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Research article

Rigidity of almost Ricci solitons with affine vector fields on semi-Riemannian manifolds

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Abstract: We investigated almost Ricci solitons on semi-Riemannian manifolds when the potential vector field was affine. Our central theorem showed that in every dimension $n \geq 3$, such almost Ricci solitons coincide with Ricci solitons, and the manifolds had constant scalar curvature. In contrast, in dimension 2, we constructed explicit examples of proper (but trivial) almost Ricci solitons, highlighting a genuine low dimensional distinction. In the Riemannian setting, this leads to rigidity: Under natural geometric assumptions, an almost Ricci soliton must be parallel, trivial, or Ricci-flat.

Keywords: semi-Riemannian manifolds; Ricci solitons; almost Ricci solitons; killing vector fields; parallel vector fields; affine vector fields; scalar curvature
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1. Introduction

In his seminal work on the Ricci flow, Hamilton [1] introduced Ricci solitons, which play a fundamental role in the study of singularity formation and the long-term behavior of geometric flows. These structures arise as self-similar solutions to the Ricci flow and are closely related to Einstein metrics and various rigidity phenomena in differential geometry.

On a semi-Riemannian manifold (N, h), a Ricci soliton is a quadruple $(N, h, \aleph, \lambda)$, where $\aleph$ is a vector field and λ is a constant satisfying

$$\mathrm{Rc} + \frac{1}{2}\mathcal{L}_{\aleph} h = \lambda h. \tag{1.1}$$

Here, Rc denotes the Ricci tensor, $\mathcal{L}_{\aleph} h$ is the Lie derivative of the metric h along $\aleph$, and $\aleph$ is called the *potential vector field*. When $\aleph = \nabla f$ for some smooth function f, the Ricci soliton is said to be gradient, and (1.1) becomes

$$\mathrm{Rc} + \mathbf{H}^f = \lambda h, \tag{1.2}$$

where $\mathbf{H}^f$ denotes the Hessian of f.

Pigola et al. [2] introduced the notion of an *almost Ricci soliton* by enabling the constant λ in (1.1) to be replaced by a smooth function. This generalization considerably enlarges the class of self-similar geometric structures while preserving many of the features of Ricci solitons. An almost Ricci soliton is called shrinking, steady, or expanding according to whether λ is positive, zero, or negative, respectively. It is called trivial when the potential vector field is Killing, and parallel when the potential vector field is parallel; in either case, the metric is Einstein.

The geometry of almost Ricci solitons has attracted considerable attention because of its close connection with geometric flows and rigidity theory [3]. In [4], several fundamental structure equations for almost Ricci solitons were established, extending analogous identities for Ricci solitons. As an application, it was shown that a compact almost Ricci soliton with either constant scalar curvature or conformal potential vector field is isometric to a sphere, generalizing a result of Petersen and Wylie [5]. Since then, numerous rigidity and classification results have been obtained under various geometric assumptions, including curvature conditions and compactness (see, for example, [6–8]), and special properties of the potential vector field (see, for example, [9–11]).

A distinguished class of vector fields in differential geometry is formed by *affine vector fields*, namely those preserving the Levi-Civita connection. A vector field $\aleph$ on (N, h) is said to be affine if

$$\mathcal{L}_\aleph \nabla = 0, \tag{1.3}$$

where ∇ is the Levi-Civita connection of h (see [12, 13]). Since every Killing vector field is affine, this class naturally extends an important family of symmetries of the underlying manifold.

Although many rigidity results for almost Ricci solitons are known, the influence of the affine condition on the potential vector field has received comparatively little attention. For Ricci solitons, it has been shown in [14] that, on a Riemannian manifold, an affine potential vector field forces the scalar curvature to be constant. In this paper, we investigate almost Ricci solitons on semi-Riemannian manifolds whose potential vector field is affine.

Our main theorem shows that the affine condition alone yields a strong rigidity phenomenon. More precisely, we prove that every almost Ricci soliton with affine potential vector field is necessarily a Ricci soliton whenever the dimension satisfies $n \geq 3$, and consequently, the scalar curvature is constant. Thus, the affine assumption provides a new rigidity criterion without imposing additional curvature, conformality, or topological hypotheses. In contrast, we show that this rigidity fails in dimension two by constructing explicit proper almost Ricci solitons, revealing a genuine low-dimensional phenomenon. Furthermore, in the Riemannian setting, we establish several rigidity results showing that, under natural geometric assumptions, such almost Ricci solitons are necessarily trivial, parallel, or Ricci-flat. In particular, in the compact case, we prove that every almost Ricci soliton with affine potential vector field is trivial.

The paper is organized as follows: In Section 2, the necessary notation and preliminaries are presented. In Section 3, we provide examples of Ricci solitons with affine potential vector fields. Sections 4 and 5 are devoted to almost Ricci solitons on semi-Riemannian manifolds, where we prove the main rigidity result for $n \geq 3$ and construct counterexamples in dimension 2. In Section 6, in the Riemannian case, if the potential vector fields of the solitons are affine, this imposes rigidity conditions.

Specifically, under natural geometric assumptions, almost Ricci solitons are constrained to be parallel, trivial, or Ricci flat. We conclude by examining the compact case.

2. Preliminaries

In this section, the key concepts and identities used throughout the paper are presented (cf. [15,16]).

For a semi-Riemannian manifold (N, h), we denote by $\mathfrak{X}(N)$ the set of smooth vector fields on (N, h). The Riemannian curvature tensor $\mathcal{R}$ is defined by

$$\mathcal{R}(\zeta, \eta)Z = \nabla_\zeta \nabla_\eta Z - \nabla_\eta \nabla_\zeta Z - \nabla_{[\zeta,\eta]} Z, \tag{2.1}$$

where $\zeta, \eta, Z \in \mathfrak{X}(N)$.

The second covariant derivative of the vector field Z with respect to the vector fields ζ and η is given by the following formula:

$$\nabla^2_Z(\zeta, \eta) = \nabla_\zeta \nabla_\eta Z - \nabla_{\nabla_\zeta \eta} Z. \tag{2.2}$$

By contraction, the Ricci tensor Rc has been obtained

$$\mathrm{Rc}(\zeta, \eta) = \mathrm{trace}\{Z \mapsto \mathcal{R}(Z, \zeta)\eta\},$$

and the scalar curvature is given by $S = \mathrm{trace}(\mathrm{Rc})$.

Let $\{e_1, \ldots, e_n\}$ be a local pseudo-orthonormal frame on (N, h), satisfying

$$h(e_i, e_j) = \epsilon_i \delta_{ij}, \qquad \epsilon_i \in \{-1, 1\}.$$

The gradient of a smooth function $f \in C^\infty(N)$ is the vector field ∇f determined by

$$h(\nabla f, X) = X(f), \qquad \text{for all } X \in \mathfrak{X}(N).$$

Equivalently,

$$\nabla f = \sum_{i=1}^{n} \epsilon_i \, e_i(f) e_i.$$

The divergence of a vector field $\zeta \in \mathfrak{X}(N)$ is defined by

$$\mathrm{div}(\zeta) = \sum_{i=1}^{n} \epsilon_i \, h(\nabla_{e_i} \zeta, e_i).$$

Consequently, the Laplacian of a smooth function f is defined by

$$\Delta f = \mathrm{div}(\nabla f) = \sum_{i=1}^{n} \epsilon_i \left[e_i(e_i(f)) - (\nabla_{e_i} e_i)(f) \right].$$

More generally, if T is a $(1, 1)$-tensor field on (N, h), its divergence is defined as

$$\mathrm{div}(T) = \sum_{i=1}^{n} \epsilon_i \, (\nabla_{e_i} T)(e_i).$$

For a vector field $\zeta \in \mathfrak{X}(N)$, the Laplacian is given by

$$\Delta\zeta = \sum_{i=1}^{n} \epsilon_i \left(\nabla_{e_i}\nabla_{e_i}\zeta - \nabla_{\nabla_{e_i}e_i}\zeta\right).$$

A vector field ζ is said to be an eigenvector of the Laplacian operator Δ if it satisfies the following:

$$\Delta\zeta = -\lambda\zeta, \tag{2.3}$$

where λ is a smooth function.

The Lie derivative with respect to a vector field $\aleph$ is given by

$$(\mathcal{L}_{\aleph}h)(\zeta, \eta) = h(\nabla_{\zeta}\aleph, \eta) + h(\zeta, \nabla_{\eta}\aleph). \tag{2.4}$$

In particular, $\aleph$ is a Killing vector field if $\mathcal{L}_{\aleph}h = 0$, a homothetic vector field if $\mathcal{L}_{\aleph}h = ch$ for some constant c, and a conformal vector field if $\mathcal{L}_{\aleph}h = 2\rho h$ for some smooth function ρ.

Now, let $\aleph$ be a vector field on (N, h) and the (1,1)-tensor $\nabla\aleph$ is defined by

$$(\nabla\aleph)(\zeta) = \nabla_{\zeta}\aleph, \tag{2.5}$$

for all vector field $\zeta \in \mathfrak{X}(N)$.

For a semi-Riemannian manifold (N, h), and $\aleph \in \mathfrak{X}(N)$. Let $\theta_{\aleph}$ be defined by $\theta_{\aleph}(\zeta) = h(\zeta, \aleph), \zeta \in \mathfrak{X}(N)$), then we can write $\mathcal{L}_{\aleph}$ as

$$\mathcal{L}_{\aleph}h(\zeta, \eta) + d\theta_{\aleph}(\zeta, \eta) = 2h(\nabla_{\zeta}\aleph, \eta), \tag{2.6}$$

for all $\zeta, \eta \in \mathfrak{X}(N)$. We have the following equations:

$$d\theta_{\aleph}(\zeta, \eta) = 2h(\phi(\zeta), \eta), \tag{2.7}$$

and

$$\mathcal{L}_{\aleph}h(\zeta, \eta) = 2h(B(\zeta), \eta), \tag{2.8}$$

where $\zeta, \eta \in \mathfrak{X}(N)$, ϕ is a skew-symmetric tensor field, and B is a symmetric tensor field of $\nabla\aleph$. We get from (2.6) that

$$\nabla\aleph = B + \phi. \tag{2.9}$$

Suppose that $(N, h, \aleph, \lambda)$ is an almost Ricci soliton. From (1.1), we deduce that

$$\nabla\aleph = \lambda I - Q + \phi, \tag{2.10}$$

where Q is the Ricci operator, determined by the relation

$$\mathrm{Rc}(\zeta, \eta) = h(Q(\zeta), \eta),$$

for all $\zeta, \eta \in \mathfrak{X}(N)$.

For a Riemannian manifold (N, h), then by (cf. [13]), an operator T, whose norm is defined by

$$\|T\|^2 = \sum_{i=1}^{n} h(T(e_i),\, T(e_i)).$$

Thus, the norm of the operator $\nabla\aleph$ equals:

$$\|\nabla\aleph\|^2 = \|\lambda I - Q\|^2 + \|\phi\|^2, \tag{2.11}$$

and trace $((\nabla\aleph)^2)$ can be written as

$$\text{trace}\,((\nabla\aleph)^2) = \|\lambda I - Q\|^2 - \|\phi\|^2. \tag{2.12}$$

Taking the trace of (1.1), in the more general setting of an almost Ricci soliton on a semi-Riemannian manifold, yields

$$S + \text{div}\,\aleph = n\lambda, \tag{2.13}$$

and (2.13) becomes

$$S + \Delta f = n\lambda. \tag{2.14}$$

On semi-Riemannian manifolds, affine vector fields are crucially defined by an equation that directly links the second covariant derivative with the curvature tensor.

$$\nabla^2_{\aleph}(\zeta,\eta) + \mathcal{R}(\aleph,\zeta)\eta = 0, \tag{2.15}$$

for all vector fields $\zeta, \eta \in \mathfrak{X}(N)$ (see [13] or [17]).

Equation (2.15) is equivalent to

$$\mathcal{R}(\aleph,\zeta) + \nabla_{\zeta}(\nabla\aleph) = 0, \tag{2.16}$$

for all $\zeta \in \mathfrak{X}(N)$.

The gradient ∇f of a smooth function is defined by,

$$\zeta(f) = h(\nabla f, \zeta),$$

for all $\zeta \in \mathfrak{X}(N)$.

The Hessian of f is the $(0,2)$-tensor

$$\mathbf{H}^f(\zeta,\eta) = h(\nabla_{\zeta}\nabla f, \eta) = \zeta(\eta(f)) - (\nabla_{\zeta}\eta)(f).$$

Accordingly, the Laplacian of f is given by the trace of its Hessian

$$\Delta f = \text{trace}(\mathbf{H}^f).$$

The following lemma is a simple but important characteristic of affine vector fields.

Lemma 2.1. *The divergence of any affine vector field on a semi-Riemannian manifold is constant.*

Proof. Suppose $\aleph$ is an affine vector field on a semi-Riemannian manifold (N,h). From Eq (2.16) and the anti-symmetric property of the Riemannian curvature tensor, it gets

$$\text{trace}(\nabla_{\zeta}(\nabla\aleph)) = 0,$$

for all $\zeta \in \mathfrak{X}(N)$. Since $\nabla_{\zeta}(\text{trace}(\nabla\aleph))\text{trace}(\nabla_{\zeta}(\nabla\aleph))$, it follows that

$$\nabla_{\zeta}(\text{trace}(\nabla\aleph)) = 0,$$

for all $\zeta \in \mathfrak{X}(N)$.

Hence, we conclude that trace$(\nabla\aleph)$ is a constant. Since $\text{div}(\aleph) = \text{trace}(\nabla\aleph)$, we conclude that div$\aleph$ is also constant. □

3. Examples

We provide explicit examples of Ricci solitons with affine potential vector fields, illustrating the results of Section 4.

Example 1. *Let* $(\mathbb{R}^n, h)$ *be the Euclidean space with standard metric h. Consider the vector field*

$$\aleph = \sum_{i=1}^{n}(a_i x_i + b_i)\frac{\partial}{\partial x_i},$$

where $a_i, b_i \in \mathbb{R}$ *are constants.*

It is straightforward to verify that $\aleph$ *is an affine vector field on* $(\mathbb{R}^n, h)$*. Moreover, a direct computation yields*

$$\frac{1}{2}\mathcal{L}_{\aleph}h = \sum_{i=1}^{n} a_i\, dx_i^2.$$

On the other hand, since $\mathrm{Rc} = 0$*,* (1.1) *reduces to*

$$\frac{1}{2}\mathcal{L}_{\aleph}h = \lambda h.$$

It follows that $(\mathbb{R}^n, h, \aleph, \lambda)$ *is a Ricci soliton precisely when* $a_i = \lambda$ *for all i. Under this condition,* $\aleph$ *is a homothetic vector field, and the Ricci soliton is trivial in the sense that the Euclidean manifold* $(\mathbb{R}^n, h)$ *is Einstein indeed, Ricci-flat.*

Example 2. *Let* $(N^n(c), h)$ *be an n-dimensional semi-Riemannian space form with constant sectional curvature c. The Ricci tensor satisfies*

$$\mathrm{Rc} = (n-1)c\, h.$$

Suppose $\aleph$ *is a homothetic vector field, i.e.,* $\mathcal{L}_{\aleph}h = 2\rho h$*.* (1.1) *becomes*

$$(n-1)c\, h + \rho h = \lambda h,$$

so that $\lambda = \rho + (n-1)c$*, and hence* λ *is a constant. Therefore,* $(N^n(c), h, \aleph, \lambda)$ *is a Ricci soliton with affine (in fact, homothetic) potential vector field.*

Example 3. *Let N be a direct product manifold* $I \times M$*, where* $I \subset \mathbb{R}$ *is an open interval and M is an n-dimensional Ricci-flat manifold,* $n \geq 2$*. Consider the metric h on N, which is given by*

$$h = -dt^2 + a^2 h_M,$$

where a is a constant.

It is straightforward to see that the vector field $\aleph = \partial_t$ *is Killing (hence affine). Consequently, we obtain a Ricci soliton with affine potential vector field. In fact, this is a steady Ricci soliton, that is* $\lambda = 0$*.*

4. Rigidity of almost Ricci solitons with affine potential vector fields

We now establish the main rigidity result for almost Ricci solitons with affine potential vector fields.

Theorem 4.1. *Let $(N, h, \aleph, \lambda)$ be an almost Ricci soliton with $\aleph$ an affine potential vector field on an n-dimensional semi-Riemannian manifold (N, h), where $n \geq 3$. Then $(N, h, \aleph, \lambda)$ is necessarily a Ricci soliton with constant scalar curvature. Moreover, the Ricci tensor is parallel, that is, $\nabla \operatorname{Rc} = 0$, (N, h) is Ricci symetric.*

Proof. If $\aleph$ is trivial, then $(N, h, \aleph, \lambda)$ is a trivial Ricci soliton and (N, h) is Einstein. Given that $n \geq 3$, it follows by the Schur Lemma that the scalar curvature is constant.

Suppose $\aleph$ is a nontrivial affine vector field. Then, for any $\zeta, \eta, Z \in \mathfrak{X}(N)$,

$$\begin{aligned}(\nabla_\zeta \mathcal{L}_\aleph h)(\eta, Z) &= h(\nabla_\zeta \nabla_\eta \aleph - \nabla_{\nabla_\zeta \eta} \aleph, Z) + h(\nabla_\zeta \nabla_Z \aleph - \nabla_{\nabla_\zeta Z} \aleph, \eta) \\ &= h(\nabla^2_\aleph(\zeta, \eta), Z) + h(\nabla^2_\aleph(\zeta, Z), \eta).\end{aligned}$$

Since $\aleph$ is affine, it follows from (2.15) that

$$\mathcal{R}(\zeta, \aleph)\eta = \nabla^2_\aleph(\zeta, \eta).$$

Hence,

$$(\nabla_\zeta \mathcal{L}_\aleph h)(\eta, Z) = h(\mathcal{R}(\zeta, \aleph)\eta, Z) + h(\mathcal{R}(\zeta, \aleph)Z, \eta) = 0,$$

which implies

$$\nabla_\zeta \mathcal{L}_\aleph h = 0, \tag{4.1}$$

for all $\zeta \in \mathfrak{X}(N)$.

From (1.1) and (4.1), we obtain

$$\nabla_\zeta \operatorname{Rc} = \zeta(\lambda) h, \tag{4.2}$$

for all $\zeta \in \mathfrak{X}(N)$.

By [16, Exercise 2.5.9], one has

$$(\nabla_\zeta \operatorname{Rc})(\eta, Z) = h((\nabla_\zeta Q)\eta, Z), \tag{4.3}$$

for all $\zeta \in \mathfrak{X}(N)$.

Combining (4.3) with (4.2) yields

$$\nabla_\zeta Q = \zeta(\lambda) I, \tag{4.4}$$

for every $\zeta \in \mathfrak{X}(N)$.

Now, for a geodesic orthonormal frame $\{e_1, \ldots, e_n\}$ on (N, h). Using (4.4), we compute

$$\operatorname{div}(Q) = \sum_{i=1}^n \epsilon_i (\nabla_{e_i} Q)(e_i) = \sum_{i=1}^n \epsilon_i e_i(\lambda) e_i = \nabla \lambda.$$

Since $\frac{1}{2}\nabla S = \operatorname{div}(Q)$, it follows that

$$\tfrac{1}{2}\nabla S = \nabla \lambda. \tag{4.5}$$

On the other hand, from (2.13) and Lemma 2.1, we obtain

$$\nabla S = n\nabla\lambda. \tag{4.6}$$

Comparing (4.5) and (4.6), we deduce

$$(n-2)\,\nabla\lambda = 0. \tag{4.7}$$

Thus, if $n \geq 3$, the function λ must be constant, which implies that S is constant. Consequently, $(N, h, \aleph, \lambda)$ is a Ricci soliton with constant scalar curvature. □

Remark 1. *Theorem 4.1 shows that with an affine potential vector field, almost Ricci solitons reduce to Ricci solitons, reflecting a rigidity phenomenon. Furthermore, if an almost Ricci soliton has a nontrivial affine potential vector field, then the manifold is necessarily Ricci symmetric.*

5. Counterexamples on 2-dimensions

In this section, we show that the rigidity result established in Theorem 4.1 does not extend to dimension 2. In fact, one can construct two-dimensional almost Ricci solitons with affine (in fact, Killing) potential vector fields that are not Ricci solitons, namely, proper almost Ricci solitons. In this setting, it is worth noting that Example 4 below, where $X = 0$, and Example 5 below, where $X = \partial_\theta$ is Killing, are trivial. They demonstrate only that λ may be nonconstant and hence that proper almost Ricci solitons can occur. We do not know of an explicit two-dimensional Lorentzian example satisfying the required nontrivial conditions, namely, a non-Killing affine potential vector field with nonconstant λ.

In dimension 2, the Ricci tensor satisfies

$$\mathrm{Rc} = \tfrac{S}{2}\, h,$$

and thus, the almost Ricci soliton equation reduces to

$$\mathcal{L}_{\aleph} h = (2\lambda - S)h, \tag{5.1}$$

which shows that $\aleph$ is a conformal vector field. Since $\aleph$ is affine, it follows that it is homothetic (see [13]).

This allows the construction of examples of proper almost Ricci solitons on surfaces with nonconstant Gaussian curvature.

Example 4. *Let (N^2, h) be a surface with nonconstant Gaussian curvature K. Then, from* (5.1)*,*

$$(N^2, h, \aleph = 0, \lambda = K)$$

defines an almost Ricci soliton with affine potential vector field $\aleph = 0$ and such that $\lambda = K$ is not constant.

Example 5. *Consider the catenoid parametrized by*

$$\mathbf{X}(t,\theta) = (\cosh t\cos\theta,\ \cosh t\sin\theta,\ t), \qquad t\in\mathbb{R},\ \theta\in[0,2\pi).$$

The induced metric is

$$h = \cosh^2 t\,(dt^2 + d\theta^2).$$

Since the metric does not depend on θ, the vector field ∂_θ is Killing, and hence an affine vector field. The Gaussian curvature and scalar curvature are

$$K(t) = -\frac{1}{\cosh^4 t}, \qquad S(t) = 2K(t) = -\frac{2}{\cosh^4 t}.$$

Hence, by (5.1)*, the structure*

$$(N^2, h, \partial_\theta, \lambda(t) = -\frac{1}{\cosh^4 t})$$

defines a proper almost Ricci soliton.

Remark 2. *The preceding examples show that, in dimension two, the potential function λ coincides with half of the scalar curvature of the surface. Since the scalar curvature of a two-dimensional manifold need not be constant, proper almost Ricci solitons arise naturally in this setting. In contrast, for dimensions greater than two, Schur's lemma implies that the scalar curvature is constant.*

6. Ricci solitons on Riemannian manifolds with affine potential vector fields

The results indicate that, in the Riemannian case, if the potential vector fields of the almost solitons are affine, this imposes rigidity conditions. Specifically, under natural geometric assumptions, almost Ricci solitons are constrained to be parallel, trivial, or Ricci flat. Throughout this section, we will assume (N,h) is an n-dimensional Riemannian manifold with $n \geq 3$.

We begin with a fundamental identity that will be used throughout this section; see [13, Theorem 2].

Lemma 6.1. *Let $\aleph$ be an affine vector field on (N,h) and let $f = \frac{1}{2}h(\aleph,\aleph)$. Then*

$$\Delta f = \|\nabla\aleph\|^2 - \mathrm{Rc}(\aleph,\aleph). \tag{6.1}$$

Using Eq (6.1), and the well-known Bochner formula (see Eq (4.2) in [18]), we have

$$\Delta f + div(\nabla_\aleph \aleph) = 2\|\lambda I - Q\|^2. \tag{6.2}$$

By Using (1.1) and (2.1), for all ζ, we have

$$\mathrm{Rc}(\aleph,\zeta) = -h(div(\phi),\zeta), \tag{6.3}$$

which means $Q(\aleph) = -div(\phi)$.

The following result shows that if the potential vector field of an almost Ricci soliton on a Riemannian manifold is an affine eigenvector of the Laplacian, then the Ricci soliton is trivial.

Theorem 6.1. *Let* $(N, h, \aleph, \lambda)$ *be an almost Ricci soliton with* $\aleph$ *an affine vector field on* (N, h)*. If* $\aleph$ *satisfies the following condition*

$$\Delta\aleph = -\lambda\aleph. \tag{6.4}$$

Then, $(N, h, \aleph, \lambda)$ *is trivial.*

Proof. Formula (3.7) in [8] is given by

$$\Delta\aleph = (2-n)\nabla\lambda - Q(\aleph).$$

Together with the condition $\Delta\aleph = -\lambda\aleph$, it yields

$$Q(\aleph) = \lambda\aleph + (2-n)\nabla\lambda.$$

Taking the divergence of both sides gives

$$\operatorname{div}(Q(\aleph)) = \lambda \operatorname{div}(\aleph) + \aleph(\lambda) + (2-n)\Delta\lambda. \tag{6.5}$$

By Theorem 4.1, λ and S are constant. Hence, (6.5) reduces to

$$\operatorname{div}(Q(\aleph)) = \lambda \operatorname{div}\aleph.$$

By virtue of (2.13), we obtain

$$\operatorname{div}(Q(\aleph)) = n\lambda^2 - \lambda S. \tag{6.6}$$

On the other hand, formula (3.5) in [8] is given by

$$\operatorname{div}(Q(\aleph)) = \frac{1}{2}h(\nabla S, \aleph) + \lambda S - \|Q\|^2.$$

Since S is constant, this yields

$$\operatorname{div}(Q(\aleph)) = \lambda S - \|Q\|^2. \tag{6.7}$$

Thus, from (6.6) and (6.7), we deduce that

$$\|Q\|^2 = 2\lambda S - n\lambda^2. \tag{6.8}$$

Moreover, by Lemma 3.3 in [8], one has

$$\left\|\mathrm{Rc} - \frac{S}{n}h\right\|^2 = \|Q\|^2 - \frac{S^2}{n}\cdot \tag{6.9}$$

Substituting (6.8) into (6.9), we obtain

$$\left\|\mathrm{Rc} - \frac{S}{n}h\right\|^2 = -\frac{1}{n}(n\lambda - S)^2. \tag{6.10}$$

Since $\left\|\mathrm{Rc} - \frac{S}{n}h\right\|^2 \geq 0$ and $-\frac{1}{n}(n\lambda - S)^2 \leq 0$, it follows that

$$\mathrm{Rc} = \frac{S}{n}h \quad \text{and} \quad S = n\lambda.$$

Since $n \geq 3$, the Schur Lemma ensures that S is constant, which in turn forces λ to be constant. Hence, $(N, h, \aleph, \lambda)$ is a trivial. □

We now derive rigidity results under a natural curvature condition.

Theorem 6.2. *Let $(N, h, \aleph, \lambda)$ be an almost Ricci soliton with $\aleph$ a nonzero geodesic affine vector field of constant length on (N, h). Then, $(N, h, \aleph, \lambda)$ is trivial.*

Proof. From (6.2), we have

$$Q = \lambda I,$$

which, by (1.1), implies that $\aleph$ is a Killing. Therefore, $(N, h, \aleph, \lambda)$ is trivial. □

Theorem 6.3. *Let $(N, h, \aleph, \lambda)$ be an almost Ricci soliton with $\aleph$ a nontrivial affine vector field of constant length on (N, h). If $\mathrm{Rc}(\aleph, \aleph) \leq 0$, then the Ricci soliton is parallel.*

Proof. Since $\aleph$ has constant length, we have $\Delta f = 0$. Using Lemma 6.1, we obtain

$$\|\nabla\aleph\|^2 - \mathrm{Rc}(\aleph, \aleph) = 0.$$

Since $\mathrm{Rc}(\aleph, \aleph) \leq 0$, it follows that $\|\nabla\aleph\|^2 = 0$ hence, $\nabla\aleph = 0$. □

Consequently, we drive the next result.

Corollary 1. *Under the assumptions of Theorem 6.3, the manifold (N, h) is Ricci flat.*

Proof. From Eq (1.1), we obtain

$$\lambda h(\aleph, \aleph) = 0.$$

Since $\aleph$ is nontrivial, it follows that $\lambda = 0$, and hence (N, h) is Ricci flat. □

Theorem 6.4. *Let $(N, h, \aleph, \lambda)$ be an almost Ricci soliton with $\aleph$ an affine vector field on (N, h). If the following inequality holds:*

$$\Delta f \leq \|\phi\|^2 - \mathrm{Rc}(\aleph, \aleph),$$

where $f = \frac{1}{2}h(\aleph, \aleph)$ and ϕ represents the skew-symmetric part of $\nabla\aleph$, then $(N, h, \aleph, \lambda)$ is trivial.

Proof. Using (6.1), we have

$$\|\nabla\aleph\|^2 \leq \|\phi\|^2.$$

From (2.11), the decomposition of $\nabla\aleph$ into symmetric and skew-symmetric parts, this implies

$$\|\lambda I - Q\|^2 \leq 0,$$

and hence $Q = \lambda I$. Therefore, by (1.1), $(N, h, \aleph, \lambda)$ is trivial. □

Theorem 6.5. *Let $(N, h, \aleph, \lambda)$ be an almost Ricci soliton with $\aleph$ a nonzero geodesic affine vector field on (N, h). If*

$$\mathrm{Rc}(\aleph, \aleph) + \|\lambda I - Q\|^2 \leq 0, \tag{6.11}$$

then $(N, h, \aleph, \lambda)$ is parallel. In particular, (N, h) is Ricci flat.

Proof. Since $\aleph$ is a geodesic, we can apply the Bochner formula, which gives us the following equation:

$$\mathrm{Rc}(\aleph, \aleph) + \mathrm{trace}\left((\nabla\aleph)^2\right) = 0.$$

Using Eq (2.12), we obtain

$$\mathrm{Rc}(\aleph, \aleph) = \|\phi\|^2 - \|\lambda I - Q\|^2. \tag{6.12}$$

By the hypothesis (6.11), we have $\phi = 0$. It then follows from (6.3) that $Q(\aleph) = -\operatorname{div}(\phi) = 0$. Since $\aleph$ is geodesic, (2.10) yields $\lambda\aleph = 0$. As we assume that $\aleph \neq 0$, we conclude that $\lambda = 0$. Since $\mathrm{Rc}(\aleph, \aleph) = 0$, it follows from (6.12) that $Q = 0$. Therefore, $(N, h, \aleph, \lambda)$ is parallel. □

Theorem 6.6. *Let $(N, h, \aleph, \lambda)$ be an almost Ricci soliton with $\aleph$ an affine vector field on a connected Riemannian manifold (N, h). If $\Delta h(\aleph, \aleph) \leq 0$ and $\mathbf{H}^f(\aleph, \aleph) \leq 0$, then $(N, h, \aleph, \lambda)$ is a trivial Ricci soliton.*

Proof. For all vector fields $\zeta, \eta \in \mathfrak{X}(N)$, we have

$$\zeta h(\nabla f, \eta) = \zeta h(\nabla_\eta \aleph, \aleph).$$

Using this result along with (2.2), we can derive

$$\mathbf{H}^f(\zeta, \eta) = h(\nabla^2_{\aleph}(\zeta, \eta), \aleph) + h(\nabla_\zeta \aleph, \nabla_\eta \aleph),$$

for all $\zeta, \eta \in \mathfrak{X}(N)$. Set $\zeta = \eta = \aleph$ in this equation, we obtain $\mathbf{H}^f(\aleph, \aleph) = h(\nabla_\aleph \aleph, \nabla_\aleph \aleph)$. By hypothesis, we conclude that $\aleph$ is a geodesic vector field. We deduce, using (6.2), that $Q = \lambda I$. Therefore, based on (1.1), we find that the Ricci soliton is trivial. □

Theorem 6.7. *Let $(N, h, \aleph, \lambda)$ be an almost Ricci soliton with $\aleph$ an affine vector field on a connected Riemannian manifold (N, h). If $\mathbf{H}^f(\aleph, \aleph) \leq 0$ and $\mathrm{Rc}(\aleph, \aleph) \geq \|\phi\|^2$, then $(N, h, \aleph, \lambda)$ is trivial.*

Proof. Since $\mathbf{H}^f(\aleph, \aleph) \leq 0$, as shown in the proof of Theorem 6.6, it follows that $\aleph$ is a geodesic. By the hypothesis and the Bochner formula, we conclude that $Q = \lambda I$. Therefore, the Ricci soliton is trivial. □

We conclude by examining the compact case. It is well known that every affine vector field is necessarily a Killing field on compact Riemannian manifolds (see [13, 19, 20]). This fundamental fact imposes restrictions on almost Ricci solitons with affine potential vector fields and leads to important consequences (see also [21]).

Corollary 2. *Every almost Ricci soliton $(N, h, \aleph, \lambda)$ on a compact connected n-dimensional Riemannian manifold (N, h), $n \geq 3$, with an affine potential vector field is necessarily a trivial Ricci soliton. In particular, the soliton function λ and the scalar curvature S are constant, satisfying*

$$S = n\lambda,$$

and the underlying manifold (N, h) is Einstein.

Proof. By Theorem 4 in [13], $\aleph$ must be a Killing. Thus, $(N, h, \aleph, \lambda)$ is trivial. Furthermore, (2.13) implies $S = n\lambda$. □

Theorem 6.8. *Let $(N, h, \aleph, \lambda)$ be an almost Ricci soliton with $\aleph$ a nonzero affine vector field on an n-dimensional compact Riemannian manifold (N, h), $n \geq 3$. If $\lambda \leq 0$, then $(N, h, \aleph, \lambda)$ is a parallel Ricci soliton.*

Proof. From (6.12), we get

$$\int_N \mathrm{Rc}(\aleph, \aleph) dv = \int_N \|\phi\|^2 dv,$$

which implies

$$\lambda \int_N g(\aleph, \aleph) dv = \int_N \|\phi\|^2 dv.$$

Since $\lambda \leq 0$ and $\aleph \neq 0$, it follows $\lambda = 0$ and $\phi = 0$. From (1.1), $Q = 0$, and $\nabla\aleph = 0$. Hence, $(N, h, \aleph, \lambda)$ is parallel. □

According to Theorem 3.9 in [8], we also present the following.

Corollary 3. *Let $(N, h, \aleph, \lambda)$ be an almost Ricci soliton with $\aleph$ an affine vector field on an n-dimensional compact Riemannian manifold (N, h), $n \geq 3$. If*

$$\int_N \left(\mathrm{Rc}(\aleph, \aleph) + n\lambda^2 \right) dv \leq 0, \tag{6.13}$$

then $(N, h, \aleph, \lambda)$ is steady and parallel.

Proof. Since N is compact, Corollary 2 ensures that the soliton is trivial. From Theorem 3.5 in [8], we deduce that (N, h) is Ricci flat. Finally, $(N, h, \aleph, \lambda)$ is parallel. □

7. Conclusions

In this work, we investigated almost Ricci solitons with affine potential vector fields on semi-Riemannian manifolds. We proved that, in all dimensions $n \geq 3$, such solitons reduce to Ricci solitons, and the manifolds must have constant scalar curvature, implying, in particular, Ricci symmetry. In contrast, in dimension 2, as in Section 5, explicit counterexamples of proper almost Ricci solitons were constructed. We also discussed several illustrative examples, including Euclidean space, space forms, and certain warped product manifolds, which highlight the rigidity imposed by the affine condition.

Future directions. It is natural to ask whether similar rigidity results hold in other settings. Thus possible directions include:

- Gradient almost Ricci solitons with concircular potential vector fields.
- Pseudo-symmetric or quasi-Einstein manifolds.
- Solvable or nilpotent Lie groups with left-invariant metrics.

Studying these cases may provide more insight into the relationship between affine vector fields and soliton-type structures in geometric analysis.

Author contributions

Norah Alshehri: Investigation, Conceptualization, Methodology, Resources, Writing-original draft, Writing-review and editing; Mohammed Guediri: Investigation, Conceptualization, Methodology, Validation, Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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