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Research article

Covering theory for strict 2-groupoids via actions, quotients, and crossed modules over groupoids

Sedat Temel*

Department of Mathematics, Recep Tayyip Erdogan University, Rize 53100, Türkiye

* **Correspondence:** Email: sedat.temel@erdogan.edu.tr.

Abstract: This paper establishes a direct correspondence between the geometric covering theory of strict 2-groupoids and the algebraic lifting theory of crossed modules over groupoids. Working entirely within an object-preserving framework, we prove a chain of categorical equivalences linking strict 2-groupoid actions, covering strict 2-groupoids, liftings of crossed modules, and covering crossed modules. Additionally, we introduce a quotient theory for strict 2-groupoids via normal 2-group bundles. We demonstrate that this quotient construction preserves the covering property and naturally induces canonical lifting crossed modules, providing a unified perspective on these geometric and algebraic structures.

Keywords: strict 2-groupoid; crossed module; covering; lifting; quotient groupoid
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1. Introduction

The algebraic structures investigated in this paper originate from the intersection of homotopy theory and higher-dimensional algebra. The concept of crossed modules was initially introduced by Whitehead in the late 1940s [1, 2] to capture non-abelian homotopical data (specifically, 2-types) that classical fundamental groups could not express. In the ensuing decades, the integration of groupoid theory into this topological framework was pioneered by Brown and Higgins, who established higher-dimensional groupoids as powerful algebraic models for topological spaces [3, 4].

A critical milestone in this historical development was achieved by Brown and Spencer [5], who algebraically translated these topological concepts by proving the equivalence between crossed modules of groups and 2-groups (also known as group-groupoids). This algebraic dictionary was subsequently generalized by İçen [6] to a broader equivalence between crossed modules over groupoids and strict 2-groupoids, opening new avenues for exploring homotopies [7] and related higher categorical formulations [8, 9]. Concurrently, the geometric concept of covering spaces was systematically algebraized into covering groupoids by Brown et al. [10], which eventually paved the way for higher-dimensional covering theories, such as coverings of topological groups [11] and, more recently, coverings of group-groupoids [12].

Covering morphisms for groupoids and topological groups were classically investigated in [10, 11]. More recently, Mucuk and Şahan [12] established an equivalence between coverings of group-groupoids and liftings of crossed modules. Such covering and action phenomena also appear broadly in Galois theories [13] and various internal categorical structures (e.g., groups with operations [14], crossed semimodules [15], and cat1-groupoids [15,16]). In the specific setting of groupoids, Şahan [17] introduced the definition of lifting crossed modules over groupoids and established their equivalence with covering crossed modules over groupoids.

Our strict categorical models can be interpreted as rigidified, object-preserving special cases of 2-group stacks and Picard 2-stacks [18–20]. Within this broader framework, our covering and lifting equivalences naturally correspond to torsors and extensions of Picard 2-stacks via length 3 complexes. Furthermore, these results complement the butterfly formalism of Aldrovandi and Noohi [21, 22]. While butterflies are designed to handle weak morphisms and localizations, our framework restricts to a strict, fixed-base environment, offering a computationally direct, algebraic approach to covering actions and normal quotients.

In non-abelian algebraic topology, groupoids and their coverings naturally encode local-to-global phenomena [3, 23]. In this paper, we study these geometric interactions for strict 2-groupoids using an object-preserving framework inspired by Brown [23]. This contrasts with Higgins' classical approach [24, 25], which typically factors the object sets themselves. Our framework leaves the underlying object set completely unchanged, requiring a rigorous treatment of normality and quotients as established in [26].

While the equivalence between strict 2-groupoids and crossed modules over groupoids is a classical result established by Brown and İçen [6, 7], and the relationship between covering crossed modules and their liftings was recently formulated by Şahan [17], the direct translation of these covering and lifting mechanisms into the internal geometry of strict 2-groupoids has remained an open problem.

The main purpose of this paper is to bridge this gap. To clarify the novelty of our work, our primary original contributions are threefold:

- A Geometric Formulation of Coverings and Actions: We introduce a novel definition of covering morphisms for strict 2-groupoids entirely in terms of vertical stars (Section 3). We then define actions of strict 2-groupoids on groupoids and prove that the category of these actions is naturally equivalent to the category of covering 2-groupoids (Theorem 4.5).

- The Three-Way Equivalence: By synthesizing our new geometric covering/action results with the classical Brown and İçen correspondence, we establish a new equivalence between strict 2-groupoid actions and liftings of crossed modules over groupoids (Theorem 5.3). This completes a fully coherent, novel chain of categorical equivalences linking coverings, actions, and liftings across these structures (Corollary 5.4).

- An Object-Preserving Quotient Theory: We develop a new quotient architecture for strict 2-groupoids based on normal 2-group bundles that leaves the underlying object set completely intact (Section 6). Crucially, we demonstrate that this original quotient construction preserves the covering property (Theorem 6.9) and naturally induces canonical lifting crossed modules (Theorem 6.10).

The remainder of this paper is organized as follows. Section 2 recalls the preliminary concepts regarding groupoids, crossed modules, liftings, and strict 2-groupoids to establish the foundational notation. In Section 3, we introduce covering morphisms of strict 2-groupoids within an object-preserving setting. Section 4 is devoted to defining actions of strict 2-groupoids on groupoids and establishing the categorical equivalence between these actions and covering morphisms. In Section 5, we utilize the Brown and İçen correspondence to prove the equivalence between strict 2-groupoid actions and liftings of crossed modules over groupoids. Section 6 presents a novel object-preserving quotient theory for strict 2-groupoids via normal 2-group bundles and demonstrates its applications to canonical liftings. Finally, Section 7 concludes the paper with a brief discussion on potential future research directions.

2. Preliminaries

2.1. Categories and groupoids

A category $\mathbb{X} = (A, X)$ consists of a class A of objects and a class X of morphisms. For every morphism $x \in X$, there are source and target maps, denoted by d_0 and d_1, respectively, such that $d_0(x) = a$ and $d_1(x) = b$, typically represented as $a \xrightarrow{x} b$. The category is equipped with an associative composition law: For any morphisms $x : a \to b$ and $y : b \to c$, there exists a composite $y \circ x : a \to c$.

$$a \xrightarrow{x} b \xrightarrow{y} c \quad \Rightarrow \quad a \xrightarrow{y \circ x} c.$$

Furthermore, for each object $a \in A$, there exists an identity morphism $1_a = \varepsilon(a)$ such that $x \circ 1_a = x$ and $1_b \circ x = x$ for any $x \in X(a, b)$, where $X(a, b)$ denotes the class of all morphisms from a to b. We denote by $X(a)$ the set of all endomorphisms at a. A category $\mathbb{X}$ is called a groupoid if every morphism $x \in X(a, b)$ is an isomorphism; that is, there exists a unique inverse $\overline{x} \in X(b, a)$ such that $x \circ \overline{x} = 1_b$ and $\overline{x} \circ x = 1_a$. A groupoid $\mathbb{X}$ is called a totally intransitive (or totally disconnected) groupoid (i.e., a bundle of groups) if $X(a, b) = \emptyset$ whenever $a \neq b$. For further details, we refer the reader to [23, 27].

In accordance with established literature [3, 4, 7], we assume all groupoids and higher-order structures throughout this study are defined over a fixed object set, namely "A". This convention ensures the necessary coherence for multi-dimensional compositions and maintains consistent alignment of domains and codomains across all layers of morphisms.

Let A be a set and X be a group. Following Mackenzie [28], the *trivial groupoid* $\mathbb{X} = (A, A \times X \times A)$ is defined where (a, x, b) is regarded as a morphism $a \xrightarrow{(a,x,b)} b$. The composition of morphisms is given by $(b, x', c) \circ (a, x, b) = (a, xx', c)$ for $a, b, c \in A$ and $x, y \in X$.

A morphism of groupoids can be defined as follows: Let $\mathbb{X}$ and $\mathbb{X}'$ be groupoids. A morphism $F = (F_0, F_1)\colon \mathbb{X} \to \mathbb{X}'$ is a functor consisting of a pair of maps—an object map $F_0\colon A \to A'$ and a morphism map $F_1\colon X \to X'$—that satisfies the following functorial conditions:

(1) For every object $a \in A$, $F_1(1_a) = 1_{F_0(a)}$;

(2) For any two composable morphisms $x, y \in X$, $F_1(y \circ x) = F_1(y) \circ F_1(x)$.

Throughout this paper, for the sake of notational brevity, we shall use F to represent both F_0 and F_1 when the distinction is clear from the context. As a direct consequence of these functorial properties, a groupoid morphism also preserves inverses, meaning $F(\bar{x}) = \overline{F(x)}$ for all $x \in X$. Groupoids and their morphisms, as defined above, form a category denoted by **Gpd**.

The notions of a normal subgroupoid and its associated quotient groupoid are adopted from the foundational framework established by Brown [23]. Let $\mathbb{X} = (A, X)$ be a groupoid and (A, N) be a wide subgroupoid of $\mathbb{X}$. We say that N is a normal subgroupoid of $\mathbb{X}$ (denoted by $N \trianglelefteq \mathbb{X}$) if it satisfies the following conditions:

(1) N is totally disconnected, meaning N is a family of groups $\{N(a)\}_{a \in A}$, where each $N(a)$ is a subgroup of the vertex group $X(a)$.

(2) N is closed under conjugation by elements of $\mathbb{X}$. That is, for any object $a \in A$, any loop $n \in N(a)$, and any morphism $a \xrightarrow{x} b$ in $\mathbb{X}$, the conjugated morphism $x \circ n \circ \overline{x}$ belongs to $N(b)$.

Let $N \trianglelefteq \mathbb{X} = (A, X)$. The quotient groupoid, denoted by $\mathbb{X}/N = (A, X/N)$, is constructed as follows:

(1) The object set of $\mathbb{X}/N$ is identical to that of $\mathbb{X}$.

(2) For any two objects a, b, the set of morphisms $(X/N)(a, b)$ consists of the equivalence classes (cosets) of the form $xN(a) = \{x \circ n \mid n \in N(a)\}$, where $a \xrightarrow{x} b$ is a morphism in $\mathbb{X}$.

(3) The composition in $\mathbb{X}/N$ is defined by $(yN(b)) \circ (xN(a)) = (y \circ x)N(a)$ whenever $a \xrightarrow{x} b \xrightarrow{y} c$.

The canonical projection $\pi\colon (A, X) \to (A, X/N)$, given by $\pi(x) = xN(d_0(x))$, forms a groupoid morphism that annihilates N by sending its elements to identity morphisms.

In what follows, we utilize a specific formulation of the first isomorphism theorem for groupoids. It is worth noting a slight but methodologically crucial refinement in our approach compared to the classical literature. In [23, p. 330], Brown established this theorem under the more general assumption that the underlying object map of the groupoid morphism is a bijection. However, since our framework for covering strict 2-groupoids is constructed strictly in an object-preserving setting—where the object sets of the domain and codomain are identical—we specialize this result to morphisms that act strictly as the identity on objects. This leads to the following precise formulation:

Theorem 2.1. *Let $\mathbb{X} = (A, X)$ and $\mathbb{X}' = (A, X')$ be groupoids, and let $F\colon \mathbb{X} \to \mathbb{X}'$ be a groupoid morphism that is the identity on objects. Then the image $(A, F(X))$ is a wide subgroupoid of $\mathbb{X}'$, and the canonical morphism $F^*\colon \mathbb{X}/\mathrm{Ker}F \to (A, F(X))$ defined on arrows by $F^*(x\mathrm{Ker}F) = F(x)$ is an isomorphism of groupoids.*

2.2. Crossed modules over groupoids and liftings

In this subsection, we review the fundamental structure of crossed modules over groupoids and establish the conceptual framework for their liftings. The notion of crossed modules over groupoids was originally introduced by Brown and Higgins [4, 29] as a higher-dimensional generalization of crossed modules of groups [1, 2], providing a powerful tool for modeling 2-types in algebraic topology. This theory was further extended by Brown and İçen [7] in the context of homotopies and automorphisms (see also [9, 30]).

Let $U = (A, U)$ be a totally intransitive groupoid and $V = (A, V)$ a groupoid over the same object set. An action of $\mathbb{V}$ on $\mathbb{U}$ (in the sense of Brown and Higgins [4,29]) is a partially defined map $(v, u) \mapsto v \cdot u$ from $V \times_A U$ to U such that the following conditions are satisfied:

[ActGd 1] $v \cdot u$ is defined if and only if $d_0(v) = d_1(u)$ and $d_1(v \cdot u) = d_1(v)$ (in which case, $d_0(v \cdot u) = d_1(v)$).

$$a \xrightarrow{u} a \xrightarrow{v} b \quad \Rightarrow \quad b \xrightarrow{v \cdot u} b.$$

[ActGd 2] $(v' \circ v) \cdot u = v' \cdot (v \cdot u)$ for all composable $v, v' \in V$ and $u \in U$.

[ActGd 3] $v \cdot (u' \circ u) = (v \cdot u') \circ (v \cdot u)$ for $u, u' \in U(a)$ and $v \in V(a, b)$.

[ActGd 4] $1_a \cdot u = u$ for all $u \in U(a)$ and $a \in A$.

Let $\mathbb{U}$ and $\mathbb{V}$ be groupoids over a common object set A. Suppose further that $\mathbb{U}$ is totally disconnected. A crossed module $C = (\mathbb{U}, \mathbb{V}, C)$ over groupoids, as defined in [7], consists of a groupoid morphism $C\colon \mathbb{U} \to \mathbb{V}$, which is the identity on objects, together with an action of $\mathbb{V}$ on $\mathbb{U}$ as defined above, satisfying the following axioms for all $u, u' \in U(a)$ and $v \in V(a, b)$:

[CMGd 1] $C(v \cdot u) = v \circ C(u) \circ \bar{v}$.

[CMGd 2] $C(u) \cdot u' = u \circ u' \circ \bar{u}$.

Let $C = (\mathbb{U}, \mathbb{V}, C)$ and $C' = (\mathbb{U}', \mathbb{V}', C')$ be crossed modules over groupoids. A morphism $\langle D_0, D_1, D_2 \rangle : C \to C'$ of crossed modules is a pair of groupoid morphisms $(D_0, D_1)\colon \mathbb{U} \to \mathbb{U}'$ and $(D_0, D_2) : \mathbb{V} \to \mathbb{V}'$, both being the identity on objects, such that

(1) $D_2 C = C' D_1$,

(2) $D_1(v \cdot u) = D_2(v) \cdot' D_1(u)$ for all $(v, u) \in V \times_A U$.

Crossed modules over groupoids and their morphisms form a category denoted by **CMGd**.

Following the developments in [17], we present the definition and some properties of liftings for these structures.

Let $C = (\mathbb{U}, \mathbb{V}, C)$ be a crossed module over groupoids, and let $\mathbb{W}$ be a groupoid defined over the same class of objects A. Let $E\colon \mathbb{W} \to \mathbb{V}$ be a groupoid morphism that acts as the identity on objects. A crossed module $(\mathbb{U}, \mathbb{W}, \widehat{C})$ is called a lifting of $(\mathbb{U}, \mathbb{V}, C)$ over E if:

[LiftCM 1] The action of $\mathbb{W}$ on $\mathbb{U}$ is induced by E; that is, $w \star u = E(w) \cdot u$ for all $w \in W$ and $u \in U$ whenever the action is defined.

[LiftCM 2] The following diagram of groupoid morphisms commutes:

$$\begin{tikzcd} & \mathbb{W} \arrow[d, "E"] \\ \mathbb{U} \arrow[ur, "\widehat{C}"] \arrow[r, "C"'] & \mathbb{V} \end{tikzcd}$$

Such a lifting is typically denoted by $(\widehat{C}, \mathbb{W}, E)$. If $(\widehat{C}, \mathbb{W}, E)$ is a lifting of $C = (\mathbb{U}, \mathbb{V}, C)$, then $\langle 1_A, 1_U, E\rangle$ is a morphism of crossed modules over groupoids. Note that every crossed module of groupoids $C = (\mathbb{U}, \mathbb{V}, C)$ canonically lifts to itself via the identity groupoid morphism $1_{\mathbb{V}} \colon \mathbb{V} \to \mathbb{V}$.

Let $C = (\mathbb{U}, \mathbb{V}, C)$ be a crossed module over groupoids. Suppose $(\mathbb{U}, \mathbb{W}, \widehat{C})$ and $(\mathbb{U}, \mathbb{W}', \widehat{C}')$ are liftings of C over E and E', respectively. A groupoid morphism $H \colon \mathbb{W} \to \mathbb{W}'$ is called a morphism of liftings if it satisfies:

(1) $H\widehat{C} = \widehat{C}'$,

(2) $E'H = E$.

The liftings of C and their corresponding morphisms form a category, denoted by $\mathbf{L\small{IFT}CMG\small{D}}/C$.

Let $\widetilde{C} = (\widetilde{\mathbb{U}}, \widetilde{\mathbb{V}}, \widetilde{C})$ and $C = (\mathbb{U}, \mathbb{V}, C)$ be two crossed modules over groupoids over the same object class A. A morphism of crossed modules $\langle D_0, D_1, D_2\rangle \colon \widetilde{C} \longrightarrow C$ is called a *covering morphism of crossed modules over groupoids* if the underlying groupoid morphism $(D_0, D_1) \colon \widetilde{\mathbb{U}} \longrightarrow \mathbb{U}$ is an isomorphism of groupoids. In this case, $\widetilde{C}$ is said to be a *covering crossed module* of C over groupoids.

The following theorem, established as the main result in [17], provides the fundamental relationship between liftings and coverings of crossed modules over groupoids:

Theorem 2.2. *For any crossed module of groupoids $C = (\mathbb{U}, \mathbb{V}, C)$, the category* $\mathbf{L\small{IFT}CMG\small{D}}/C$ *of liftings is naturally equivalent to the category* $\mathbf{Cov(CMG\small{D})}/C$ *of covering crossed modules over* C.

2.3. 2-groupoids

Following the formalisms in [31, 32], we define a strict 2-category $\mathbb{G} = (A, X, G)$ as a structure consisting of a class A of objects (0-cells), a class X of 1-morphisms (1-cells), and a class G of 2-morphisms (2-cells). A 2-morphism g is visualized as follows:

$$\begin{tikzcd} a \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow g"] & b, \end{tikzcd}$$

where the maps $d_0^h(g) = a$ and $d_1^h(g) = b$ denote the horizontal source and target, while $d_0^v(g) = x$ and $d_1^v(g) = y$ denote the vertical source and target, respectively. The composition of 1-morphisms follows the axioms of an ordinary category. The 2-morphisms are subject to two associative composition laws:

- **Horizontal composition:** For 2-morphisms $g : x \Rightarrow y : a \to b$ and $g' : x' \Rightarrow y' : b \to c$, their horizontal composite $g' * g : x' \circ x \Rightarrow y' \circ y : a \to c$ is given by

$$\begin{tikzcd} a \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow g"] & b \arrow[r, bend left, "x'"] \arrow[r, bend right, "y'"'] \arrow[r, phantom, "\Downarrow g'"] & c & = & a \arrow[r, bend left, "x' \circ x"] \arrow[r, bend right, "y' \circ y"'] \arrow[r, phantom, "\Downarrow g' * g"] & z. \end{tikzcd}$$

- **Vertical composition:** For 2-morphisms $g : x \Rightarrow y : a \to b$ and $h : y \Rightarrow z : a \to b$, their vertical composite $h \bullet g : x \Rightarrow z : a \to b$ is given by

$$\begin{tikzcd} a \arrow[r, bend left=60, "x"] \arrow[r, "y" description] \arrow[r, bend right=60, "z"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow g"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow h"] & b & = & a \arrow[r, bend left, "x"] \arrow[r, bend right, "z"'] \arrow[r, phantom, "\Downarrow h \bullet g"] & b. \end{tikzcd}$$

These operations must satisfy the interchange rule, stating that $(h' * h) \bullet (g' * g) = (h' \bullet g') * (h \bullet g)$ whenever the compositions are well-defined. The structure also includes horizontal identity maps $\varepsilon^h(a) = 1_{1_a}$ and vertical identity maps $\varepsilon^v(x) = 1_x$, satisfying the standard identity axioms. A 2-functor $F = (F_0, F_1, F_2)\colon \mathbb{G} \to \mathbb{G}'$ is a mapping that strictly preserves objects, 1-morphisms, 2-morphisms, and all their respective compositions and identities:

$$\begin{tikzcd} a \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow g"] & b \end{tikzcd} \mapsto \begin{tikzcd} F_0(a) \arrow[r, bend left, "F_1(x)"] \arrow[r, bend right, "F_1(y)"'] \arrow[r, phantom, "\Downarrow F_2(g)"] & F_0(b). \end{tikzcd}$$

The collection of all strict 2-categories and 2-functors forms a category denoted by **2Cat** [9]. A strict 2-groupoid is a 2-category where every 1-morphism is an isomorphism and every 2-morphism is invertible with respect to both composition laws:

$$\begin{tikzcd} a \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow g"] & b \arrow[r, bend left, "\bar{x}"] \arrow[r, bend right, "\bar{y}"'] \arrow[r, phantom, "\Downarrow \bar{g}^h"] & a \end{tikzcd} = \begin{tikzcd} a \arrow[r, bend left, "1_a"] \arrow[r, bend right, "1_a"'] \arrow[r, phantom, "\Downarrow 1_{1_a}"] & a, \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, bend left=50, "x"] \arrow[r, "y"] \arrow[r, bend right=50, "x"'] \arrow[r, phantom, shift left=3, "\Downarrow g"] \arrow[r, phantom, shift right=3, "\Downarrow \bar{g}^v"] & b \end{tikzcd} = \begin{tikzcd} a \arrow[r, bend left, "x"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "\Downarrow 1_x"] & b. \end{tikzcd}$$

Strict 2-groupoids and their morphisms (2-functors) constitute the category **2Gpd** [9]. For a 1-morphism $x \in X$, the vertical star of $\mathbb{G}$ at x is defined as $St^v_{\mathbb{G}}(x) = (d^v_0)^{-1}(x)$.

A 2-groupoid structure $\mathbb{G} = (A, X, G)$ inherently encapsulates compatible groupoid structures. Specifically, these consist of the underlying groupoid (A, X) formed by objects and 1-morphisms, the groupoid (A, G) formed by objects and 2-morphisms under horizontal composition, and the groupoid (X, G) formed by 1-morphisms and 2-morphisms under vertical composition. The mutual compatibility of these three structures is precisely governed by the interchange law between the vertical and horizontal compositions.

The trivial groupoid $\mathbb{X}$ can be extended to a strict 2-groupoid $\mathbb{G}$. In this construction, 2-morphisms are defined as quadruples $(a, x, y, b) : (a, x, b) \Rightarrow (a, y, b)$ for $x, y \in X$, where the vertical composition of 2-morphisms is given by $(a, y, z, b) \circ_v (a, x, y, b) = (a, x, z, b)$ and the horizontal composition is given by $(b, x', y', c) \circ_h (a, x, y, b) = (a, xx', yy', c)$. This construction equips $\mathbb{G}$ with a strict 2-groupoid structure, which is called the *trivial* 2*-groupoid*.

The following fundamental result was established by İçen in [7]. We provide a sketch of the proof using our current notation to highlight certain details required in the subsequent sections.

Theorem 2.3. *The category of strict 2-groupoids is naturally equivalent to the category of crossed modules over groupoids.*

Proof. We first define a functor $\Delta\colon$ **2Gpd** $\to$ **CMGd**. Let $\mathbb{G} = (A, X, G)$ be a strict 2-groupoid. The associated crossed module $\Delta(\mathbb{G}) = (\mathbb{U}, \mathbb{V}, C)$ is constructed such that $\mathbb{V} = (A, X)$, $\mathbb{U} = (A, \mathrm{Ker} d^v_0)$, where $\mathrm{Ker} d^v_0 = \{g \in G \mid d^v_0(g) = 1_{d^h_0(g)}\}$, $C\colon \mathbb{U} \to \mathbb{V}$ is defined by $C(g) = d^v_1(g)$, where the action of $\mathbb{V}$ on $\mathbb{U}$ is given by conjugation: $x \cdot g = 1_x * g * 1_{\bar{x}}$ for $x \in X(a, b)$ and $g \in U(a)$. For a morphism of 2-groupoids $F = (F_0, F_1, F_2)$, the induced morphism of crossed modules is $\Delta(F) = \langle F_0, F_1, F_2|_{\mathrm{Ker} d^v_0}\rangle$.

Conversely, we define a functor $\Gamma\colon \mathbf{CMGd} \to \mathbf{2Gpd}$. Let $C = (\mathbb{U}, \mathbb{V}, C)$ be a crossed module over groupoids $\mathbb{U} = (A, U)$ and $\mathbb{V} = (A, V)$. The associated 2-groupoid $\Gamma(C) = (A, V, V \ltimes U)$ is constructed using the semi-direct product $V \ltimes U = \{(v, u) \mid v \in V, u \in U, d_0(u) = d_1(u) = d_1(v)\}$. A 2-morphism $(v, u) \in V \ltimes U$ is represented as

$$\begin{tikzcd} a \arrow[r, bend left, "v"{name=U}] \arrow[r, bend right, "C(u)\circ v"'{name=D}] & b \arrow[from=U, to=D, phantom, "\Downarrow (v,u)"] \end{tikzcd} .$$

The structural maps and compositions in $\Gamma(C)$ are defined as follows:

- Source/Target: $d_0^h(v, u) = d_0(v)$, $d_1^h(v, u) = d_1(v)$, $d_0^v(v, u) = v$ and $d_1^v(v, u) = C(u) \circ v$.
- Horizontal Composition: $(v', u') * (v, u) = (v' \circ v, u' \circ (v' \cdot u))$ for $(v, u), (v', u') \in V \ltimes U$, where $d_0(v') = d_1(v)$.
- Vertical Composition: $(v_1, u_1) \bullet (v, u) = (v, u_1 \circ u)$ for $(v, u), (v_1, u_1) \in V \ltimes U$, where $v_1 = C(u) \circ v$.
- Identities: $\varepsilon^h(a) = (1_a, 1_a)$, $\varepsilon^v(v) = (v, 1_{d_1(v)})$.
- Inverses: $\overline{(v, u)}^v = (C(u) \circ v, \bar{u})$ and $\overline{(v, u)}^h = (\bar{v}, \bar{v} \cdot \bar{u})$.

A natural equivalence $\eta\colon \mathbf{1_{2Gpd}} \Rightarrow \Gamma\Delta$ is defined for each 2-groupoid $\mathbb{G}$ by a morphism $\eta_{\mathbb{G}}\colon \mathbb{G} \to \Gamma\Delta(\mathbb{G})$. This morphism is the identity on objects and 1-morphisms, and on 2-morphisms it maps $g\colon x \Rightarrow y$ to $(x, g * 1_{\bar{x}})$. Similarly, a natural equivalence $\mu\colon \mathbf{1_{CMGd}} \Rightarrow \Delta\Gamma$ is defined for each crossed module C by $\mu_C\colon C \to \Delta\Gamma(C)$, which is the identity on objects and the groupoid $\mathbb{V}$, and maps $u \in U$ to $(1_{d_1(u)}, u)$. Both η and μ are natural isomorphisms, establishing the equivalence of categories. $\square$

3. Covering morphisms of strict 2-groupoids

In this section, we introduce and study covering morphisms of strict 2-groupoids in an object-preserving setting. Motivated by the classical covering theories of groupoids and 2-groups [12], we restrict our attention to morphisms that act identically on the underlying object set. This viewpoint is consistent with the strict correspondence between crossed modules over groupoids and strict 2-groupoids developed in the previous section.

The notion of covering adopted here is formulated in terms of vertical stars of 2-morphisms. More precisely, a morphism of strict 2-groupoids is called a covering whenever it induces a bijection on each vertical star. We investigate the basic properties of such coverings, establish their relationship with coverings of crossed modules over groupoids, and show that the covering property is preserved under suitable quotient constructions. These results provide the structural foundation for the categorical equivalences involving coverings, actions, and liftings that will be developed in the subsequent sections.

Definition 3.1. *Let $\mathbb{G} = (A, X, G)$ and $\widetilde{\mathbb{G}} = (A, \widetilde{X}, \widetilde{G})$ be two strict 2-groupoids. A morphism of 2-groupoids $P\colon \widetilde{\mathbb{G}} \to \mathbb{G}$ is called a covering morphism if it acts as the identity on objects and satisfies the following condition: For every 1-morphism $\tilde{x} \in \widetilde{X}$ in the covering structure, the restriction $P_* = P\big|_{St^v_{\widetilde{\mathbb{G}}}(\tilde{x})}$ on the vertical stars*

$$P_*\colon St^v_{\widetilde{\mathbb{G}}}(\tilde{x}) \longrightarrow St^v_{\mathbb{G}}(P(\tilde{x}))$$

is a bijection. In this case, $\widetilde{\mathbb{G}}$ is referred to as a covering 2-groupoid of $\mathbb{G}$.

Example 3.2. *For every strict 2-groupoid $\mathbb{G} = (A, X, G)$, the identity morphism $1_{\mathbb{G}} : \mathbb{G} \to \mathbb{G}$ is a covering morphism, since it induces the identity bijection on each vertical star.*

Now, utilizing the trivial 2-groupoid construction, we provide an illustrative example of a covering morphism.

Example 3.3. *Let $p : \tilde{X} \to X$ be a group isomorphism. We denote by $\tilde{\mathbb{G}}$ and $\mathbb{G}$ the trivial 2-groupoids constructed over the same object set A using the groups $\tilde{X}$ and X, respectively. Consider the induced object-preserving strict 2-groupoid morphism $P = (id_A, p, p) : \tilde{\mathbb{G}} \to \mathbb{G}$, which maps each 2-morphism $(a, \tilde{x}, \tilde{y}, b)$ in $\tilde{\mathbb{G}}$ to $(a, p(\tilde{x}), p(\tilde{y}), b)$ in $\mathbb{G}$. Since p is an isomorphism, the induced map on the vertical stars*

$$P_* : St^v_{\tilde{\mathbb{G}}}(a, \tilde{x}, b) \longrightarrow St^v_{\mathbb{G}}(a, p(\tilde{x}), b),$$

defined by $P_*(a, \tilde{x}, \tilde{y}, b) = (a, p(\tilde{x}), p(\tilde{y}), b)$, is a bijection for all $(a, \tilde{x}, b)$. Consequently, $P : \tilde{\mathbb{G}} \to \mathbb{G}$ constitutes a covering morphism of strict 2-groupoids.

The following example may be regarded as an object-preserving analogue of Higgins' classical product covering construction [25, Example 2.5].

Example 3.4. *Let $\mathbb{G} = (A, X, G)$ be a 2-groupoid. Consider the trivial 2-groupoid over the same object set A given by $\mathbb{T}_A = (A, \{1_a\}_{a\in A}, \{1_{1_a}\}_{a\in A})$. Then the canonical projection $P : \mathbb{G} \times_A \mathbb{T}_A \longrightarrow \mathbb{G}$ acts identically on objects, and for every 1-cell $x \in X$, the induced map $St^v_{\mathbb{G}\times_A\mathbb{T}_A}(x, 1_{d_0(x)}) \to St^v_{\mathbb{G}}(x)$ is clearly bijective. Hence, P is a covering morphism of strict 2-groupoids.*

The next result shows that the notion of covering is compatible with the passage from crossed modules over groupoids to the associated strict 2-groupoids.

Proposition 3.5. *Let $\widetilde{C} = (\widetilde{\mathbb{U}}, \widetilde{\mathbb{V}}, \widetilde{C})$ and $C = (\mathbb{U}, \mathbb{V}, C)$ be crossed modules over groupoids, and let $\langle D_0, D_1, D_2\rangle\colon \widetilde{C} \to C$ be a covering morphism of crossed modules over groupoids, i.e., D_1 is an isomorphism. Then, via Theorem 2.3, $\langle D_0, D_1, D_2\rangle$ induces a well-defined morphism of strict 2-groupoids $P\colon (A, \widetilde{V}, \widetilde{V} \ltimes \widetilde{U}) \to (A, V, V \ltimes U)$, which acts as the identity on the object set A such that $P(\widetilde{v}) = D_2(\widetilde{v})$ and $P(\widetilde{v}, \widetilde{u}) = (D_2(\widetilde{v}), D_1(\widetilde{u}))$. Moreover, P is a covering morphism of strict 2-groupoids.*

Proof. By Theorem 2.3, the morphism $\langle D_0, D_1, D_2\rangle\colon \widetilde{C} \to C$ induces a strict 2-groupoid morphism $P : (A, \widetilde{V}, \widetilde{V} \ltimes \widetilde{U}) \to (A, V, V \ltimes U)$. For every $\widetilde{v} \in \widetilde{V}$, the vertical star $St^v_{(A,\widetilde{V},\widetilde{V}\ltimes\widetilde{U})}(\widetilde{v})$ consists of pairs $(\widetilde{v}, \widetilde{u})$ with $\widetilde{u} \in \widetilde{U}(d_0(\widetilde{v}))$. The induced map on vertical stars is given by $P_*(\widetilde{v}, \widetilde{u}) = (D_2(\widetilde{v}), D_1(\widetilde{u}))$. Since $\widetilde{v}$ is fixed, the map P_* is completely determined by D_1. As $D_1 : \widetilde{U} \to U$ is an isomorphism, P_* is bijective. Hence, P is a covering morphism of strict 2-groupoids. □

Example 3.6. *Let $\mathbb{G} = (A, X, G)$ be a strict 2-groupoid, and let $(X, N) \trianglelefteq (X, G)$ be the trivial normal subgroupoid of the underlying vertical groupoid, that is, $N(x) = \{1_x\}_{x\in X}$. Then the quotient groupoid $(X, G/N)$ is naturally isomorphic to (X, G), and hence $\mathbb{G}/N := (A, X, G/N)$ inherits a strict 2-groupoid structure. The canonical projection $P : \mathbb{G} \longrightarrow \mathbb{G}/N$ is defined by $P(a) = a$, $P(x) = x$, and $P(g) = gN(d^v_0(g))$. Since N is trivial, each coset consists of a single element, and therefore the induced map $P_* : St^v_{\mathbb{G}}(x) \longrightarrow St^v_{\mathbb{G}/N}(P(x))$ is bijective for every $x \in X$. Hence, P is a covering morphism of strict 2-groupoids.*

This trivial construction provides the fundamental example of a covering morphism of strict 2-groupoids and may be regarded as the algebraic analogue of a one-sheeted covering space in topology. Although elementary, it illustrates the covering condition in its simplest form and serves as a baseline for the non-trivial constructions developed below.

The previous example represents the simplest possible covering. The next result shows how new covering morphisms may be obtained by passing to suitable quotient 2-groupoids.

Proposition 3.7. *Let $\widetilde{\mathbb{G}} = (A, \widetilde{X}, \widetilde{G})$, $\mathbb{G} = (A, X, G)$ be strict 2-groupoids and $P\colon \widetilde{\mathbb{G}} \to \mathbb{G}$ be a covering morphism. Suppose that $\widetilde{N} \trianglelefteq (\widetilde{X}, \widetilde{G})$ and $N \trianglelefteq (X, G)$ are normal subgroupoids of their respective underlying vertical groupoids such that $P(\widetilde{N}) \subseteq N$. Then, P induces a strict 2-groupoid morphism $\hat{P}\colon (A, \widetilde{X}, \widetilde{G}/\widetilde{N}) \to (A, X, G/N)$ defined by $\hat{P}(a) = a$, $\hat{P}(\widetilde{x}) = P(\widetilde{x})$, and $\hat{P}(\widetilde{g}\widetilde{N}(\widetilde{x})) = P(\widetilde{g})N(P(\widetilde{x}))$. If, moreover, $\widetilde{N} = P^{-1}(N)$, then $\hat{P}$ inherits the covering property from P, meaning that the induced map on vertical stars $\hat{P}_*\colon St^{v}_{\widetilde{\mathbb{G}}/\widetilde{N}}(\widetilde{x}) \to St^{v}_{\mathbb{G}/N}(\hat{P}(\widetilde{x}))$ is bijective for every $\widetilde{x} \in \widetilde{X}$.*

Proof. First, $\hat{P}$ is well-defined. Indeed, if $\widetilde{g}\widetilde{N}(\widetilde{x}) = \widetilde{h}\widetilde{N}(\widetilde{x})$ for $\widetilde{x} \in \widetilde{X}$ and $\widetilde{g}, \widetilde{h} \in \widetilde{G}$, where $d_0^v(\widetilde{g}) = d_0^v(\widetilde{h}) = \widetilde{x}$, then $\overline{\widetilde{h}}^{v} \bullet \widetilde{g} \in \widetilde{N}(\widetilde{x})$. Applying P and using $P(\widetilde{N}) \subseteq N$, we obtain $P(\overline{\widetilde{h}}^{v} \bullet \widetilde{g}) = \overline{P(\widetilde{h})}^{v} \bullet P(\widetilde{g}) \in N(P(\widetilde{x}))$, and hence $P(\widetilde{g})N(P(\widetilde{x})) = P(\widetilde{h})N(P(\widetilde{x}))$. Assume now that $\widetilde{N} = P^{-1}(N)$. Let $\widetilde{x} \in \widetilde{X}$. Since P is a covering morphism, the map $P_* : St^{v}_{\widetilde{\mathbb{G}}}(\widetilde{x}) \to St^{v}_{\mathbb{G}}(P(\widetilde{x}))$ is bijective.

To prove surjectivity of $\hat{P}_*$, let $gN(P(\widetilde{x})) \in St^{v}_{\mathbb{G}/N}(P(\widetilde{x}))$. Since P_* is surjective, there exists $\widetilde{g} \in St^{v}_{\widetilde{\mathbb{G}}}(\widetilde{x})$ such that $P(\widetilde{g}) = g$. Hence, $\hat{P}_*(\widetilde{g}\widetilde{N}(\widetilde{x})) = P(\widetilde{g})N(P(\widetilde{x})) = gN(P(\widetilde{x}))$, showing that $\hat{P}_*$ is surjective. For injectivity, suppose $\hat{P}_*(\widetilde{g}\widetilde{N}(\widetilde{x})) = \hat{P}_*(\widetilde{h}\widetilde{N}(\widetilde{x}))$. Then $P(\widetilde{g})N(P(\widetilde{x})) = P(\widetilde{h})N(P(\widetilde{x}))$, which implies $P(\overline{\widetilde{h}}^{v} \bullet \widetilde{g}) \in N(P(\widetilde{x}))$. Since $\widetilde{N} = P^{-1}(N)$, it follows that $\overline{\widetilde{h}}^{v} \bullet \widetilde{g} \in \widetilde{N}(\widetilde{x})$, and therefore $\widetilde{g}\widetilde{N}(\widetilde{x}) = \widetilde{h}\widetilde{N}(\widetilde{x})$. Thus, $\hat{P}_*$ is injective. Hence, $\hat{P}_*$ is bijective for every $\widetilde{x} \in \widetilde{X}$, and consequently $\hat{P}$ is a covering morphism of strict 2-groupoids. □

Definition 3.8. *Let $P\colon \widetilde{\mathbb{G}} \to \mathbb{G}$ and $P' : \widetilde{\mathbb{G}}' \to \mathbb{G}$ be two covering morphisms of a strict 2-groupoid $\mathbb{G}$. A morphism of 2-groupoids $F\colon \widetilde{\mathbb{G}} \to \widetilde{\mathbb{G}}'$ is called a morphism of coverings if it acts as the identity on objects and the following triangle commutes:*

$$\begin{tikzcd}
\widetilde{\mathbb{G}} \arrow[rr, "F"] \arrow[dr, "P"'] & & \widetilde{\mathbb{G}}' \arrow[dl, "P'"] \\
& \mathbb{G} &
\end{tikzcd}$$

i.e., $P'F = P$. The covering 2-groupoids over $\mathbb{G}$, together with these morphisms, form a category denoted by **Cov(2Gpd)**$/\mathbb{G}$.

4. Actions of strict 2-groupoids

Following the covering theory developed in the previous section, we now introduce the notion of an action of a strict 2-groupoid on a groupoid. This concept provides an algebraic counterpart to covering morphisms and allows the vertical structure of a strict 2-groupoid to be encoded through suitable groupoid actions. Working throughout in an object-preserving setting, we formulate these actions via a natural fiber product construction and investigate their basic categorical properties. The main result of this section establishes a natural equivalence between coverings and actions of strict 2-groupoids.

Definition 4.1. *Let $\mathbb{G} = (A, X, G)$ be a 2-groupoid and $\mathbb{M} = (A, M)$ be a groupoid. Let $R\colon \mathbb{M} \to \mathbb{X}$ be a groupoid morphism that is the identity on objects, where $\mathbb{X} = (A, X)$ is the underlying groupoid of $\mathbb{G}$. An* action of $\mathbb{G}$ on $\mathbb{M}$ via R *is a mapping $G\,{}_{d_0^v}\!\times_R M \to M$, denoted by $(g, m) \mapsto g \centerdot m$, where the domain is the fiber product $G\,{}_{d_0^v}\!\times_R M = \{(g, m) \mid g \in G, m \in M, d_0^v(g) = R(m)\}$. For a morphism $a \xrightarrow{m} b$, the action is represented by the configuration*

$$\begin{tikzcd} a \arrow[r, bend left, "R(m)"] \arrow[r, bend right, "R(g\centerdot m)"'] \arrow[r, phantom, "\Downarrow g"] & b \end{tikzcd},$$

which satisfies the following conditions whenever all compositions and actions are defined:

[Act2Gpd 1] $R(g \centerdot m) = d_1^v(g)$.

[Act2Gpd 2] $1_{R(m)} \centerdot m = m$, *where $1_{R(m)}$ is the identity 2-cell on the 1-morphism $R(m)$.*

[Act2Gpd 3] $(g_1 \bullet g) \centerdot m = g_1 \centerdot (g \centerdot m)$ *for vertically composable 2-morphisms $g, g_1 \in G$.*

[Act2Gpd 4] $(g' * g) \centerdot (m' \circ m) = (g' \centerdot m') \circ (g \centerdot m)$ *for 2-morphisms $g, g' \in G$ and morphisms $m, m' \in M$.*

This action of $\mathbb{G}$ on $\mathbb{M}$ via R is denoted by $(\mathbb{M}, R)$. In this context, we say that $\mathbb{G}$ acts on $\mathbb{M}$ via R or that $\mathbb{M}$ is a $\mathbb{G}$-groupoid.

Example 4.2. *Let $\mathbb{G} = (A, X, G)$ be a strict 2-groupoid. Then $\mathbb{G}$ acts on the underlying vertical groupoid (A, G) via vertical composition:*

$$g \centerdot h = g \bullet h$$

whenever $d_0^v(g) = d_1^v(h)$. The action axioms follow immediately from the associativity and identity properties of the vertical composition.

The following example demonstrates that any strict 2-groupoid possesses an intrinsic, canonical action on a suitable quotient of its own vertical structural cells. This construction highlights the internal algebraic symmetry of higher groupoids and provides a concrete testbed for the abstract action axioms introduced above.

Example 4.3. *Let $\mathbb{G} = (A, X, G)$ be a strict 2-groupoid, and let $N = \mathrm{Ker} d_1^v = \{g \in G \mid d_1^v(g) = 1_{d_0^h(g)}\}$ be the kernel subgroupoid of the underlying vertical groupoid. Since every 1-morphism $x \in X$ admits the identity 2-cell 1_x, the morphism $d_1^v\colon G \to X$ is surjective. Hence, by the first isomorphism theorem (Theorem 2.1), there is a canonical isomorphism of groupoids $R\colon (A, G/N) \longrightarrow (A, X)$ defined on arrows by $R(gN(d_0^v(g))) = d_1^v(g)$. We define an action of $\mathbb{G}$ on the groupoid $(A, G/N)$ via R by*

$$g \centerdot (hN(d_0^v(h))) := (g \bullet h)N(d_0^v(h))$$

whenever $d_0^v(g) = R(hN) = d_1^v(h)$, so that the vertical composite $g \bullet h$ is well-defined. The axioms [Act2Gpd 1–4] *follow directly from the properties of vertical composition, identity 2-morphisms, and the interchange law in the strict 2-groupoid $\mathbb{G}$. Therefore, $(A, G/N)$ becomes a $\mathbb{G}$-groupoid via R, which we call the* natural action *associated with $\mathbb{G}$.*

Definition 4.4. *Let $(\mathbb{M}, R)$ and $(\mathbb{M}', R')$ be two actions of a strict 2-groupoid $\mathbb{G} = (A, X, G)$ on groupoids $\mathbb{M} = (A, M)$ and $\mathbb{M}' = (A, M')$ via the groupoid morphisms $R\colon \mathbb{M} \to \mathbb{X}$ and $R'\colon \mathbb{M}' \to \mathbb{X}$, respectively. A morphism of actions $F\colon (\mathbb{M}, R) \to (\mathbb{M}', R')$ is a groupoid morphism $F\colon \mathbb{M} \to \mathbb{M}'$, which acts as the identity on the object set A and satisfies $R'F = R$ and $F(g \centerdot m) = g \centerdot F(m)$ whenever the action $g \centerdot m$ is defined. Consequently, actions of the 2-groupoid $\mathbb{G}$ on groupoids, together with these morphisms, form a well-defined category denoted by* **Act(2Gpd)**$/\mathbb{G}$.

We now state the main result of this section.

Theorem 4.5. *The categories* **Cov(2Gpd)**$/\mathbb{G}$ *and* **Act(2Gpd)**$/\mathbb{G}$ *are naturally equivalent.*

Proof. We define a functor $\Theta\colon$ **Act(2Gpd)**$/\mathbb{G} \to$ **Cov(2Gpd)**$/\mathbb{G}$, which provides the equivalence. Let $(\mathbb{M}, R)$ be an object in **Act(2Gpd)**$/\mathbb{G}$, where $\mathbb{G} = (A, X, G)$. The associated covering 2-groupoid is constructed as $\widetilde{\mathbb{G}} = (A, M, G \ltimes M)$, where the set of 2-morphisms is the semi-direct product:

$$G \ltimes M = \{(g, m) \mid g \in G, m \in M, d_0^v(g) = R(m)\}.$$

The structural maps for $\widetilde{\mathbb{G}}$ are defined as $d_0^v(g, m) = m$, $d_1^v(g, m) = g \centerdot m$, $d_0^h(g, m) = d_0(m)$ and $d_1^h(g, m) = d_1(m)$ for any $(g, m) \in G \ltimes M$. The vertical and horizontal compositions in $\widetilde{\mathbb{G}}$ are given by $(h, g \centerdot m) \bullet (g, m) = (h \bullet g, m)$, and $(g', m') * (g, m) = (g' * g, m' \circ m)$. Due to axiom [Act2Gpd 4], the vertical target respects the horizontal composition:

$$\begin{aligned} d_1^v\Big((g', m') * (g, m)\Big) &= d_1^v(g' * g, m' \circ m) \\ &= (g' * g) \centerdot (m' \circ m) \\ &= (g' \centerdot m') \circ (g \centerdot m) \\ &= d_1^v(g', m') \circ d_1^v(g, m). \end{aligned}$$

Using the interchange law in $\mathbb{G}$ and axiom [Act2Gpd 4], we verify the interchange rule in $\widetilde{\mathbb{G}}$:

$$\begin{aligned} \Big[(h', g' \centerdot m') \bullet (g', m')\Big] * \Big[(h, g \centerdot m) \bullet (g, m)\Big] &= (h' \bullet g', m') * (h \bullet g, m) \\ &= \Big((h' \bullet g') * (h \bullet g), m' \circ m\Big) \\ &= \Big((h' * h) \bullet (g' * g), m' \circ m\Big) \\ &= \Big(h' * h, (g' * g) \centerdot (m' \circ m)\Big) \bullet (g' * g, m' \circ m) \\ &= \Big(h' * h, (g' \centerdot m') \circ (g \centerdot m)\Big) \bullet (g' * g, m' \circ m) \\ &= \Big[(h', g' \centerdot m') * (h, g \centerdot m)\Big] \bullet \Big[(g', m') * (g, m)\Big] \end{aligned}$$

whenever the compositions and actions are defined. The associativity of the horizontal composition in $G \ltimes M$ follows trivially from the associativity of the horizontal products in M and G. For the vertical composition, the associativity is directly guaranteed by the action axiom [Act2Gpd 3], which dictates the compatibility between the action and vertical compositions of 2-morphisms. Thus, all composition laws in the induced structure are associative. Since R preserves identities on objects (i.e., $R(1_a) = 1_a$), the identity 2-cells in the semidirect structure are naturally given by $1_m = (1_{R(m)}, m)$ for any $m \in M$ and $1_{1_a} = (1_{1_a}, 1_a)$ for the identity object $1_a \in M$. The covering morphism $P\colon \widetilde{\mathbb{G}} \to \mathbb{G}$ is defined to be the identity on objects, $P(m) = R(m)$ on 1-morphisms, and $P(g, m) = g$ on 2-morphisms.

Given a morphism of actions $F\colon (\mathbb{M}, R) \to (\mathbb{M}', R')$, $\Theta(F) = (1_A, F, 1_G \times F)$ is a morphism of **Cov(2Gpd)**$/\mathbb{G}$, which satisfies $P'\Theta(F) = P$.

Let $P = (1_A, P_1, P_2)\colon \widetilde{\mathbb{G}} = (A, \widetilde{X}, \widetilde{G}) \to \mathbb{G} = (A, X, G)$ be a covering morphism of 2-groupoids. Now we define a functor

$$\Omega\colon \mathbf{Cov(2G{\scriptstyle PD})}/\mathbb{G} \to \mathbf{A{\scriptstyle CT}(2G{\scriptstyle PD})}/\mathbb{G}$$

as a weak inverse of Θ such that $\Omega(\widetilde{\mathbb{G}}, P) = (\mathbb{M}, R)$ and $\Omega(\widetilde{F}) = \widetilde{F}_1$, where $\widetilde{F}$ is a morphism of coverings. Here, $\mathbb{M} = \widetilde{\mathbb{X}}$ is the underlying groupoid of $\widetilde{\mathbb{G}}$, and $R = P_1$ is the restriction of the covering morphism to 1-morphisms. The action of $\mathbb{G}$ on $\mathbb{M}$ is defined by

$$g \centerdot \widetilde{x} = d_1^v(\widetilde{g}),$$

where $\widetilde{g}$ is the unique lift of the 2-morphism g starting at the 1-morphism x (i.e., $P(\widetilde{g}) = g$ and $d_0^v(\widetilde{g}) = \widetilde{x}$) for $\widetilde{g} \in \widetilde{G}, \widetilde{x} \in \widetilde{X}$.

$$\begin{tikzcd} a \arrow[r, bend left, "\widetilde{x}"{name=U}] \arrow[r, bend right, "g\centerdot\widetilde{x}"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow \widetilde{g}"] & b \end{tikzcd} \xrightarrow{\quad P \quad} \begin{tikzcd} a \arrow[r, bend left, "x"{name=U}] \arrow[r, bend right, "R(g\centerdot\widetilde{x})"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow g"] & b \end{tikzcd} .$$

Let $g \in G$ and $\tilde{x} \in M$ such that $d_0^v(g) = R(\tilde{x})$. By Definition 3.1, P is a covering morphism of strict 2-groupoids, meaning that the restriction map

$$P_* = P|_{St^v_{\tilde{G}}(\tilde{x})} : St^v_{\tilde{G}}(\tilde{x}) \longrightarrow St^v_G(P_1(\tilde{x}))$$

is a bijection. Thus, for the given $g \in St^v_G(R(\tilde{x}))$, there exists a unique lift $\tilde{g} \in St^v_{\tilde{G}}(\tilde{x})$ satisfying $d_0^v(\tilde{g}) = \tilde{x}$ and $P_2(\tilde{g}) = g$. If $\tilde{g}' \in St^v_{\tilde{G}}(\tilde{x})$ were another element such that $P_2(\tilde{g}') = g$, the injectivity of P immediately forces $\tilde{g} = \tilde{g}'$. Hence, the assignment $g \centerdot \tilde{x} := d_1^v(\tilde{g})$ is well-defined and independent of any choices.

We now verify that $\centerdot$ satisfies the axioms of an action of $\mathbb{G}$ on $\mathbb{M}$ whenever all following compositions and actions are defined:

[Act2Gpd 1] $R(g \centerdot \widetilde{x}) = P_1(d_1^v(\widetilde{g})) = d_1^v(P(\widetilde{g})) = d_1^v(g)$.

[Act2Gpd 2] Let $R(\widetilde{x}) = x$. Since $d_0^v(1_{R(\widetilde{x})}) = d_0^v(1_x) = x = R(\widetilde{x})$ and $P(1_{\widetilde{x}}) = 1_x$, we write
$1_{R(\widetilde{x})} \centerdot \widetilde{x} = 1_x \centerdot \widetilde{x} = d_1^v(1_{\widetilde{x}}) = \widetilde{x}$.

[Act2Gpd 3] $h \centerdot (g \centerdot m) = h \centerdot d_1^v(\widetilde{g}) = d_1^v(\widetilde{h}) = d_1^v(\widetilde{h} \bullet \widetilde{g}) = (h \bullet g) \centerdot m$, where $P(\widetilde{h}) = h, P(\widetilde{g}) = g$.

[Act2Gpd 4] $(g' * g) \centerdot (m' \circ m) = d_1^v(\widetilde{g' * g}) = d_1^v(\widetilde{g'}) \circ d_1^v(\widetilde{g}) = (g' \centerdot m') \circ (g \centerdot m)$.

In the last axiom, we use the fact that in a covering of 2-groupoids, the lift of a horizontal composition is the horizontal composition of the lifts, and the target of a horizontal composition in $\widetilde{\mathbb{G}}$ follows the groupoid composition $\circ$ in $\mathbb{M}$.

Clearly, $\Omega\Theta \cong \mathbf{1}$. To prove $\mathbf{1} \cong \Theta\Omega$, a natural equivalence $\xi\colon \mathbf{1}_{\mathbf{Cov(2G{\scriptstyle PD})}/\mathbb{G}} \Rightarrow \Theta\Omega$ is defined via a map $\xi_{\widetilde{\mathbb{G}}}$ such that it is the identity on objects and on 1-morphisms. For any 2-morphism $\widetilde{g} \in \widetilde{G}$, we define

$$\xi_{\widetilde{\mathbb{G}}}(\widetilde{g}) = (P_2(\widetilde{g}), d_0^v(\widetilde{g})).$$

Since P is a covering morphism, for every pair (g,m) with $d_0^v(g) = R(m)$, there exists a unique lift $\widetilde{g}$ such that $P_2(\widetilde{g}) = g$ and $d_0^v(\widetilde{g}) = m$. This implies that $\xi_{\widetilde{\mathbb{G}}}$ is a bijection on 2-morphisms and thus an isomorphism. We now prove that $\xi_{\widetilde{\mathbb{G}}}$ preserves the vertical and horizontal compositions:

$$\begin{aligned}
\xi_{\widetilde{\mathbb{G}}}(\widetilde{h} \bullet \widetilde{g}) &= (P_2(\widetilde{h} \bullet \widetilde{g}), d_0^v(\widetilde{h} \bullet \widetilde{g})) \\
&= (P_2(\widetilde{h}) \bullet P_2(\widetilde{g}), d_0^v(\widetilde{g})) \\
&= (P_2(\widetilde{h}), P_2(\widetilde{g}) \centerdot d_0^v(\widetilde{g})) \bullet (P_2(\widetilde{g}), d_0^v(\widetilde{g})) \\
&= (P_2(\widetilde{h}), d_0^v(\widetilde{h})) \bullet (P_2(\widetilde{g}), d_0^v(\widetilde{g})) \\
&= \xi_{\widetilde{\mathbb{G}}}(\widetilde{h}) \bullet \xi_{\widetilde{\mathbb{G}}}(\widetilde{g}), \\
\xi_{\widetilde{\mathbb{G}}}(\widetilde{g}' * \widetilde{g}) &= (P_2(\widetilde{g}' * \widetilde{g}), d_0^v(\widetilde{g}' * \widetilde{g})) \\
&= (P_2(\widetilde{g}') * P_2(\widetilde{g}), d_0^v(\widetilde{g}') \circ d_0^v(\widetilde{g})) \\
&= (P_2(\widetilde{g}'), d_0^v(\widetilde{g}')) * (P_2(\widetilde{g}), d_0^v(\widetilde{g})) \\
&= \xi_{\widetilde{\mathbb{G}}}(\widetilde{g}') * \xi_{\widetilde{\mathbb{G}}}(\widetilde{g}).
\end{aligned}$$

□

Theorem 4.5 shows that covering morphisms and actions of strict 2-groupoids encode the same mathematical information. This correspondence forms the conceptual bridge between the covering theory developed in Section 3 and the lifting theory of crossed modules over groupoids investigated in the next section.

5. Liftings crossed modules via strict 2-groupoids

In this section, we investigate the relationship between actions of strict 2-groupoids and liftings of crossed modules over groupoids. The correspondence established in the previous section enables us to transfer structural information between these two settings. We first show that every action of a strict 2-groupoid naturally determines a lifting of the associated crossed module. Conversely, every lifting of a crossed module induces an action of the corresponding strict 2-groupoid. These constructions lead to an equivalence between the categories of actions and liftings. To begin, we establish the following proposition.

Proposition 5.1. *Let $(\mathbb{M}, R)$ be an action of a 2-groupoid $\mathbb{G} = (A, X, G)$ on a groupoid $\mathbb{M} = (A, M)$, and let $(\mathbb{U}, \mathbb{V}, C)$ be the crossed module over groupoids corresponding to $\mathbb{G}$. Then the triple $(\mathbb{U}, \mathbb{M}, \widehat{C})$ constitutes a lifting of $(\mathbb{U}, \mathbb{V}, C)$ over $R = E$, where the boundary map $\widehat{C}\colon \mathbb{U} \to \mathbb{M}$ is defined by $\widehat{C}(g) = g \centerdot 1_a$ and the action $\star$ of $\mathbb{M}$ on $\mathbb{U}$ is given by $m \star g = 1_{R(m)} * g * 1_{\overline{R(m)}}$, where $a \xrightarrow{m} b$ is a 1-morphism in $\mathbb{M}$ and* $\begin{tikzcd} a \arrow[r, bend left, "1_a"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow g"] & a \end{tikzcd}$ *is a 2-morphism in $\mathbb{G}$.*

Proof. By axiom [Act2Gpd 4], we write

$$\widehat{C}(g' * g) = (g' * g) \centerdot 1_a = (g' * g) \centerdot (1_a \circ 1_a) = (g' \centerdot 1_a) \circ (g \centerdot 1_a) = \widehat{C}(g') \circ \widehat{C}(g),$$

where

$$
\begin{tikzcd} a \arrow[r, bend left, "1_a"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow g"] & a \arrow[r, bend left, "1_a"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow g'"] & a . \end{tikzcd}
$$

Hence, $\widehat{C}$ is a morphism of groupoids. Since the horizontal inverse in $\mathbb{G}$ satisfies $\overline{1_{R(m)}}^h = 1_{\overline{R(m)}} = 1_{R(\overline{m})}$, and recalling the relation between the horizontal composition in 2-groupoids and the crossed module action, we obtain

$$m \star g = R(m) \centerdot g. \tag{5.1}$$

This shows that the action of the groupoid $\mathbb{M}$ on $\mathbb{U}$ is defined via the morphism R and the inherited action $\centerdot$ of the crossed module. Now we show that $(\mathbb{U}, \mathbb{M}, \widehat{C})$ is a well-defined crossed module over groupoids.

[CMGd 1] From the conditions [Act2Gpd 2] and [Act2Gpd 4], we have

$$
\begin{aligned}
\widehat{C}(m \star g) &= \widehat{C}(1_{R(m)} * g * 1_{R(\overline{m})}) \\
&= (1_{R(m)} * g * 1_{R(\overline{m})}) \centerdot 1_a \\
&= (1_{R(m)} * g * 1_{R(\overline{m})}) \centerdot (m \circ 1_a \circ \overline{m}) \\
&= (1_{R(m)} \centerdot m) \circ (g \centerdot 1_a) \centerdot (1_{R(\overline{m})} \centerdot \overline{m}) \\
&= m \circ \widehat{C}(g) \circ \overline{m}.
\end{aligned}
$$

[CMGd 2] From the condition [Act2Gpd 1], we write $R\widehat{C}(g) = R(g \centerdot 1_a) = d_1^v(g) = C(g)$, where $C\colon \mathbb{U} \to \mathbb{V}$ is the boundary map of the crossed module associated with $\mathbb{G}$. Then, using (5.1), we have

$$\widehat{C}(g) \star g_1 = R\widehat{C}(g) \cdot g_1 = C(g) \cdot g_1 = g * g_1 * g^{-1}.$$

□

Conversely, every lifting of a crossed module gives rise to a natural action of the associated strict 2-groupoid.

Proposition 5.2. *Let $(\mathbb{U}, \mathbb{V}, C)$ be a crossed module over groupoids, and let $(\mathbb{U}, \mathbb{W}, E)$ be a lifting of C. Then, this configuration induces a well-defined action $(\mathbb{M}, R)$ of the corresponding 2-groupoid, where $R = E$, $\mathbb{M} = \mathbb{W}$, and the action is defined on the fiber product by $(v, u) \centerdot w = \widehat{C}(u) \circ w$ whenever $d_0^v(v, u) = v = E(w)$ for elements represented by $a \xrightarrow{v} b \xrightarrow{u} b$ in the associated semidirect product groupoid.*

Proof. We must prove that $\centerdot$ is an action of $\mathbb{G}$ on $\mathbb{M}$:

[Act2Gpd 1] $R((v, u) \centerdot w) = E(\widehat{C}(u) \circ w) = E\widehat{C}(u) \circ E(w) = C(u) \circ w = d_1^v(v, u)$.

[Act2Gpd 2] Let $a \xrightarrow{w} b$ be a morphism of $\mathbb{W}$. Then, we write

$$1_{R(w)} \centerdot w = (E(w), 1_b) \centerdot w = \widehat{C}(1_b) \circ w = 1_b \circ w = w.$$

[Act2Gpd 3] $((v_1,u_1)\bullet(v,u))\,.\,w=(v,u_1\circ u)\,.\,w=\widehat{C}(u_1\circ u)\circ w=\widehat{C}(u_1)\circ\widehat{C}(u)\circ w=(v_1,u_1)\,.\,\big((v,u)\,.\,w\big)$, where $v_1=C(u)\circ v$.

[Act2Gpd 4] Let $a\xrightarrow{v'}b\xrightarrow{u'}b$ and $v'=E(w')$. Then

$$\begin{aligned}\big((v',u')*(v,u)\big)\,.\,(w'\circ w)&=(v'\circ v,u'\circ(v'\cdot u))\,.\,(w'\circ w)\\&=\widehat{C}(u'\circ(v'\cdot u))\circ w'\circ w\\&=\widehat{C}(u')\circ\widehat{C}(E(w')\cdot u)\circ w'\circ w\\&=\widehat{C}(u')\circ\widehat{C}(w'\star u)\circ w'\circ w\\&=\widehat{C}(u')\circ w'\circ\widehat{C}(u)\circ\overline{w'}\circ w'\circ w\\&=\widehat{C}(u')\circ w'\circ\widehat{C}(u)\circ w\\&=\big((v',u')\,.\,w'\big)\circ\big((v,u)\,.\,w\big).\end{aligned}$$

□

Combining the previous two propositions yields the main result of this section.

Theorem 5.3. *Let* $(\mathbb{U},\mathbb{V},C)$ *be the crossed module over groupoids corresponding to the strict 2-groupoid* $\mathbb{G}$*. Then the categories* **LiftCMGd**$/(\mathbb{U},\mathbb{V},C)$ *and* **Act(2Gpd)**$/\mathbb{G}$ *are equivalent.*

Proof. We extend the equivalence functor given in Theorem 2.3 by Proposition 5.1,

$$\Delta\colon \textbf{Act(2Gpd)}/\mathbb{G}\to\textbf{LiftCMGd}/(\mathbb{U},\mathbb{V},C),$$

such that $\Delta(\mathbb{M},R)=(\widehat{C},\mathbb{M},E)$, where $E=R$. To verify that $R\hat{C}=C$, let $g\in\ker d_0^v$. By definition of the action, we have $\hat{C}(g)=g\,.\,1_a$, where $g\,.\,1_a=d_1^v(\tilde{g})$ for the unique lift $\tilde{g}$ corresponding to g. Applying R, we obtain

$$R(\hat{C}(g))=R(g\,.\,1_a)=d_1^v(\tilde{g})=C(g).$$

Thus, $R\hat{C}=C$ holds, which proves the commutativity of the diagram. Let $F\colon\mathbb{M}\to\mathbb{M}'$ be a morphism of **Act(2Gpd)**$/\mathbb{G}$. Since

$$F\widehat{C}(g)=F(g\,.\,1_a)=g\,.'\,F(1_a)=g\,.'\,1_{F(a)}=\widehat{C}'(g),$$

then $\Delta(F)=F$ is a morphism of **LiftCMGd**$/(\mathbb{U},\mathbb{V},C)$.

Conversely, the functor Γ: **LiftCMGd**$/(\mathbb{U},\mathbb{V},C)\to$ **Act(2Gpd)**$/\mathbb{G}$ given in Theorem 2.3 can be extended such that $\Gamma(\widehat{C},\mathbb{W},E)=(\mathbb{M},R)$ via Proposition 5.2. Let $H\colon\mathbb{W}\to\mathbb{W}'$ be a morphism of liftings. Since $H\widehat{C}=\widehat{C'}$, we write

$$H((v,u)\,.\,w)=H(\widehat{C}(u)\circ w)=H\widehat{C}(u)\circ H(w)=\widehat{C}'(u)\circ H(w)=(v,u)\,.'\,H(w).$$

Hence, $\Gamma(H)=H$ is a morphism of **Act(2Gpd)**$/\mathbb{G}$.

For the natural equivalence $\eta\colon \mathbf{1}_{\textbf{Act(2Gpd)}/\mathbb{G}}\Rightarrow\Gamma\Delta$, $\eta_{\mathbb{M}}\colon(\mathbb{M},R)\to\Gamma\Delta(\mathbb{M},R)$ is defined to be the identity on elements of M. To verify that $\eta_M:(M,R)\to\Gamma\Delta(M,R)$ is a valid morphism in the slice category **Act(2Gpd)**$/\mathbb{G}$, we must show that $\Gamma\Delta(R)\circ\eta_M=R$. Indeed, for any $m\in M$, we have

$$(\Gamma\Delta(R)\circ\eta_M)(m)=\Gamma\Delta(R)(\eta_M(m))=\Gamma E(m)=R(m).$$

This guarantees that the unit component η_M commutes with the structural projections to X, establishing that η is well-defined in **Act(2Gpd)**.

We must prove that the action is preserved, i.e., $\eta_{\mathbb{G}}(g) \centerdot m = g \centerdot m$, where $d_0^v(g) = R(m)$,

$$\begin{aligned}
\eta_{\mathbb{G}}(g) \centerdot m &= (d_0^v(g), g * 1_{\overline{R(m)}}) \centerdot m \\
&= \widehat{C}(g * 1_{\overline{R(m)}}) \circ m \\
&= \Big((g * 1_{\overline{R(m)}}) \centerdot 1_{d_0^h(g)}\Big) \circ (1_{R(m)} \centerdot m) \\
&= \Big((g * 1_{\overline{R(m)}}) * 1_{R(m)}\Big) \centerdot (1_{d_0^h(g)} \circ m) \\
&= \Big(g * (1_{\overline{R(m)}} * 1_{R(m)})\Big) \centerdot m \\
&= (g * 1_{d_0^h(g)}) \centerdot m \\
&= g \centerdot m.
\end{aligned}$$

For the natural equivalence $\mu\colon \mathbf{1}_{\mathbf{LiftCMGd}/(\mathbb{U},\mathbb{V},C)} \Rightarrow \Delta\Gamma$, we define $\mu(\widehat{C}, \mathbb{M}, E) = (\widehat{C'}, \mathbb{M}, E)$, since the new boundary map $\widehat{C'}$ satisfies

$$\widehat{C'}(1_{d_0(u)}, u) = (1_{d_0(u)}, u) \centerdot 1_{d_0(u)} = \widehat{C}(u) \circ 1_{d_0(u)} = \widehat{C}(u).$$

Dually, to verify that the counit component $\mu_W : (\hat{C}, W, E) \to \Delta\Gamma(\hat{C}, W, E)$ is a valid morphism in the slice category over V, we must show that the corresponding triangle commutes. Indeed, for any $w \in W$, evaluating the composition with the structural projection map yields

$$(\Delta\Gamma(E) \circ \mu_W)(w) = \Delta\Gamma(E)(\mu_W(w)) = \Delta R(w) = E(w).$$

Hence, $\Delta\Gamma(E) \circ \mu_W = E$, which confirms the commutativity of the triangle over V and completes the well-definedness proof for the counit μ. □

Corollary 5.4. *Let $\mathbb{G}$ be a strict 2-groupoid, and let $C = (\mathbb{U}, \mathbb{V}, C)$ be its corresponding crossed module over groupoids. By synthesizing Theorems 5.3, 4.5, and the preceding equivalence, we obtain a fully coherent chain of categorical equivalences:*

$$\mathbf{Act(2Gpd)}/\mathbb{G} \simeq \mathbf{Cov(2Gpd)}/\mathbb{G} \simeq \mathbf{LiftCMGd}/C \simeq \mathbf{Cov(CMGd)}/C.$$

The above equivalence allows constructions in lifting theory to be translated directly into actions of strict 2-groupoids. As an application, we introduce a canonical lifting associated with an arbitrary crossed module over groupoids. The following construction is a direct consequence of the first isomorphism theorem for groupoids (Theorem 2.1). Since the quotient groupoid $\mathbb{U}/\mathrm{Ker}C$ is naturally isomorphic to the image subgroupoid $(A, C(U))$, the crossed module morphism C admits a canonical factorization through $\mathbb{U}/\mathrm{Ker}C$.

Proposition 5.5 (Natural lifting)**.** *Let $C = (\mathbb{U}, \mathbb{V}, C)$ be a crossed module over groupoids, and let $N = \mathrm{Ker}C$. Then C admits a canonical factorization via the commutative diagram*

$$\begin{tikzcd}
 & \mathbb{U}/N \arrow[d, "E"] \\
\mathbb{U} \arrow[ur, "\widehat{C}"] \arrow[r, "C"'] & \mathbb{V}
\end{tikzcd}$$

where $\widehat{C}(u) = uN(d_0(u))$ is the natural quotient morphism and $E\colon \mathbb{U}/N \to \mathbb{V}$ is the canonical embedding defined by $E(uN(d_0(u))) = C(u)$. Moreover, the triple $(\widehat{C}, \mathbb{U}/N, E)$ constitutes a well-defined lifting of the crossed module $\mathcal{C}$ via the morphism E. This structure is called the natural lifting of $\mathcal{C}$.

The natural lifting provides a canonical example of a lifting associated with any crossed module over groupoids. Via Theorem 5.3, it also determines a canonical action of the associated strict 2-groupoid.

Corollary 5.6. *Let $\mathcal{C} = (\mathbb{U}, \mathbb{V}, C)$ be a crossed module over groupoids and $(\widehat{C}, \mathbb{U}/N, E)$ be its natural lifting. Then, this natural lifting configuration directly determines a canonical action of the strict 2-groupoid $\mathbb{G}_{\mathcal{C}}$ associated with $\mathcal{C}$ on the quotient groupoid $\mathbb{U}/N$ via Theorem 5.3, which consequently yields a well-defined covering morphism of strict 2-groupoids $P\colon \widetilde{\mathbb{G}_{\mathcal{C}}} \longrightarrow \mathbb{G}_{\mathcal{C}}$ via Theorem 4.5.*

We now return to the internal structure of strict 2-groupoids. The next theorem shows that the first isomorphism theorem applied to the vertical source map naturally yields a lifting crossed module.

Theorem 5.7. *Let $\mathbb{G} = (A, X, G)$ be a 2-groupoid and $N = \mathrm{Ker} d_0^v$. Applying the first isomorphism theorem to the vertical groupoid morphism $F = (id, d_0^v)\colon (A, G) \to (A, X)$, the induced canonical isomorphism $E = F^*\colon (A, G/N) \to (A, X)$ defined on arrows by $E(gN(a)) = d_0^v(g)$ determines a lifting crossed module over groupoids via the commutative diagram*

$$\begin{tikzcd} & (A, G/\mathrm{Ker} d_0^v) \arrow[d, "E"] \\ (A, \mathrm{Ker} d_0^v) \arrow[ur, "\widehat{C}"] \arrow[r, "C"'] & (A, X) \end{tikzcd}$$

where $\widehat{C}(g) = gN(a)$ and the action is given by $g'N(a) \star g = 1_x * g * 1_{\overline{x}} = d_0^v(g') \cdot g$ for 2-morphisms

$$\begin{tikzcd} a \arrow[r, bend left, "1_a"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow g"] & a \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow g'"] & b \end{tikzcd} .$$

Proof. Since $d_0^v(G) = X$, by Theorem 2.1, the induced functor $E\colon (A, G/N) \to (A, X)$ given by $E(gN(a)) = d_0^v(g)$ is a well-defined groupoid isomorphism. To show that it determines a lifting, we verify the commutativity of the diagram and the crossed module axioms. For any $g \in N$, we have $d_0^v(g) = d_1^v(g) = 1_a$ since g belongs to the kernel $\mathrm{Ker} d_0^v$. Thus,

$$E\widehat{C}(g) = E(gN(a)) = d_0^v(g) = d_1^v(g) = C(g),$$

confirming that the diagram strictly commutes. Next, we verify that $\widehat{C}\colon (A, N) \to (A, G/N)$ is a crossed module over groupoids.

[CMGd 1] Since $d_0^v(g' * g * \overline{g'}) = d_0^v(1_x * g * 1_{\overline{x}})$, their corresponding cosets in the quotient groupoid coincide, giving $(1_x * g * 1_{\overline{x}})N(b) = (g' * g * \overline{g'})N(b)$. Therefore,

$$\begin{aligned} \widehat{C}(g'N(a) \star g) &= \widehat{C}(1_x * g * 1_{\overline{x}}) \\ &= (1_x * g * 1_{\overline{x}})N(b) \\ &= (g' * g * \overline{g'})N(b) \\ &= g'N(a) * gN(a) * \overline{g'}N(b) \\ &= g'N(a) * \widehat{C}(g) * \overline{g'}N(b). \end{aligned}$$

[CMGd 2] $\widehat{C}(g) \star h = E\widehat{C}(g) \cdot h = C(g) \cdot h = g * h * \overline{g}$.

□

Applying the same argument to the target map d_1^v yields the following dual construction.

Corollary 5.8. *Let $\mathbb{G} = (A, X, G)$ be a strict 2-groupoid, and let $N' = \mathrm{Ker}d_1^v$. Then, by applying Theorem 2.1 to the surjective vertical groupoid morphism $F' = (id, d_1^v)\colon (A, G) \to (A, X)$, the canonical factorization induces a dual lifting crossed module $(\mathrm{Ker}d_1^v, G/\mathrm{Ker}d_1^v, \widehat{C'})$ over groupoids via the target-induced isomorphism $E'\colon (A, G/N') \to (A, X)$ via the commutative diagram:*

$$\begin{tikzcd} & (A, G/\mathrm{Ker}d_1^v) \arrow[d, "E'"] \\ (A, \mathrm{Ker}d_1^v) \arrow[ur, "\widehat{C'}"] \arrow[r, "C"'] & (A, X) \end{tikzcd}$$

6. Quotient 2-groupoids via 2-group bundles and some applications

In this section, we introduce a quotient construction for strict 2-groupoids based on object-preserving normal sub-2-groupoids. Unlike the quotient construction considered by Higgins [25] and adopted by Temel [26], where objects are identified via equivalence classes, our approach follows Brown's object-preserving framework [23], in which the object set remains fixed.

After defining normal sub-2-groupoids and the corresponding quotient structures, we investigate the behavior of covering morphisms under quotient operations. We then apply these constructions to the lifting theory of crossed modules over groupoids through the Brown and İçen correspondence. The main result of the section shows that suitable quotient structures of strict 2-groupoids naturally induce lifting crossed modules over groupoids.

First, we introduce the concept of a strict 2-group bundle within our object-preserving framework.

Definition 6.1. *Let $\mathbb{G} = (A, X, G)$ be a strict 2-groupoid. A wide sub-2-groupoid $\mathbb{N} = (A, N_X, N_G)$ is called a* totally disconnected 2-groupoid *(or a* strict 2-group bundle *indexed over the object set A) if (A, N_X) is a totally disconnected subgroupoid of the underlying 1-groupoid (A, X), and (A, N_G) is a totally disconnected subgroupoid of the underlying horizontal groupoid (A, G), meaning that $N_X(a, b) = \varnothing$ and $N_G(a, b) = \varnothing$ for all $a, b \in A$ with $a \neq b$. In this configuration, for each object $a \in A$, the local cell structures over a canonically inherit the structure of a strict 2-group (or equivalently, a crossed module of groups).*

Next, by equipping this bundle structure with suitable conjugation conditions, we define normality and construct the corresponding quotient 2-groupoid.

Definition 6.2. *Let $\mathbb{G} = (A, X, G)$ be a strict 2-groupoid, and let $\mathbb{N} = (A, N_X, N_G)$ be a strict 2-group bundle. Then, $\mathbb{N}$ is called a* normal sub-2-groupoid *of $\mathbb{G}$ if the totally disconnected subgroupoids (A, N_X) and (A, N_G) are normal in (A, X) and (A, G), respectively, such that N_G is closed under both vertical and horizontal conjugations of $\mathbb{G}$. In this case, the corresponding* quotient 2-groupoid *is defined by*

$$\mathbb{G}/\mathbb{N} := (A, X/N_X, G/N_G),$$

where the object set A is strictly preserved. Here, the 1-morphisms are given by the cosets $xN_X(d_0(x))$, and the 2-morphisms are given by the cosets $gN_G(d_0^h(g))$. The source and target assignments for these 2-morphisms are well-defined by

$$d_0^v(gN_G(d_0^h(g))) = d_0^v(g)N_X(d_0^h(g)) \quad \text{and} \quad d_1^v(gN_G(d_0^h(g))) = d_1^v(g)N_X(d_0^h(g)),$$

while the horizontal and vertical compositions are directly induced from those of $\mathbb{G}$ on coset representatives. The normality assumptions ensure that the source and target maps, as well as the horizontal and vertical compositions, are independent of the chosen coset representatives. Consequently, $\mathbb{G}/\mathbb{N}$ inherits a well-defined strict 2-groupoid structure.

Example 6.3. *Let A be a set, X a group, and $N \trianglelefteq X$ a normal subgroup. Consider the trivial strict 2-groupoid $\mathbb{G}$. Define the subset of 1-morphisms as $\mathbb{N}_1 = \{(a, n, a) \mid a \in A,\ n \in N\}$ and the subset of 2-morphisms as $\mathbb{N}_2 = \{(a, n, n', a) \mid a \in A,\ n, n' \in N\}$. Then $\mathbb{N} = (A, \mathbb{N}_1, \mathbb{N}_2)$ forms a normal 2-group bundle of $\mathbb{G}$. Moreover, the quotient 2-groupoid $\mathbb{G}/\mathbb{N}$ is naturally isomorphic to the trivial strict 2-groupoid constructed from the quotient group X/N over the same object set A. Thus, the object-preserving quotient construction successfully recovers the classical group quotient within the trivial 2-groupoid setting.*

Example 6.4. *Let $F : \mathbb{G} \to \mathbb{H}$ be an identity-on-objects morphism of strict 2-groupoids. Then, the kernel $\mathrm{Ker}F$ is a normal sub-2-groupoid of $\mathbb{G}$.*

By lifting the refined, object-preserving framework established for groupoids in Theorem 2.1 to the higher-dimensional setting, we naturally obtain the first isomorphism theorem for strict 2-groupoids. This ensures that the identity-on-objects quotient architecture behaves entirely compatibly with categorical factorization at both the 1-morphism and 2-morphism levels.

Theorem 6.5. *Let $F : \mathbb{G} \to \mathbb{H}$ be an identity-on-objects morphism of strict 2-groupoids. Then, there exists a unique identity-on-objects induced isomorphism of strict 2-groupoids*

$$F^* : \mathbb{G}/\mathrm{Ker}F \to F(\mathbb{G})$$

defined on 1-morphisms by $F^*(x\mathrm{Ker}F_1) = F_1(x)$ and on 2-morphisms by $F^*(g\mathrm{Ker}F_2) = F_2(g)$.

Corollary 6.6. *Let $F : \mathbb{G} \to \mathbb{H}$ be an identity-on-objects morphism of strict 2-groupoids. Then, the lifting of crossed modules over groupoids obtained from the quotient 2-groupoid $\mathbb{G}/\mathrm{Ker}F$ via Theorem 6.10 is naturally determined by (and isomorphic to) the lifting structure induced by the image 2-groupoid $F(\mathbb{G})$.*

Remark 6.7. *In general, the canonical quotient 2-functor $F : \mathbb{G} \longrightarrow \mathbb{G}/\mathbb{N}$ is not a covering 2-functor. Indeed, distinct 1-cells and 2-cells belonging to the same coset are identified under F, and consequently the induced maps on local stars need not be bijective. Thus, quotient projections typically fail to satisfy the local lifting properties required of coverings. Nevertheless, the quotient 2-groupoid $\mathbb{G}/\mathbb{N}$ plays a fundamental role in the subsequent theory. Through the Brown and İçen correspondence between strict 2-groupoids and crossed modules over groupoids, quotient 2-groupoids provide a natural source of algebraic data from which covering and lifting crossed modules may be constructed.*

As a natural extension of the induced covering mechanisms, the following proposition demonstrates how a covering morphism of strict 2-groupoids passes to the quotient structures under compatible normal sub-2-groupoids.

Lemma 6.8. *Let $P : \widetilde{\mathbb{G}} = (A, \widetilde{X}, \widetilde{G}) \longrightarrow \mathbb{G} = (A, X, G)$ be a strict 2-groupoid morphism acting identically on objects. Let $\widetilde{\mathbb{N}} = (A, N_{\widetilde{X}}, N_{\widetilde{G}})$ and $\mathbb{N} = (A, N_X, N_G)$ be normal sub-2-groupoids of $\widetilde{\mathbb{G}}$ and $\mathbb{G}$, respectively. If $P(N_{\widetilde{X}}) \subseteq N_X$ and $P(N_{\widetilde{G}}) \subseteq N_G$, then P induces a well-defined strict 2-functor*

$$\widehat{P} : \widetilde{\mathbb{G}}/\widetilde{\mathbb{N}} \longrightarrow \mathbb{G}/\mathbb{N}$$

given by $\widehat{P}(\widetilde{x}N_{\widetilde{X}}(a)) = P(\widetilde{x})N_X(a)$ for any $\widetilde{x} \in \widetilde{X}$, where $\widetilde{x} : a \to b$, and $\widehat{P}(\widetilde{g}N_{\widetilde{G}}(a)) = P(\widetilde{g})N_G(a)$ for any $\widetilde{g} \in \widetilde{G}$, where $d_0^h(\widetilde{g}) = a$.

Proof. Suppose that $\widetilde{x}N_{\widetilde{X}}(a) = \widetilde{y}N_{\widetilde{X}}(a)$ for suitable $\widetilde{x}, \widetilde{y} \in \widetilde{X}$. Then there exists $\widetilde{n} \in N_{\widetilde{X}}(a)$ such that $\widetilde{y} = \widetilde{x} \circ \widetilde{n}$. Applying P, we obtain $P(\widetilde{y}) = P(\widetilde{x}) \circ P(\widetilde{n})$. Since $P(N_{\widetilde{X}}) \subseteq N_X$, it follows that $P(\widetilde{n}) \in N_X(a)$, and therefore $P(\widetilde{y})N_X(a) = P(\widetilde{x})N_X(a)$. Hence, the assignment $\widetilde{x}N_{\widetilde{X}}(a) \mapsto P(\widetilde{x})N_X(a)$ is well-defined. Similarly, suppose $\widetilde{g}N_{\widetilde{G}}(a) = \widetilde{h}N_{\widetilde{G}}(a)$ for suitable $\widetilde{g}, \widetilde{h} \in \widetilde{G}$. Then $\widetilde{h} = \widetilde{g} * \widetilde{m}$ for some $\widetilde{m} \in N_{\widetilde{G}}(a)$. Since $P(N_{\widetilde{G}}) \subseteq N_G$, we obtain $P(\widetilde{h})N_G(a) = P(\widetilde{g})N_G(a)$, showing that $\widetilde{g}N_{\widetilde{G}}(a) \mapsto P(\widetilde{g})N_G(a)$ is also well-defined. Since P preserves sources, targets, horizontal compositions, and vertical compositions, the induced maps on quotient classes preserve the corresponding structure maps. Consequently, $\widehat{P} : \widetilde{\mathbb{G}}/\widetilde{\mathbb{N}} \longrightarrow \mathbb{G}/\mathbb{N}$ is a well-defined strict 2-functor. □

Building upon the well-definedness of this induced functor, we are now in a position to establish that the covering property behaves conditionally under quotient structures.

Theorem 6.9. *Let $P : \widetilde{\mathbb{G}} \longrightarrow \mathbb{G}$ be a covering morphism of strict 2-groupoids acting identically on objects. Let $\widetilde{\mathbb{N}} = (A, N_{\widetilde{X}}, N_{\widetilde{G}})$ and $\mathbb{N} = (A, N_X, N_G)$ be normal sub-2-groupoids satisfying $N_{\widetilde{X}} = P^{-1}(N_X)$ and $N_{\widetilde{G}} = P^{-1}(N_G)$. Then the induced strict 2-functor $\widehat{P} : \widetilde{\mathbb{G}}/\widetilde{\mathbb{N}} \longrightarrow \mathbb{G}/\mathbb{N}$ of Lemma 6.8 is a covering morphism of strict 2-groupoids.*

Proof. Assume that $N_{\widetilde{X}} = P^{-1}(N_X)$ and $N_{\widetilde{G}} = P^{-1}(N_G)$. Let $\widetilde{x} : a \to b$ be a 1-morphism in $\widetilde{X}$. To prove that $\widehat{P}$ is a covering morphism of strict 2-groupoids, it suffices to show that the induced map on the vertical local stars over the 1-cell class

$$\widehat{P}_* : St^v_{\widetilde{\mathbb{G}}/\widetilde{\mathbb{N}}}(\widetilde{x}N_{\widetilde{X}}(a)) \longrightarrow St^v_{\mathbb{G}/\mathbb{N}}(P(\widetilde{x})N_X(a))$$

is bijective.

For surjectivity, let $gN_G(a) \in St^v_{\mathbb{G}/\mathbb{N}}(P(\widetilde{x})N_X(a))$, where $g \in G$ is a 2-cell whose vertical source is the 1-cell $P(\widetilde{x})$. Since P is a covering morphism, the corresponding induced map on the vertical stars of the total space, $P_* : St^v_{\widetilde{\mathbb{G}}}(\widetilde{x}) \to St^v_{\mathbb{G}}(P(\widetilde{x}))$, is bijective. Hence, there exists a unique 2-cell $\widetilde{g} \in St^v_{\widetilde{\mathbb{G}}}(\widetilde{x})$ such that $P(\widetilde{g}) = g$. By applying the definition of $\widehat{P}$, we obtain

$$\widehat{P}_*(\widetilde{g}N_{\widetilde{G}}(a)) = P(\widetilde{g})N_G(a) = gN_G(a),$$

which proves that $\widehat{P}_*$ is surjective.

For injectivity, suppose that $\widetilde{g}N_{\widetilde{G}}(a)$ and $\widetilde{h}N_{\widetilde{G}}(a)$ are elements in $St^v_{\widetilde{\mathbb{G}}/\widetilde{\mathbb{N}}}(\widetilde{x}N_{\widetilde{X}}(a))$ such that $\widehat{P}_*(\widetilde{g}N_{\widetilde{G}}(a)) = \widehat{P}_*(\widetilde{h}N_{\widetilde{G}}(a))$. Then, we have

$$P(\widetilde{g})N_G(a) = P(\widetilde{h})N_G(a).$$

Therefore,

$$\overline{P(\widetilde{h})}^{v} \bullet P(\widetilde{g}) = P(\overline{\widetilde{h}}^{v} \bullet \widetilde{g}) \in N_G(a).$$

Using the pullback-type assumption $N_{\widetilde{G}} = P^{-1}(N_G)$, it follows that $\overline{\widetilde{h}}^{v} \bullet \widetilde{g} \in N_{\widetilde{G}}(a)$. This relation implies that $\widetilde{g} N_{\widetilde{G}}(a) = \widetilde{h} N_{\widetilde{G}}(a)$, which establishes that $\widehat{P}_*$ is injective. □

Finally, by translating the quotient construction through the Brown and İçen correspondence, we obtain a canonical lifting crossed module over groupoids associated with a normal sub-2-groupoid.

Theorem 6.10. *Let $\mathbb{N} = (A, N_X, N_G)$ be a normal sub-2-groupoid of $\mathbb{G} = (A, X, G)$. Then, the triple $(\widehat{C}, \mathbb{X}/N_X, E)$ constitutes a lifting of the crossed module over groupoids, where*

- *$\widehat{C} \colon (A, \mathrm{Ker} d_0^v) \to (A, X/N_X)$ is defined by $\widehat{C}(g) = d_1^v(g)N(a)$, where*

$$\begin{tikzcd} a \arrow[r, bend left, "1_a"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow g"] & a \end{tikzcd} ;$$

- *the action of $(A, X/N_X)$ on $(A, \mathrm{Ker} d_0^v)$ is given by $xN(a) \star g = 1_x * g * 1_{\overline{x}}$ where*

$$\begin{tikzcd} a \arrow[r, bend left, "1_a"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow g"] & a \arrow[r, bend left, "x"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "\Downarrow 1_x"] & b \end{tikzcd} ;$$

- *the lifting functor $E : (A, X/N_X) \to (A, X)$ is given by $E(xN(a)) = x$ for all $x \in X$.*

Proof. Since $E\widehat{C}(g) = E(d_1^v(g)N(a)) = d_1^v(g) = C(g)$, the following diagram is commutative:

$$\begin{tikzcd} & X/N_X \arrow[d, "E"] \\ \mathrm{Ker} d_0^v \arrow[ur, "\widehat{C}"] \arrow[r, "C"] & X \end{tikzcd}$$

The groupoid action of $(A, X/N_X)$ on the totally disconnected groupoid $(A, \mathrm{Ker} d_0^v)$ can be written in terms of the action of (A, X) on $(A, \mathrm{Ker} d_0^v)$ via E as follows:

$$xN(a) \star g = 1_x * g * 1_{\overline{x}} = x \cdot g = E(xN(a)) \cdot g.$$

Now we show that $\widehat{C} \colon (A, \mathrm{Ker} d_0^v) \to (A, X/N_X)$ is a crossed module over groupoids:

[CMGd 1]

$$\begin{aligned} \widehat{C}(xN(a) \star g) &= \widehat{C}(1_x * g * 1_{\overline{x}}) = d_1^v(1_x * g * 1_{\overline{x}})N(b) \\ &= d_1^v(1_x)N(a) \circ d_1^v(g)N(a) \circ d_1^v(1_{\overline{x}})N(b) = xN(a) \circ \widehat{C}(g) \circ \overline{x}N(b). \end{aligned}$$

[CMGd 2] $\widehat{C}(g) \star g' = E\widehat{C}(g) \cdot g' = C(g) \cdot g' = g * g' * \overline{g}$.

□

Theorem 6.10 shows that normal sub-2-groupoids provide a natural source of lifting crossed modules over groupoids. Consequently, quotient constructions in the category of strict 2-groupoids not only preserve essential covering structures but also generate canonical lifting configurations. This establishes a direct connection between quotient theory, covering theory, and lifting theory within the framework developed throughout this paper. Moreover, by integrating the first isomorphism theorem for strict 2-groupoids (Theorem 6.5), we can significantly generalize the scope of Theorem 6.10. Instead of restrictedly considering a static, isolated normal sub-2-groupoid, the following result lifts the construction to a dynamic, functorial framework regulated by identity-on-objects morphisms.

Corollary 6.11. *Let $F : \mathbb{G} \to \mathbb{H}$ be an identity-on-objects morphism of strict 2-groupoids. Then, the lifting of crossed modules over groupoids obtained from the quotient 2-groupoid $\mathbb{G}/\mathrm{Ker}F$ via Theorem 6.10 is naturally determined by (and isomorphic to) the lifting structure induced by the image 2-groupoid $F(\mathbb{G})$.*

7. Conclusions

In this paper, we have successfully established a unified framework connecting the geometric covering theory of strict 2-groupoids with the algebraic lifting theory of crossed modules over groupoids. By working within a strictly object-preserving setting, we demonstrated that covering morphisms, strict 2-groupoid actions, and liftings of crossed modules are categorically equivalent structures. Furthermore, our introduction of an object-preserving quotient architecture via normal 2-group bundles provided a direct mechanism for constructing canonical lifting crossed modules. These results offer a computationally direct, strict algebraic alternative that naturally complements the weak equivalences found in the butterfly formalism and stack theories.

The quotient constructions developed in this paper suggest several directions for further investigation. Indeed, for a normal sub-2-groupoid $\mathbb{N}$ of a strict 2-groupoid $\mathbb{G}$, the canonical inclusion and quotient projection naturally determine a sequence

$$\mathbb{T}_A \longrightarrow \mathbb{N} \longrightarrow \mathbb{G} \longrightarrow \mathbb{G}/\mathbb{N} \longrightarrow \mathbb{T}_A,$$

where $\mathbb{T}_A = (A, \{1_a\}_{a\in A}, \{1_{1_a}\}_{a\in A})$ denotes the trivial 2-groupoid on the common object set A. This sequence may be viewed as an object-preserving analogue of a short exact sequence and suggests the possibility of developing notions of exactness, extensions, and homological invariants for strict 2-groupoids within the identity-on-objects framework.

Furthermore, the results established in this paper suggest that analogous covering-action-lifting correspondences may exist in broader categorical settings. In particular, it would be natural to investigate similar constructions for Lie groupoids [28], cat^1-groupoids [16], crossed modules of categories [30, 33] and of 2-groups [34, 35], and other higher-dimensional internal categorical structures. Another interesting direction is to examine whether the object-preserving quotient framework developed here can be extended to these settings and whether corresponding notions of coverings, actions, and liftings can be characterized through analogous categorical equivalences. We expect that the quotient, covering, and lifting constructions established here will provide useful tools for such developments, and these questions will be pursued in future work.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest.

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