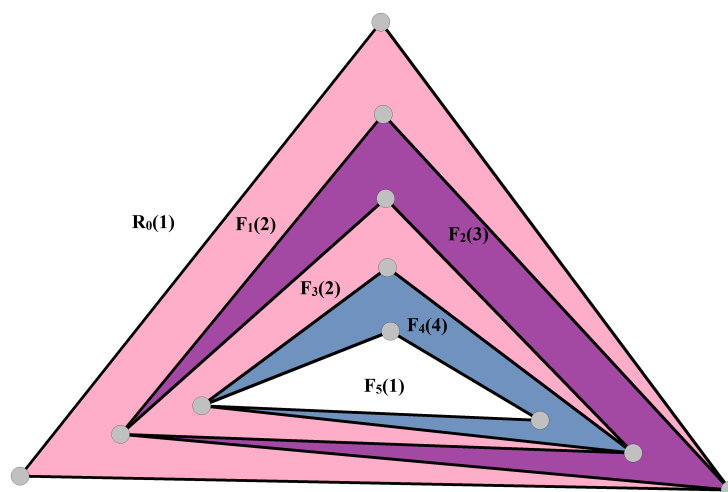
$$\begin{aligned} &\{1, 2\}, \{1, 2, 3\}, \{2, 3\}, \{2, 3, 4\}, \\ &\{1, 2, 4\}, \{1, 4\}, \{1, 3, 4\}, \{3, 4\}, \end{aligned}$$

and these are pairwise distinct, so (2.2) holds. Hence,  $R\pi\chi(TS_7, \Phi_1) = 4$ . Theorem 3.2 gives  $R\pi\chi(TS_7, \Phi_0) = 7 + 1 = 8$ .  $\square$

(a) Standard drawing of  $TS_5$ :  $R\pi\chi = 6$



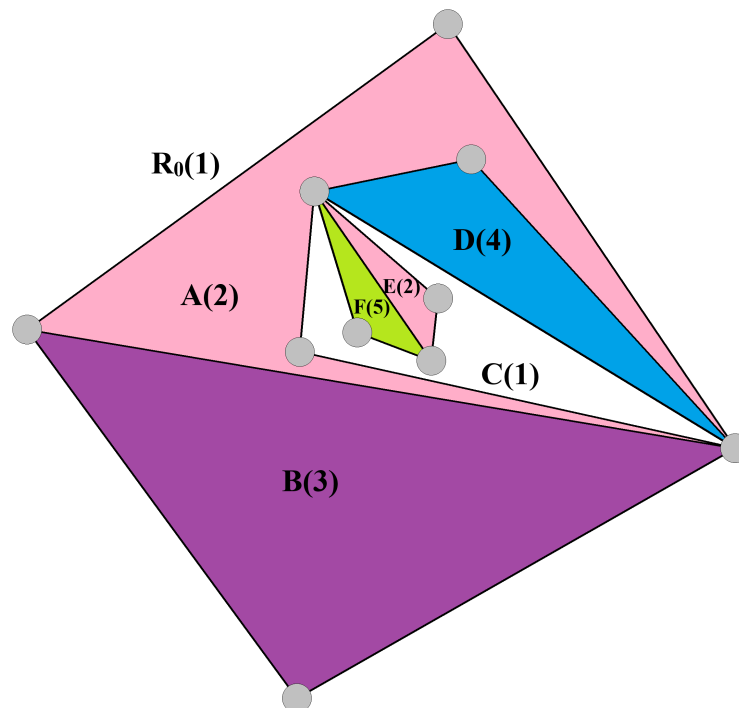
(b) Nested drawing of  $TS_5$ :  $R\pi\chi = 4$



**Figure 2.** Two planar embeddings of the same graph  $TS_5$ . The number on each region is its color.

**Remark 3.4.** Theorem 3.3 shows that  $R\pi\chi$  is an invariant of a plane graph and not of the underlying planar graph. Theorem 3.2 shows that the standard embedding is the one for which the value is largest. Every result in the sections that follow is stated for the standard embedding, and  $R\pi\chi(G)$  abbreviates  $R\pi\chi(G, \Phi_0)$ . The linear and alternate families are cacti, so by Theorem 3.2, the values reported for them are the maxima of  $R\pi\chi$  over all embeddings. The cyclic families are not cacti, but their value does not depend on the embedding at all, by Remark 3.5. The double families are not cacti, and for them, the standard embedding is not maximal, as Remark 3.5 shows. The exact order of growth of  $\min_{\Phi} R\pi\chi(G, \Phi)$  is not determined here. For the nested embeddings of  $TS_n$ , inequality (3.1) gives a lower bound of order  $n^{1/3}$ , and no matching upper bound is known.

**Remark 3.5.** The cyclic families of Section 7 are not cacti, as each base edge lies in both the base cycle and a petal. Their region structure is nevertheless the same for all embeddings. Each petal is attached to the base cycle at the two ends of a base edge, and the petal path together with that edge bounds the petal region. Drawing the petal path outside or inside the base cycle exchanges which of the two large regions meets the petal along the base edge, but the petal remains adjacent to both of them. The region adjacency graph is therefore unchanged, and by the observation at the start of this section, the value of  $R\pi\chi$  is unchanged, as well. The double families of Section 8 are not cacti either, as the two faces of a kite unit share a diagonal edge. Their kite units meet at cut vertices and can be nested in the manner of Theorem 3.3, and for them, nesting *raises* the value. Consider  $DT_3$  with kite units  $\{A, B\}$ ,  $\{C, D\}$ , and  $\{E, F\}$ , drawn so that the second unit lies inside the region  $A$  and the third inside the region  $C$ . The adjacent region sets are  $P_{R_0} = P_B = \{R_0, A, B\}$ ,  $P_A = \{R_0, A, B, C, D\}$ ,  $P_D = \{A, C, D\}$ ,  $P_C = \{A, C, D, E, F\}$  and  $P_E = P_F = \{C, E, F\}$ . Suppose four colors suffice. The regions  $R_0$ ,  $A$  and  $B$  are pairwise adjacent, so  $\delta(P_{R_0})$  has exactly three colors. Because  $P_{R_0} \subset P_A$  we have  $\delta(P_{R_0}) \subseteq \delta(P_A)$ , and  $P_{R_0} \neq P_A$  forces  $\delta(P_{R_0}) \neq \delta(P_A)$  by (2.2), so  $\delta(P_A)$  uses all four colors. The same argument applied to  $P_D \subset P_C$  shows that  $\delta(P_C)$  uses all four colors. Then,  $\delta(P_A) = \delta(P_C)$  when  $P_A \neq P_C$ , contradicting (2.2). Five colors do suffice: Assign 1, 2, 3, 1, 4, 2, 5 to  $R_0, A, B, C, D, E, F$ , as Figure 3 shows. Hence, this embedding of  $DT_3$  needs five colors, whereas Theorem 8.1 gives four for the standard embedding. The maximum of  $R\pi\chi$  over all embeddings of a double snake is not determined here.



**Figure 3.** A nested embedding of  $DT_3$  needing five colors, against four for the standard embedding. Letters name the regions, and numbers give their colors.

#### 4. A general framework

Every family treated in Sections 5 to 8 exploits one of two region adjacency patterns. In the first pattern, the interior regions fall into groups that are attached to the exterior region and to nothing else. In the second pattern, the interior regions are attached to two large regions. This section proves a formula for each pattern. The family results that follow are instances of these two theorems, and Remark 4.10 records which instance each family is.

**Definition 4.1.** Let  $d \geq 1$ , and  $n \geq 1$  be integers. A plane graph  $G$  with exterior region  $R_0$  is of *hub type with parameters*  $(n, d)$  if its interior regions can be partitioned into  $n$  groups  $U_1, U_2, \dots, U_n$ , each of size  $d$ , such that

- (H1) every interior region shares an edge with  $R_0$ ,
- (H2) two regions in the same group share an edge,
- (H3) two regions in different groups share no edge.

**Theorem 4.2.** Let  $G$  be a plane graph of hub type with parameters  $(n, d)$ . Then,

$$R\pi\chi(G) = 1 + \min\left\{k \in \mathbb{N} : \binom{k}{d} \geq n\right\}. \quad (4.1)$$

*Proof.* Write  $k_0$  for the minimum in (4.1). By (H1), the exterior region meets every interior region, so  $P_0 = \{R_0\} \cup U_1 \cup \dots \cup U_n$ . By (H2) and (H3), a region  $R_i \in U_j$  meets  $R_0$  and the other members of  $U_j$  and nothing else, so  $P_i = \{R_0\} \cup U_j$ . Write  $P^{(j)} = \{R_0\} \cup U_j$ . The distinct adjacent region sets are  $P_0$  together with  $P^{(1)}, \dots, P^{(n)}$ , and  $P_0 = P^{(1)}$  exactly when  $n = 1$ .

*Sufficiency.* Put  $h = 1 + k_0$  and  $\mathcal{K} = \{1, 2, \dots, h\}$ , and set  $c(R_0) = 1$ . The set  $\{2, 3, \dots, h\}$  has  $k_0$  elements and  $\binom{k_0}{d} \geq n$ , so there are  $n$  pairwise distinct  $d$ -element subsets  $S_1, \dots, S_n$  of it. Color the  $d$  regions of  $U_j$  bijectively by the  $d$  colors of  $S_j$ . Two regions of a common group then carry different colors, two regions of different groups share no edge by (H3), and no interior region carries the color 1. Condition (2.1) therefore holds. The delta values are

$$\delta(P_0) = \{1\} \cup \bigcup_{j=1}^n S_j, \quad \delta(P^{(j)}) = \{1\} \cup S_j.$$

For  $j \neq l$ , the sets  $S_j$  and  $S_l$  differ, so the corresponding delta values differ. If  $n \geq 2$ , then  $S_1 \neq S_2$ , and both have  $d$  elements, so  $|S_1 \cup S_2| \geq d + 1$ , and hence,  $|\delta(P_0)| > d + 1 = |\delta(P^{(j)})|$  for every  $j$ . If  $n = 1$ , then  $P_0 = P^{(1)}$ , and no instance of (2.2) applies. In both cases, (2.2) holds, so  $R\pi\chi(G) \leq 1 + k_0$ .

*Necessity.* Let  $c$  be a region  $\pi$  coloring of  $G$  with  $h$  colors and put  $a = c(R_0)$ . By (H1) and (2.1), no interior region carries the color  $a$ , so all interior colors lie in  $\mathcal{K} \setminus \{a\}$ , a set of  $h - 1$  colors. By (H2), the  $d$  regions of  $U_j$  are pairwise adjacent and therefore carry  $d$  distinct colors, so they determine a  $d$ -element subset  $S_j \subseteq \mathcal{K} \setminus \{a\}$  with  $\delta(P^{(j)}) = \{a\} \cup S_j$ . For  $j \neq l$ , the sets  $P^{(j)}$  and  $P^{(l)}$  differ, so (2.2) gives  $\{a\} \cup S_j \neq \{a\} \cup S_l$ , and hence  $S_j \neq S_l$ . The sets  $S_1, \dots, S_n$  are therefore  $n$  pairwise distinct  $d$ -element subsets of a set of size  $h - 1$ , which requires  $\binom{h-1}{d} \geq n$ . By the minimality of  $k_0$ , this gives  $h - 1 \geq k_0$ , so  $R\pi\chi(G) \geq 1 + k_0$ .  $\square$

**Corollary 4.3.** A plane graph of hub type with parameters  $(n, 1)$  satisfies  $R\pi\chi(G) = n + 1 = |F(G)|$ .

*Proof.* Here,  $\binom{k}{1} = k$ , so  $k_0 = n$ , and (4.1) gives  $n + 1$ . The graph has  $n$  interior regions and one exterior region.  $\square$

**Corollary 4.4.** *A plane graph of hub type with parameters  $(n, 2)$  satisfies*

$$R\pi\chi(G) = 2 + \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor = 2 - \left\lfloor \frac{1-\sqrt{8n+1}}{2} \right\rfloor. \quad (4.2)$$

The value equals  $m + 2$  on the range  $T_{m-1} < n \leq T_m$ , where  $T_m = m(m+1)/2$ , and  $T_0 = 0$ . It is therefore constant between consecutive triangular numbers and increases by one as  $n$  passes a triangular number.

*Proof.* The condition  $\binom{k}{2} \geq n$  reads  $k(k-1) \geq 2n$ , whose least integer solution is  $k_0 = \lceil (1 + \sqrt{8n+1})/2 \rceil$ . Because  $(1+x)/2 = (x-1)/2 + 1$  for every real  $x$ , and  $\lceil y+1 \rceil = \lceil y \rceil + 1$ , formula (4.1) gives  $1 + k_0 = 2 + \lceil (\sqrt{8n+1}-1)/2 \rceil$ . The identity  $\lceil y \rceil = -\lfloor -y \rfloor$  yields the second expression. If  $n = T_m$ , then  $\sqrt{8n+1} = 2m+1$ , and  $k_0 = m+1$ , whereas  $n = T_m - 1$  gives  $k_0 = m$ , so the value steps up exactly at the triangular numbers.  $\square$

**Corollary 4.5.** *For fixed  $d$ , a plane graph of hub type with parameters  $(n, d)$  satisfies  $R\pi\chi(G) = (d!n)^{1/d}(1 + o(1))$  as  $n \rightarrow \infty$ . The growth is linear in  $n$  for  $d = 1$  and of order  $(6n)^{1/3}$  for  $d = 3$ . For  $d = 2$ , the error term is explicit, namely*

$$\sqrt{2n} + \frac{3}{2} \leq R\pi\chi(G) \leq \sqrt{2n} + \frac{5}{2} \quad \text{for every } n \geq 1. \quad (4.3)$$

*Proof.* For fixed  $d$ , one has  $\binom{k}{d} = k^d/d! + O(k^{d-1})$ , so the least  $k$  with  $\binom{k}{d} \geq n$  satisfies  $k_0 = (d!n)^{1/d}(1 + o(1))$ . The additive constant in (4.1) is absorbed into the error term.

For  $d = 2$ , Corollary 4.4 gives  $R\pi\chi(G) = 1 + \lceil (1 + \sqrt{8n+1})/2 \rceil$ . From  $\sqrt{8n} \leq \sqrt{8n+1} \leq \sqrt{8n} + 1/(2\sqrt{8n})$ , the argument of the ceiling lies between  $\sqrt{2n} + 1/2$  and  $\sqrt{2n} + 1/2 + 1/(8\sqrt{2n})$ . Adding one, and bounding the ceiling below by its argument and above by its argument plus one, gives (4.3).  $\square$

**Definition 4.6.** A plane graph  $G$  is annular with  $p$  petals if its regions are  $H_1, H_2$ , and  $W_1, \dots, W_p$ , where every petal  $W_m$  shares an edge with  $H_1$  and an edge with  $H_2$ , and no two petals share an edge. The regions  $H_1$  and  $H_2$  may or may not share an edge.

**Theorem 4.7.** *Let  $G$  be annular with  $p \geq 1$  petals. Then,  $R\pi\chi(G) = p + 2 = |F(G)|$ .*

*Proof. Sufficiency.* Assign  $c(H_1) = 1$ ,  $c(W_m) = m + 1$  for  $1 \leq m \leq p$ , and  $c(H_2) = p + 2$ . All  $p + 2$  regions receive different colors, so Lemma 2.4 applies, and  $p + 2$  colors suffice.

*Necessity.* Let  $c$  be a region  $\pi$  coloring with  $h$  colors. Every petal shares an edge with both  $H_1$  and  $H_2$ , so (2.1) gives  $c(W_m) \neq c(H_1)$  and  $c(W_m) \neq c(H_2)$  for every  $m$ . The petal-adjacent region sets are  $P_{W_m} = \{H_1, W_m, H_2\}$ , so  $\delta(P_{W_m}) = \{c(H_1), c(W_m), c(H_2)\}$ . For  $m \neq m'$ , these sets differ by (2.2), which forces  $c(W_m) \neq c(W_{m'})$ . The  $p$  petal colors are therefore pairwise distinct, and all differ from  $c(H_1)$ , so  $h \geq p + 1$ .

It remains to show  $c(H_1) \neq c(H_2)$ . If  $H_1$  and  $H_2$  share an edge this is immediate from (2.1). If they do not, then  $P_{H_1} = \{H_1, W_1, \dots, W_p\}$  and  $P_{H_2} = \{H_2, W_1, \dots, W_p\}$  are different sets of regions, so (2.2) requires  $\delta(P_{H_1}) \neq \delta(P_{H_2})$ . Were  $c(H_1) = c(H_2)$ , both delta values would equal  $\{c(H_1), c(W_1), \dots, c(W_p)\}$ , a contradiction. In either case, the two colors differ, and  $h \geq p + 2$ .  $\square$

**Corollary 4.8.** *Let  $G$  be a connected plane graph that is either of hub type with parameters  $(n, 1)$  or annular with  $p$  petals. Then,*

$$R\pi\chi(G) = |F(G)| = |E(G)| - |V(G)| + 2.$$

*Proof.* Corollary 4.3 and Theorem 4.7 give  $R\pi\chi(G) = |F(G)|$  in the two cases, and Euler's formula for a connected plane graph gives  $|F| = |E| - |V| + 2$ .  $\square$

**Remark 4.9.** Corollary 4.8 accounts for the difference between the linear families and the cyclic families in structural terms. The quantity  $|E| - |V| + 1$  is the dimension of the cycle space of  $G$ . Closing the base path of a linear snake into a ring adds one independent cycle and therefore one region. The value of  $R\pi\chi$  moves from  $n + 1$  to  $n + 2$  for that reason alone. The size of the polygon enters nowhere, which is why the triangular, quadrilateral, square, quintet, and heptagonal versions all return the same answer.

**Remark 4.10.** Table 1 records the instance of Theorem 4.2 or Theorem 4.7 to which each family of Sections 5 to 8 belongs. In every case, the parameter  $n$  or  $p$  of the general theorem is the number of groups or petals in the standard embedding. The table serves as a map from each family to the general result that governs it, and to the place where the value is computed. The values themselves, together with the parity cases and the ranges of validity, are listed separately in Table 6 of the concluding section.

**Table 1.** Each snake family as an instance of Theorem 4.2 or Theorem 4.7. Values are collected in Table 6.

| Family                           | Type    | Parameters                | Proved in |
|----------------------------------|---------|---------------------------|-----------|
| $TS_n, Ql_n, SqS_n, Qt_n, Hts_n$ | hub     | $(n, 1)$                  | Cor. 4.3  |
| $ATS_n, AQS_n$ (CE/OE/1E)        | hub     | $(q, 1)$                  | Cor. 4.3  |
| $CqS_n, Ls_n$                    | annular | $p = n$                   | Thm. 4.7  |
| $ATCS_n, AQCS_n$                 | annular | $p = \lfloor n/2 \rfloor$ | Thm. 4.7  |
| $DT_n, Dqls_n, ADT_n, ADQ_n$     | hub     | $(n, 2)$                  | Cor. 4.4  |

## 5. Linear snake graph families

A linear  $k$ -gonal snake graph  $G_n$  is constructed by taking a path  $v_0v_1 \cdots v_n$  and attaching a copy of the cycle  $C_k$  at each internal edge  $v_{i-1}v_i$  so that the two endpoints are shared with the path. The resulting planar embedding produces  $n$  bounded interior faces  $R_1, R_2, \dots, R_n$ , and one unbounded exterior face  $R_0$ . Consecutive interior faces  $R_i$  and  $R_{i+1}$  share exactly one vertex (the attachment vertex  $v_i$ ), so they are *not* edge-adjacent. Each interior face  $R_i$  shares a base edge with the exterior face  $R_0$ .

### 5.1. Triangular snake graph $TS_n$

The triangular snake graph  $TS_n$  consists of  $n$  copies of the cycle  $C_3$  joined along a base path. The planar embedding has  $n + 1$  faces:  $R_0$  (exterior) and  $R_1, \dots, R_n$  (interior triangles).

**Theorem 5.1.** *For every integer  $n \geq 1$ , the region  $\pi$  chromatic number of the triangular snake graph  $TS_n$  is  $R\pi\chi(TS_n) = n + 1$ .*

*Proof.* The graph  $TS_n$  has  $n + 1$  faces in its planar embedding: the exterior face  $R_0$  and the interior triangular faces  $R_1, R_2, \dots, R_n$ . In the standard embedding, each triangle  $R_j$  shares its base edge with  $R_0$  and meets its neighbors only at path vertices. No two interior faces therefore share a common edge. *Adjacent region sets.* Define the collection  $T = \{T_0, T_1, \dots, T_n\}$  by

$$\begin{aligned} T_0 &= \{R_0, R_1, R_2, \dots, R_n\}, \\ T_j &= \{R_0, R_j\}, \quad 1 \leq j \leq n. \end{aligned}$$

The set  $T_0$  lists every edge-neighbor of  $R_0$ , and each  $T_j$  records the two faces that share the base edge of triangle  $R_j$ .

*Color assignment.* Let  $\mathcal{K} = \{1, 2, \dots, n + 1\}$ . As shown in Figure 4, assign colors to regions as

$$R_0 \rightarrow 1, \quad R_f \rightarrow f + 1, \quad 1 \leq f \leq n.$$

Each interior face receives a distinct color from  $\{2, 3, \dots, n + 1\}$ .

*Delta sets.* The induced delta function yields

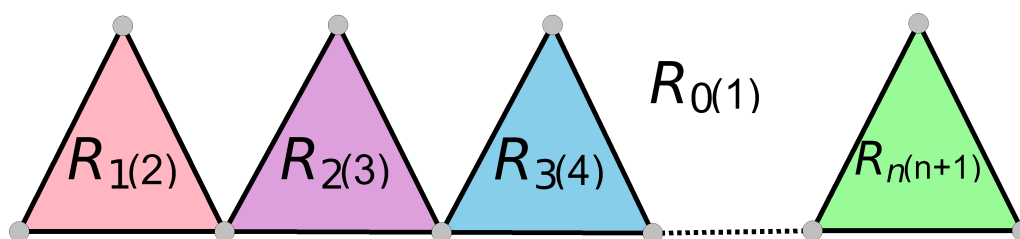
$$\begin{aligned} \delta(T_0) &= \{1, 2, 3, \dots, n + 1\}, \\ \delta(T_j) &= \{1, j + 1\}, \quad 1 \leq j \leq n. \end{aligned}$$

*Pairwise distinctness.* Because  $|\delta(T_0)| = n + 1$  and  $|\delta(T_j)| = 2$  for all  $j \geq 1$ , it is immediate that  $\delta(T_0) \neq \delta(T_j)$ . For any two indices  $1 \leq p < q \leq n$ , suppose  $\delta(T_p) = \delta(T_q)$ , so  $\{1, p + 1\} = \{1, q + 1\}$ . This forces  $p + 1 = q + 1$ , and hence,  $p = q$ , contradicting  $p < q$ . Therefore,

$$\delta(T_p) \neq \delta(T_q) \quad \text{for all } p \neq q.$$

*Lower bound.* Let  $c$  be any region  $\pi$  coloring of  $TS_n$ , and suppose first that  $n \geq 2$ . Every interior face shares its base edge with  $R_0$ , so (2.1) gives  $c(R_j) \neq c(R_0)$  for  $1 \leq j \leq n$ , and therefore,  $\delta(T_j) = \{c(R_0), c(R_j)\}$  has exactly two elements. For  $p \neq q$ , the sets  $T_p$  and  $T_q$  are different, so (2.2) gives  $\delta(T_p) \neq \delta(T_q)$ , which is possible only if  $c(R_p) \neq c(R_q)$ . The  $n$  interior colors are thus pairwise distinct and none of them equals  $c(R_0)$ , so  $c$  uses at least  $n + 1$  colors. If  $n = 1$ , then  $T_0 = T_1$ , no instance of (2.2) applies, and (2.1) alone forces the two faces to differ, again giving  $n + 1 = 2$  colors. This lower bound is the case  $d = 1$  of Theorem 4.2.

The construction uses exactly  $n + 1$  colors, establishing that the minimum number of colors required for region  $\pi$  coloring of  $TS_n$  is  $n + 1$ . Therefore,  $R\pi\chi(TS_n) = n + 1$ .  $\square$

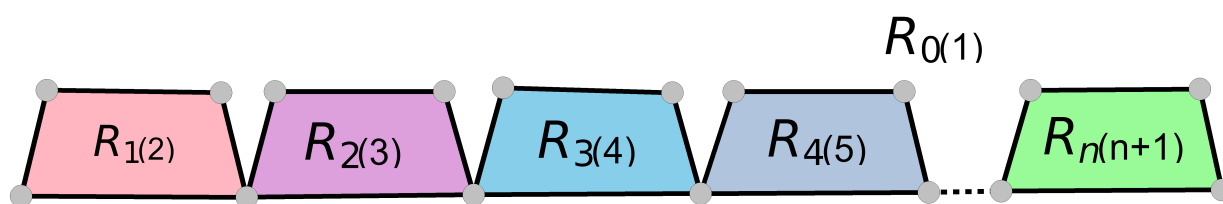


**Figure 4.** Triangular snake graph  $TS_n$  with region  $\pi$  coloring;  $R\pi\chi(TS_n) = n + 1$ .

The proof of Theorem 5.1 rests entirely on the star-type adjacency pattern: Every interior face of  $TS_n$  is edge-adjacent to  $R_0$  and to no other interior face. This pattern holds in every linear  $k$ -gonal snake graph for any  $k \geq 3$ , regardless of how large or small the polygon is. The following four corollaries are therefore immediate.

**Corollary 5.2.** *For every integer  $n \geq 1$ ,  $R\pi\chi(Ql_n) = n + 1$ , where  $Ql_n$  is the quadrilateral snake graph formed by  $n$  copies of  $C_4$ .*

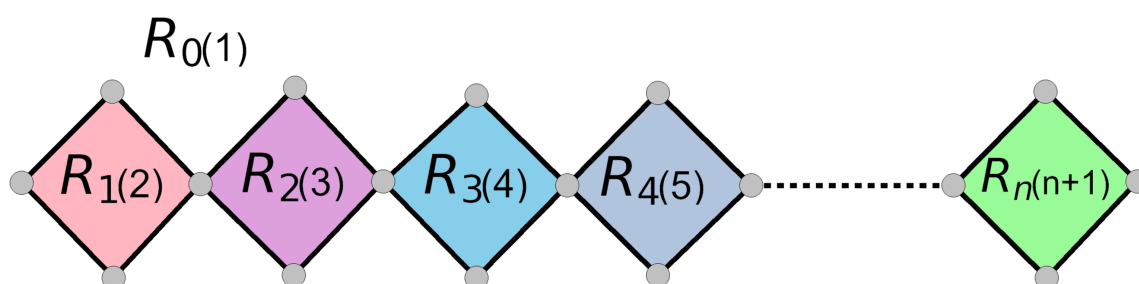
*Proof.* Figure 5 shows that each quadrilateral interior region of  $Ql_n$  shares its bottom edge with  $R_0$ , and two interior regions meet only at a corner vertex of the base path. Conditions (H1)–(H3) of Definition 4.1 therefore hold with the singleton groups  $U_j = \{R_j\}$ , so  $Ql_n$  is of hub type with parameters  $(n, 1)$ . Corollary 4.3 gives  $R\pi\chi(Ql_n) = n + 1$ .  $\square$



**Figure 5.** Quadrilateral snake graph  $Ql_n$  with region  $\pi$  coloring;  $R\pi\chi(Ql_n) = n + 1$ .

**Corollary 5.3.** *For every integer  $n \geq 1$ ,  $R\pi\chi(Sqs_n) = n + 1$ , where  $Sqs_n$  is the square snake graph formed by  $n$  axis-aligned copies of  $C_4$ .*

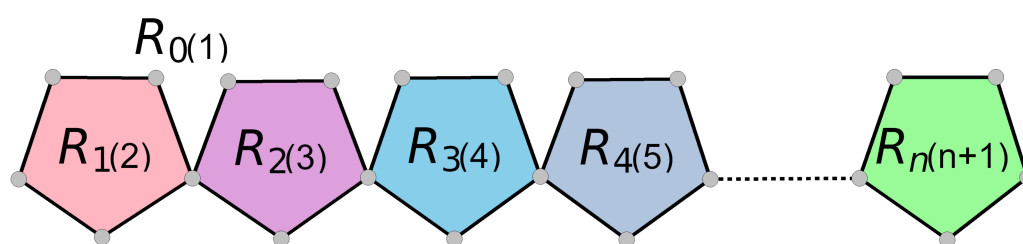
*Proof.* Figure 6 shows  $Sqs_n$ , in which each  $C_4$ -region is placed in an axis-aligned orientation. The change of orientation does not alter which regions share edges. Each interior region still shares one edge with  $R_0$  and no edge with another interior region, so  $Sqs_n$  is of hub type with parameters  $(n, 1)$ , and Corollary 4.3 gives  $R\pi\chi(Sqs_n) = n + 1$ .  $\square$



**Figure 6.** Square snake graph  $Sqs_n$  with region  $\pi$  coloring;  $R\pi\chi(Sqs_n) = n + 1$ .

**Corollary 5.4.** *For every integer  $n \geq 1$ ,  $R\pi\chi(Qt_n) = n + 1$ , where  $Qt_n$  is the quintet snake graph formed by  $n$  copies of  $C_5$ .*

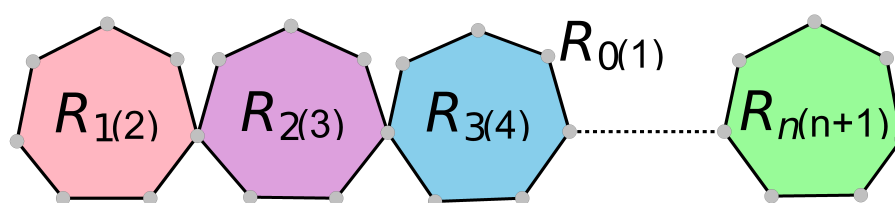
*Proof.* As shown in Figure 7, each pentagonal interior region of  $Qt_n$  contributes exactly one edge shared with  $R_0$ , and consecutive pentagons are joined at single vertices of the base path. The graph is of hub type with parameters  $(n, 1)$ , so Corollary 4.3 gives  $R\pi\chi(Qt_n) = n + 1$ .  $\square$



**Figure 7.** Quintet snake graph  $Q_t_n$  with region  $\pi$  coloring;  $R\pi\chi(Q_t_n) = n + 1$ .

**Corollary 5.5.** For every integer  $n \geq 1$ ,  $R\pi\chi(Hts_n) = n + 1$ , where  $Hts_n$  is the heptagonal snake graph formed by  $n$  copies of  $C_7$ .

*Proof.* Figure 8 depicts  $Hts_n$ . Each heptagonal interior region shares one edge with  $R_0$  and meets its neighbors only at vertices of the base path, so  $Hts_n$  is of hub type with parameters  $(n, 1)$ . Corollary 4.3 gives  $R\pi\chi(Hts_n) = n + 1$ .  $\square$



**Figure 8.** Heptagonal snake graph  $Hts_n$  with region  $\pi$  coloring;  $R\pi\chi(Hts_n) = n + 1$ .

Table 2 records the region  $\pi$  chromatic number for selected values of  $n$  across the five linear families.

**Table 2.** Region  $\pi$  chromatic numbers for linear snake graph families at selected values of  $n$ .

| $n$ | $TS_n$  | $Ql_n$  | $Sqs_n$ | $Qt_n$  | $Hts_n$ |
|-----|---------|---------|---------|---------|---------|
| 1   | 2       | 2       | 2       | 2       | 2       |
| 2   | 3       | 3       | 3       | 3       | 3       |
| 3   | 4       | 4       | 4       | 4       | 4       |
| 4   | 5       | 5       | 5       | 5       | 5       |
| 5   | 6       | 6       | 6       | 6       | 6       |
| $n$ | $n + 1$ | $n + 1$ | $n + 1$ | $n + 1$ | $n + 1$ |

## 6. Alternate snake graph families

The alternate triangular snake graph  $ATS_n$  is obtained from a path  $P_n$  on  $n$  vertices by attaching a triangle at every alternate edge of the path. Depending on whether the path length is even or odd, and on whether both, one, or neither endpoint is a triangle vertex, the number and adjacency structure of the bounded faces varies. Three distinct subcases arise.

**Theorem 6.1.** Let  $ATS_n$  be the alternate triangular snake graph on a path of  $n$  vertices.

$$R\pi\chi(ATS_n) = \begin{cases} \lfloor n/2 \rfloor + 1, & n \equiv 0 \pmod{2}, (CE), \\ n/2, & n \equiv 0 \pmod{2}, (OE), \\ \lceil n/2 \rceil, & n \equiv 1 \pmod{2}, (1E). \end{cases} \quad (6.1)$$

Here, CE = closed end, OE = open end, and 1E = one-end open.

*Proof.* Three configurations are distinguished according to the parity of  $n$  and whether one or both endpoints of the path are attached to a triangle.

The first configuration covers even  $n$  when both path endpoints are attachment vertices, so that triangles appear at the first and last positions (**Case 1 (a):** Closed-end  $ATS_n$ ,  $n \equiv 0 \pmod{2}$ ). Figure 9 shows that the closed-end graph contains exactly  $\lfloor n/2 \rfloor + 1$  faces: the exterior face  $R_0$  and interior triangular faces  $R_1, R_2, \dots, R_{\lfloor n/2 \rfloor}$ . Define the adjacent region collection  $S = \{S_0, S_1, \dots, S_{\lfloor n/2 \rfloor}\}$  by

$$\begin{aligned} S_0 &= \{R_0, R_1, R_2, \dots, R_{\lfloor n/2 \rfloor}\}, \\ S_f &= \{R_0, R_f\}, \quad 1 \leq f \leq \lfloor n/2 \rfloor. \end{aligned}$$

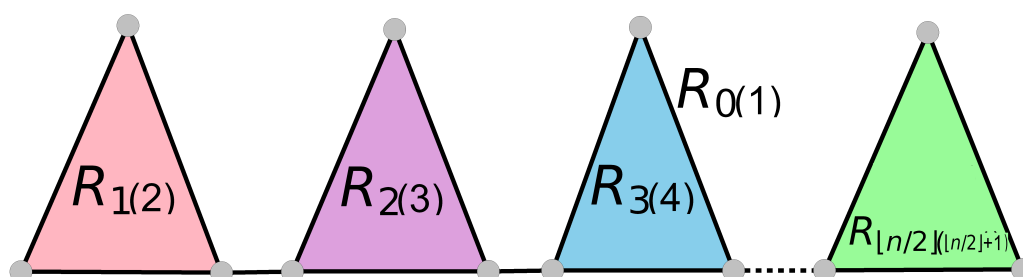
Every alternate triangle shares its base edge with  $R_0$  and touches its neighbors only at path vertices, so no two interior faces are edge-adjacent to each other. Assign colors as

$$R_0 \rightarrow 1, \quad R_f \rightarrow f + 1, \quad 1 \leq f \leq \lfloor n/2 \rfloor.$$

The delta function then gives

$$\begin{aligned} \delta(S_0) &= \{1, 2, \dots, \lfloor n/2 \rfloor + 1\}, \\ \delta(S_f) &= \{1, f + 1\}, \quad 1 \leq f \leq \lfloor n/2 \rfloor. \end{aligned}$$

For any two indices  $p \neq q$  with  $1 \leq p, q \leq \lfloor n/2 \rfloor$ , suppose  $\delta(S_p) = \delta(S_q)$ ; that is,  $\{1, p + 1\} = \{1, q + 1\}$ . It follows immediately that  $p = q$ , a contradiction. Because  $|\delta(S_0)| = \lfloor n/2 \rfloor + 1 > 2 = |\delta(S_f)|$ , the set  $\delta(S_0)$  is distinct from every  $\delta(S_f)$ . All delta sets are pairwise distinct, and exactly  $\lfloor n/2 \rfloor + 1$  colors are used. For the lower bound, let  $c$  be any region  $\pi$  coloring. Each interior face shares its base edge with  $R_0$ , so (2.1) gives  $c(R_f) \neq c(R_0)$ , and hence,  $\delta(S_f) = \{c(R_0), c(R_f)\}$ . For  $p \neq q$ , the sets  $S_p$  and  $S_q$  differ, so (2.2) forces  $c(R_p) \neq c(R_q)$ . The  $\lfloor n/2 \rfloor$  interior colors are therefore pairwise distinct, and all differ from  $c(R_0)$ , so at least  $\lfloor n/2 \rfloor + 1$  colors are used. This is the case  $d = 1$  of Theorem 4.2. Therefore,  $R\pi\chi(ATS_n) = \lfloor n/2 \rfloor + 1$  for closed-end  $ATS_n$  with even  $n$ .



**Figure 9.** Closed-end  $ATS_n$  ( $n \equiv 0 \pmod{2}$ );  $R\pi\chi = \lfloor n/2 \rfloor + 1$ .

The second configuration has even  $n$ , but both path endpoints are free vertices, so triangles appear only at strictly interior alternate edges (**Case 1 (b):** Open-end  $ATS_n$ ,  $n \equiv 0 \pmod{2}$ ). As Figure 10 illustrates, the open-end graph has exactly  $n/2$  faces in total, namely the exterior face  $R_0$  together with the  $n/2 - 1$  bounded triangular faces  $R_1, \dots, R_{n/2-1}$ . Defining

$$S_0 = \{R_0, R_1, \dots, R_{n/2-1}\},$$

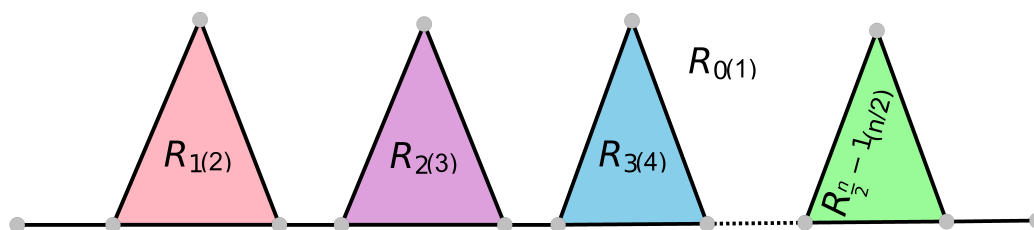
$$S_f = \{R_0, R_f\}, \quad 1 \leq f \leq n/2 - 1,$$

and assigning  $R_0 \rightarrow 1, R_f \rightarrow f + 1$  for  $1 \leq f \leq n/2 - 1$  yields

$$\delta(S_0) = \{1, 2, \dots, n/2\},$$

$$\delta(S_f) = \{1, f + 1\}, \quad 1 \leq f \leq n/2 - 1.$$

The same distinctness argument as in Case 1 (a) applies, using  $n/2$  colors in total. The lower bound follows exactly as in Case 1 (a): Properness makes every  $\delta(S_f)$  a two-element set containing  $c(R_0)$ , and (2.2) forces the  $n/2 - 1$  interior colors to be pairwise distinct, giving at least  $n/2$  colors. Hence,  $R\pi\chi(AT S_n) = n/2$  for open-end  $ATS_n$  with even  $n$ .



**Figure 10.** Open-end  $ATS_n$  ( $n \equiv 0 \pmod{2}$ , both endpoints free);  $R\pi\chi = n/2$ .

The third configuration covers odd  $n$  when exactly one path endpoint is an attachment vertex (**Case 2:** One-end-open  $ATS_n$ ,  $n \equiv 1 \pmod{2}$ ). referring to Figure 11, this graph contains  $\lceil n/2 \rceil$  bounded triangular faces  $R_1, \dots, R_{\lceil n/2 \rceil - 1}$  together with the exterior face  $R_0$ , giving  $\lceil n/2 \rceil$  faces in total. The adjacent region collection is

$$S_0 = \{R_0, R_1, \dots, R_{\lceil n/2 \rceil - 1}\},$$

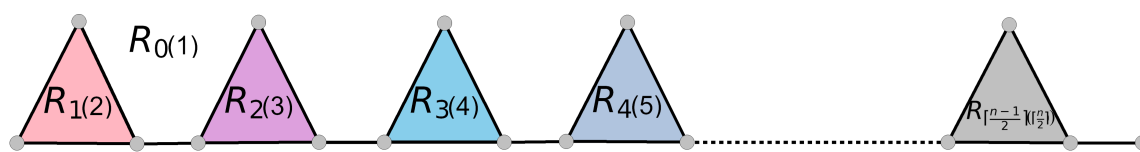
$$S_f = \{R_0, R_f\}, \quad 1 \leq f \leq \lceil n/2 \rceil - 1,$$

and the color assignment  $R_0 \rightarrow 1, R_f \rightarrow f + 1$  for  $1 \leq f \leq \lceil n/2 \rceil - 1$  produces

$$\delta(S_0) = \{1, 2, \dots, \lceil n/2 \rceil\},$$

$$\delta(S_f) = \{1, f + 1\}, \quad 1 \leq f \leq \lceil n/2 \rceil - 1.$$

Distinctness holds by the same contrapositive argument, and the lower bound is  $\lceil n/2 \rceil$  because  $|S_0| = \lceil n/2 \rceil$ . Therefore,  $R\pi\chi(AT S_n) = \lceil n/2 \rceil$  for one-end-open  $ATS_n$  with odd  $n$ .



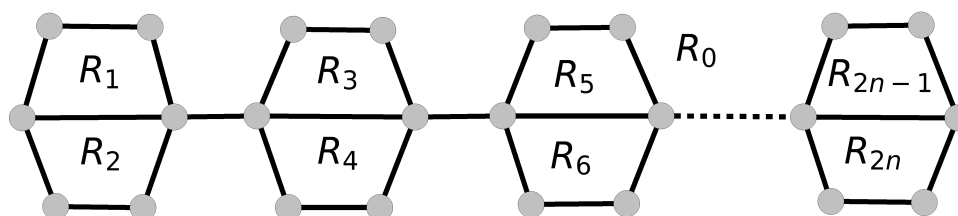
**Figure 11.** One-end-open  $ATS_n$  ( $n \equiv 1 \pmod{2}$ );  $R\pi\chi = \lceil n/2 \rceil$ .

In all three configurations, the minimum number of colors equals the value stated in (6.1), completing the proof.  $\square$

The same adjacency structure governs the quadrilateral variant, as the edge-adjacency pattern between interior faces and the exterior face is determined by the base-path structure, not by the polygon size.

**Corollary 6.2.** *For the alternate quadrilateral snake graph  $AQS_n$ , the region  $\pi$  chromatic number satisfies the same three-case formula as in (6.1), with  $n$  denoting the number of path vertices.*

*Proof.* Replacing each triangle of  $ATS_n$  by a quadrilateral leaves the number of interior regions unchanged and leaves every edge-adjacency unchanged. In each of the three configurations,  $AQS_n$  is therefore of hub type with parameters  $(q, 1)$ , where  $q$  is the number of interior regions recorded in the proof of Theorem 6.1. Corollary 4.3 gives  $R\pi\chi(AQS_n) = q + 1$ , which is the value stated in (6.1) in each case. Figure 12 shows a region  $\pi$  coloring of  $AQS_n$ .  $\square$



**Figure 12.** Region  $\pi$  coloring of the alternate quadrilateral snake  $AQS_n$ .

Table 3 summarizes the region  $\pi$  chromatic number for selected values of  $n$ .

**Table 3.** Region  $\pi$  chromatic numbers for alternate snake graphs. C = closed end, O = open end, P = one-end-open.

| $n$ | parity | type | $R\pi\chi(ATS_n)$ | $R\pi\chi(AQS_n)$ |
|-----|--------|------|-------------------|-------------------|
| 2   | even   | C    | 2                 | 2                 |
| 2   | even   | O    | 1                 | 1                 |
| 3   | odd    | P    | 2                 | 2                 |
| 4   | even   | C    | 3                 | 3                 |
| 4   | even   | O    | 2                 | 2                 |
| 5   | odd    | P    | 3                 | 3                 |
| 6   | even   | C    | 4                 | 4                 |
| 6   | even   | O    | 3                 | 3                 |

## 7. Cyclic snake graph families

Cyclic snake graphs are formed by closing the base path of a linear snake into a cycle. The resulting planar embedding introduces a large central interior face  $R_{n+1}$  in addition to the  $n$  petal-like interior faces  $R_1, \dots, R_n$  and the exterior face  $R_0$ . Each petal face is adjacent to both  $R_0$  and  $R_{n+1}$ , yielding a three-element adjacent region set for interior petals and two  $(n+1)$ -element adjacent region sets for  $R_0$  and  $R_{n+1}$ .

### 7.1. Cycle quadrilateral snake graph $Cqs_n$

**Theorem 7.1.** *For every integer  $n \geq 1$ , the region  $\pi$  chromatic number of the cycle quadrilateral snake graph  $Cqs_n$  is  $R\pi\chi(Cqs_n) = n + 2$ .*

*Proof.* The graph  $Cqs_n$  is a planar graph with  $n+2$  faces: the exterior face  $R_0$ , the  $n$  interior quadrilateral petal faces  $R_1, R_2, \dots, R_n$ ; and the central interior face  $R_{n+1}$ . Each petal  $R_m$  shares one edge with  $R_0$  along the outer boundary and one edge with  $R_{n+1}$  along the inner ring; no two petals share an edge with each other.

*Adjacent region sets.* Define the collection  $Q = \{Q_0, Q_1, \dots, Q_n, Q_{n+1}\}$  by

$$\begin{aligned} Q_0 &= \{R_0, R_1, R_2, \dots, R_n\}, \\ Q_{n+1} &= \{R_1, R_2, \dots, R_n, R_{n+1}\}, \\ Q_m &= \{R_0, R_m, R_{n+1}\}, \quad 1 \leq m \leq n. \end{aligned}$$

*Color assignment.* Let  $\mathcal{K} = \{1, 2, \dots, n+2\}$ . Following Figure 13, assign

$$R_0 \rightarrow 1, \quad R_l \rightarrow l+1 \quad (1 \leq l \leq n), \quad R_{n+1} \rightarrow n+2.$$

*Delta sets.* The induced delta function yields

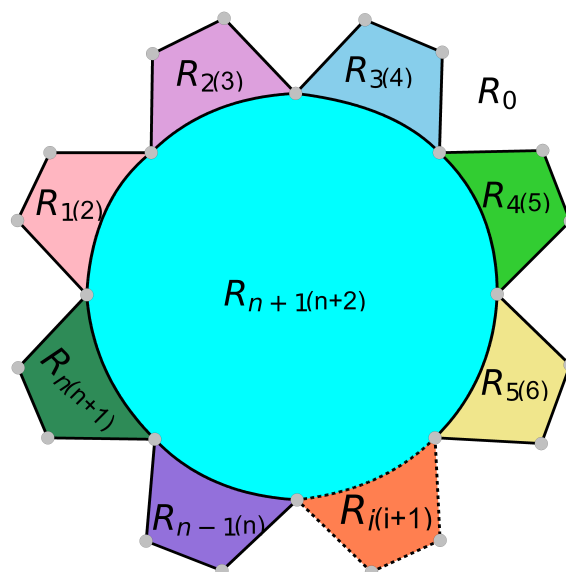
$$\begin{aligned} \delta(Q_0) &= \{1, 2, 3, \dots, n+1\}, \\ \delta(Q_{n+1}) &= \{2, 3, \dots, n+1, n+2\}, \\ \delta(Q_m) &= \{1, m+1, n+2\}, \quad 1 \leq m \leq n. \end{aligned}$$

*Pairwise distinctness.* Four separation arguments cover all pairs. Because  $1 \in \delta(Q_0)$ , but  $1 \notin \delta(Q_{n+1})$ , and conversely,  $n+2 \in \delta(Q_{n+1})$ , but  $n+2 \notin \delta(Q_0)$ , these two sets are distinct. For any  $1 \leq m \leq n$ , the set  $\delta(Q_m)$  has exactly three elements. Because  $|\delta(Q_0)| = |\delta(Q_{n+1})| = n+1 \geq 2$  and equals 3 only when  $n=2$ , one checks directly that  $\delta(Q_0) \neq \delta(Q_m)$  (color  $n+2 \in \delta(Q_m)$  but  $n+2 \notin \delta(Q_0)$ ), and  $\delta(Q_{n+1}) \neq \delta(Q_m)$  (color  $1 \in \delta(Q_m)$ , but  $1 \notin \delta(Q_{n+1})$ ). Finally, for any two petal indices  $1 \leq j < k \leq n$ , suppose  $\delta(Q_j) = \delta(Q_k)$ , that is,  $\{1, j+1, n+2\} = \{1, k+1, n+2\}$ . This forces  $j+1 = k+1$ , giving  $j = k$ , a contradiction. Hence, all  $n+2$  delta sets are pairwise distinct.

*Lower bound.* Let  $c$  be any region  $\pi$  coloring of  $Cqs_n$ . Every petal shares one edge with  $R_0$  and one edge with  $R_{n+1}$ , so (2.1) gives  $c(R_m) \notin \{c(R_0), c(R_{n+1})\}$  for  $1 \leq m \leq n$ , and therefore,  $\delta(Q_m) =$

$\{c(R_0), c(R_m), c(R_{n+1})\}$ . For  $p \neq q$ , the sets  $Q_p$  and  $Q_q$  differ, so (2.2) forces  $c(R_p) \neq c(R_q)$ , and the  $n$  petal colors are pairwise distinct. The faces  $R_0$  and  $R_{n+1}$  share no edge, so  $Q_0 = \{R_0, R_1, \dots, R_n\}$  and  $Q_{n+1} = \{R_1, \dots, R_n, R_{n+1}\}$  are different sets of faces. Were  $c(R_0) = c(R_{n+1})$ , both delta values would equal  $\{c(R_0), c(R_1), \dots, c(R_n)\}$ , contradicting (2.2). Hence,  $c(R_0) \neq c(R_{n+1})$ , and together with the  $n$  distinct petal colors, this gives at least  $n + 2$  colors. This lower bound is Theorem 4.7 with  $p = n$ .

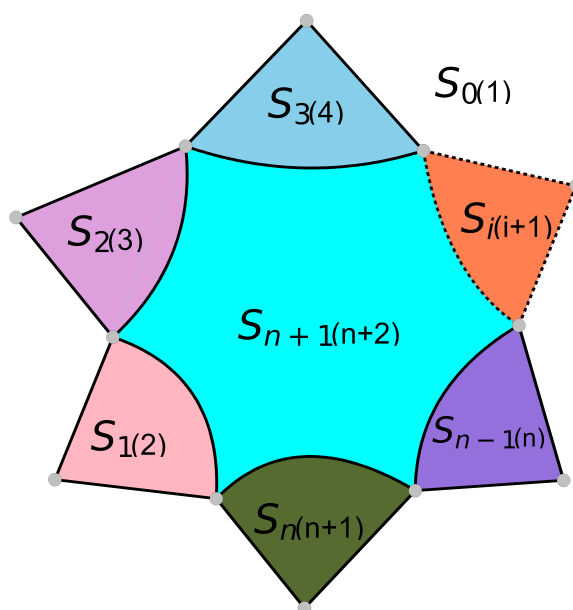
Therefore, the minimum number of colors required for region  $\pi$  coloring of  $Cqs_n$  is  $n + 2$ , establishing  $R\pi\chi(Cqs_n) = n + 2$ .  $\square$



**Figure 13.** Cycle quadrilateral snake graph  $Cqs_n$  with region  $\pi$  coloring;  $R\pi\chi(Cqs_n) = n + 2$ .

**Corollary 7.2.** *The region  $\pi$  chromatic number of the line of sunlet graph  $Ls_n$  on  $n$  triangular petals is  $R\pi\chi(Ls_n) = n + 2$ .*

*Proof.* In Figure 14, the line of sunlet graph  $Ls_n$  has the exterior region  $S_0$ ; the  $n$  triangular petal regions  $S_1, \dots, S_n$  on the outer rim; and the inner region  $S_{n+1}$ . Each petal shares one edge with  $S_0$  and one edge with  $S_{n+1}$ , and no two petals share an edge. Taking  $H_1 = S_0$  and  $H_2 = S_{n+1}$  shows that  $Ls_n$  is annular with  $n$  petals in the sense of Definition 4.6, so Theorem 4.7 gives  $R\pi\chi(Ls_n) = n + 2$ .  $\square$



**Figure 14.** Line of sunlet graph  $Ls_n$  with region  $\pi$  coloring;  $R\pi\chi(Ls_n) = n + 2$ .

## 7.2. Alternate triangular cycle snake graph $ATCS_n$

**Theorem 7.3.** Let  $ATCS_n$  be the alternate triangular cycle snake graph on a circular path of  $n$  vertices. Its region  $\pi$  chromatic number is

$$R\pi\chi(ATCS_n) = \begin{cases} \frac{n}{2} + 2, & n \equiv 0 \pmod{2}, \\ \left\lfloor \frac{n}{2} \right\rfloor + 2, & n \equiv 1 \pmod{2}. \end{cases} \quad (7.1)$$

*Proof.* Two parity configurations arise depending on whether  $n$  is even or odd.

When  $n \equiv 0 \pmod{2}$ , the graph  $ATCS_n$  carries triangular petals at every alternate vertex of the circular path. By observing Figure 15, this produces exactly  $n/2 + 2$  faces: the exterior face  $R_0$ ; the petal triangles  $R_1, R_2, \dots, R_{n/2}$ ; and the central circular face  $R_{n/2+1}$ . Both  $R_0$  and  $R_{n/2+1}$  are edge-adjacent to all petals; no two petals share a common edge. Define the adjacent region collection  $C = \{C_0, C_1, \dots, C_{n/2}, C_{n/2+1}\}$  by

$$\begin{aligned} C_0 = C_{n/2+1} &= \{R_0, R_1, \dots, R_{n/2}, R_{n/2+1}\}, \\ C_j &= \{R_0, R_j, R_{n/2+1}\}, \quad 1 \leq j \leq n/2. \end{aligned}$$

Note that  $C_0 = C_{n/2+1}$  as sets; their delta values are therefore automatically equal, and the region  $\pi$  coloring condition imposes no constraint between them. Assign colors as

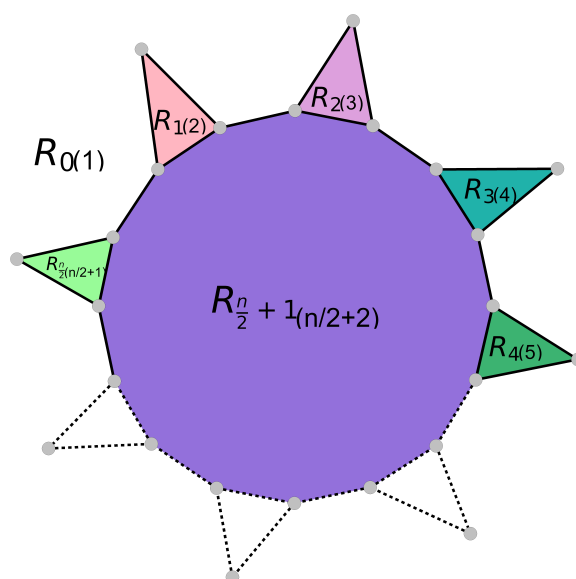
$$\begin{aligned} R_0 &\rightarrow 1, \\ R_j &\rightarrow j + 1, \quad 1 \leq j \leq n/2, \\ R_{n/2+1} &\rightarrow n/2 + 2. \end{aligned}$$

The delta function then yields

$$\delta(C_0) = \delta(C_{n/2+1}) = \{1, 2, 3, \dots, n/2 + 2\},$$

$$\delta(C_j) = \{1, j+1, n/2+2\}, \quad 1 \leq j \leq n/2.$$

Because  $|\delta(C_0)| = n/2+2$  and  $|\delta(C_j)| = 3$ , the set  $\delta(C_0)$  differs from every petal set in cardinality (for  $n \geq 2$ ; the case  $n/2 = 1$  is checked directly). For any two petal indices  $1 \leq p < q \leq n/2$ , if  $\delta(C_p) = \delta(C_q)$ , then  $\{1, p+1, n/2+2\} = \{1, q+1, n/2+2\}$ , forcing  $p = q$ , a contradiction. All delta sets are pairwise distinct. For the lower bound, let  $c$  be any region  $\pi$  coloring. Every petal shares an edge with  $R_0$  and an edge with  $R_{n/2+1}$ , so (2.1) gives  $c(R_j) \notin \{c(R_0), c(R_{n/2+1})\}$  and  $\delta(C_j) = \{c(R_0), c(R_j), c(R_{n/2+1})\}$ . For  $p \neq q$ , the sets  $C_p$  and  $C_q$  differ, so (2.2) forces the  $n/2$  petal colors to be pairwise distinct. The faces  $R_0$  and  $R_{n/2+1}$  share the base edges that carry no triangle, so (2.1) gives  $c(R_0) \neq c(R_{n/2+1})$ . At least  $n/2 + 2$  colors are therefore used, which is Theorem 4.7 with  $p = n/2$ . Therefore,  $R\pi\chi(ATCS_n) = n/2 + 2$  when  $n \equiv 0 \pmod{2}$ .



**Figure 15.** Alternate triangular cycle snake  $ATCS_n$  ( $n \equiv 0 \pmod{2}$ );  $R\pi\chi = n/2 + 2$ .

When  $n \equiv 1 \pmod{2}$ , the asymmetry of the circular path means one fewer triangle is attached compared with the even case. As shown in Figure 16, the graph has  $\lfloor n/2 \rfloor + 2$  faces:  $R_0$ ; the petal triangles  $R_1, \dots, R_{\lfloor n/2 \rfloor}$ ; and the central face  $R_{\lfloor n/2 \rfloor + 1}$ . The adjacent region collection is

$$C_0 = C_{\lfloor n/2 \rfloor + 1} = \{R_0, R_1, \dots, R_{\lfloor n/2 \rfloor}, R_{\lfloor n/2 \rfloor + 1}\},$$

$$C_j = \{R_0, R_j, R_{\lfloor n/2 \rfloor + 1}\}, \quad 1 \leq j \leq \lfloor n/2 \rfloor.$$

Again,  $C_0 = C_{\lfloor n/2 \rfloor + 1}$  as sets, so equality of their delta values is not a conflict. Assigning

$$R_0 \rightarrow 1, \quad R_j \rightarrow j+1 \quad (1 \leq j \leq \lfloor n/2 \rfloor),$$

$$R_{\lfloor n/2 \rfloor + 1} \rightarrow \lfloor n/2 \rfloor + 2,$$

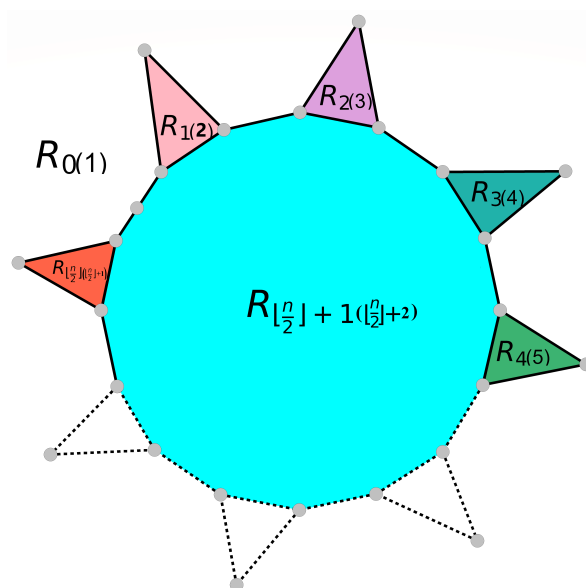
one obtains

$$\delta(C_0) = \delta(C_{\lfloor n/2 \rfloor + 1}) = \{1, 2, \dots, \lfloor n/2 \rfloor + 2\},$$

$$\delta(C_j) = \{1, j+1, \lfloor n/2 \rfloor + 2\},$$

$$1 \leq j \leq \lfloor n/2 \rfloor.$$

The same cardinality argument and contrapositive distinctness proof apply. The lower bound follows exactly as in the even case: Properness forces every petal color to avoid both  $c(R_0)$  and  $c(R_{\lfloor n/2 \rfloor + 1})$ , (2.2) forces the  $\lfloor n/2 \rfloor$  petal colors to be pairwise distinct, and  $R_0$  and  $R_{\lfloor n/2 \rfloor + 1}$  share an edge and so carry different colors. This gives at least  $\lfloor n/2 \rfloor + 2$  colors, which is Theorem 4.7 with  $p = \lfloor n/2 \rfloor$ . Therefore,  $R\pi\chi(ATCS_n) = \lfloor n/2 \rfloor + 2$  when  $n \equiv 1 \pmod{2}$ .



**Figure 16.** Alternate triangular cycle snake  $ATCS_n$  ( $n \equiv 1 \pmod{2}$ );  $R\pi\chi = \lfloor n/2 \rfloor + 2$ .

In both parity cases, the minimum number of colors equals the value stated in (7.1), completing the proof.  $\square$

**Corollary 7.4.** *For the alternate quadrilateral cycle snake graph  $AQCS_n$ , the region  $\pi$  chromatic number satisfies the same two-case formula as in (7.1), with  $n$  denoting the number of circular path vertices.*

*Proof.* The graph  $AQCS_n$  is formed by attaching quadrilateral regions at every alternate vertex of a circular path, so it is the quadrilateral analog of  $ATCS_n$ . Replacing each triangular petal by a quadrilateral petal changes neither the number of petals nor any edge-adjacency. In both parity cases,  $AQCS_n$  is annular with  $\lfloor n/2 \rfloor$  petals, and Theorem 4.7 returns the same two-case formula as in (7.1).  $\square$

Table 4 gives  $R\pi\chi$  values for the cyclic families at selected  $n$ .

**Table 4.** Region  $\pi$  chromatic numbers for cyclic snake graph families.

| $n$        | $Cqs_n$ | $ATCS_n$                  | $AQCS_n$                  |
|------------|---------|---------------------------|---------------------------|
| 1          | 3       | 3                         | 3                         |
| 2          | 4       | 3                         | 3                         |
| 3          | 5       | 3                         | 3                         |
| 4          | 6       | 4                         | 4                         |
| 5          | 7       | 4                         | 4                         |
| 6          | 8       | 5                         | 5                         |
| $n$ (even) | $n + 2$ | $n/2 + 2$                 | $n/2 + 2$                 |
| $n$ (odd)  | $n + 2$ | $\lfloor n/2 \rfloor + 2$ | $\lfloor n/2 \rfloor + 2$ |

## 8. Double snake graph families

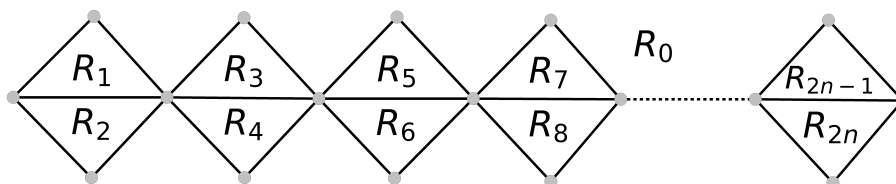
A double snake graph is formed by replacing each edge of the base path by a pair of triangles (or quadrilaterals) sharing that edge as a common diagonal, creating kite-shaped units. The number of kite units is denoted by  $n$ , and the combinatorial relationship between  $n$  and the minimum number of colors is governed by the sequence of triangular numbers  $T_m = m(m + 1)/2$ .

### 8.1. Double triangular snake graph $DT_n$

**Theorem 8.1.** For every integer  $n \geq 1$ , the region  $\pi$  chromatic number of the double triangular snake graph  $DT_n$  is

$$R\pi\chi(DT_n) = 2 + \left\lceil \frac{\sqrt{8n+1} - 1}{2} \right\rceil. \quad (8.1)$$

*Proof.* The graph  $DT_n$  has  $2n+1$  faces: the exterior face  $R_0$  and the interior faces  $R_1, R_2, \dots, R_{2n}$ . In the planar embedding of Figure 17, the  $i$ th kite unit consists of faces  $R_{2i-1}$  and  $R_{2i}$  ( $i = 1, \dots, n$ ), which share the diagonal edge of their kite; both faces are edge-adjacent to  $R_0$  through outer boundary edges. Faces belonging to different kite units share only a path vertex, so no two faces from distinct kites are edge-adjacent.



**Figure 17.** Double triangular snake graph  $DT_n$ ;  $R\pi\chi(DT_n) = 2 + \lceil (\sqrt{8n+1} - 1)/2 \rceil$ .

*Adjacent region collection.* Set  $\mathcal{X} = \{\mathcal{K}, \mathcal{K}_1, \dots, \mathcal{K}_n\}$ , where

$$\begin{aligned} \mathcal{K} &= \{R_0, R_1, \dots, R_{2n}\}, \\ \mathcal{K}_i &= \{R_0, R_{2i-1}, R_{2i}\}, \quad i = 1, \dots, n. \end{aligned}$$

The faces  $R_{2i-1}$  and  $R_{2i}$  share the same adjacent region set  $\mathcal{K}_i$ , so their delta values coincide automatically and impose no constraint on the coloring.

*Coloring strategy.* Assign  $R_0 \rightarrow 1$ . For each kite unit  $i$ , select an unordered pair  $\{a_i, b_i\} \subset \{2, 3, \dots, k\}$  with  $a_i \neq b_i$ , and set  $R_{2i-1} \rightarrow a_i$ ,  $R_{2i} \rightarrow b_i$ . The resulting delta sets are

$$\begin{aligned}\delta(\mathcal{K}) &= \{1, 2, \dots, k\}, \\ \delta(\mathcal{K}_i) &= \{1, a_i, b_i\}, \quad i = 1, \dots, n.\end{aligned}$$

The set  $\delta(\mathcal{K})$  has cardinality  $k$ , and each  $\delta(\mathcal{K}_i)$  has cardinality 3; for  $k > 3$  (i.e.  $n \geq 2$ ), they differ by cardinality alone, and for  $n = 1$ , a direct check confirms  $\delta(\mathcal{K}) = \{1, a_1, b_1\} = \delta(\mathcal{K}_1)$  only when  $k = 3$ , in which case, the single pair  $\{a_1, b_1\}$  satisfies the condition trivially. For distinct kite indices  $i \neq j$ , the condition  $\delta(\mathcal{K}_i) \neq \delta(\mathcal{K}_j)$  reduces to  $\{a_i, b_i\} \neq \{a_j, b_j\}$  as unordered pairs. Hence, all  $n$  pairs must be distinct 2-element subsets of  $\{2, \dots, k\}$ .

*Minimum colors.* The number of distinct 2-element subsets of a  $(k-1)$ -element set is  $(k-1)$ -element set is  $\binom{k-1}{2} = (k-1)(k-2)/2$ . A valid assignment of  $n$  distinct pairs exists if and only if  $\binom{k-1}{2} \geq n$ . It is claimed that the minimum integer  $k$  satisfying this inequality is  $k = 2 + \lceil \gamma(n) \rceil$ , where

$$\gamma(n) = \frac{\sqrt{8n+1} - 1}{2}.$$

*Proof of the formula by strong induction.* Let  $f(n) = 2 + \lceil \gamma(n) \rceil$ .

**Base case  $n = 1$ :**  $\gamma(1) = (3-1)/2 = 1$ , so  $f(1) = 3$ . With  $k = 3$ , there is  $\binom{2}{2} = 1$  distinct pair available for the single kite. With  $k = 2$ ,  $\binom{1}{2} = 0 < 1$ , so two colors do not suffice. Therefore,  $R\pi\chi(DT_1) = 3$ , confirming the formula.

**Inductive step:** Assume the formula holds for  $n = m$ . The formula is to be established for  $n = m + 1$ . Two subcases arise according to whether  $m + 1$  is a triangular number.

*Subcase (i):*  $m + 1 = T_p = p(p+1)/2$  for some  $p \geq 1$ . Substituting into  $\gamma$ :

$$\begin{aligned}\gamma(T_p) &= \frac{\sqrt{8 \cdot \frac{p(p+1)}{2}} + 1 - 1}{2} = \frac{\sqrt{4p^2 + 4p + 1} - 1}{2} \\ &= \frac{(2p+1) - 1}{2} = p.\end{aligned}$$

Hence,  $\lceil \gamma(T_p) \rceil = p$ , and  $f(T_p) = 2 + p$ . With  $k = 2 + p$ ,  $\binom{k-1}{2} = \binom{p+1}{2} = T_p = m + 1$  pairs are available—exactly enough. With  $k = 2 + p - 1 = p + 1$ ,  $\binom{p}{2} = T_{p-1} < T_p = m + 1$ , so fewer than  $m + 1$  pairs exist, and the coloring fails. The formula holds for  $n = m + 1 = T_p$ .

*Subcase (ii):*  $m + 1$  is not a triangular number, that is,  $T_p < m + 1 < T_{p+1}$  for some  $p \geq 1$ . The bounds  $T_p < m + 1 < T_{p+1}$  imply  $4p^2 + 4p + 1 < 8(m+1) + 1 < 4p^2 + 12p + 9$ , so  $(2p+1)^2 < 8(m+1) + 1 < (2p+3)^2$ . Taking positive square roots:  $2p + 1 < \sqrt{8(m+1) + 1} < 2p + 3$ , giving

$$p < \gamma(m+1) < p + 1,$$

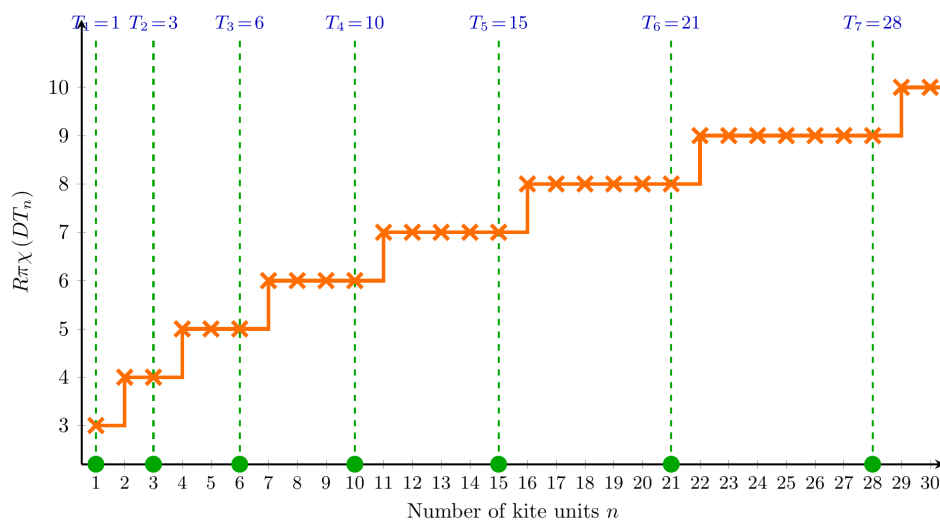
and therefore,  $\lceil \gamma(m+1) \rceil = p + 1$ , so  $f(m+1) = 2 + (p+1) = p + 3$ . With  $k = p + 3$ ,  $\binom{p+2}{2} = T_{p+1} > m + 1$  pairs are available—sufficient. With  $k = p + 2$ ,  $\binom{p+1}{2} = T_p < m + 1$ , so the coloring fails. The formula

holds for all nontriangular  $m + 1$ .

The counting premise used in both subcases is justified as follows. Every face of a kite unit shares an edge with  $R_0$ , so (2.1) keeps the color  $c(R_0)$  off all interior faces, and the two faces of a unit share the diagonal edge and so carry two distinct colors. Each unit therefore determines a 2-element subset of the remaining  $k - 1$  colors, and (2.2) makes the map from units to these subsets injective. A coloring with  $k$  colors exists precisely when  $\binom{k-1}{2} \geq n$ .

By strong induction,  $R\pi\chi(DT_n) = 2 + \lceil \gamma(n) \rceil$  for all  $n \geq 1$ .  $\square$

The step graph in Figure 18 illustrates the staircase growth of  $R\pi\chi(DT_n)$ : The chromatic number equals  $m + 2$  on the interval  $T_{m-1} < n \leq T_m$  and increases by exactly one as  $n$  passes each triangular number. This staircase pattern is a direct consequence of the binomial structure of the lower bound.



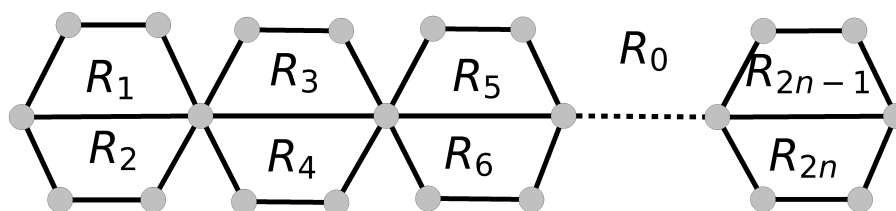
**Figure 18.** Step graph of  $R\pi\chi(DT_n)$  against the number  $n$  of kite units. Dashed lines mark the triangular numbers  $T_m$ .

**Corollary 8.2.** *The region  $\pi$  chromatic number of the double quadrilateral snake graph  $Dqls_n$  satisfies*

$$R\pi\chi(Dqls_n) = 2 + \left\lceil \frac{\sqrt{8n+1} - 1}{2} \right\rceil, \quad n \geq 1,$$

where  $n$  counts the hexagonal kite units of  $Dqls_n$ .

*Proof.* In Figure 19, each kite unit of  $Dqls_n$  consists of two quadrilateral regions sharing a diagonal edge, rather than the two triangular regions of  $DT_n$ . Both regions of a unit share an edge with  $R_0$ , the two regions of a unit share the diagonal, and regions of different units meet only at a base vertex. Conditions (H1)–(H3) hold with  $U_i = \{R_{2i-1}, R_{2i}\}$ , so  $Dqls_n$  is of hub type with parameters  $(n, 2)$ , and Corollary 4.4 gives the stated value.  $\square$



**Figure 19.** Double quadrilateral snake graph  $Dqls_n$ ;  $R\pi\chi(Dqls_n) = 2 + \lceil (\sqrt{8n+1} - 1)/2 \rceil$ .

## 8.2. Alternate double triangular snake graph $ADT_n$

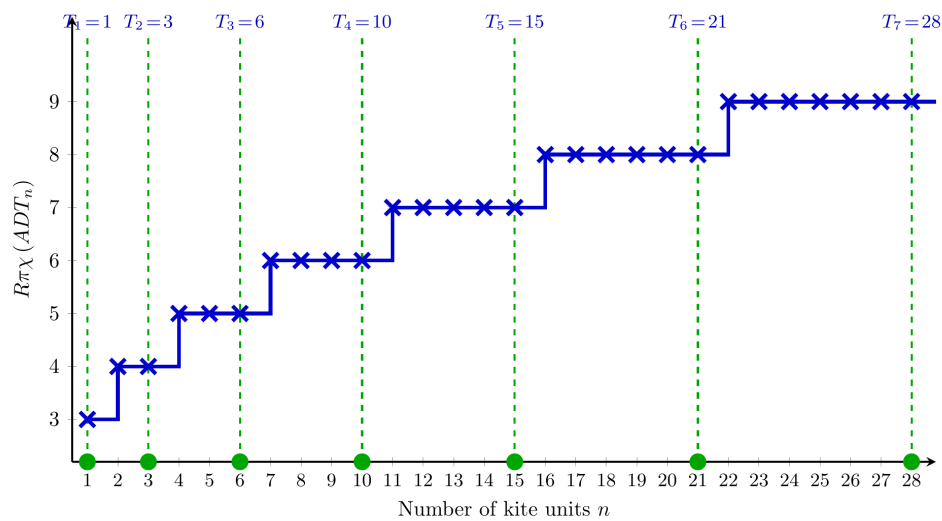
**Theorem 8.3.** For every integer  $n \geq 1$ , the region  $\pi$  chromatic number of the alternate double triangular snake graph  $ADT_n$  is

$$R\pi\chi(ADT_n) = R\pi\chi(DT_n) = 2 + \left\lceil \frac{\sqrt{8n+1} - 1}{2} \right\rceil. \quad (8.2)$$

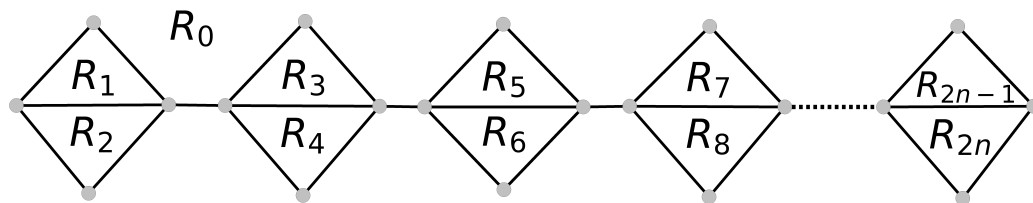
The value coincides with that of  $DT_n$  by Corollary 4.4, so it is recorded here in the single form used for  $DT_n$ .

*Proof.* Figure 21 shows that the graph  $ADT_n$  has  $2n + 1$  regions and the same kite structure as  $DT_n$ . The alternating construction changes where the kites sit along the base path, but not the number of kites and not any edge-adjacency. Both regions of the  $i$ th kite share an edge with the exterior region  $R_0$ , the two regions of a kite share the diagonal edge, and regions of different kites meet only at a base vertex. Conditions (H1)–(H3) of Definition 4.1 hold with  $U_i = \{R_{2i-1}, R_{2i}\}$ , so  $ADT_n$  is of hub type with parameters  $(n, 2)$ . Corollary 4.4 gives the value in (8.2), which is that of  $DT_n$ . Figure 20 shows the resulting staircase, whose value is  $m + 2$  on the range  $T_{m-1} < n \leq T_m$  and which rises by one as  $n$  passes a triangular number.  $\square$

Figure 20 shows the step structure of  $R\pi\chi(ADT_n)$  for  $1 \leq n \leq 28$ , in which the dashed lines mark the triangular numbers  $T_1, \dots, T_7$ . The value rises immediately after the first six of them, at  $n = 2, 4, 7, 11, 16$ , and  $22$ . The rise following  $T_7 = 28$  falls outside the plotted range.

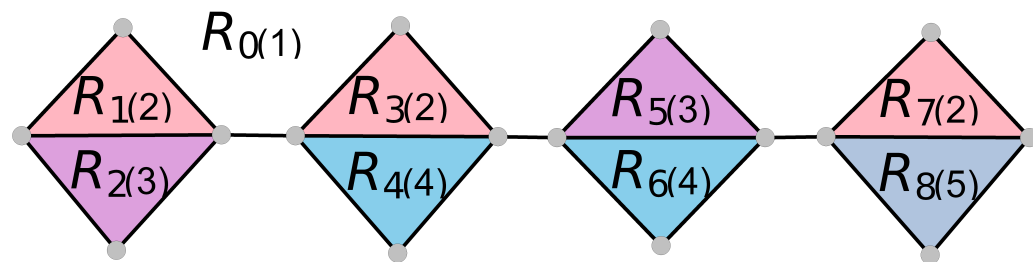


**Figure 20.** Step graph of  $R\pi\chi(ADT_n)$  against the number  $n$  of kite units. Dashed lines mark the triangular numbers  $T_m$ .



**Figure 21.** Alternate double triangular snake graph  $ADT_n$ .

The explicit coloring for  $n = 4$  (four kite units, eight interior faces) is shown in Figure 22. For this case,  $R\pi\chi(ADT_4) = 2 + \lceil (\sqrt{33} - 1)/2 \rceil = 5$ . The four unordered pairs of colors assigned to the kite units are  $\{2, 3\}$ ,  $\{2, 4\}$ ,  $\{3, 4\}$ , and  $\{2, 5\}$ , four distinct 2-element subsets of  $\{2, 3, 4, 5\}$ , each producing a distinct delta set  $\delta(\mathcal{K}_i) = \{1, a_i, b_i\}$ .



**Figure 22.** Region  $\pi$  coloring of  $ADT_4$ , giving  $R\pi\chi(ADT_4) = 5$ .

**Remark 8.4.** Formulas (8.1) and (8.2) are algebraically identical. Because  $\lceil x \rceil = -\lfloor -x \rfloor$  for all  $x \in \mathbb{R}$ , one verifies

$$2 + \left\lceil \frac{\sqrt{8n+1} - 1}{2} \right\rceil = 2 - \left\lfloor \frac{1 - \sqrt{8n+1}}{2} \right\rfloor, \quad n \geq 1.$$

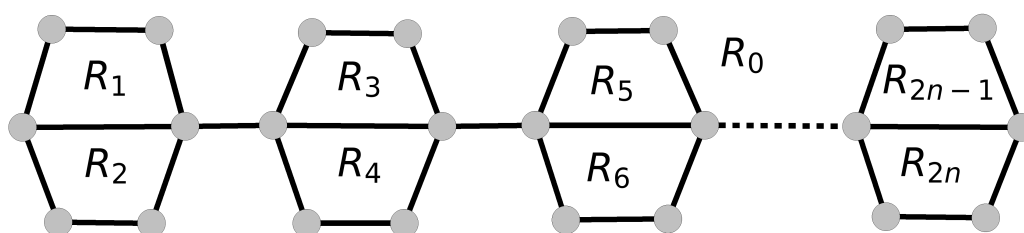
The two theorems therefore express the same step function; the distinction is presentational rather than mathematical.

**Corollary 8.5.** *The region  $\pi$  chromatic number of the alternate double quadrilateral snake graph  $ADQ_n$  satisfies*

$$R\pi\chi(ADQ_n) = R\pi\chi(Dqls_n) = 2 + \left\lceil \frac{\sqrt{8n+1}-1}{2} \right\rceil, \quad n \geq 1,$$

where  $n$  counts the alternate kite units of  $ADQ_n$ .

*Proof.* As Figure 23 shows, the alternate double quadrilateral snake  $ADQ_n$  is obtained from  $ADT_n$  by replacing each triangular kite unit with a quadrilateral kite unit. The number of kite units is  $n$ , both regions of a unit share an edge with  $R_0$  and the diagonal with each other, and regions of different units share no edge. The graph is of hub type with parameters  $(n, 2)$ , so Corollary 4.4 gives the stated value.  $\square$



**Figure 23.** Region  $\pi$  coloring of the alternate double quadrilateral snake  $ADQ_n$ .

Table 5 records  $R\pi\chi$  values for all four double snake families.

**Table 5.** Region  $\pi$  chromatic numbers for double snake graph families. The value jumps as  $n$  passes a triangular number  $T_m = m(m+1)/2$ .

| $n$                    | $DT_n$ | $Dqls_n$ | $ADT_n$ | $ADQ_n$ |
|------------------------|--------|----------|---------|---------|
| 1                      | 3      | 3        | 3       | 3       |
| 2                      | 4      | 4        | 4       | 4       |
| 3                      | 4      | 4        | 4       | 4       |
| 4                      | 5      | 5        | 5       | 5       |
| 5                      | 5      | 5        | 5       | 5       |
| 6                      | 5      | 5        | 5       | 5       |
| 7                      | 6      | 6        | 6       | 6       |
| 10                     | 6      | 6        | 6       | 6       |
| 11                     | 7      | 7        | 7       | 7       |
| 15                     | 7      | 7        | 7       | 7       |
| 16                     | 8      | 8        | 8       | 8       |
| $T_{m-1} < n \leq T_m$ | $m+2$  | $m+2$    | $m+2$   | $m+2$   |

## 9. Computation and complexity

Theorems 4.2 and 4.7 give closed formulas once a plane graph is known to have one of the two adjacency patterns. This section shows that the recognition itself is cheap, and it locates the general problem in a complexity class.

Throughout, a plane graph is given by a rotation system, that is, by the cyclic order of the edges around each vertex. From this, the region boundaries are obtained by the standard face-tracing walk, in which each edge is traversed once in each direction. Building the region adjacency graph  $H(G, \Phi)$  of Section 3 in this way costs  $O(|E|)$  time, and  $H$  has  $|F| = |E| - |V| + 2$  vertices.

**Theorem 9.1.** *Algorithm 1 is correct, and it runs in  $O(|E(G)|)$  time.*

*Proof. Correctness.* By (H1), a hub-type plane graph has an exterior region adjacent to every other region, which is the test on line 3. Conditions (H2) and (H3) say exactly that the components of  $H - x$  are the groups  $U_1, \dots, U_n$  and that each is a clique. A graph on  $d$  vertices is a clique precisely when it has  $\binom{d}{2}$  edges, which is the test on line 5. When these tests pass,  $G$  is of hub type with parameters  $(n, d)$ , and line 8 returns the value given by Theorem 4.2. The coloring built on line 7 is the one used in the sufficiency half of that proof, so it is optimal. For the annular test, every petal of an annular plane graph is adjacent to  $H_1$  and  $H_2$  and to nothing else, so every petal has degree 2 in  $H$ , and line 11 finds one. Its two neighbors are the two hub regions, and line 14 verifies Definition 4.6 directly. The coloring on line 16 assigns distinct colors to all  $p + 2$  regions, which is optimal by Theorem 4.7.

*Running time.* Face tracing visits each edge twice, so line 1 costs  $O(|E|)$ , and  $H$  has  $O(|E|)$  edges. Line 3 is a degree lookup. The components on line 4 are found by one search of  $H$ , and the orders and edge counts needed on line 5 are accumulated during that search, so lines 4 and 5 cost  $O(|E|)$ . Because  $\binom{k}{d}$  is nondecreasing in  $k$ , the value  $k_0$  on line 6 is found by scanning  $k = d, d + 1, \dots$  until the bound is met, which terminates at  $k_0 \leq n + d$  and costs  $O(|F|)$  arithmetic operations. Line 7 writes  $dn = |F| - 1$  colors. Lines 11 to 16 are two further passes over the adjacency lists of  $H$ . The total is  $O(|E|)$ .  $\square$

**Remark 9.2.** Algorithm 1 recognizes every family treated in Sections 5 to 8, by Remark 4.10. It returns UNRECOGNIZED on plane graphs outside the two patterns, for instance, on the nested embeddings of Theorem 3.3, whose region adjacency graph is a path. No polynomial algorithm for general plane graphs follows from the results of this article.

The general decision problem is stated as follows.

**Problem 9.3** (REGION  $\pi$  COLORING). Given a connected plane graph  $G$  and a positive integer  $h$ , decide whether  $R\pi\chi(G) \leq h$ .

**Proposition 9.4.** REGION  $\pi$  COLORING belongs to NP.

*Proof.* Take a coloring  $c : F(G) \rightarrow \{1, \dots, h\}$  as the certificate. Its size is  $O(|F| \log h)$ , which is polynomial in the size of  $G$  because  $|F| = |E| - |V| + 2$ . Verification builds  $H$  and the sets  $P_i$  by face tracing in time  $O(|E|)$ , tests condition (2.1) by one pass over the edges of  $H$  in time  $O(|E|)$ , forms each  $\delta(P_i)$  as a sorted list of at most  $h$  colors, and compares the  $|F|$  delta values pairwise. The comparison costs  $O(|F|^2 h)$  in the worst case. The whole verification is therefore polynomial, so the problem lies in NP.  $\square$

Algorithm 1 recognizes the two structured patterns and fails on every other plane graph. The exhaustive procedure below applies to an arbitrary plane graph, at the cost of exponential running time in general.

---

**Algorithm 1** Region  $\pi$  chromatic number of a hub-type or annular plane graph

---

**Require:** a connected plane graph  $G$  given by a rotation system

**Ensure:**  $R\pi\chi(G)$  and an optimal coloring, or UNRECOGNIZED

```

1: trace the region boundaries and build  $H \leftarrow H(G, \Phi)$ 
2:  $x \leftarrow$  the vertex of  $H$  for the exterior region  $R_0$ 
3: if  $\deg_H(x) = |V(H)| - 1$  then                                      $\triangleright$  test for hub type
4:    $C_1, \dots, C_n \leftarrow$  connected components of  $H - x$ 
5:   if every  $C_j$  has the same order  $d$  and  $|E(C_j)| = \binom{d}{2}$  then
6:      $k_0 \leftarrow \min\{k : \binom{k}{d} \geq n\}$ 
7:      $c(R_0) \leftarrow 1$ ; give the  $j$ th component the  $j$ th  $d$ -subset of  $\{2, \dots, k_0 + 1\}$  in lexicographic order
8:     return  $1 + k_0$  and  $c$ 
9:   end if
10: end if
11:  $w \leftarrow$  a vertex of  $H$  of degree 2, if one exists
12:  $\{H_1, H_2\} \leftarrow$  the two neighbors of  $w$                               $\triangleright$  test for annular
13: if  $H - \{H_1, H_2\}$  has no edges and every other vertex is adjacent to both  $H_1$  and  $H_2$  then
14:    $p \leftarrow |V(H)| - 2$ 
15:   color  $H_1, W_1, \dots, W_p, H_2$  with  $1, 2, \dots, p + 2$ 
16:   return  $p + 2$  and  $c$ 
17: end if
18: return UNRECOGNIZED

```

---

**Proposition 9.5.** *For every plane graph  $G$ , the value  $R\pi\chi(G)$  can be computed exactly in time  $|F|^{O(|F|)}$ , where  $|F| = |E| - |V| + 2$ . In particular,  $R\pi\chi$  is computable in polynomial time on any class of plane graphs whose cycle rank is bounded by a constant.*

*Proof.* By Lemma 2.4 it suffices to test  $h = 1, 2, \dots, |F|$  in turn and to return the first value for which a region  $\pi$  coloring exists. For a fixed  $h$ , enumerate all  $h^{|F|}$  maps  $c : F(G) \rightarrow \{1, \dots, h\}$ , and verify each one by the procedure of Proposition 9.4, which costs  $O(|E| + |F|^2 h)$ . The total is  $|F|^{O(|F|)}$ . Euler's formula gives  $|F| = |E| - |V| + 2$ , which is one more than the cycle rank of  $G$ . If the cycle rank is at most a constant  $c$ , then  $|F| \leq c + 1$ , and the bound is a constant.  $\square$

Thus, Algorithm 1 is the linear-time recognizer for the structured families of Sections 5 to 8, whereas Algorithm 2 applies to every plane graph and, by Proposition 9.5, runs in polynomial time whenever the cycle rank is bounded. Algorithm 2 is an exhaustive search and is not efficient in general.

**Algorithm 2** Region  $\pi$  chromatic number of an arbitrary plane graph**Require:** a connected plane graph  $G$  given by a rotation system**Ensure:**  $R\pi\chi(G)$  and an optimal region  $\pi$  coloring

```

1: trace the region boundaries to obtain  $F(G)$  and the adjacent region sets  $P_i$ 
2: for  $h = 1, 2, \dots, |F(G)|$  do
3:   for each map  $c : F(G) \rightarrow \{1, \dots, h\}$  do
4:     if  $c$  satisfies (2.1) and (2.2) then
5:       return  $h$  and  $c$ 
6:     end if
7:   end for
8: end for

```

**Remark 9.6.** Condition (2.2) quantifies over pairs of regions of the given plane graph, not over colorings. For a fixed coloring, it is therefore checkable directly, and no alternation of quantifiers arises beyond the single existential quantifier over colorings. This is why Proposition 9.4 places the problem in NP rather than higher in the polynomial hierarchy. Whether the problem is NP-complete is not settled here. Related coloring parameters are hard. Sánchez-Arroyo proved that deciding the total chromatic number is NP-hard [26], and Conjecture 11.7 below records the expected answer for the present problem. For a plane graph with  $|F|$  regions, exhaustive search over all colorings settles the question in  $O(h^{|F|} |F|^2 h)$  time, which is the only general upper bound offered here. On the families treated in Sections 5 to 8, Algorithm 1 returns the exact value in linear time, so no approximation is needed there. Whether the general problem admits a polynomial-time approximation scheme is not settled here and is recorded in Problem 11.6.

**10. Relations to classical parameters**

The two conditions of Definition 2.1 contribute very differently to the value of  $R\pi\chi$ . The following bounds separate their effects.

**Proposition 10.1.** *Let  $G$  be a connected plane graph with dual  $G^*$ . Then,*

$$\chi(G^*) \leq R\pi\chi(G) \leq |F(G)|.$$

If, in addition,  $G$  is simple and 2-connected, then  $\chi(G^*) \leq 4$  by the four color theorem [1].

*Proof.* Two faces of  $G$  are adjacent exactly when the corresponding vertices of  $G^*$  are adjacent, so Condition (2.1) says that a region  $\pi$  coloring is a proper vertex coloring of  $G^*$ . This gives the left inequality, and Lemma 2.4 gives the right one. A simple 2-connected plane graph has a simple planar dual, so the four color theorem applies to  $G^*$ .  $\square$

**Corollary 10.2.** *The gap between the two bounds of Proposition 10.1 is unbounded. For the triangular snake  $TS_n$  in its standard embedding,  $\chi(G^*) = 2$  when  $R\pi\chi(TS_n) = n + 1$ . The growth of  $R\pi\chi$  is therefore caused by the distinguishing condition (2.2) and not by properness.*

*Proof.* The graph  $TS_n$  is of hub type with parameters  $(n, 1)$ , as recorded in Table 1. Every interior face therefore meets  $R_0$ , and no two interior faces meet each other, so the region adjacency graph is

the star with center  $R_0$ , and the  $n$  interior faces as leaves. Every edge of  $TS_n$  lies on a triangle and so is not a bridge, which means no face of  $TS_n$  appears on both sides of an edge, and  $G^*$  has no loop. Discarding parallel edges leaves the simple graph  $K_{1,n}$ , which is bipartite, so  $\chi(G^*) = 2$  for every  $n \geq 1$ . Theorem 5.1 gives  $R\pi\chi(TS_n) = n + 1$ . The difference  $R\pi\chi(TS_n) - \chi(G^*)$  therefore equals  $n - 1$ , which is unbounded.  $\square$

**Proposition 10.3.** *There is no function  $f$  with  $R\pi\chi(G) \leq f(s)$  for every plane graph  $G$ , where  $s$  denotes the largest size of an adjacent region set among the interior faces of  $G$ .*

*Proof.* In  $TS_n$ , every interior face satisfies  $|T_j| = 2$ , so  $s = 2$  for every  $n$ , and Theorem 5.1 gives  $R\pi\chi(TS_n) = n + 1$ , which is unbounded.  $\square$

**Proposition 10.4.** *None of the following quantities bounds  $R\pi\chi$ : the maximum face degree  $\Delta_f$ , the chromatic number  $\chi(G^{*2})$  of the square of the dual, and the degeneracy of  $G^{*2}$ . In particular the inequality  $R\pi\chi(G) \leq \chi(G^{*2})$  is false.*

*Proof.* Take the nested embedding  $(TS_n, \Phi_1)$  of Theorem 3.3. The region carried by  $T_i$  is bounded by the three edges of  $T_i$  together with the three edges of  $T_{i+1}$ , so it has face degree 6. The innermost region and the exterior region are each bounded by three edges. Hence,  $\Delta_f = 6$  for every  $n$ .

By Theorem 3.3, the region adjacency graph is the path on  $n + 1$  vertices, so the underlying simple graph of  $G^*$ , is that path. In its square  $G^{*2}$  two vertices are adjacent when their indices differ by at most two. Coloring the  $i$ th vertex by  $i \bmod 3$  is proper, and the first three vertices are pairwise adjacent, so  $\chi(G^{*2}) = 3$  whenever  $n + 1 \geq 3$ . In the natural order, every vertex has at most two earlier neighbors, so the degeneracy of  $G^{*2}$  is 2. All three quantities are bounded independently of  $n$ , and inequality (3.1) forces  $R\pi\chi(TS_n, \Phi_1) \rightarrow \infty$ .

For the stated inequality, it is enough to take  $n = 7$ . Theorem 3.3 gives  $R\pi\chi(TS_7, \Phi_1) = 4$ , whereas  $\chi(G^{*2}) = 3$ .  $\square$

**Remark 10.5.** Proposition 10.3 shows that no purely local quantity controls  $R\pi\chi$ . In  $TS_n$ , the obstruction is the single face  $R_0$ , whose adjacent region set has  $n + 1$  members. The correct general lower bound is Lemma 2.5, which involves both the largest adjacent region set and the number of distinct adjacent region sets. The two kinds of embedding studied in Section 3 sit at opposite ends of that bound. A standard drawing has one very large adjacent region set and many small ones, which drives  $R\pi\chi$  up to  $|F|$ . A nested drawing has uniformly small adjacent region sets, and Lemma 2.5 is then much weaker, as Theorem 3.3 illustrates.

## 11. Conclusions

The current study determines the precise region of the  $\pi$  chromatic number for planar snake graph families which belong to three different structural categories.

The five linear families, which include triangular, quadrilateral, square, quintet, and heptagonal snakes have one complete proof which combines Theorem 5.1 and Corollaries 5.2–5.5 to establish the general formula  $R\pi\chi = n + 1$ . The result holds for all polygon sizes  $k \geq 3$  and all  $n \geq 1$  because the decisive structural property is not the shape of the interior face but the star-type adjacency pattern which requires every interior face to share exactly one edge with the exterior face while sharing no edge with any other interior face. The coloring problem can be solved through this observation because it

requires the identification of star-type adjacent region collections whose minimum color requirements depend on their arm count.

The alternate and cyclic families introduce a second boundary face or a parity dependence, raising the chromatic number by one or two units. Alternate snake graphs (Theorem 6.1 and Corollary 6.2) yield a parity-sensitive formula:  $\lfloor n/2 \rfloor + 1$  or  $n/2$  for even  $n$ , and  $\lceil n/2 \rceil$  for odd  $n$ , depending on the endpoint configuration. Cyclic snake graphs, in which the base path closes into a ring and introduces a central interior face, obey  $R\pi\chi = n + 2$  (Theorem 7.1 and Corollary 7.2). The alternate triangular cycle snake graph and its quadrilateral analog exhibit a combined effect: The parity-driven floor/ceiling structure of the alternate family shifted upward by 2 (Theorem 7.3 and Corollary 7.4).

The double snake families occupy a qualitatively different regime. Because each kite unit contributes a pair of interior faces sharing the same adjacent region set, the region  $\pi$  coloring problem reduces to selecting  $n$  distinct unordered 2-element subsets of a color set  $\{2, \dots, k\}$ . The minimum  $k$  is governed by the binomial coefficient  $\binom{k-1}{2} \geq n$ , whose threshold values are exactly the triangular numbers  $T_m = m(m+1)/2$ . The resulting formula  $R\pi\chi(DT_n) = 2 + \lceil (\sqrt{8n+1} - 1)/2 \rceil$  (Theorem 8.1) produces a staircase function that increases by exactly one unit as  $n$  passes each triangular number, taking the value  $m+2$  on the range  $T_{m-1} < n \leq T_m$ . Remark 8.4 records that the floor-based formulation of Theorem 8.3 for  $ADT_n$  is algebraically equivalent; both expressions encode the same step function. The rate of this growth is made precise in Corollary 4.5, which gives  $R\pi\chi = (d!n)^{1/d}(1 + o(1))$  for a hub-type family with parameters  $(n, d)$ . For the double families, this is  $\sqrt{2n} + O(1)$ , and the sharper two-sided bound  $\sqrt{2n} + \frac{3}{2} \leq R\pi\chi \leq \sqrt{2n} + \frac{5}{2}$  holds for every  $n \geq 1$ . The double quadrilateral variants (Corollaries 8.2 and 8.5) inherit the identical formula, as the polygon shape does not alter the kite adjacency structure.

Table 6 gathers all results.

**Table 6.** Region  $\pi$  chromatic numbers of all families studied here;  $T_m = m(m+1)/2$ , CE = closed end, OE = open end, 1E = one-end open. Classification in Table 1.

| Graph            | Parameter    | $R\pi\chi$                              |
|------------------|--------------|-----------------------------------------|
| $TS_n$           | $n \geq 1$   | $n + 1$                                 |
| $Ql_n$           | $n \geq 1$   | $n + 1$                                 |
| $Sqs_n$          | $n \geq 1$   | $n + 1$                                 |
| $Qt_n$           | $n \geq 1$   | $n + 1$                                 |
| $Hts_n$          | $n \geq 1$   | $n + 1$                                 |
| $ATS_n, AQS_n$   | $n$ even, CE | $\lfloor n/2 \rfloor + 1$               |
| $ATS_n, AQS_n$   | $n$ even, OE | $n/2$                                   |
| $ATS_n, AQS_n$   | $n$ odd, 1E  | $\lceil n/2 \rceil$                     |
| $Cqs_n$          | $n \geq 1$   | $n + 2$                                 |
| $LS_n$           | $n \geq 1$   | $n + 2$                                 |
| $ATCS_n, AQCS_n$ | $n$ even     | $n/2 + 2$                               |
| $ATCS_n, AQCS_n$ | $n$ odd      | $\lfloor n/2 \rfloor + 2$               |
| $DT_n, Dqls_n$   | $n \geq 1$   | $2 + \lceil (\sqrt{8n+1} - 1)/2 \rceil$ |
| $ADT_n, ADQ_n$   | $n \geq 1$   | $2 + \lceil (\sqrt{8n+1} - 1)/2 \rceil$ |

### 11.1. Open problems and conjectures

The following problems and conjecture are left open.

**Problem 11.1.** Determine  $\min_{\Phi} R\pi\chi(G, \Phi)$  for a cactus  $G$  with  $n$  cycles. Theorem 3.2 gives  $n + 1$  for the maximum over all embeddings. For the nested embeddings of  $TS_n$ , inequality (3.1) gives a lower bound of order  $n^{1/3}$ , and no matching upper bound is known. Is the minimum of order  $n^{1/3}$ ?

**Problem 11.2.** A  $d$ -fold snake graph carries  $d$  stacked polygon regions on each base edge. Theorem 4.2 settles every case in which the  $d$  regions of a unit are pairwise adjacent, and returns  $R\pi\chi = 1 + \min\{k : \binom{k}{d} \geq n\}$ . For  $d = 2$ , the two regions of a kite unit share the diagonal edge, so the hypothesis holds, and Corollary 4.4 applies. For  $d \geq 3$ , the stacked construction places the  $d$  regions between  $d + 1$  internally disjoint paths joining the two base vertices, and then only consecutive regions share an edge. Condition (H2) of Definition 4.1 fails, and the value is not determined by Theorem 4.2. Determine  $R\pi\chi$  for these graphs.

**Problem 11.3.** Replace the base path by a tree or by a cycle of cycles. The region adjacency graph is then neither a star nor an annulus, so neither Theorem 4.2 nor Theorem 4.7 applies. Determine  $R\pi\chi$  for snake graphs over such bases.

**Problem 11.4.** Corollary 4.8 exhibits two classes of plane graphs with  $R\pi\chi(G) = |F(G)|$ , the largest value allowed by Lemma 2.4. Characterize the plane graphs that attain this bound.

**Problem 11.5.** The cyclic chromatic number  $\chi_c$  of a plane graph counts the colors needed so that any two vertices lying on a common face boundary receive different colors [16]. It is a vertex parameter, but duality turns it into a face parameter of  $G$ : Because the vertices of  $G^*$  are the faces of  $G$ , and the faces of  $G^*$  are the vertices of  $G$ , the quantity  $\chi_c(G^*)$  is the least number of colors on the faces of  $G$  for which any two faces meeting at a common vertex receive different colors. This is a stronger requirement than the properness condition (2.1), which constrains only faces sharing an edge. Establish general inequalities relating  $R\pi\chi(G)$  and  $\chi_c(G^*)$ , for the snake families or for plane graphs at large.

A further direction is comparative rather than internal. Corollary 10.2 shows that the distinguishing condition, and not properness, drives the growth of  $R\pi\chi$ . Similar separation requirements arise where spectral and higher-order structures are used to distinguish the parts of a graph. Berahmand and coauthors surveyed graph structure learning for spectral clustering and discussed anchor-based and hypergraph-based constructions [27]. Berahmand and coauthors also developed a one-step, anchor-adaptive hypergraph method for the same purpose [28]. Whether the region adjacency hypergraph of a plane graph admits eigenvalue bounds on  $R\pi\chi$  in that spirit is worth investigating.

**Problem 11.6.** Determine whether the optimization version of Problem 9.3 admits a polynomial-time approximation scheme, or a constant-factor approximation, on plane graphs in general or on snakelike plane graphs in particular.

**Conjecture 11.7.** *The decision problem REGION  $\pi$  COLORING of Problem 9.3 is NP-complete for general plane graphs.*

### Author contributions

Sulochana Aradhe: Conceptualization, Formal analysis, Investigation, Writing—original draft; Archana A. Bhangre: Methodology, Validation, Supervision, Writing—review and editing; Haribhau R.

Bhaskar: Conceptualization, Formal analysis, Supervision, Writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

### Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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### Conflict of interest

The authors declare that there is no conflict of interest.

### Data availability

No data were used to support this research.

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