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Research article

On subordination relations of analytic functions associated with linear operators

Adel A. Attiya[1,2,*], **Nak Eun Cho**[3] **and Mansour F. Yassen**[4,5]

[1] Department of Mathematics, College of Science, University of Ha'il, Ha'il 81451, Saudi Arabia
[2] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[3] Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea
[4] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia
[5] Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Damietta, Egypt

* **Correspondence:** Email: aattiy@mans.edu.eg.

Abstract: This paper investigates important subordination properties of β-convex functions defined on the open unit disk, in relation to a family of linear operators. By exploring these subordination properties, we identify certain constraints of these functions. In addition, applications to integral operators are also considered. Overall, the study contributes to both the theory of β-convex functions and the general field of operator theory.

Keywords: subordination relation; univalent function; convex functions; β-convex functions; Hadamard product; subordination chain; integral operator
Mathematics Subject Classification: 30C80, 30C55

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{D})$ denote the class of all analytic functions defined in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

For a constant $a \in \mathbb{C}$ and a positive integer $n \in \mathbb{N} = \{1, 2, \ldots\}$, we define the subclass

$$\mathcal{H}[a, n] = \left\{ f \in \mathcal{H} : f(z) = a + \sum_{k=n}^{\infty} a_k z^k \right\}.$$

Hence, $\mathcal{H}[a, n]$ represents the family of analytic functions in $\mathbb{D}$ whose power series expansion begins with the constant term a, while the first nonconstant term appears at order n.

Let f and F belong to $\mathcal{H}$. We say that f is *subordinate* to H (or equivalently that F is *superordinate* to f) if there exists an analytic mapping w on $\mathbb{D}$ such that

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad \text{for all } z \in \mathbb{D},$$

with

$$f(z) = F(w(z)) \quad \text{for all } z \in \mathbb{D}.$$

In this situation, we write

$$f \prec F \quad \text{or} \quad f(z) \prec F(z).$$

If F is univalent in $\mathbb{D}$, then it follows that (cf. [10, 15])

$$f \prec F \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{D}) \subset F(\mathbb{D}).$$

Let Q denote the family of functions that are analytic and univalent on $\overline{\mathbb{D}} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{D} : \lim_{z \to \zeta} f(z) = \infty \right\}, \tag{1.1}$$

and satisfy

$$f'(\zeta) \neq 0 \quad \text{for all } \zeta \in \partial\mathbb{D} \setminus E(f).$$

A function $f \in \mathcal{H}$ satisfying $f(0) = 0$, $f(z)/z \neq 0$, $f'(z)/z \neq 0$, and $f'(z) \neq 0$ is called a *β-convex function* (note that $f(z)$ is not necessarily normalized in the standard sense, since $f'(z)$ is not required to equal 1) if it satisfies

$$\operatorname{Re}\left((1-\beta)\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0, \quad (\beta \in \mathbb{R},\ z \in \mathbb{D}). \tag{1.2}$$

Then, we denote the corresponding class by $\mathcal{H}_\beta$. The notion of β-convex functions was introduced by Mocanu [11]. It has been noted (cf. [8]) that all functions in this class are univalent and starlike, and

$$\mathcal{H}_\beta \subset \mathcal{H}_\alpha \subset \mathcal{H}_0, \quad 0 \leq \frac{\alpha}{\beta} \leq 1.$$

Moreover, the class $\mathcal{H}_0$ coincides with the class of normalized starlike functions in $\mathbb{D}$, while the class $\mathcal{H}_1$ coincides with the class of normalized convex functions in $\mathbb{D}$. Thus, the family $\mathcal{H}_\lambda$ provides a unified framework that connects these two fundamental subclasses of univalent functions. In addition, several other subclasses of analytic functions associated with various notions of convexity and their geometric properties have been extensively investigated in the literature; see, for example, [1, 20].

Example 1.1. By means of the subordination relation and the definition of β-convex functions, one can readily obtain the following examples of functions belonging to the class of β-convex functions.

(1) $f(z) = \left(\tanh^{-1} \sqrt{z}\right) \in \mathcal{H}_2$,

(2) $f(z) = \sqrt{\dfrac{z^2(3-z)}{3(1-z)^3}} \in \mathcal{H}_{\frac{1}{2}}$,

(3) $f(z) = z\,{}_2F_1(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; z) \in \mathcal{H}_3$,
(4) $f(z) = z\,{}_2F_1(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; z) \in \mathcal{H}_4$,

where ${}_2F_1(a, b; c; z)$ denotes the Gaussian hypergeometric function.

An analytic function f in the unit disk $\mathbb{D}$ is called *p-valent* in $\mathbb{D}$ if every value is assumed by f at most p times in $\mathbb{D}$ (counting multiplicities), and there exists a value that is assumed exactly p times.

Consider the class $\mathcal{A}_p$ consisting of functions given by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \qquad (p \in \mathbb{N}),$$

which are analytic in the open unit disk $\mathbb{D}$ and p-valent in this domain.

Moreover, the function $\phi_p(b, c; z)$ is defined through the power series

$$\phi_p(b, c; z) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} z^{k+p}, \quad (b, c \in \mathbb{R} \text{ and } c \notin \mathbb{Z}^- \cup \{0\}),$$

where $(a)_n$ denotes the Pochhammer symbol (or shifted factorial) given in terms of the Gamma function by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \text{ and } a \in \mathbb{C} \setminus \{0\}, \\ a(a+1)\cdots(a+n-1), & \text{if } n \in \mathbb{N} \text{ and } a \in \mathbb{C}. \end{cases}$$

For any $f \in \mathcal{A}_p$, we introduce the linear operator $H_p(b, c) : \mathcal{A}_p \to \mathcal{A}_p$ defined by

$$H_p(b, c)f(z) = \phi_p(b, c; z) * f(z), \quad z \in \mathbb{D},$$

where $*$ represents the Hadamard product (or convolution). It can be observed that

$$H_p(p+1, p)f(z) = \frac{zf'(z)}{p}, \quad H_p(n+p, 1)f(z) = D^{n+p-1}f(z),$$

where n is any real number greater than $-p$ and D^n denotes the Ruscheweyh derivative [17] (see also [5]). The operator $H_p(b, c)$ was introduced by Saitoh [18] as an extension of the classical Carlson-Shaffer operator $H_1(b, c)$, which has been extensively studied in the space of analytic and univalent functions in $\mathbb{D}$ (cf. [3, 19]).

Corresponding to $\phi_p(b, c; z)$, let $\phi^{\dagger}_{p,\lambda}(b, c; z)$ be defined such that

$$\phi_p(b, c; z) * \phi^{\dagger}_{p,\lambda}(b, c; z) = \frac{z^p}{(1-z)^{\lambda+p}}, \quad (\lambda > -p).$$

Similarly, we define the linear operator $\mathcal{I}^{\lambda}_p(b, c)$ on $\mathcal{A}_p$ by

$$\mathcal{I}^{\lambda}_p(b, c)f(z) = \phi^{\dagger}_{p,\lambda}(b, c; z) * f(z), \quad (b, c \neq 0, -1, -2, \ldots;\ \lambda > -p;\ z \in \mathbb{D}). \tag{1.3}$$

Throughout this paper, for the operator $\mathcal{I}^{\lambda}_p(b, c)f(z)$, we assume that the parameters satisfy the conditions

$$b, c \neq 0, -1, -2, \ldots, \qquad \lambda > -p,$$

unless otherwise stated.

From the definitions of $\mathcal{I}_p^\lambda(b,c)$ and $\phi_{p,\lambda}^\dagger(b,c;z)$, together with the expansion of $\frac{z^p}{(1-z)^{\lambda+p}}$, $(\lambda > -p)$, in terms of the Pochhammer symbol, and by applying the Hadamard (convolution) product, we have

$$\mathcal{I}_p^1(p+1,1)f(z) = f(z), \quad \mathcal{I}_p^1(p,1)f(z) = \frac{zf'(z)}{p}.$$

Moreover, it can be verified that

$$z(\mathcal{I}_p^\lambda(b+1,c)f(z))' = b\,\mathcal{I}_p^\lambda(b,c)f(z) - (b-p)\,\mathcal{I}_p^\lambda(b+1,c)f(z), \tag{1.4}$$

$$z(\mathcal{I}_p^\lambda(b,c)f(z))' = (\lambda+p)\,\mathcal{I}_p^{\lambda+1}(b,c)f(z) - \lambda\,\mathcal{I}_p^\lambda(b,c)f(z). \tag{1.5}$$

In particular, the operators $\mathcal{I}_1^\lambda(\mu+2,1)$ with $\mu > -2$ and $\lambda > -1$ were introduced by [4], which also examined inclusion properties for classes defined via $\mathcal{I}_1^\lambda(\mu+2,1)$. For the case $p = 1$, $b = n+1$ ($n \in \mathbb{N}_0$), and $c = \lambda = 1$, the operator $\mathcal{I}_p^\lambda(b,c)$ coincides with the Noor integral operator of n^{th} order studied by Liu [7] (see also [12, 13]).

This paper studies subordination properties of β-convex functions in the open unit disk using the linear operator $\mathcal{I}_p^\lambda(b,c)$ defined in (1.3). The motivation for this study stems from the central role of subordination theory in geometric function theory, where it serves as a powerful tool for investigating the geometric behavior of analytic functions and establishing inclusion relationships among their subclasses. By combining the framework of β-convexity with the operator $\mathcal{I}_p^\lambda(b,c)$, we develop new subordination criteria that provide further insight into the structural properties of this family of functions. Our aim is to derive sufficient conditions under which β-convex functions preserve desirable geometric characteristics when acted upon by the operator. In addition, we investigate several consequences involving the associated integral operator and demonstrate how the obtained results naturally extend to this setting. The analysis presented herein reveals new connections between generalized linear operators and β-convex functions, leading to a broader perspective on their interaction within the theory of analytic functions. Consequently, the results obtained in this paper enrich the existing framework of differential subordination and contribute to a deeper understanding of the relationship between β-convexity, integral transforms, and operator-theoretic methods in geometric function theory.

The following lemmas will be required in our investigation.

Lemma 1.2. *[9] Suppose that $p \in \mathcal{Q}$ with the condition $p(0) = a$. Let $q \in \mathcal{H}[a,n]$ in the form*

$$q(z) = a + a_n z^n + \cdots$$

be analytic in $\mathbb{D}$, where

$$q(z) \not\equiv a \quad \text{and} \quad n \in \mathbb{N}.$$

If $q \not\prec p$, then there exist points

$$z_0 = r_0\, exp(i\theta) \in \mathbb{D} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{D} \setminus E(f),$$

such that

$$q(\mathbb{D}_{r_0}) \subset p(\mathbb{D}), \quad q(z_0) = p(\zeta_0), \quad \text{and} \quad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0), \quad (m \geq n).$$

Definition 1. *The function $L(z,t)$ defined on $\mathbb{D}\times[0,\infty)$ is said to be a* subordination chain *(also known as a Löwner chain) provided that it satisfies the following conditions:*

- *For each fixed $t\in[0,\infty)$, the function $L(\cdot,t)$ is analytic and univalent in $\mathbb{D}$.*
- *For each fixed $z\in\mathbb{D}$, the mapping $t\mapsto L(z,t)$ is continuously differentiable on $[0,\infty)$.*
- *For $z\in\mathbb{D}$ and $0\le s<t$, the subordination property*

$$L(z,s)\prec L(z,t)$$

is satisfied.

Lemma 1.3. *[16] Consider the function*

$$L(z,t)=a_1(t)z+\cdots$$

with

$$a_1(t)\neq 0 \quad \text{and} \quad \lim_{t\to\infty}|a_1(t)|=\infty.$$

Assume that $L(\cdot,t)$ is an analytic function in the open unit disk $\mathbb{D}$ for all $t\ge 0$, and let $L(z,\cdot)$ be continuously differentiable on $[0,\infty)$ for each $z\in\mathbb{D}$.

If the function $L(z,t)$ satisfies the conditions:

$$\mathrm{Re}\left(\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right)>0, \quad (z\in\mathbb{D};\ 0\le t<\infty),$$

and

$$|L(z,t)|\le K_0|a_1(t)|, \quad (|z|<r_0<1;\ 0\le t<\infty),$$

where K_0 and r_0 are positive constants, then $L(z,t)$ is a subordination chain.

2. Main results

We start by establishing a subordination theorem concerning the multiplier operator $\mathcal{I}_p^{\lambda}(b,c)$ defined in (1.3).

Theorem 2.1. *Let $f\in\mathcal{A}_p$ and $k\in\mathcal{H}_\beta$. Suppose that the subordination*

$$\left(\mathcal{I}_p^{\lambda}(b+1,c)f(z)\right)^{1-\beta}\left(\mathcal{I}_p^{\lambda}(b,c)f(z)\right)^{\beta}\prec k(z), \tag{2.1}$$

holds for

$$\mathrm{Re}(b)\ge p, \quad \beta\ge 0, \quad z\in\mathbb{D}.$$

Then, it follows that

$$\mathcal{I}_p^{\lambda}(b+1,c)f(z)\prec k(z), \quad z\in\mathbb{D}.$$

Proof. Define

$$q(z):=\mathcal{I}_p^{\lambda}(b+1,c)f(z), \quad f\in\mathcal{A}_p,\ z\in\mathbb{D}. \tag{2.2}$$

Using (1.4) together with (2.2) and simple computations, we obtain

$$\mathcal{I}_p^{\lambda}(b,c)f(z) = \frac{(b-p)q(z) + zq'(z)}{b}. \tag{2.3}$$

Hence, combining (2.2) and (2.3), we rewrite (2.1) as

$$\left(\mathcal{I}_p^{\lambda}(b+1,c)f(z)\right)^{1-\beta}\left(\mathcal{I}_p^{\lambda}(b,c)f(z)\right)^{\beta} = q(z)\left(\frac{b-p+\frac{zq'(z)}{q(z)}}{b}\right)^{\beta}. \tag{2.4}$$

Thus, we need to prove the subordination implication

$$q(z)\left(\frac{b-p+\frac{zq'(z)}{q(z)}}{b}\right)^{\beta} \prec k(z) \quad \Longrightarrow \quad q(z) \prec k(z), \quad z \in \mathbb{D}. \tag{2.5}$$

Without loss of generality, we suppose that the function k fulfills the hypotheses of Theorem 2.1 on the closed unit disk $\overline{\mathbb{D}}$ with

$$k'(\zeta) \neq 0, \quad \zeta \in \partial\mathbb{D}.$$

Otherwise, we may consider

$$f_r(z) = f(rz), \quad k_r(z) = k(rz), \quad 0 < r < 1,$$

so that k_r is univalent on $\overline{\mathbb{D}}$ and

$$q_r(z)\left(\frac{b-p+\frac{zq_r'(z)}{q_r(z)}}{b}\right)^{\beta} \prec k_r(z), \quad q_r(z) = q(rz),\ z \in \mathbb{D}.$$

By letting $r \to 1^-$, the subordination (2.5) for q_r implies it for q.

Suppose, to the contrary, that (2.5) does not hold. By Lemma 1.2, there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D}$ such that

$$q(z_0) = k(\zeta_0), \quad z_0 q'(z_0) = m\zeta_0 k'(\zeta_0), \quad m \geq 1. \tag{2.6}$$

To complete the proof, define the function

$$L(z,t) = k(z)\left(\frac{b-p+(1+t)\frac{zk'(z)}{k(z)}}{b}\right)^{\beta}, \quad (z,t) \in \mathbb{D} \times [0,\infty). \tag{2.7}$$

A direct computation gives

$$a_1(t) := \frac{\partial L}{\partial z}(0,t) = k'(0)\left(\frac{b-p+1+t}{b}\right)^{\beta} \neq 0, \quad \lim_{t\to\infty}|a_1(t)| = \infty. \tag{2.8}$$

Moreover, one can verify

$$\mathrm{Re}\left(\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right) = \frac{\mathrm{Re}(b)-p}{\beta} + \beta(1+t)\,\mathrm{Re}\left((1-\beta)\frac{zk'(z)}{k(z)} + \beta\left(1+\frac{zk''(z)}{k'(z)}\right)\right), \tag{2.9}$$

Since $k \in \mathcal{H}_\beta$ with $\beta > 0$ and $k'(z) \neq 0$, we have

$$\operatorname{Re}\left(\frac{z \dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}} \right) > 0.$$

Using the sharp growth and distortion bounds for k (by applying the result established in [16]),

$$\frac{r}{(1+r)^2} \leq |k(z)| \leq \frac{r}{(1-r)^2}, \quad \frac{1-r}{(1+r)^3} \leq |k'(z)| \leq \frac{1+r}{(1-r)^3}, \quad |z| = r < 1,$$

and we conclude that the second condition of Lemma 1.3 is satisfied. Consequently, all the hypotheses of Lemma 1.3 are fulfilled, which implies that the function $L(z,t)$ constitutes a subordination chain in $\mathbb{D}$. In particular,

$$k(z) = L(z,0) \prec L(z,t), \quad z \in \mathbb{D}, \ t \geq 0. \tag{2.10}$$

Finally, combining (2.6) with (2.1) and applying Lemma 1.2, we arrive at a contradiction to our initial assumption. This contradiction shows that the assumption that (2.5) fails is untenable. Hence, (2.5) must hold, thereby completing the proof of the theorem. □

Theorem 2.2. *Let $f \in \mathcal{A}_p$ and $k \in \mathcal{H}_1$. Assume the subordination*

$$(1-\beta)\mathcal{I}_p^\lambda(b,c)f(z) + \beta\mathcal{I}_p^\lambda(b+1,c)f(z) \prec k(z), \quad 0 \leq \beta \leq 1, \ z \in \mathbb{D}, \ \operatorname{Re}(b) \geq p.$$

Then, it follows that

$$\mathcal{I}_p^\lambda(b+1,c)f(z) \prec k(z), \quad z \in \mathbb{D}.$$

Proof. Define $q(z)$ as in (2.2) and proceed similarly using (1.4) to get (2.3). Combining (2.2) and (2.3), we have

$$(1-\beta)\mathcal{I}_p^\lambda(b,c)f(z) + \beta\mathcal{I}_p^\lambda(b+1,c)f(z) = q(z)\left(\frac{(1-\beta)\left(b - p + \frac{zq'(z)}{q(z)}\right)}{b} + \beta \right), \tag{2.11}$$

so for the subordination we have to prove that

$$q(z)\left(\frac{(1-\beta)\left(b - p + \frac{zq'(z)}{q(z)}\right)}{b} + \beta \right) \prec k(z) \implies q(z) \prec k(z).$$

The remaining steps of the proof are identical, in essence, to those employed in the proof of Theorem 2.1. Since no new arguments or techniques are required, the details are omitted to avoid unnecessary repetition. This completes the proof.

Using relation (1.5) and proceeding in a manner analogous to the proofs of Theorems 2.1 and 2.2, we can derive Theorems 2.3 and 2.4. The details are therefore omitted as they follow by the same line of reasoning. □

Theorem 2.3. *Let $f \in \mathcal{A}_p$ and $k \in \mathcal{H}_\beta$. If*

$$\left(\mathcal{I}_p^\lambda(b,c)f(z)\right)^{1-\beta}\left(\mathcal{I}_p^{\lambda+1}(b,c)f(z)\right)^\beta \prec k(z), \quad \lambda \geq 0, \ \beta \geq 0, \ z \in \mathbb{D},$$

then

$$\mathcal{I}_p^\lambda(b,c)f(z) \prec k(z), \quad z \in \mathbb{D}.$$

Theorem 2.4. *Let $f \in \mathcal{A}_p$ and $k \in \mathcal{H}_\beta$. Assume*

$$(1-\beta)\mathcal{I}_p^{\lambda+1}(b,c)f(z) + \beta\mathcal{I}_p^{\lambda}(b,c)f(z) \prec k(z), \quad \lambda \geq 0,\ 0 \leq \beta \leq 1,\ z \in \mathbb{D}.$$

Then, it follows that

$$\mathcal{I}_p^{\lambda}(b,c)f(z) \prec k(z), \quad z \in \mathbb{D}.$$

By Setting $\beta = 1$ in Theorem 2.1, we have the flowing example.

Example 2.5. Let $f \in \mathcal{A}_p$ and $k(z)$ is a convex function. Suppose that the subordination

$$\mathcal{I}_p^{\lambda}(b,c)f(z) \prec k(z), \quad (\mathrm{Re}\{b\} \geq p,\ \beta \geq 0,\ z \in \mathbb{D}), \tag{2.12}$$

holds. Then,

$$\mathcal{I}_p^{\lambda}(b+1,c)f(z) \prec k(z), \quad z \in \mathbb{D}.$$

By Setting $\beta = 1$ in Theorem 2.3, we have the following example.

Example 2.6. Let $f \in \mathcal{A}_p$ and $k(z)$ be a convex function. If

$$\mathcal{I}_p^{\lambda+1}(b,c)f(z) \prec k(z), \quad (\mathrm{Re}\{b\} \geq p,\ \beta \geq 0,\ \lambda \geq 0; z \in \mathbb{D}), \tag{2.13}$$

holds. Then,

$$\mathcal{I}_p^{\lambda}(b,c)f(z) \prec k(z).$$

By setting $\beta = 1/2$ and $q(z) = \sqrt{\dfrac{z^2(3-z)}{3(1-z)^3}}$ in Theorem 2.1, we have

Example 2.7. Let $f \in \mathcal{A}_p$. Suppose that

$$\mathcal{I}_p^{\lambda}(b+1,c)f(z)\ \mathcal{I}_p^{\lambda}(b,c)f(z) \prec \sqrt{\frac{z^2(3-z)}{3(1-z)^3}},\ (\mathrm{Re}(b) \geq p,\ \beta \geq 0,\ z \in \mathbb{D}). \tag{2.14}$$

Then, it follows that

$$\mathcal{I}_p^{\lambda}(b+1,c)f(z) \prec \sqrt{\frac{z^2(3-z)}{3(1-z)^3}}.$$

3. Subordination results for the generalized Libera operator

In this section, we examine the generalized Libera integral operator F_ζ for $\mathrm{Re}(\zeta) > -p$, defined as (cf. [2, 6, 14]):

$$F_\zeta(f)(z) := \frac{\zeta+p}{z^\zeta}\int_0^z t^{\zeta-1}f(t)\,dt, \quad (f \in \mathcal{A}_p,\ \mathrm{Re}\{\zeta\} > -p). \tag{3.1}$$

First, we establish a subordination property for this operator.

Theorem 3.1. *Let $f \in \mathcal{A}_p$ and $k \in \mathcal{H}_\beta$. Assume the subordination*

$$\left(\mathcal{I}_p^\lambda F_\zeta(f)(z)\right)^{1-\beta} \left(\mathcal{I}_p^\lambda(b,c)f(z)\right)^\beta \prec k(z),$$

for

$$\mathrm{Re}(\zeta) > 0, \quad \beta \geq 0, \quad z \in \mathbb{D}.$$

Then, it follows that

$$\mathcal{I}_p^\lambda(b,c)F_\zeta(f)(z) \prec k(z), \quad (z \in \mathbb{D}; \lambda \geq 0).$$

Proof. Define

$$q(z) := \mathcal{I}_p^\lambda(b,c)F_\zeta(f)(z), \quad f \in \mathcal{A}_p, \ z \in \mathbb{D}. \tag{3.2}$$

From the definition of F_ζ in (3.1), a direct computation yields

$$z\frac{d}{dz}\left(\mathcal{I}_p^\lambda(b,c)F_\zeta(f)(z)\right) = (\zeta + p)\mathcal{I}_p^\lambda(b,c)f(z) - \zeta\mathcal{I}_p^\lambda(b,c)F_\zeta(f)(z). \tag{3.3}$$

Hence, applying (3.2), (3.3), and using arguments similar to the proof of Theorem 2.1, we obtain the desired subordination. The detailed steps are omitted. □

Next, we present a related subordination result using a convex combination, analogous to Theorems 2.3 and 2.4.

Theorem 3.2. *Let $f \in \mathcal{A}_p$ and $k \in \mathcal{H}_1$. Suppose that*

$$(1-\beta)\mathcal{I}_p^\lambda F_\zeta(f)(z) + \beta\mathcal{I}_p^\lambda(b,c)f(z) \prec k(z),$$

for

$$\zeta > 0, \quad 0 \leq \beta \leq 1, \quad z \in \mathbb{D}.$$

Then, it follows that

$$\mathcal{I}_p^\lambda F_\zeta(f)(z) \prec k(z), \quad (z \in \mathbb{D}; \lambda \geq 0).$$

Finally, by taking $b = p + 1$, $c = \lambda = 1$, and $\beta = 1$ in Theorems 3.1 and 3.2, we obtain the following corollary as a special case.

Corollary 3.3. *Let $f \in \mathcal{A}_p$ and $k \in \mathcal{H}_1$. Then*

$$f(z) \prec k(z), \quad z \in \mathbb{D} \quad \Longrightarrow \quad F_\zeta(f)(z) \prec k(z).$$

4. Conclusions

In this paper, we have established several new subordination results for the class of β-convex functions in the open unit disk associated with the family of linear operators $\mathcal{I}_p^\lambda(b,c)$. The main results were obtained by combining the theory of differential subordination with Miller and Mocanu's lemma [9] and the method of subordination chains, yielding sufficient conditions that characterize the behavior of β-convex functions under the action of the considered operator. These results provide a unified approach for investigating the geometric properties of this class of functions and further demonstrate the effectiveness of subordination techniques in the study of operator-induced transformations. As an application of the established theory, we have derived corresponding results for a related class of integral operators, illustrating the flexibility and wider applicability of the developed approach. The obtained results also indicate that the techniques employed in this paper are not restricted to the present setting and may be adapted to investigate other families of linear and integral operators arising in geometric function theory. Furthermore, it is worth noting that the results presented herein can be extended to various subclasses of analytic and meromorphic functions whose associated operators satisfy suitable recurrence relations such as those given in (1.4) and (1.5). Consequently, the methods developed in this work provide a useful framework for studying subordination and related geometric properties in a broader operator-theoretic context. We expect that the present investigation will stimulate further research on generalized linear operators, differential subordinations, and their applications to other subclasses of analytic and meromorphic functions.

Author contributions

Adel A. Attiya, Nak Eun Cho and Mansour F. Yassen: Conceptualization, Investigation, Writing original draft, Writing–review & editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Confilict of interest

The authors declare no conflict of interests.

References

1. A. A. Attiya, T. M. Seoudy, M. K. Aouf, A. M. Albalahi, Certain analytic functions defined by generalized Mittag-Leffler function associated with conic domain, *J. Funct. Space.*, **2022** (2022), 1688741. https://doi.org/10.1155/2022/1688741
2. S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.*, **135** (1969), 429–446. https://doi.org/10.1090/S0002-9947-1969-0232920-2
3. B. C. Carlson, D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15** (1984), 737–745. https://doi.org/10.1137/0515057
4. J. H. Choi, M. Saigo, H. M. Srivastava, Some inclusion properties of a certain family of integral operators, *J. Math. Anal. Appl.*, **276** (2002), 432–445. https://doi.org/10.1016/S0022-247X(02)00500-0

5. R. M. Goel, N. S. Sohi, A new criterion for p-valent functions, *Proc. Amer. Math. Soc.*, **78** (1980), 353–357. https://doi.org/10.1090/S0002-9939-1980-0553375-4

6. R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, **16** (1965), 755–758. https://doi.org/10.2307/2033917

7. J. L. Liu, The Noor integral and strongly starlike functions, *J. Math. Anal. Appl.*, **261** (2001), 441–447. https://doi.org/10.1006/jmaa.2001.7489

8. S. S. Miller, P. T. Mocanu, M. O. Reade, Bazilevič functions and generalized convexity, *Rev. Roumaine Math. Pures Appl.*, **19** (1974), 213–224.

9. S. S. Miller, P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.*, **28** (1981), 157–172. https://doi.org/10.1307/mmj/1029002507

10. S. S. Miller, P. T. Mocanu, *Differential subordinations: theory and applications*, Boca Raton: CRC Press, 2000. https://doi.org/10.1201/9781482289817

11. P. T. Mocanu, Une propriété de convexité généralisée dans la théorie de la représentation conforme, *Mathematica*, **11** (1969), 127–133.

12. K. I. Noor, On new classes of integral operators, *Journal of Natural Geometry*, **16** (1999), 71–80.

13. K. I. Noor, M. A. Noor, On integral operators, *J. Math. Anal. Appl.*, **238** (1999), 341–352. https://doi.org/10.1006/jmaa.1999.6501

14. S. Owa, H. M. Srivastava, Some applications of the generalized Libera integral operator, *Proc. Japan Acad. Ser. A Math. Sci.*, **62** (1986), 125–128. https://doi.org/10.3792/pjaa.62.125

15. S. Owa, A. A. Attiya, An application of differential subordinations to the class of certain analytic functions, *Taiwanese J. Math.*, **13** (2009), 369–375. https://doi.org/10.11650/twjm/1500405342

16. Ch. Pommerenke, *Univalent functions*, Gottingen: Vanderhoeck and Ruprecht, 1975.

17. S. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, **49** (1975), 109–115. https://doi.org/10.1090/S0002-9939-1975-0367176-1

18. H. Saitoh, A linear operator and its applications of first order differential subordinations, *Math. Japon.*, **44** (1996), 31–38.

19. H. M. Srivastava, S. Owa, Some characterizations and distortions theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions, *Nagoya Math. J.*, **106** (1987), 1–28. https://doi.org/10.1017/S0027763000000854

20. H. M. Srivastava, N. E. Cho, A. A. Alderremy, A. A. Lupas, E. E. Mahmoud, S. Khan, Sharp inequalities for a class of novel convex functions associated with Gregory polynomials, *J. Inequal. Appl.*, **2024** (2024), 140. https://doi.org/10.1186/s13660-024-03210-5