AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 27351–27369.
DOI: 10.3934/math.20261094
Received: 10 June 2026
Revised: 14 August 2026
Accepted: 18 August 2026
Published: 31 August 2026

*Research article*

# Fuzzy strong vertex connectivity and fuzzy strong edge connectivity with their applications

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**Abstract:** Conventional connectivity analysis of fuzzy graphs provides a crucial basis for evaluating the transmission reliability of networks. However, the separate consideration of strong paths and strongest paths has become the primary bottleneck in accurately measuring the overall network connectivity performance. This paper takes the strongest strong path as a core starting point. It breaks through the one-dimensional limitations of conventional connectivity evaluation in fuzzy graphs. Accordingly, the concepts of fuzzy strong vertex connectivity and fuzzy strong edge connectivity are defined. On this basis, this paper investigates the quantitative laws of these two types of connectivity in typical fuzzy graphs, constructs standardized calculation algorithms, and clarifies practical application pathways. The research provides solid scientific support for the health evaluation of regional ecosystems and further extends the application foundation of fuzzy graph network simulation methods.

**Keywords:** fuzzy graph; strongest strong path; strong vertex connectivity; strong edge connectivity
**Mathematics Subject Classification:** 05C72, 05C40, 05C90, 92D40

## 1. Introduction

### 1.1. Research background

As a major research direction in graph theory, connectivity serves as a core metric for quantifying the invulnerability and stability of networks. Corresponding research in this field has been extensively adopted in pivotal domains including communication networks, intelligent transportation systems, and distributed systems. In classical graph theory, vertex connectivity and edge connectivity characterize the resilience of network connectivity. These metrics are defined as the cardinality of minimum vertex cuts and minimum edge cuts, which provide a solid theoretical foundation for the optimal design of deterministic networks. However, networks in practical scenarios generally exhibit fuzziness and uncertainty. Factors such as signal attenuation in communication links, dynamic traffic capacity, and varying reliability of data nodes restrict the application of classical connectivity metrics. These indicators only focus on the critical state of structural disconnection, so they fail to precisely describe the dynamic support mechanism of paths and the threshold range of network invulnerability in fuzzy environments. Consequently, the identification of core hubs and critical links lacks a solid quantitative foundation. As a result, traditional methods cannot meet the demand for refined optimization in complex fuzzy networks.

The emergence of fuzzy graph theory has paved a new way to address this problem. In 1975, Rosenfeld [1] first incorporated fuzzy set theory into graph theory and established the basic framework of fuzzy graphs. Since then, extensive research on fuzzy connectivity has been continuously conducted worldwide. Bhutani and Rosenfeld [2] defined strong edges and uncovered their relationship to fuzzy cut bridges. This work laid the foundation for edge structure analysis in fuzzy graphs. Sunitha and Vijayakumar [3] conducted an in-depth analysis of the properties of fuzzy cut edges and fuzzy cut vertices. By combining these with the unique maximum spanning tree of a fuzzy tree, they proposed multiple equivalence criteria for fuzzy trees. Yeh and Bang first put forward the fundamental definitions of fuzzy vertex connectivity and fuzzy edge connectivity [4]. Based on this foundational achievement, Mathew and Sunitha revised and generalized the original fuzzy definitions of vertex connectivity and edge connectivity [5]. This advance has promoted the practical development of invulnerability research on fuzzy networks. Ali et al. [6] constructed $t$-connected fuzzy graphs for any real number $t$. They introduced the concepts of average fuzzy vertex connectivity and uniformly $t$-connected fuzzy graphs and characterized their properties in fuzzy forests, fuzzy cycles, and complete fuzzy graphs. This provided critical theoretical support for identifying core network hubs. Additional studies on fuzzy graph connectivity can be found in [7–12].

### *1.2. Motivation*

In the study of fuzzy graph theory, connectivity serves as the core metric for characterizing the transmission reliability of networks. It also acts as a critical indicator for evaluating the overall operational performance of networks. Strong paths and strongest paths are two core transmission carriers in fuzzy graphs. These two types of paths feature complementary functions and correlated properties. The strong paths provide fundamental guarantees for stable network transmission, while the strongest paths determine the optimal threshold for network traffic capacity. Their coupling relationship acts as an essential prerequisite for accurately evaluating network connectivity quality.

Nevertheless, current studies of fuzzy connectivity exhibit evident limitations in the theoretical framework. Based on the dual properties of strong paths and strongest paths, Reference [13] defined fuzzy strong vertex cuts and fuzzy strong edge cuts. It realized the accurate localization of bottleneck nodes and bottleneck links on a network's core transmission paths and provided a new qualitative research paradigm for network bottleneck identification. However, this research remains confined to qualitative analysis. An adaptive quantitative model for fuzzy strong connectivity has not yet been established. As a result, standardized connectivity metrics cannot systematically characterize the inherent properties of strong paths and strongest paths, which impedes fine-grained quantitative evaluation of network connectivity quality. Reference [14] developed a dual-parameter fuzzy edge connectivity framework based on strongest paths. Even so, this modeling approach still presents evident limitations. First, the framework relies exclusively on strongest paths and focuses on edge connectivity analysis alone. It overlooks the stable transmission properties inherent to strong paths and cannot reflect the transmission stability of networks. Second, systematic analysis of vertex connectivity is lacking in this modeling approach, which impedes comprehensive connectivity assessment covering network nodes and links. Overall, existing research tends to rely on single-dimensional modeling and lacks comprehensive evaluation systems. A quantitative analytical framework incorporating dual-path coupling and full-domain coverage has not yet been developed.

To address the aforementioned limitations and theoretical gaps of existing studies, this paper leverages the dual coupling characteristics of strong paths and strongest paths. On this basis, a complete quantitative system for fuzzy strong vertex connectivity and fuzzy strong edge connectivity is innovatively established. Theoretically, this paper builds upon the foundational qualitative research on strong cuts in [13] and fills the existing gap in quantitative analysis of fuzzy strong connectivity. This work addresses the theoretical limitation of [14], which only explores single edge connectivity based on strongest paths. For the first time, this study constructs a coupled model of strong paths and strongest paths, achieves full coverage of strong vertex connectivity and strong edge connectivity, and realizes multi-dimensional theoretical integration of qualitative bottleneck identification and dual-parameter quantitative characterization. This research optimizes and refines the conventional dual-parameter theoretical framework. By utilizing the stable transmission characteristics of strong paths and the optimal flow-carrying capacity of strongest paths, the framework balances transmission reliability and efficiency of networks. It supports multidimensional, systematic, and high-precision characterization of strong connectivity quality for nodes and links within fuzzy networks and further advances the theoretical analysis framework for strong connectivity in fuzzy graphs. Practically, this paper establishes a complete theoretical toolchain covering indicator quantification, bottleneck localization, and resilience evaluation. The toolchain can be directly applied to engineering scenarios including network topology optimization, logistics route planning, and anti-interference analysis of data transmission. It delivers reliable quantitative support for engineering decisions involving network optimization, fault prevention and control and efficiency improvement and effectively boosts the engineering practicability and practical application value of fuzzy graph theory research.

### *1.3. Framework of this study*

This paper systematically studies the strong connectivity of fuzzy graphs. It is organized in the structure of theoretical basis, core definitions and derivations, algorithm design, and practical applications. The core content of each section is outlined below. First, Section 2 systematically summarizes fundamental notations and key terms. This helps consolidate the theoretical foundation and clarify conceptual boundaries, ensuring rigor and consistency in subsequent analysis. On this basis, Sections 3 and 4 focus on two core concepts: fuzzy strong vertex connectivity and fuzzy strong edge connectivity. By means of theoretical formalization and rigorous derivation, this paper establishes their explicit computational formulas. These formulas are applied to three typical fuzzy graphs: fuzzy trees, complete fuzzy graphs, and fuzzy cycles. The quantitative characteristics of the fuzzy strong connectivity across different fuzzy graphs are clearly identified. To facilitate the computability of the connectivity parameters, Section 5 proposes dedicated algorithms for these three typical fuzzy graphs. These algorithms compute fuzzy strong vertex connectivity and fuzzy strong edge connectivity, linking theoretical results to practical implementation. Finally, to verify the application value of the research results, Section 6 adopts real-world application scenarios. This section explores the application paths and practical significance of fuzzy strong vertex connectivity and fuzzy strong edge connectivity. The analysis focuses on the following field: regional ecosystem health assessment. The present study completes a closed loop from theoretical research to practical application.

## 2. Preliminaries

All fuzzy graphs discussed in this paper are loopless, finite, undirected, and connected. The basic notations and terminology relevant to this study are given below, as detailed in [15].

**Definition 2.1.** *Let $V$ denote a set. A fuzzy subset of $V$ is a mapping from $V$ to* $[0, 1]$.

**Definition 2.2.** *A fuzzy graph $F = (V, \rho, \eta)$ is defined as a triple consisting of a non-empty vertex set $V$, a fuzzy vertex subset $\rho : V \to [0, 1]$, and a fuzzy edge subset $\eta : V \times V \to [0, 1]$ such that $\eta(y, z) \leq \rho(y) \wedge \rho(z)$ holds for all vertices $y, z \in V$. In this definition, $\rho(y)$ denotes the membership value of vertex $y$, and $\eta(y, z)$ denotes the membership value of edge $(y, z)$. When $V$ is finite, $F$ is referred to as a finite fuzzy graph. When $\eta$ is symmetric, $F$ is termed an undirected fuzzy graph. Otherwise, it is termed a directed fuzzy graph. If $\eta(y, y) \neq 0$, then there exists a loop on vertex $y$.*

**Definition 2.3.** *For a fuzzy graph $F = (V, \rho, \eta)$, its underlying graph $F^*$ is given by $F^* = (\rho^*, \eta^*)$, where $\rho^* = \{y \in V \mid \rho(y) > 0\}$ and $\eta^* = \{(y, z) \in V \times V \mid \eta(y, z) > 0\}$. We use $|\rho^*|$ to represent the cardinality of the vertex set $\rho^*$ and write $yz$ as a shorthand for the edge $(y, z)$ in $\eta^*$.*

**Definition 2.4.** *For a path $P = x_1 \to x_2 \to \cdots \to x_n \to x_{n+1}$ in $F$, its path strength $s(P)$ is defined as the minimum edge membership value along the path, formally expressed as $s(P) = \wedge_{i=1}^{n} \eta(x_i, x_{i+1})$. If $x_1 = x_{n+1}$ and $n \geq 2$, then $P$ is a cycle.*

**Definition 2.5.** *If a cycle contains at least two weakest edges, then it is called a fuzzy cycle.*

**Definition 2.6.** *The strength of connectedness between two vertices $y$ and $z$ in $F$, denoted by $CONN_F(y, z)$, is defined as the maximum strength of all $y - z$ paths in $F$.*

**Definition 2.7.** *A path $P$ in $F$ is called a strongest path between $y$ and $z$ if $P$ is a $y - z$ path in $F$ and $s(P) = CONN_F(y, z)$.*

**Definition 2.8.** *A fuzzy graph $F = (V, \rho, \eta)$ is called a connected fuzzy graph if for every pair of vertices $y, z \in \rho^*$, we have $CONN_F(y, z) > 0$.*

**Definition 2.9.** *A fuzzy graph $F = (V, \rho, \eta)$ is called a complete fuzzy graph if $\eta(y, z) = \rho(y) \wedge \rho(z)$ for any pair of vertices $y, z \in \rho^*$.*

**Definition 2.10.** *Let $F = (V, \rho, \eta)$ and $G = (V, \sigma, \mu)$ be two fuzzy graphs. If $\sigma(y) < \rho(y)$ holds for all vertices $y \in \sigma^*$ and $\mu(y, z) < \eta(y, z)$ for every edge $yz \in \mu^*$, then $G$ is referred to as a partial fuzzy subgraph of $F$. If $\sigma = \rho$ and $\mu(y, z) = \eta(y, z)$ for every edge $yz \in \mu^*$, then $G$ is termed a spanning fuzzy subgraph of $F$.*

**Definition 2.11.** *Let $F = (V, \rho, \eta)$ be a connected fuzzy graph. Suppose there exists a spanning subgraph $H$ of $F$ that is a tree, and for every edge $yz$ not contained in $H$, there is a $y - z$ path in $H$ with strength greater than $\eta(y, z)$. Then $F$ is called a fuzzy tree.*

**Definition 2.12.** *Let $yz \in \eta^*$. If $\eta(y, z) > CONN_{F-\{yz\}}(y, z)$, then $yz$ is called an $\alpha$-strong edge. If $\eta(y, z) = CONN_{F-\{yz\}}(y, z)$, then $yz$ is called a $\beta$-strong edge. If $\eta(y, z) < CONN_{F-\{yz\}}(y, z)$, then $yz$ is called a $\delta$-edge. An edge is called a strong edge if it is either $\alpha$-strong or $\beta$-strong.*

**Definition 2.13.** *If every edge in a path $P$ is strong, then $P$ is called a strong path.*

## 3. Fuzzy strong vertex connectivity

In this section, we first define fuzzy strong vertex connectivity and then derive its formulas for fuzzy trees, complete fuzzy graphs, and fuzzy cycles.

**Definition 3.1.** *[15] Let $F = (V, \rho, \eta)$ be a fuzzy graph and $y, z \in \rho^*$. Let $P$ be a strongest $y - z$ path in $F$. If $P$ is a strong path, then $P$ is called a fuzzy strongest strong $y - z$ path.*

**Definition 3.2.** *[13] Let $V^* \subseteq V$. If there exist vertices $y, z \in \rho^* \setminus V^*$ such that fuzzy strongest strong $y - z$ paths exist in $F$, while no fuzzy strongest strong $y - z$ paths exist in $F - V^*$, then $V^*$ is called a fuzzy strong vertex cut (referred to as FSVC) of $F$. In particular, if there is only one vertex $w$ in $V^*$, then $w$ is called a fuzzy strong cut vertex (referred to as FSCV) of $F$.*

**Definition 3.3.** *Let $F = (V, \rho, \eta)$ be a fuzzy graph and $V^* \subseteq V$. The strong weight of $V^*$ is denoted by $\varsigma(V^*)$ and is defined as $\varsigma(V^*) = \sum_{x \in V^*} \min\{\eta(x, y) \mid xy \text{ is a strong edge in } F\}$.*

**Definition 3.4.** *Let $M$ be an FSVC of $F$. If $\varsigma(M) = \tau(F)$, where $\tau(F) = \wedge\{\varsigma(V^*) \mid V^* \text{ is an FSVC of } F\}$, then $M$ is called a minimal fuzzy strong vertex cut (referred to as MFSVC) of $F$. In particular, if there is only one vertex $w$ in $M$, then $w$ is called a minimal fuzzy strong cut vertex (referred to as MFSCV) of $F$.*

When there is no FSVC in $F$, this implies that there exists a strong edge between any two vertices in $F$. Furthermore, the original strong structure of $F$ can only be destroyed by deleting $|\rho^*| - 1$ vertices. In this case, $F$ degenerates into a trivial fuzzy graph with exactly one vertex. Let $q$ be the minimum membership value of all edges. Then the strong weight of any vertex subset containing $|\rho^*| - 1$ vertices is $(|\rho^*| - 1)q$. For this scenario, we stipulate that $\tau(F) = (|\rho^*| - 1)q$.

**Definition 3.5.** *Let $V^*$ be an FSVC of $F$. The impact of $V^*$ is denoted by $I(V^*)$ and is defined as $I(V^*) = \vee_{y,z \in \rho^* \setminus V^*}\{1 - \frac{CONN_{F-V^*}(y,z)}{CONN_F(y,z)} \mid$ there exist fuzzy strongest strong $y - z$ paths in $F$, but no such paths exist in $F - V^*\}$.*

**Definition 3.6.** *The fuzzy strong vertex connectivity of $F$ is denoted by $\gamma(F)$ and is defined as $\gamma(F) = (\tau(F), \psi(F))$, where $\tau(F) = \wedge\{\varsigma(V^*) \mid V^* \text{ is an FSVC of } F\}$ and $\psi(F) = \vee\{I(M) \mid M \text{ is an MFSVC of } F\}$.*

When there is no FSVC in $F$, we define $\gamma(F) = ((|\rho^*| - 1)q, 0)$.

**Definition 3.7.** *Let $\gamma(F_1) = (\tau(F_1), \psi(F_1))$ and $\gamma(F_2) = (\tau(F_2), \psi(F_2))$ be fuzzy strong vertex connectivity of fuzzy graphs $F_1$ and $F_2$, respectively.*

*1. If either of the following conditions is fulfilled:*
*(a) $\tau(F_1) < \tau(F_2)$;*
*(b) $\tau(F_1) = \tau(F_2)$ and $\psi(F_1) > \psi(F_2)$,*
*then the fuzzy strong vertex connectivity of $F_2$ is greater than that of $F_1$, denoted by $\gamma(F_1) < \gamma(F_2)$.*
*2. Given that*
*(c) $\tau(F_1) = \tau(F_2)$ and $\psi(F_1) = \psi(F_2)$,*
*the fuzzy strong vertex connectivity of $F_1$ is equal to that of $F_2$, denoted by $\gamma(F_1) = \gamma(F_2)$.*
*3. If any one of conditions (a), (b), and (c) holds, the fuzzy strong vertex connectivity of $F_1$ is no greater than that of $F_2$, denoted by $\gamma(F_1) \leq \gamma(F_2)$.*

**Definition 3.8.** *Let $F = (V, \rho, \eta)$ be a fuzzy cycle and $h \in \rho^*$. A vertex $h$ is classified as a type-1 vertex of $F$ if it is incident to two $\alpha$-strong edges in $F$. A vertex $h$ is referred to as a type-2 vertex of $F$ if it is incident to exactly one $\alpha$-strong edge and one $\beta$-strong edge in $F$. A vertex $h$ is referred to as a type-3 vertex of $F$ if it is incident to two $\beta$-strong edges in $F$.*

In every example of a fuzzy graph $F = (V, \rho, \eta)$ considered in this paper, we set $\rho(h) = 1$ for each $h \in \rho^*$.

**Example 3.1.** *Let $F = (V, \rho, \eta)$ be a fuzzy graph with vertex set $V = \{a, b, c, d, e, f, g, h, x, y\}$, where the edge membership values are specified as $\eta(a, b) = \eta(c, d) = \eta(g, h) = \eta(e, y) = 0.5$, $\eta(c, x) = \eta(f, g) = \eta(g, y) = 0.2$, $\eta(b, c) = \eta(x, y) = 0.1$, $\eta(d, e) = \eta(h, a) = 0.6$, $\eta(e, f) = 0.4$, and $\eta(a, x) = 0.7$, as shown in Figure 1. It can be deduced that $\{c, g\}$, $\{g, x\}$, and $\{e\}$ are MFSVCs of $F$, with $\tau(F) = 0.4$. Given that $I(\{c, g\}) = I(\{g, x\}) = 0.5$ and $I(\{e\}) = 0.6$, we derive $\psi(F) = 0.6$. Consequently, $\gamma(F) = (0.4, 0.6)$.*

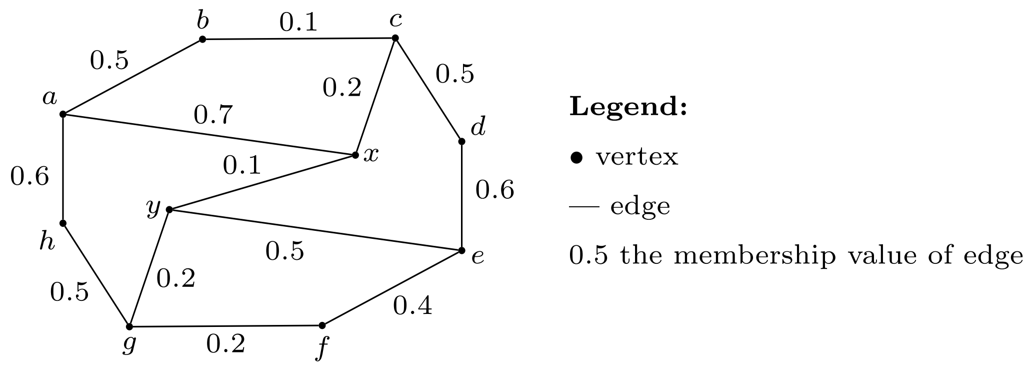


**Figure 1.** Fuzzy graph in Example 3.1.

**Lemma 3.1.** *[13] Let $F = (V, \rho, \eta)$ be a fuzzy tree and $h \in \rho^*$. Let $H$ be the unique maximum spanning tree of $F$. Then $h$ qualifies as an FSCV of $F$ if and only if $h$ is an internal vertex of $H$.*

**Lemma 3.2.** *[2] Let $F = (V, \rho, \eta)$ be a fuzzy tree. Then an edge of $F$ is a strong edge if and only if it is contained in the unique maximum spanning tree of $F$.*

**Theorem 3.1.** *Let $F = (V, \rho, \eta)$ be a fuzzy tree satisfying $|\rho^*| = 2$, and let $q$ be the membership value of the unique edge in $F$. Then $\gamma(F) = (q, 0)$.*

*Proof.* Assume that $|\rho^*| = 2$. Then there is no FSVC in $F$. In accordance with the definition of $\gamma(F)$, we conclude that $\gamma(F) = (q, 0)$. □

**Theorem 3.2.** *Let $F = (V, \rho, \eta)$ be a fuzzy tree such that $F^*$ is a tree and $|\rho^*| \geq 3$. Let $q$ denote the minimum membership value of all strong edges in $F$. Then $\gamma(F) = (q, 1)$.*

*Proof.* Let $V^*$ be an arbitrary FSVC of $F$. The definition of strong weight gives $\varsigma(V^*) \geq q$, so $\tau(F) \geq q$. By Lemma 3.2, it follows that the subgraph induced by all strong edges in $F$ is its unique maximum spanning tree $H$. Let $e = x_0y_0$ be a strong edge in $F$ with membership value $q$. Since $|\rho^*| \geq 3$, we may take $x_0$ to be an internal vertex of $H$, which makes $x_0$ an FSCV of $F$ with strong weight $q$. By the minimality of $\tau(F)$, we have $\tau(F) \leq \varsigma(\{x_0\}) = q$. Therefore, $\tau(F) = q$.

Since $\varsigma(\{x_0\}) = \tau(F)$, $\{x_0\}$ is an MFSVC of $F$. From the definition of FSVCs, we have a pair of vertices $y, z \in \rho^* \setminus \{x_0\}$ such that fuzzy strongest strong $y - z$ paths exist in $F$, but no such paths exist in

$F - \{x_0\}$. Since $F^*$ is a tree, $F - \{x_0\}$ contains no strongest $y - z$ paths, meaning $CONN_{F-\{x_0\}}(y, z) = 0$. By the definition of impact, we have $I(\{x_0\}) = 1$. Furthermore, from the definition of $\psi(F)$, we obtain $\psi(F) = 1$. Thus, $\gamma(F) = (q, 1)$. □

**Lemma 3.3.** *[16] Let $F = (V, \rho, \eta)$ be a complete fuzzy graph. Then $F$ contains no $\delta$-edges.*

**Lemma 3.4.** *Let $F = (V, \rho, \eta)$ be a complete fuzzy graph. Then there exists no FSVC in $F$ .*

*Proof.* Assume that $V^*$ is an FSVC of $F$. By the definition of FSVCs, we have a pair of vertices $y, z \in \rho^* \setminus V^*$ such that fuzzy strongest strong $y - z$ paths exist in $F$, but no such paths exist in $F - V^*$. This implies that every fuzzy strongest strong $y - z$ path in $F$ contains at least one vertex belonging to $V^*$. By the definition of complete fuzzy graphs and Lemma 3.3, $y \to z$ is a fuzzy strongest strong $y - z$ path in $F$. As this path does not contain any vertex from $V^*$, it directly contradicts the condition that all such paths pass through at least one vertex in $V^*$. Therefore, the assumption does not hold, and there is no FSVC in $F$. □

**Theorem 3.3.** *Let $F = (V, \rho, \eta)$ be a complete fuzzy graph with $\rho^* = \{a_1, a_2, \ldots, a_n\}$ and the membership values of vertices satisfy $\rho(a_1) \geq \rho(a_2) \geq \cdots \geq \rho(a_n)$. Then $\gamma(F) = ((n - 1)\rho(a_n), 0)$.*

*Proof.* From Lemma 3.4, it follows that there is no FSVC in $F$. In accordance with the definition of $\gamma(F)$, it follows immediately that $\gamma(F) = ((n - 1)\rho(a_n), 0)$. □

**Lemma 3.5.** *[13] Let $F = (V, \rho, \eta)$ be a fuzzy cycle and $h \in \rho^*$. Then $h$ is an FSCV of $F$ if and only if $h$ is incident to two $\alpha$-strong edges in $F$.*

**Theorem 3.4.** *Let $F = (V, \rho, \eta)$ be a fuzzy cycle satisfying $|\rho^*| = 3$, and let $q$ be the minimum membership value of all edges. Then $\gamma(F) = (2q, 0)$.*

*Proof.* By the conditions of this theorem and Lemma 3.5, there exists no FSCV in $F$. Furthermore, it can be easily shown that there is no FSVC in $F$. By the definition of $\gamma(F)$, we thus conclude that $\gamma(F) = (2q, 0)$. □

**Theorem 3.5.** *Let $F = (V, \rho, \eta)$ be a fuzzy cycle with $\rho^* = \{a_1, a_2, \ldots, a_n\}$ satisfying $n \geq 4$. Let $q$ denote the minimum membership value of all edges, $B$ a set of type-1 vertices in $F$, and $p$ the minimum membership value of edges incident to type-1 vertices.Then the fuzzy strong vertex connectivity of $F$ is given by*

$$\gamma(F) = \begin{cases} \left(p, \dfrac{p-q}{p}\right), & B \neq \emptyset \text{ and } 2q > p, \\ (2q, 1), & \text{otherwise}. \end{cases}$$

*Proof.* By Lemma 3.5, $a_l \in B$ if and only if $a_l$ is an FSCV of $F$, where $l \in \{1, 2, \ldots, n\}$. We analyze three distinct cases below.

Case 1: $B \neq \emptyset$ and $2q > p$.

Then there exists an FSCV $a_r$ in $B$ with strong weight $p$, where $r \in \{1, 2, \ldots, n\}$. By the minimality of $p$, $a_r$ is an MFSCV of $F$.

It is straightforward to show that, given $B \neq \emptyset$ and $2q > p$, only MFSCVs exist in $F$. Suppose $X$ is an MFSVC of $F$ and $|X| \geq 2$. Since $|X| \geq 2$, $\varsigma(X) \geq 2q > p$ . This contradicts the definition of MFSVCs. Therefore, there are only MFSCVs in $F$.

Thus, $\tau(F) = p$.

There is no loss of generality in taking $a_1$ to be an MFSCV of $F$. By Lemma 3.5 and the definition of FSCVs, for vertices $a_2$, $a_n$, there exists a fuzzy strongest strong $a_n - a_2$ path $a_n \to a_1 \to a_2$ in $F$, but no such paths exist in $F - \{a_1\}$, and $CONN_{F-\{a_1\}}(a_n, a_2) = q < p = CONN_F(a_n, a_2)$. Therefore, by the definition of $I(\{a_1\})$, we have $I(\{a_1\}) = \frac{p-q}{p}$. Additionally, in accordance with the definition of $\psi(F)$, it follows that $\psi(F) = \frac{p-q}{p}$. Hence, $\gamma(F) = (p, \frac{p-q}{p})$.

Case 2: $B \neq \emptyset$ and $2q \leq p$.

Since every vertex in $B$ is an FSCV of $F$, the minimum strong weight among FSVCs of cardinality 1 in $F$ is $p$. Any proper subset $Y$ of $V$ with $|Y| \geq 2$ that contains a pair of non-adjacent vertices constitutes an FSVC of $F$. Further analysis indicates that the minimum strong weight of such FSVCs is $2q$. Given $2q \leq p$, we conclude that $\tau(F) = 2q$.

Without loss of generality, take $T = \{a_i, a_j\}$ as an MFSVC, where $1 \leq i < j \leq n$. By the definition of FSVCs, we have a pair of vertices $a_{r_1}, a_{r_2} \in \rho^* \setminus T$ such that fuzzy strongest strong $a_{r_1} - a_{r_2}$ paths exist in $F$, but no such paths exist in $F - T$, and $CONN_{F-T}(a_{r_1}, a_{r_2}) = 0 < q = CONN_F(a_{r_1}, a_{r_2})$, where $r_1 = (i+1) \bmod n$ and $r_2 = (j+1) \bmod n$. Therefore, $I(T) = 1$. Additionally, in accordance with the definition of $\psi(F)$, it follows that $\psi(F) = 1$. Thus, $\gamma(F) = (2q, 1)$.

Case 3: $B = \emptyset$.

Then any set formed by a pair of non-adjacent type-2 or type-3 vertices in $F$ constitutes an FSVC of $F$, and each such FSVC has strong weight $2q$. It follows that $\tau(F) = 2q$. The proof for $\psi(F) = 1$ proceeds analogously to Case 2, so we conclude that $\gamma(F) = (2q, 1)$.

To summarize, when $B \neq \emptyset$ and $2q > p$, we have $\gamma(F) = (p, \frac{p-q}{p})$. In all other cases, $\gamma(F) = (2q, 1)$. □

## 4. Fuzzy strong edge connectivity

In this section, we first define fuzzy strong edge connectivity and then derive its formulas for fuzzy trees, complete fuzzy graphs, and fuzzy cycles.

**Definition 4.1.** *[13] Let $F = (V, \rho, \eta)$ denote a fuzzy graph and $D^* \subseteq E(F)$. If there exist vertices $y, z \in \rho^*$ such that fuzzy strongest strong $y - z$ paths exist in $F$, while no fuzzy strongest strong $y - z$ paths exist in $F - D^*$, then $D^*$ is called a fuzzy strong edge cut (referred to as FSEC) of $F$. In particular, if $D^*$ contains only one edge $xy$, then $xy$ is called a fuzzy strong cut edge (referred to as FSCE) of $F$.*

**Definition 4.2.** *Let $F = (V, \rho, \eta)$ be a fuzzy graph and $D^* \subseteq E(F)$. The strong weight of $D^*$, denoted by $\varsigma'(D^*)$, is defined as $\varsigma'(D^*) = \sum_{xy \in D^*} \eta(x, y)$.*

**Definition 4.3.** *Let $U$ be an FSEC of $F$. If $\varsigma'(U) = \tau'(F)$, where $\tau'(F) = \wedge\{\varsigma'(D^*) \mid D^*$ is an FSEC of $F\}$, then $U$ is called a minimal fuzzy strong edge cut (referred to as MFSEC) of $F$. In particular, if there is only one edge $xy$ in $U$, then $xy$ is called a minimal fuzzy strong cut edge (referred to as MFSCE) of $F$.*

**Definition 4.4.** *Let $D^*$ be an FSEC of $F$. The impact of $D^*$ is denoted by $I'(D^*)$ and is defined as $I'(D^*) = \vee_{y,z \in \rho^*}\{1 - \frac{CONN_{F-D^*}(y,z)}{CONN_F(y,z)} \mid$ there exist fuzzy strongest strong $y - z$ paths in $F$, but no such paths exist in $F - D^*\}$.*

**Definition 4.5.** *The fuzzy strong edge connectivity of $F$ is denoted by $\gamma'(F)$ and is defined as $\gamma'(F) = (\tau'(F), \psi'(F))$, where $\tau'(F) = \wedge\{\varsigma'(D^*) \mid D^*$ is an FSEC of $F\}$ and $\psi'(F) = \vee\{I'(U) \mid U$ is an MFSEC of $F\}$.*

**Definition 4.6.** *Let $\gamma'(F_1) = (\tau'(F_1), \psi'(F_1))$ and $\gamma'(F_2) = (\tau'(F_2), \psi'(F_2))$ be fuzzy strong edge connectivity of fuzzy graphs $F_1$ and $F_2$, respectively.*

*1. If either of the following conditions is fulfilled:*

*(a) $\tau'(F_1) < \tau'(F_2)$;*

*(b) $\tau'(F_1) = \tau'(F_2)$ and $\psi'(F_1) > \psi'(F_2)$,*

*then the fuzzy strong edge connectivity of $F_2$ is greater than that of $F_1$, denoted by $\gamma'(F_1) < \gamma'(F_2)$.*

*2. Given that*

*(c) $\tau'(F_1) = \tau'(F_2)$ and $\psi'(F_1) = \psi'(F_2)$,*

*the fuzzy strong edge connectivity of $F_1$ is equal to that of $F_2$, denoted by $\gamma'(F_1) = \gamma'(F_2)$.*

*3. If any one of conditions (a), (b), and (c) holds, the fuzzy strong edge connectivity of $F_1$ is no greater than that of $F_2$, denoted by $\gamma'(F_1) \leq \gamma'(F_2)$.*

**Example 4.1.** *Let $F = (V, \rho, \eta)$ be a fuzzy graph with vertex set $V = \{a, b, c, d, e, f, g\}$, where the edge membership values are specified as $\eta(a, b) = 0.7$, $\eta(b, c) = \eta(g, a) = 0.5$, $\eta(c, d) = \eta(b, e) = 0.2$, $\eta(d, e) = 0.4$, $\eta(b, d) = \eta(d, g) = \eta(c, a) = \eta(e, f) = \eta(e, g) = \eta(f, a) = 0.05$, and $\eta(f, g) = 0.6$, as illustrated in Figure 2. It can be deduced that $\{be, cd\}$ and $\{de\}$ are MFSECs of $F$, with $\tau'(F) = 0.4$. Since $I'(\{be, cd\}) = 0.75$ and $I'(\{de\}) = 0.5$, it follows that $\psi'(F) = 0.75$. Hence, $\gamma'(F) = (0.4, 0.75)$.*

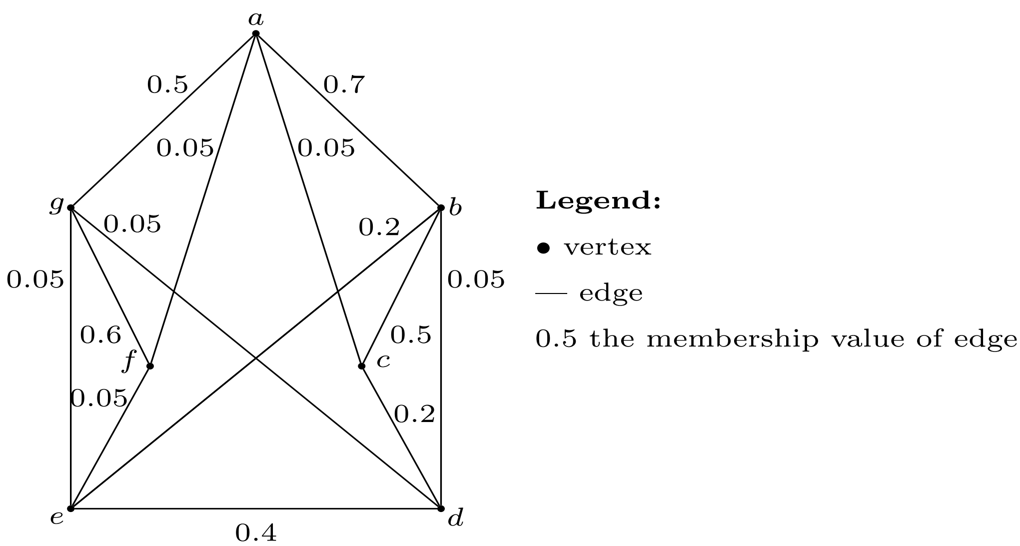


**Figure 2.** Fuzzy graph in Example 4.1.

**Lemma 4.1.** *[13] Let $F = (V, \rho, \eta)$ be a fuzzy tree with $H$ as its unique maximum spanning tree. An edge $yz \in \eta^*$ is an FSCE of $F$ if and only if $yz \in E(H)$.*

**Theorem 4.1.** *Let $F = (V, \rho, \eta)$ be a fuzzy tree and $F^*$ be a tree. Let $q$ be the minimum membership value of all strong edges in $F$. Then $\gamma'(F) = (q, 1)$.*

*Proof.* By Lemma 3.2 and Lemma 4.1, an edge $e$ is an FSCE of $F$ if and only if $e$ is a strong edge in $F$. A proper subset $D^*$ of $E(F)$ is an FSEC of $F$ provided that there exists a strong edge contained in $D^*$, which further yields $\tau'(F) = q$.

Take a strong edge $e_0 = yz$ of $F$ with membership value $q$. Evidently, $e_0$ is an MFSCE of $F$. By the definition of FSCEs, there exist vertices $y, z \in \rho^*$ for which a fuzzy strongest strong $y - z$ path $y \to z$ exists in $F$, yet no such paths exist in $F - \{e_0\}$. Since $F^*$ is a tree, there is exactly one $y - z$ path $y \to z$. Hence, there is no $y - z$ path in $F - \{e_0\}$, which implies that $CONN_{F-\{e_0\}}(y, z) = 0$. Furthermore, by the definition of impact, we have $I'(\{e_0\}) = 1$. Based on the definition of $\psi'(F)$, we conclude that $\psi'(F) = 1$, and thus $\gamma'(F) = (q, 1)$. □

**Lemma 4.2.** *[17] Let $F = (V, \rho, \eta)$ be a complete fuzzy graph. Then $CONN_F(y, z) = \eta(y, z)$ for $yz \in \eta^*$.*

**Theorem 4.2.** *Let $F = (V, \rho, \eta)$ be a complete fuzzy graph with $\rho^* = \{a_1, a_2, \dots, a_n\}$ and the membership values of vertices satisfy $\rho(a_1) \geq \rho(a_2) \geq \cdots \geq \rho(a_n)$. Let $m_l$ be the number of vertices with membership value at least $\rho(a_l)$, where $l \in \{2, 3, \dots, n\}$. Then $\tau'(F) = \wedge\{(m_2 - 1)\rho(a_2), (m_3 - 1)\rho(a_3), \dots, (m_n - 1)\rho(a_n)\}$. Furthermore, if $\tau'(F) = (n-1)\rho(a_n)$, then $\psi'(F) = 1$. Otherwise, $\psi'(F) = \vee\{\frac{\rho(a_t) - \rho(a_{m_t+1})}{\rho(a_t)} \mid \tau'(F) = (m_t - 1)\rho(a_t),\ where\ t \in \{2, \dots, n-1\}\}$.*

*Proof.* We begin by proving $\tau'(F) = \wedge\{(m_2 - 1)\rho(a_2), (m_3 - 1)\rho(a_3), \dots, (m_n - 1)\rho(a_n)\}$.

Let $a_i, a_j \in \rho^*$, where $1 \leq i < j \leq n$. By Lemma 3.3 and Definition 2.12, all edges in $F$ are strong edges. From Lemma 4.2, we have $CONN_F(a_i, a_j) = \eta(a_i, a_j) = \rho(a_j)$.

Let $D_{a_ia_j}$ be an FSEC of $F$ such that removing $D_{a_ia_j}$ from $F$ results in no fuzzy strongest strong $a_i - a_j$ path existing in $F - D_{a_ia_j}$. Two cases are considered below.

Case 1: $m_j = j$.

Let $P_r$ denote the path $a_i \to a_r \to a_j$, where $1 \leq r < j$ and $r \neq i$, and let $P_i$ denote the path $a_i \to a_j$. Clearly, $P_1, P_2, \dots, P_{j-1}$ are $j - 1$ edge-disjoint fuzzy strongest strong $a_i - a_j$ paths in $F$. It follows that $s(P_1) = s(P_2) = \cdots = s(P_{j-1}) = \rho(a_j)$. Every edge on the fuzzy strongest strong $a_i - a_j$ path $P_m$ possesses a membership value no less than $\rho(a_j)$, where $m = 1, 2, \dots, j - 1$. Since $P_1, P_2, \dots, P_{j-1}$ are edge-disjoint fuzzy strongest strong paths between $a_i$ and $a_j$ in $F$, at least one edge of each path $P_m$ is contained in $D_{a_ia_j}$. Therefore, $\varsigma'(D_{a_ia_j}) \geq (j - 1)\rho(a_j)$.

Case 2: $m_j > j$.

Let $P_r$ denote the path $a_i \to a_r \to a_j$, where $1 \leq r \leq m_j$ and $r \neq i, j$, and let $P_i$ denote the path $a_i \to a_j$. Clearly, $P_1, \dots, P_{j-1}, P_{j+1}, \dots, P_{m_j}$ are $m_j - 1$ edge-disjoint fuzzy strongest strong $a_i - a_j$ paths in $F$. It follows that $s(P_1) = \cdots = s(P_{j-1}) = s(P_{j+1}) = \cdots = s(P_{m_j}) = \rho(a_j)$. Every edge on the fuzzy strongest strong $a_i - a_j$ path $P_m$ possesses a membership value no less than $\rho(a_j)$, where $m = 1, \dots, j - 1, j + 1, \dots, m_j$. Since $P_1, \dots, P_{j-1}, P_{j+1}, \dots, P_{m_j}$ are edge-disjoint fuzzy strongest strong paths between $a_i$ and $a_j$ in $F$, at least one edge of each path $P_m$ is contained in $D_{a_ia_j}$. Therefore, $\varsigma'(D_{a_ia_j}) \geq (m_j - 1)\rho(a_j)$.

To summarize, if $D_{a_ia_j}$ is an FSEC of $F$, and there exist no fuzzy strongest strong $a_i - a_j$ paths in $F - D_{a_ia_j}$, then $\varsigma'(D_{a_ia_j}) \geq (m_j - 1)\rho(a_j)$.

If $m_j = j$, let $D$ be the set consisting of edges $a_1a_j, a_2a_j, \dots, a_{j-1}a_j$; otherwise, let $D$ be the set consisting of edges $a_1a_j, \dots, a_{j-1}a_j, a_{j+1}a_j, \dots, a_{m_j}a_j$. By the definition of FSECs, $D$ is an FSEC of $F$. Moreover, fuzzy strongest strong $a_i - a_j$ paths exist in $F$, but no such paths exist in $F - D$.

Hence, $\varsigma'(D) = (m_j - 1)\rho(a_j)$. Furthermore, the strong weight of $D$ is the minimum strong weight over all FSECs in $F$ for which no fuzzy strongest strong $a_i - a_j$ paths exist in $F - D$. Therefore, by the arbitrariness of $a_i$ and $a_j$, $\tau'(F) = \wedge\{(m_2 - 1)\rho(a_2), (m_3 - 1)\rho(a_3), \ldots, (m_n - 1)\rho(a_n)\}$.

Subsequently, we conduct a casewise analysis to derive $\psi'(F)$ corresponding to distinct values of $\tau'(F)$.

Case 1: $\tau'(F) = (n - 1)\rho(a_n)$.

Let $D_1$ be the set of all edges incident to vertex $a_n$. By the definition of FSECs, $D_1$ is an FSEC of $F$. Moreover, fuzzy strongest strong $a_1 - a_n$ paths exist in $F$, but no such paths exist in $F - D_1$. It follows that $\varsigma'(D_1) = (n - 1)\rho(a_n)$. Since $\varsigma'(D_1) = \tau'(F)$, $D_1$ is an MFSEC of $F$. Furthermore, $CONN_{F-D_1}(a_1, a_n) = 0$. From the definition of impact, we obtain $I'(D_1) = 1$. Applying the definition of $\psi'(F)$ yields $\psi'(F) = 1$.

Case 2: $\tau'(F) \neq (n - 1)\rho(a_n)$.

At this point, there exists some positive integer $t \in [2, n - 1]$ such that $\tau'(F) = (m_t - 1)\rho(a_t)$. Let $D_2$ be an MFSEC of $F$ such that there exist fuzzy strongest strong $a_i - a_t$ paths in $F$, but no such paths exist in $F - D_2$, where $1 \leq i < t$. By the minimality of $D_2$, $D_2$ contains neither edge $a_i a_{m_t+1}$ nor edge $a_t a_{m_t+1}$. Since $CONN_F(a_i, a_t) = \rho(a_t)$, we have $i \leq m_t$. Hence, $\eta(a_i, a_{m_t+1}) = \eta(a_t, a_{m_t+1}) = \rho(a_{m_t+1})$. The path $a_i \to a_{m_t+1} \to a_t$ is a path of strength $\rho(a_{m_t+1})$ between $a_i$ and $a_t$ in $F - D_2$. From the definition of $m_t$, we have $CONN_{F-D_2}(a_i, a_t) = \rho(a_{m_t+1})$. It follows that $I'(D_2) = \frac{\rho(a_t)-\rho(a_{m_t+1})}{\rho(a_t)}$. By the arbitrariness of $D_2$ and the definition of $\psi'(F)$, we obtain $\psi'(F) = \vee\{\frac{\rho(a_t)-\rho(a_{m_t+1})}{\rho(a_t)} \mid \tau'(F) = (m_t - 1)\rho(a_t),\ \text{where } t \in \{2, \ldots, n-1\}\}$.

In summary, the conclusion holds. □

**Lemma 4.3.** *[13] Let $F = (V, \rho, \eta)$ be a fuzzy cycle and $yz \in \eta^*$. Then $yz$ is an FSCE of $F$ if and only if $yz$ is a non-weakest edge in $F$.*

**Theorem 4.3.** *Let $F = (V, \rho, \eta)$ be a fuzzy cycle where $\eta$ is constant. Let $q$ be the membership value of an edge in $F$. Then $\gamma'(F) = (2q, 1)$.*

*Proof.* As established in Lemma 4.3, there are no FSCEs in $F$. Given that $F$ is a fuzzy cycle, the edge set formed by any two edges of $F$ is an FSEC of $F$, with its strong weight equal to $2q$ and its impact equal to 1. Based on the definition of $\gamma'(F)$, we obtain that $\gamma'(F) = (2q, 1)$. □

**Theorem 4.4.** *Let $F = (V, \rho, \eta)$ be a fuzzy cycle where $\eta$ is non-constant. Let $q$ and $p$ denote the minimum and second minimum membership values of all edges in $F$, respectively. Then the fuzzy strong edge connectivity of $F$ is given by*

$$\gamma'(F) = \begin{cases} (2q, 1), & 2q \leq p, \\ \left(p, \dfrac{p-q}{p}\right), & 2q > p. \end{cases}$$

*Proof.* As removing any two edges disconnects $F$, every pair of edges in $F$ constitutes an FSEC of $F$. For the case $2q \leq p$, every pair of weakest edges in $F$ is an MFSEC of $F$, with strong weight $2q$ and impact 1. From the definition of $\gamma'(F)$, we have $\gamma'(F) = (2q, 1)$.

When $2q > p$, Lemma 4.3 guarantees that every edge in $F$ with membership value $p$ is an MFSCE of $F$. Therefore, $\tau'(F) = p$. Let $yz$ be an MFSCE in $F$ with $I'(\{yz\}) = \psi'(F)$. In the fuzzy cycle, $y \to z$ is the unique fuzzy strongest strong $y - z$ path in $F$. This leads to $CONN_{F-\{yz\}}(y, z) = q < p = CONN_F(y, z)$, which shows that $\psi'(F) = \frac{p-q}{p}$. Thus, $\gamma'(F) = (p, \frac{p-q}{p})$. □

## 5. Algorithms for fuzzy strong vertex connectivity and fuzzy strong edge connectivity

Based on the previous study of two connectivity parameters for fuzzy trees, complete fuzzy graphs, and fuzzy cycles, this section further presents relevant algorithms. It also analyzes the time complexity of these algorithms for determining parameters of the three types of fuzzy graphs.

**Algorithm 5.1** Calculate fuzzy strong vertex connectivity of a fuzzy tree with a tree as its underlying graph.
**Input:** A fuzzy tree $F = (V, \rho, \eta)$, where the underlying graph $F^*$ is a tree
**Output:** $\gamma(F)$
1: Set $q$ to be the minimum membership value of all edges in $F^*$;
2: Enumerate all vertices of $\rho^*$ consecutively from 1 to $n$, where $|\rho^*| = n$;
3: **If** $n = 2$ **then**
4: $\quad\gamma(F) = (q, 0)$;
5: **else**
6: $\quad\gamma(F) = (q, 1)$;
7: **End**
8: **Return** $\gamma(F)$.

In Algorithm 5.1, Step 1 requires traversing $n - 1$ edges, with a time complexity of $O(n)$. Step 2 completes the labeling by traversing $n$ vertices, with a time complexity of $O(n)$. Thus, the total time complexity of Algorithm 5.1 is $O(n)$.

**Algorithm 5.2** Calculate fuzzy strong vertex connectivity of a complete fuzzy graph.
**Input:** A complete fuzzy graph $F = (V, \rho, \eta)$
**Output:** $\gamma(F)$
1: Index the vertices of $\rho^*$ from $a_1$ to $a_n$ subject to $\rho(a_1) \geq \rho(a_2) \geq \cdots \geq \rho(a_n)$, where $n$ is the number of vertices in $F$;
2: Set $\gamma(F) = ((n - 1)\rho(a_n), 0)$;
3: **Return** $\gamma(F)$.

In Algorithm 5.2, Step 1 labels vertices by sorting their membership values in descending order. This step is implemented with merge sort, and its time complexity is $O(n \log n)$. Thus, the time complexity of Algorithm 5.2 is $O(n \log n)$.

**Algorithm 5.3** Calculate fuzzy strong vertex connectivity of a fuzzy cycle.
**Input:** A fuzzy cycle $F = (V, \rho, \eta)$
**Output:** $\gamma(F)$
1: Enumerate all vertices of $\rho^*$ consecutively from $a_1$ to $a_n$, where $|\rho^*| = n$;
2: Set $q$ to be the minimum membership value of all edges in $F$;
3: Set $\omega(\{a_i\})$ to be the minimum membership value of edges incident to vertex $a_i$ in $F$;
4: Set $B = \emptyset$, $p = +\infty$;
5: **For** $l = 1$ **to** $n$ **do**
6: $\quad t_l = \omega(\{a_l\})$;
7: $\quad$**If** $t_l \neq q$ **then**

8: $B = B \cup \{a_l\}$;
9: **If** $t_l < p$ **then**
10: $p = t_l$;
11: **End**
12: **End**
13: **End**
14: **If** $n = 3$ **then**
15: $\gamma(F) = (2q, 0)$;
16: **else**
17: **If** $B \neq \emptyset$ and $2q > p$ **then**
18: $\gamma(F) = \left(p, \dfrac{p-q}{p}\right)$;
19: **else**
20: $\gamma(F) = (2q, 1)$;
21: **End**
22: **End**
23: **Return** $\gamma(F)$.

In Steps 1, 3, and 5 of Algorithm 5.3, the computation process needs to traverse all vertices, so their time complexity is $\mathrm{O}(n)$. Step 2 finds the minimum value among $n$ numbers, with a time complexity of $\mathrm{O}(n)$. Therefore, the total time complexity of Algorithm 5.3 is $\mathrm{O}(n)$.

**Algorithm 5.4** Calculate fuzzy strong edge connectivity of a fuzzy tree with a tree as its underlying graph.
**Input:** A fuzzy tree $F = (V, \rho, \eta)$, where the underlying graph $F^*$ is a tree
**Output:** $\gamma^{'}(F)$
1: Set $q$ to be the minimum membership value of all edges in $F^*$;
2: Set $\gamma^{'}(F) = (q, 1)$;
3: **Return** $\gamma^{'}(F)$.

For Algorithm 5.4, given that the edge count of $F^*$ is $n-1$, Step 1 requires $\mathrm{O}(n)$ time. As a result, the overall time complexity of Algorithm 5.4 is $\mathrm{O}(n)$.

**Algorithm 5.5** Calculate fuzzy strong edge connectivity of a complete fuzzy graph.
**Input:** A complete fuzzy graph $F = (V, \rho, \eta)$
**Output:** $\gamma^{'}(F)$
1: Index the vertices of $\rho^*$ from $a_1$ to $a_n$ subject to $\rho(a_1) \geq \rho(a_2) \geq \cdots \geq \rho(a_n)$, where $n$ is the number of vertices in $F$;
2: Set $k = 1$;
3: **For** $l = 2$ **to** $n$ **do**
4: **If** $l = 2$ **then**
5: $m_l = 0$;
6: **else**
7: $m_l = m_{l-1}$;
8: **while** $\rho(a_k) \geq \rho(a_l)$ **do**

9: $m_l = m_l + 1, k = k + 1$;
10: **End**
11: **End**
12: Set $\tau^{'}(F) = \min\{(m_2 - 1)\rho(a_2), (m_3 - 1)\rho(a_3), \ldots, (m_n - 1)\rho(a_n)\}$;
13: **If** $\tau^{'}(F) = (n - 1)\rho(a_n)$ **then**
14: $\psi^{'}(F) = 1$;
15: **else**
16: set $\psi^{'}(F) = 0$;
17: **For** $t = 2$ **to** $n - 1$ **do**
18: **If** $\tau^{'}(F) = (m_t - 1)\rho(a_t)$ **then**
19: $\psi^{'}(F) = \max\left\{\psi^{'}(F), \dfrac{\rho(a_t) - \rho(a_{m_t+1})}{\rho(a_t)}\right\}$;
20: **End**
21: **End**
22: **End**
23: Set $\gamma^{'}(F) = (\tau^{'}(F), \psi^{'}(F))$;
24: **Return** $\gamma^{'}(F)$.

In Algorithm 5.5, Step 1 labels vertices by sorting their membership values in descending order. This step is implemented with merge sort, and its time complexity is $O(n \log n)$. For subsequent steps, Step 3 is an outer loop with $O(n)$ iterations. Step 8 runs in amortized $O(n)$ time overall. Steps 12 and 17 each take linear $O(n)$ time. Thus, the time complexity of Algorithm 5.5 is $O(n \log n)$.

**Algorithm 5.6** Calculate fuzzy strong edge connectivity of a fuzzy cycle.
**Input:** A fuzzy cycle $F = (V, \rho, \eta)$
**Output:** $\gamma^{'}(F)$
1: Set $q$ to be the minimum value of all edges in $F$;
2: Set $p^{'}$ to be the maximum value of all edges in $F$;
3: **If** $q = p^{'}$ **then**
4: $\gamma^{'}(F) = (2q, 1)$;
5: **else**
6: set $p$ to be the second minimum value of all edges in $F$;
7: **If** $p < 2q$ **then**
8: $\gamma^{'}(F) = \left(p, \dfrac{p - q}{p}\right)$;
9: **else**
10: $\gamma^{'}(F) = (2q, 1)$;
11: **End**
12: **End**
13: **Return** $\gamma^{'}(F)$.

Computing the maximum or minimum value from a set of $n$ elements requires $O(n)$ time, which means the overall time complexity of Algorithm 5.6 is $O(n)$.

## 6. Applications of fuzzy strong vertex connectivity and fuzzy strong edge connectivity in the health assessment of regional ecosystems

Global ecological frameworks such as the United Nations Sustainable Development Goals and the Paris Agreement are continually evolving. Consequently, nations have adopted ecological health as a core indicator for regional development. Fuzzy strong vertex connectivity and fuzzy strong edge connectivity provide a new technical tool for addressing fuzziness and correlation issues in ecological evaluation. This method accurately characterizes the health status of ecological elements with membership values. The membership values of fuzzy edges quantify intercomponent connection strength. Fuzzy strong connectivity then integrates such information to assess the overall stability of ecosystems. This effectively compensates for the limitations of traditional evaluation methods.

This section develops fuzzy graph models for research on various regional ecosystems. Vertices in the fuzzy graph represent ecological elements within the region. These include forests, rivers, wetlands, wildlife populations, and so on. The membership value of a vertex represents the health level of an individual element, such as forest vegetation coverage and river water quality rating. Edges in the fuzzy graph represent interdependent and interactive relationships among ecological elements. Examples include water conservation between forests and rivers and habitat dependence between wildlife and forests. The membership value of an edge represents the closeness of the connection between ecological elements. For example, it can be measured by the contribution of forests to river conservation. A higher value indicates a stronger interdependent relationship.

The core connotation of fuzzy strong vertex connectivity consists in identifying the key vertex set that affects the overall stability of regional ecosystems. This set represents the core ecological elements. Using the calculation method proposed in this paper, researchers can identify priority restoration elements in ecologically degraded areas. It also provides plans for maintaining ecosystem health and a basis for recognizing potential risk factors in well-protected ecosystems. A higher fuzzy strong vertex connectivity indicates greater overall stability of a regional ecosystem and stronger self-healing ability under sudden disturbances. The core connotation of fuzzy strong edge connectivity lies in identifying the key edge set that determines the overall stability of regional ecosystems. This set refers to the core ecological relationship. Using the calculation method proposed in this paper, researchers can identify priority restoration relationships in ecologically degraded areas. It also provides strategies for maintaining core relationships and a basis for identifying potential relational risks in healthy ecosystems. A higher fuzzy strong edge connectivity indicates stronger overall stability of a regional ecosystem. Even if the connections between elements are subjected to sudden disturbances, the ecosystem still has a strong self-repair ability. In conclusion, fuzzy strong vertex connectivity and fuzzy strong edge connectivity form a dual quantitative analysis system. This framework supports accurate monitoring and scientific assessment of ecosystems across different regions. It provides robust support for decision-making in ecosystem management and protection.

This paper constructs fuzzy graphs $F_1$ and $F_2$ for Region $I$ and Region $II$, respectively. These fuzzy graphs characterize the element network topology and interdependencies of the two ecosystems, as detailed in Figure 3.

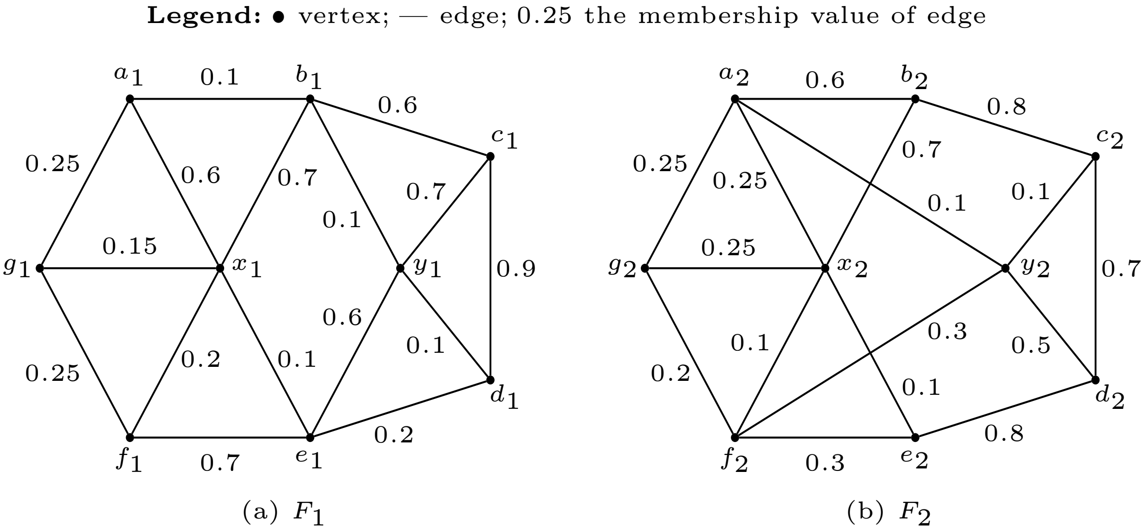


**Figure 3.** Fuzzy graphs $F_1$ and $F_2$.

To verify the effectiveness and superiority of the proposed fuzzy strong connectivity, we carry out comparative experiments with the existing dual-parameter fuzzy edge connectivity method proposed in [14] on the two regional ecosystem cases. The comparative results are analyzed as follows.

According to the definition of fuzzy edge connectivity in [14], the fuzzy edge connectivity of $F_1$ and $F_2$ is $(0.5, 0.4)$. This result indicates that fuzzy edge connectivity has certain limitations in distinguishing the stability levels of two types of regional ecosystems. It fails to provide an effective basis for judging the stability difference between them. Given this fact, this paper adopts the proposed fuzzy strong connectivity index. It carries out a new round of quantitative analysis and calculation on the connectivity of $F_1$ and $F_2$.

In the calculation of fuzzy strong vertex connectivity, we first identify all MFSVCs of $F_1$ and $F_2$. The MFSVC of $F_1$ is $\{a_1, f_1\}$. The MFSVCs of $F_2$ are $\{a_2, x_2\}$ and $\{d_2\}$. From $\tau(F_1) = 0.5$ and $\psi(F_1) = I(\{a_1, f_1\}) = 0.4$, it can be deduced that the fuzzy strong vertex connectivity of $F_1$ is $\gamma(F_1) = (0.5, 0.4)$. Similarly, we obtain that the fuzzy strong vertex connectivity of $F_2$ is $\gamma(F_2) = (0.5, 0.71)$. Clearly, $\gamma(F_1) > \gamma(F_2)$.

Notice that $\tau(F_1)$ and $\tau(F_2)$ are both equal to 0.5. A single-parameter definition of strong vertex connectivity would assign equal values to the two fuzzy graphs. This demonstrates that our newly proposed two-parameter strong vertex connectivity offers higher discriminative accuracy for structural robustness differences among fuzzy graphs. While single-parameter evaluation cannot detect subtle structural and stability gaps between $F_1$ and $F_2$ and leads to defective ecosystem assessments, the presented two-parameter joint evaluation differentiates the ecosystems' heterogeneous stability and generates more coherent evaluation outputs. Quantitatively, the dual-index system lessens evaluation bias and achieves greater reliability and discriminability in ecosystem health assessment than single-index methods.

A comparative analysis of the above fuzzy strong vertex connectivity index leads to a clear conclusion. The ecosystem corresponding to Region *I* exhibits superior overall stability. From a microscopic perspective of network structure, this quantitative finding reveals a key insight. The core nodes of the ecosystem in Region *I* possess higher connectivity redundancy and anti-disturbance potential. They can sustain the continuity of ecological functions via more robust node linkages under sudden disturbances like extreme climate and human interference. This finding not only provides a quantifiable basis for judging the health of regional ecosystems, but also offers an accurate approach for formulating differentiated ecological management strategies. For Region $I$, protection of key ecological nodes can be further strengthened based on this conclusion. A preventive ecological risk early-warning system can also be established. For Region $II$, fragile nodes $a_2$, $x_2$, and $d_2$ can be identified and prioritized for restoration based on the calculated fuzzy strong vertex connectivity. This will systematically improve the overall resilience of the ecosystem.

The above comparative experiment demonstrates the obvious advantages of the proposed method. The fuzzy edge connectivity in [14] only focuses on the connection relationships between ecological elements and fails to distinguish the stability difference between the two ecosystems. Moreover, it cannot identify critical ecological nodes that threaten ecosystem stability. In contrast, our constructed fuzzy strong vertex connectivity can quantitatively distinguish ecosystem stability levels, accurately locate fragile core nodes, and support differentiated ecological restoration strategies.

For comparative analysis, we further calculate the fuzzy strong edge connectivity of the two graphs. The results show that both $F_1$ and $F_2$ yield the same fuzzy strong edge connectivity value of $(0.5, 0.4)$. The findings from the calculations further substantiate the fundamental importance of fuzzy strong vertex connectivity and fuzzy strong edge connectivity, rendering both indicators essential. Fuzzy strong edge connectivity determines the basic resilience at the level of ecological element relationships. Fuzzy strong vertex connectivity dominates the risk prevention and control capacity at the level of core ecological element nodes. Therefore, a single indicator cannot fully reflect the overall stability of a regional ecosystem. Only the combination of the two indicators can achieve an accurate assessment of its overall stability. These results provide a new technical approach for accurate assessment and targeted management of ecosystem stability. They also offer important support for the scientific construction of regional ecological security patterns.

## 7. Conclusions

This paper systematically explores fuzzy strong vertex connectivity and fuzzy strong edge connectivity in fuzzy graphs. It is of great value in both theoretical deepening and practical empowerment dimensions. At the theoretical level, this study provides precise mathematical definitions for fuzzy strong vertex connectivity and fuzzy strong edge connectivity. It effectively links the existing research on fuzzy strong vertex cuts and fuzzy strong edge cuts based on strongest strong paths. Such an endeavor fills the research gap in the quantitative characterization of fuzzy strong connectivity. Meanwhile, it systematically reveals the essential characteristics and quantitative rules of these two indices in fuzzy trees, complete fuzzy graphs, and fuzzy cycles. Corresponding standard calculation algorithms are thereby established. This breaks through the one-sidedness of traditional single-index evaluation. It establishes a quantitative system that balances reliability and efficiency for assessing the connectivity quality of fuzzy graph networks. This enriches the research scope of connectivity and core network capabilities in fuzzy graph theory. It provides new support for improving the theoretical system of this field. At the practical level, this study constructs a complete analysis framework that follows the logic of indicator quantification, bottleneck identification, and resilience assessment. This framework can accurately identify critical nodes, link bottlenecks in scenarios including regional ecosystem health assessment, and quantify the impact of node or link failures. As a directly applicable tool for relevant practical engineering and application scenarios, it provides a scientific quantitative basis for decision-making including planning optimization and risk prevention and control. This work effectively enhances the engineering application value of fuzzy graph theory and promotes its in-depth application in practical network analysis and optimization.

## Author contributions

Junye Ma: Conceptualization, Methodology, Writing-original draft, Writing-review & editing, Supervision, Funding acquisition; Xiaoke Han: Formal analysis, Writing-original draft, Visualization; Tong Ning: Supervision, Funding acquisition, Validation, Writing-review & editing; Suping Wang: Supervision, Funding acquisition, Validation, Writing-review & editing. All authors reviewed and approved the final manuscript.

## Acknowledgments

The work is supported by National Natural Science Foundation of China (Grant No. 12301591), Taiyuan University of Science and Technology Scientific Research Initial Funding (Grant No. 20242121 and No. 2022085), Fundamental Research Program of Shanxi Province (Grant No. 202203021222190), the Reward Fund for Excellent Doctors Working in Shanxi Province (Grant No. 20252049 and No. 20242059), Key Scientific Research Project of Institutions of Higher Education in Henan Province (Grant No. 23B110008), and Natural Science Foundation of Henan Province (Grant No. 262300420714).

## Conflict of interest

The authors declare that they have no competing interests.

## Use of Generative-AI tools declaration

The authors declare that no Generative AI tools were used in the creation of this article.

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