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Research article

Orbit-rank simplicial complexes generated by Lie algebras arising from smooth Lie group actions

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Abstract: In this paper, we introduce orbit-rank simplicial complexes arising from smooth Lie group actions on manifolds. The proposed construction is based on the infinitesimal geometry of orbit maps together with common intersections of stabilizer Lie algebras, yielding a simplicial complex whose vertices correspond to the non-trivial orbits of the action. This framework provides a unified description of orbit geometry and infinitesimal symmetry by combining orbit ranks with common stabilizer directions. We establish that the proposed construction is intrinsic and investigate its fundamental geometric, algebraic, and combinatorial properties. In particular, we study maximal simplices, facets, dimension bounds, and rank-induced filtrations and characterize facets through common stabilizer directions and their associated zero sets. Furthermore, we relate orbit-map differentials, stabilizer Lie algebras, tangent space decompositions, and embedded submanifolds to the simplicial structure, revealing new interactions between differential geometry and simplicial topology generated by Lie group actions. Additionally, we derive formulas for the f-vector and Euler characteristic, together with their decomposition through the rank filtration, providing refined topological invariants of the proposed complexes. Several illustrative examples are presented to demonstrate the effectiveness of the construction and to verify the developed theory. The obtained results establish a new connection between differential geometry, Lie group actions, algebraic topology, and combinatorial structures, providing a systematic framework to translate infinitesimal geometric information into higher-dimensional simplicial models.

Keywords: Lie group action; smooth manifold; orbit rank; stabilizer Lie algebra; simplicial complex; rank filtration; algebraic topology
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1. Introduction

The study of smooth Lie group actions constitutes a fundamental area of differential geometry, providing a unified framework to describe continuous symmetries on smooth manifolds. Through orbit decompositions, stabilizer subgroups, and infinitesimal actions, Lie group actions have become indispensable in geometry, topology, and mathematical physics. The differential-geometric foundations of smooth manifolds, Lie groups, homogeneous spaces, orbit maps, and stabilizer Lie algebras are systematically developed in classical monographs of Warner [1] and Arvanitogeorgos [2]. In parallel with the development of Lie theory, simplicial complexes have become one of the principal combinatorial models to describe higher-order interactions that cannot be represented by ordinary graphs. Their flexibility has led to numerous applications across topology, geometry, network science, and data analyses. For example, Singh et al. [3] investigated topological properties of simplicial complexes generated from β-sparsified d-uniform hypergraphs. Pan and Liu [4] introduced new families of few-weight ternary linear codes constructed from simplicial complexes. Iqbal et al. [5] introduced a new class of Lie groups and investigated their fundamental geometric properties. Bagchi [6] studied topological properties of braid-path connected simplicial structures in covering spaces. Ditmarsch et al. [7] demonstrated the role of simplicial complexes in modeling distributed knowledge and epistemic systems. Sharma et al. [8] introduced weighted simplicial complexes to predict the evolution of small groups, while Courtney and Bianconi [9] proposed weighted growing simplicial complexes to model higher-order evolving networks. More recently, Wu et al. [10] developed simplicial complex neural networks to learn higher-order data structures. Additional geometric developments include the study of occupants in simplicial complexes by Tillmann [11] and the construction of complex line bundles over simplicial complexes by Knöppel and Pinkall [12]. Additionally, connections between transformation groups and topology have also attracted considerable attention. In particular, Chen et al. [13] investigated orbit configuration spaces associated with small covers and quasitoric manifolds, revealing deep interactions between orbit spaces and algebraic topology. Nevertheless, the existing literature does not provide a simplicial construction generated directly from the infinitesimal geometry of smooth Lie group actions through orbit maps and stabilizer Lie algebras. In particular, a systematic framework that combines orbit ranks, common stabilizer directions, and simplicial topology has remained unavailable.

Motivated by this observation, we introduce orbit-rank simplicial complexes arising from smooth Lie group actions. The proposed construction is generated from orbit ranks determined by the differentials of orbit maps together with common intersections of stabilizer Lie algebras. Consequently, each simplex simultaneously reflects the infinitesimal geometry of the action and the common symmetry shared by a family of orbits, providing a natural bridge between Lie theory, differential geometry, algebraic topology, and combinatorial topology.

The main contribution of this paper is the introduction of the orbit-rank simplicial complex associated with a smooth Lie group action on a manifold. The proposed construction is generated from orbit ranks induced by the differentials of orbit maps together with common intersections of stabilizer Lie algebras. This approach combines infinitesimal geometric information with higher-dimensional simplicial structures and produces a new topological model to study orbit families. We establish that the construction is intrinsic and investigate its fundamental geometric, algebraic, and combinatorial properties. In particular, we study maximal simplices, facets, dimension bounds, and rank-induced filtrations and characterize facets through common stabilizer directions and their associated zero sets. Furthermore, we relate orbit-map differentials, tangent space decompositions, and embedded submanifolds to the simplicial structure. Finally, we derive formulas for the f-vector and Euler characteristic, together with its decomposition through the rank filtration, revealing new interactions between differential geometry, Lie group actions, algebraic topology, and combinatorial structures. Section 3 introduces the orbit-rank simplicial complex and establishes its fundamental geometric and combinatorial properties. Section 4 investigates facets, dimension bounds, and geometric structures induced by common stabilizer directions. Section 5 develops rank filtrations and topological invariants of orbit-rank simplicial complexes, including f-vectors, Euler characteristics, and illustrative examples demonstrating the effectiveness of the proposed framework.

2. Preliminaries

Throughout this paper, all manifolds, Lie groups, and group actions are assumed to be smooth. Let $\mathcal{G}$ be a Lie group acting smoothly on a smooth manifold $\mathcal{A}$. The identity element of $\mathcal{G}$ is denoted by e, and its Lie algebra by $\mathfrak{g} = T_e(\mathcal{G})$. We briefly recall several standard notions from differential geometry, Lie group theory, and simplicial topology that will be used throughout the paper; for example, see the works [1, 2, 14, 15].

A smooth manifold of dimension n is a Hausdorff, second countable topological space equipped with a smooth atlas. A Lie group is a smooth manifold $\mathcal{G}$ endowed with a group structure such that both the multiplication and inversion maps are smooth. A smooth action of $\mathcal{G}$ on $\mathcal{A}$ is a smooth map $\mathcal{G}\times\mathcal{A} \to \mathcal{A}$, given by $(g, a) \mapsto g\cdot a$, which satisfies $e\cdot a = a$ and $(g_1g_2)\cdot a = g_1\cdot(g_2\cdot a)$ for all $g_1, g_2 \in \mathcal{G}$ and $a \in \mathcal{A}$.

For each $a \in \mathcal{A}$, the orbit of a is

$$\mathcal{O}_a = \mathcal{G}\cdot a = \{g\cdot a : g \in \mathcal{G}\},$$

while the stabilizer subgroup of a is $\mathcal{G}_a = \{g \in \mathcal{G} : g\cdot a = a\}$. The orbits are precisely the equivalence classes of the relation $a \sim b$ whenever $b = g\cdot a$ for some $g \in \mathcal{G}$. For each $a \in \mathcal{A}$, the orbit map is the smooth map θ_a: $\mathcal{G} \to \mathcal{A}$, $\theta_a(g) = g\cdot a$. Its differential at the identity element is the linear map $(d\theta_a)_e$: $\mathfrak{g} \to T_a(\mathcal{A})$. For every $X \in \mathfrak{g}$, the associated fundamental vector field is defined the following

$$X_{\mathcal{A}}(a) = \left.\frac{d}{dt}\right|_{t=0}(\exp(tX)\cdot a).$$

Hence, $(d\theta_a)_e(X) = X_{\mathcal{A}}(a)$. The stabilizer Lie algebra of a is as follows:

$$\mathfrak{g}_a = \{X \in \mathfrak{g} : X_{\mathcal{A}}(a) = 0\}.$$

The tangent space to the orbit $\mathcal{O}_a$ at a is as follows:

$$T_a(\mathcal{O}_a) = \{X_{\mathcal{A}}(a) : X \in \mathfrak{g}\}.$$

Therefore, $\mathrm{Im}((d\theta_a)_e) = T_a(\mathcal{O}_a)$ and $\ker((d\theta_a)_e) = \mathfrak{g}_a$. Consequently, $\mathrm{rank}((d\theta_a)_e) = \dim(\mathcal{O}_a)$ and the rank–nullity theorem yields the following: $\dim(\mathcal{O}_a) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_a)$. An orbit $\mathcal{O}_a$ is called non-trivial whenever $\dim(\mathcal{O}_a) > 0$.

For smooth vector fields V and W on $\mathcal{A}$, their Lie bracket is denoted by $[V, W]$. For $X, Y \in \mathfrak{g}$, the corresponding fundamental vector fields satisfy $[X_{\mathcal{A}}, Y_{\mathcal{A}}] = \pm[X, Y]_{\mathcal{A}}$, where the sign depends on the convention adopted for the action.

A distribution on $\mathcal{A}$ is an assignment $a \mapsto \mathcal{D}_a \subseteq T_a(\mathcal{A})$. A distribution is called involutive if the Lie bracket of any two local sections remains a local section of the distribution. By Frobenius' theorem, every smooth involutive distribution admits maximal connected integral manifolds, called its integral leaves.

The rank of the orbit O_a is defined by $\rho(O_a) = \mathrm{rank}((d\theta_a)_e) = \dim(O_a)$. For each positive integer r, the rank stratum is

$$\mathcal{A}_r = \{a \in \mathcal{A} : \mathrm{rank}((d\theta_a)_e) = r\},$$

and the corresponding rank level is $\mathcal{V}_r = \{O_a : a \in \mathcal{A}_r\}$. The set $\Sigma_\rho = \{r \geq 1 : \mathcal{V}_r \neq \emptyset\}$ is called the rank spectrum of the action.

A simplicial complex is a collection Δ of finite subsets of a vertex set V such that whenever $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of Δ are called simplices. A simplex containing $k + 1$ vertices is called a k-simplex and has a dimension of k. The dimension of Δ is $\dim(\Delta) = \max\{\dim(\sigma) : \sigma \in \Delta\}$. A maximal simplex with respect to inclusion is called a facet of Δ. The set of all k-dimensional simplices of Δ is denoted by Δ_k. For each $k \geq 0$, let f_k denote the number of k-simplices in Δ. The sequence $(f_0, f_1, \ldots, f_d)$ is called the f-vector of Δ, where $d = \dim(\Delta)$. The Euler characteristic of Δ is defined the following:

$$\chi(\Delta) = \sum_{k=0}^{d} (-1)^k f_k.$$

The 1-skeleton of Δ is the graph $\Delta^{(1)}$ whose vertices are the 0-simplices and whose edges are the 1-simplices of Δ. Let $\Delta_1 \subseteq \Delta_2 \subseteq \cdots \subseteq \Delta_m$ be an increasing sequence of simplicial complexes. Such a sequence is called a simplicial filtration. Filtrations provide a natural framework for studying the evolution of topological structures across different geometric or algebraic levels.

3. Orbit-rank simplicial complexes

Throughout this paper, we assume that $\mathcal{G}$ is a finite-dimensional Lie group acting smoothly on a smooth manifold $\mathcal{A}$. The purpose of this section is to introduce the orbit-rank simplicial complex associated with the action of $\mathcal{G}$ on $\mathcal{A}$. The construction combines orbit ranks with common stabilizer directions and yields a simplicial model encoding higher-order geometric relations among orbit classes.

Let $\mathcal{V} = \{O_a : \rho(O_a) > 0\}$ denote the collection of all non-trivial orbits of the action. For every finite family of orbit classes $\sigma = \{O_{a_0}, O_{a_1}, \ldots, O_{a_k}\} \subseteq \mathcal{V}$, define the following:

$$\Gamma(\sigma) = \max_{p_i \in O_{a_i}} \dim\left(\bigcap_{i=0}^{k} \mathfrak{g}_{p_i}\right).$$

In particular, for two orbit classes O_a and O_b, we define the following:

$$\omega(O_a, O_b) = \Gamma(\{O_a, O_b\}) = \max_{p \in O_a,\, q \in O_b} \dim(\mathfrak{g}_p \cap \mathfrak{g}_q).$$

The quantity $\Gamma(\sigma)$ measures the largest dimension of a common infinitesimal symmetry simultaneously shared by all orbit classes contained in σ. Its two-orbit specialization is the quantity $\omega(O_a, O_b)$, which measures the largest dimension of a common infinitesimal symmetry shared by the two orbit classes.

Definition 3.1. Let $\mathcal{G}$ be a Lie group acting smoothly on a manifold $\mathcal{A}$. The orbit-rank simplicial complex associated with the action is the simplicial complex $\Delta(\mathcal{G}, \mathcal{A})$. Its vertex set is $\mathcal{V} = \{O_a : \rho(O_a) > 0\}$, namely the collection of all non-trivial orbits of the action. Every singleton $\{O_a\}$ is a simplex. A finite subset $\sigma = \{O_{a_0}, O_{a_1}, \ldots, O_{a_k}\} \subseteq \mathcal{V},\ k \geq 1$, is called an orbit-rank simplex whenever $\Gamma(\sigma) > 0$. The simplices of $\Delta(\mathcal{G}, \mathcal{A})$ are called orbit-rank simplices.

Example 3.2. Let $\mathcal{G} = (\mathbb{R}^3, +)$, and let $\mathfrak{g} = \mathbb{R}^3$ with basis e_1–e_3. Let $\mathcal{A}$ be the disjoint union of four copies of $\mathbb{R}$, identified with $\{1, 2, 3, 4\} \times \mathbb{R}$. For $g = (x, y, z) \in \mathcal{G}$ and $(i, u) \in \mathcal{A}$, define the following: $g \cdot (i, u) = (i, u + \alpha_i(g))$, where

$$\alpha_1(x, y, z) = z,\ \ \alpha_2(x, y, z) = y,\ \ \alpha_3(x, y, z) = y - z$$

and $\alpha_4(x, y, z) = x$. For $i = 1, 2, 3, 4$, let $a_i = (i, 0)$ and $O_i = \mathcal{G} \cdot a_i$. Since each α_i is nonzero, each orbit is non-trivial and $O_i = \{i\} \times \mathbb{R}$. Thus, $\mathcal{V} = \{O_1, O_2, O_3, O_4\}$. The stabilizer Lie algebras are as follows:

$$\mathfrak{g}_{a_1} = \operatorname{span}\{e_1, e_2\},\ \ \mathfrak{g}_{a_2} = \operatorname{span}\{e_1, e_3\},\ \ \mathfrak{g}_{a_3} = \operatorname{span}\{e_1, e_2 + e_3\},\ \ \mathfrak{g}_{a_4} = \operatorname{span}\{e_2, e_3\}.$$

Since the action is by translations on each component, these stabilizer Lie algebras are constant along the corresponding orbits. Hence, in this example, the value of $\Gamma(\sigma)$ is directly obtained from the common intersection of the corresponding stabilizer Lie algebras. For the two-element subsets, we have

$$\begin{aligned}
&\mathfrak{g}_{a_1} \cap \mathfrak{g}_{a_2} = \operatorname{span}\{e_1\}, &&\mathfrak{g}_{a_1} \cap \mathfrak{g}_{a_3} = \operatorname{span}\{e_1\}, &&\mathfrak{g}_{a_2} \cap \mathfrak{g}_{a_3} = \operatorname{span}\{e_1\},\\
&\mathfrak{g}_{a_1} \cap \mathfrak{g}_{a_4} = \operatorname{span}\{e_2\}, &&\mathfrak{g}_{a_2} \cap \mathfrak{g}_{a_4} = \operatorname{span}\{e_3\}, &&\mathfrak{g}_{a_3} \cap \mathfrak{g}_{a_4} = \operatorname{span}\{e_2 + e_3\}.
\end{aligned}$$

Hence, every two-element subset of $\mathcal{V}$ is a 1-simplex. For the three-element subsets, we obtain the following:

$$\mathfrak{g}_{a_1} \cap \mathfrak{g}_{a_2} \cap \mathfrak{g}_{a_3} = \operatorname{span}\{e_1\},$$

while

$$\mathfrak{g}_{a_1} \cap \mathfrak{g}_{a_2} \cap \mathfrak{g}_{a_4} = \{0\},\ \ \mathfrak{g}_{a_1} \cap \mathfrak{g}_{a_3} \cap \mathfrak{g}_{a_4} = \{0\},\ \ \mathfrak{g}_{a_2} \cap \mathfrak{g}_{a_3} \cap \mathfrak{g}_{a_4} = \{0\}.$$

Therefore, the only 2-simplex is $\{O_1, O_2, O_3\}$. Finally, $\mathfrak{g}_{a_1} \cap \mathfrak{g}_{a_2} \cap \mathfrak{g}_{a_3} \cap \mathfrak{g}_{a_4} = \{0\}$. Thus, $\{O_1, O_2, O_3, O_4\}$ is not a 3-simplex. Consequently, the orbit-rank simplicial complex is as follows:

$$\Delta(\mathcal{G}, \mathcal{A}) = \{\{O_i\}, \{O_i, O_j\}, \{O_1, O_2, O_3\} : 1 \leq i < j \leq 4\}.$$

Hence, $f_0 = 4$, $f_1 = 6$, $f_2 = 1$, and the Euler characteristic is $\chi(\Delta) = f_0 - f_1 + f_2 = -1$. Thus, this example produces a simplicial complex with four orbit vertices, six edges and one filled triangle, but no three-dimensional simplex.

Theorem 3.3. Let $\mathcal{G}$ be a finite-dimensional Lie group acting smoothly on a smooth manifold $\mathcal{A}$. For a finite family of non-trivial orbits $\sigma = \{O_{a_0}, \ldots, O_{a_k}\}$, define the following:

$$\Gamma(\sigma) = \max_{p_i \in O_{a_i}} \dim \left(\bigcap_{i=0}^{k} \mathfrak{g}_{p_i} \right).$$

Then, $\Gamma(\sigma)$ is well-defined on orbit classes. In particular, the condition $\Gamma(\sigma) > 0$ does not depend on the choice of representatives of the orbits. Consequently, the orbit-rank simplices of $\Delta(\mathcal{G}, \mathcal{A})$ are intrinsic objects associated with the action of $\mathcal{G}$ on $\mathcal{A}$.

Proof. Let $\sigma = \{O_{a_0}, \ldots, O_{a_k}\}$. For each i, every point $p_i \in O_{a_i}$ can be written as $p_i = g_i \cdot a_i$ for some $g_i \in \mathcal{G}$. The stabilizer subgroup satisfies

$$\mathcal{G}_{p_i} = g_i \mathcal{G}_{a_i} g_i^{-1},$$

and therefore $\mathfrak{g}_{p_i} = \mathrm{Ad}_{g_i}(\mathfrak{g}_{a_i})$. Hence

$$\bigcap_{i=0}^{k} \mathfrak{g}_{p_i} = \bigcap_{i=0}^{k} \mathrm{Ad}_{g_i}(\mathfrak{g}_{a_i}).$$

Since each adjoint map $\mathrm{Ad}_{g_i} \colon \mathfrak{g} \to \mathfrak{g}$ is a linear isomorphism, the family of stabilizer Lie algebras obtained by varying the points along each orbit only depends on the orbit itself and not on the chosen representative.

Consequently, the collection

$$\left\{ \bigcap_{i=0}^{k} \mathfrak{g}_{p_i} : p_i \in O_{a_i} \right\}$$

only depends on the orbit classes $O_{a_0}, \ldots, O_{a_k}$. Since every intersection has dimension belonging to the finite set $\{0, 1, \ldots, \dim \mathfrak{g}\}$, the maximum defining $\Gamma(\sigma)$ exists and is independent of the chosen representatives. Therefore $\Gamma(\sigma)$ is well defined on the orbit classes. In particular, the condition $\Gamma(\sigma) > 0$ only depends on the orbit classes and hence, the orbit-rank simplices of $\Delta(\mathcal{G}, \mathcal{A})$ are intrinsic objects associated with the action. □

Theorem 3.4. Let $\mathcal{G}$ be a finite-dimensional Lie group acting smoothly on a smooth manifold $\mathcal{A}$, and let $\sigma = \{O_{a_0}, \ldots, O_{a_k}\}$ be an orbit-rank simplex with $\Gamma(\sigma) = r > 0$. Then, there exist points $p_i \in O_{a_i}$ and an r-dimensional Lie subalgebra $\mathfrak{c}_\sigma \subseteq \mathfrak{g}$ such that $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_{p_i}$ for every $i = 0, \ldots, k$. Consequently, every $X \in \mathfrak{c}_\sigma$ generates a fundamental vector field satisfying $X_{\mathcal{A}}(p_i) = 0$ for all $i = 0, \ldots, k$. Moreover, $\rho(O_{a_i}) \leq \dim(\mathfrak{g}) - r$ for every $i = 0, \ldots, k$.

Proof. Since σ is an orbit-rank simplex and $\Gamma(\sigma) = r > 0$, by the definition of $\Gamma(\sigma)$, there exist points $p_i \in O_{a_i}$ for $i = 0, \ldots, k$ such that

$$r = \dim \left(\bigcap_{i=0}^{k} \mathfrak{g}_{p_i} \right).$$

Define the following:

$$\mathfrak{c}_\sigma = \bigcap_{i=0}^{k} \mathfrak{g}_{p_i}.$$

Then $\dim(\mathfrak{c}_\sigma) = r$. First, we show that $\mathfrak{c}_\sigma$ is a Lie subalgebra of $\mathfrak{g}$. For every i, the stabilizer Lie algebra $\mathfrak{g}_{p_i}$ is the kernel of the differential of the orbit map at the identity, $\mathfrak{g}_{p_i} = \ker((d\theta_{p_i})_e)$. Equivalently, $\mathfrak{g}_{p_i} = \{X \in \mathfrak{g} : X_{\mathcal{A}}(p_i) = 0\}$. Since $\mathfrak{g}_{p_i}$ is the Lie algebra of the stabilizer subgroup $\mathcal{G}_{p_i}$, it is closed under the Lie bracket of $\mathfrak{g}$. Thus, if $X, Y \in \mathfrak{g}_{p_i}$, then $[X, Y] \in \mathfrak{g}_{p_i}$.

Now, let $X, Y \in \mathfrak{c}_\sigma$. Then $X, Y \in \mathfrak{g}_{p_i}$ for every $i = 0, \ldots, k$. Since each $\mathfrak{g}_{p_i}$ is a Lie subalgebra, we get $[X, Y] \in \mathfrak{g}_{p_i}$ for every i. Therefore

$$[X, Y] \in \bigcap_{i=0}^{k} \mathfrak{g}_{p_i} = \mathfrak{c}_\sigma.$$

Hence $\mathfrak{c}_\sigma$ is a Lie subalgebra of $\mathfrak{g}$. Next, by construction, $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_{p_i}$ for every i. Thus, for every $X \in \mathfrak{c}_\sigma$, we have the following: $X \in \mathfrak{g}_{p_i}$. By the definition of the stabilizer Lie algebra,

$$X \in \mathfrak{g}_{p_i} \iff X_{\mathcal{A}}(p_i) = 0.$$

Therefore $X_{\mathcal{A}}(p_i) = 0$, $i = 0, \ldots, k$. This means that the infinitesimal symmetry generated by X simultaneously vanishes at the selected points $p_0, \ldots, p_k$. Equivalently, the one-parameter subgroup $\exp(tX)$ fixes each p_i. Indeed,

$$X_{\mathcal{A}}(p_i) = \left.\frac{d}{dt}\right|_{t=0} (\exp(tX) \cdot p_i) = 0.$$

Since $X \in \mathfrak{g}_{p_i}$ is an element of the Lie algebra of the stabilizer subgroup $\mathcal{G}_{p_i}$, the one-parameter subgroup generated by X is contained in the identity component $(\mathcal{G}_{p_i})^0$ of $\mathcal{G}_{p_i}$. Hence $\exp(tX) \cdot p_i = p_i$ for every $t \in \mathbb{R}$. It remains to prove the rank estimate. Since $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_{p_i}$, we have the following:

$$r = \dim(\mathfrak{c}_\sigma) \leq \dim(\mathfrak{g}_{p_i}).$$

Using the rank–nullity identity for the orbit map,

$$\rho(O_{a_i}) = \rho(O_{p_i}) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_{p_i}),$$

we obtain the following: $\rho(O_{a_i}) \leq \dim(\mathfrak{g}) - r$. This holds for every $i = 0, \ldots, k$. □

Corollary 3.5. If $\sigma = \{O_{a_0}, \ldots, O_{a_k}\}$ is an orbit-rank simplex with $\Gamma(\sigma) = r > 0$, then no orbit in σ can have a rank larger than $\dim(\mathfrak{g}) - r$. In particular, if an orbit O_a satisfies $\rho(O_a) = \dim(\mathfrak{g})$, then it cannot belong to any orbit-rank simplex containing another vertex.

Proof. The first assertion directly follows from Theorem 3.4. Indeed, for every $O_{a_i} \in \sigma$, we have the following:

$$\rho(O_{a_i}) \leq \dim(\mathfrak{g}) - r.$$

If $\rho(O_a) = \dim(\mathfrak{g})$, then $\dim(\mathfrak{g}_a) = 0$. Hence, no nonzero common stabilizer direction can be shared with another orbit. Therefore, such an orbit cannot occur in any simplex containing another vertex. □

For orbit-rank simplices σ and τ, we denote their corresponding common stabilizer Lie subalgebras by $\mathfrak{c}_\sigma$ and $\mathfrak{c}_\tau$. Whenever $\mathfrak{c}_\sigma \cap \mathfrak{c}_\tau = \{0\}$, their vector-space direct sum is denoted by $\mathfrak{c}_\sigma \oplus \mathfrak{c}_\tau$.

Theorem 3.6. Let $\sigma, \tau \in \Delta(\mathcal{G}, \mathcal{A})$ and suppose that $\mathfrak{c}_\sigma \cap \mathfrak{c}_\tau = \{0\}$. Assume that $\mathfrak{c}_\sigma$ and $\mathfrak{c}_\tau$ are fixed common stabilizer Lie subalgebras realizing $\Gamma(\sigma)$ and $\Gamma(\tau)$, respectively, with $\dim(\mathfrak{c}_\sigma) = r$ and $\dim(\mathfrak{c}_\tau) = s$. If there exist representatives p_a of all orbit classes contained in $\sigma \cup \tau$ such that $\mathfrak{c}_\sigma \oplus \mathfrak{c}_\tau \subseteq \mathfrak{g}_{p_a}$, then $\Gamma(\sigma \cup \tau) \geq r + s$. Consequently, $\sigma \cup \tau$ is an orbit-rank simplex. Moreover, for every orbit $O_a \in \sigma \cup \tau$, one has $\rho(O_a) \leq \dim(\mathfrak{g}) - (r + s)$.

Proof. Since $\mathfrak{c}_\sigma$ and $\mathfrak{c}_\tau$ are finite-dimensional vector subspaces of $\mathfrak{g}$ and $\mathfrak{c}_\sigma \cap \mathfrak{c}_\tau = \{0\}$, the direct sum theorem gives the following:

$$\dim(\mathfrak{c}_\sigma \oplus \mathfrak{c}_\tau) = \dim(\mathfrak{c}_\sigma) + \dim(\mathfrak{c}_\tau) = r + s.$$

Let O_a be an arbitrary orbit belonging to $\sigma \cup \tau$, and let p_a be the representative specified in the assumptions. By hypothesis, $\mathfrak{c}_\sigma \oplus \mathfrak{c}_\tau \subseteq \mathfrak{g}_{p_a}$. Recall that the stabilizer Lie algebra at p_a is given by $\mathfrak{g}_{p_a} = \ker((d\theta_{p_a})_e)$, where θ_{p_a} denotes the orbit map $\theta_{p_a}(g) = g \cdot p_a$. Hence, every element $X \in \mathfrak{c}_\sigma \oplus \mathfrak{c}_\tau$ satisfies $(d\theta_{p_a})_e(X) = 0$. Using the standard identification between orbit-map differentials and fundamental vector fields, we obtain the following: $(d\theta_{p_a})_e(X) = X_{\mathcal{A}}(p_a)$. Therefore, $X_{\mathcal{A}}(p_a) = 0$ for every $X \in \mathfrak{c}_\sigma \oplus \mathfrak{c}_\tau$. Consequently, every infinitesimal direction contained in $\mathfrak{c}_\sigma \oplus \mathfrak{c}_\tau$ simultaneously vanishes at the chosen representatives of all orbit classes belonging to $\sigma \cup \tau$. This implies the following:

$$\mathfrak{c}_\sigma \oplus \mathfrak{c}_\tau \subseteq \bigcap_{O_a \in \sigma \cup \tau} \mathfrak{g}_{p_a}.$$

Taking dimensions and using the direct-sum formula yields the following:

$$r + s = \dim(\mathfrak{c}_\sigma \oplus \mathfrak{c}_\tau) \leq \dim\left(\bigcap_{O_a \in \sigma \cup \tau} \mathfrak{g}_{p_a}\right).$$

Since $\Gamma(\sigma \cup \tau)$ is defined as the maximum possible dimension of such common stabilizer intersections over all admissible representatives, we obtain the following:

$$\Gamma(\sigma \cup \tau) \geq \dim\left(\bigcap_{O_a \in \sigma \cup \tau} \mathfrak{g}_{p_a}\right) \geq r + s.$$

In particular, $\Gamma(\sigma \cup \tau) > 0$. Hence, the family $\sigma \cup \tau$ determines an orbit-rank simplex of $\Delta(\mathcal{G}, \mathcal{A})$. It remains to prove the rank estimate. Since $\mathfrak{c}_\sigma \oplus \mathfrak{c}_\tau \subseteq \mathfrak{g}_{p_a}$, we have the following:

$$r + s = \dim(\mathfrak{c}_\sigma \oplus \mathfrak{c}_\tau) \leq \dim(\mathfrak{g}_{p_a}).$$

By applying the rank-nullity theorem to the differential $(d\theta_{p_a})_e$, we obtain the following:

$$\rho(O_a) = \operatorname{rank}((d\theta_{p_a})_e) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_{p_a}).$$

Combining the previous two inequalities gives the following:

$$\rho(O_a) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_{p_a}) \leq \dim(\mathfrak{g}) - (r + s).$$

Since O_a was arbitrary, the estimate holds for every orbit belonging to $\sigma \cup \tau$. □

Corollary 3.7. Under the assumptions of Theorem 3.6, if $r + s = \dim(\mathfrak{g})$, then the hypotheses of Theorem 3.6 cannot be satisfied by a union of non-trivial orbit vertices.

Proof. By Theorem 3.6,

$$\rho(\mathcal{O}_a) \leq \dim(\mathfrak{g}) - (r + s).$$

If $r + s = \dim(\mathfrak{g})$, then $\rho(\mathcal{O}_a) \leq 0$. Since every vertex of $\Delta(\mathcal{G}, \mathcal{A})$ represents a non-trivial orbit, one has $\rho(\mathcal{O}_a) > 0$, which contradicts the above inequality. Therefore, the hypotheses of Theorem 3.6 cannot be satisfied by a union of non-trivial orbit vertices. $\square$

Theorem 3.8. Let

$$\sigma = \{\mathcal{O}_{a_0}, \ldots, \mathcal{O}_{a_k}\} \in \Delta(\mathcal{G}, \mathcal{A})$$

be an orbit-rank simplex with $\Gamma(\sigma) = r > 0$. Let $\mathfrak{c}_\sigma$ be an r-dimensional common stabilizer Lie subalgebra realizing $\Gamma(\sigma)$. Then, for each representative $p_i \in \mathcal{O}_{a_i}$ with $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_{p_i}$, the differential of the orbit map factors through the quotient vector space $\mathfrak{g}/\mathfrak{c}_\sigma$. More precisely, there exists a linear map $\overline{d\theta}_{p_i}\colon \mathfrak{g}/\mathfrak{c}_\sigma \to T_{p_i}(\mathcal{A})$ such that $(d\theta_{p_i})_e = \overline{d\theta}_{p_i} \circ \pi_\sigma$, where $\pi_\sigma\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{c}_\sigma$ is the canonical projection. Consequently,

$$T_{p_i}(\mathcal{O}_{a_i}) = \mathrm{Im}(\overline{d\theta}_{p_i}) \subseteq T_{p_i}(\mathcal{A}) \text{ and } \rho(\mathcal{O}_{a_i}) \leq \dim(\mathfrak{g}/\mathfrak{c}_\sigma) = \dim(\mathfrak{g}) - r.$$

Proof. Since σ is an orbit-rank simplex and $\Gamma(\sigma) = r > 0$, there exist representatives $p_i \in \mathcal{O}_{a_i}$ such that

$$\mathfrak{c}_\sigma = \bigcap_{i=0}^{k} \mathfrak{g}_{p_i}$$

with $\dim(\mathfrak{c}_\sigma) = r$. In particular, $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_{p_i}$ for every $i = 0, \ldots, k$. Fix an index i. The orbit map at p_i is $\theta_{p_i}\colon \mathcal{G} \to \mathcal{A}$, $\theta_{p_i}(g) = g \cdot p_i$. Its differential at the identity is $(d\theta_{p_i})_e\colon \mathfrak{g} \to T_{p_i}(\mathcal{A})$. By the standard relation between the orbit-map differentials and the fundamental vector fields, $(d\theta_{p_i})_e(X) = X_{\mathcal{A}}(p_i)$ for every $X \in \mathfrak{g}$. Since $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_{p_i}$, every $X \in \mathfrak{c}_\sigma$ satisfies $X_{\mathcal{A}}(p_i) = 0$. Hence, $(d\theta_{p_i})_e(X) = 0$ for all $X \in \mathfrak{c}_\sigma$. Therefore, $\mathfrak{c}_\sigma \subseteq \ker((d\theta_{p_i})_e)$.

Now, let $\pi_\sigma\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{c}_\sigma$ be the canonical quotient map. Since $\mathfrak{c}_\sigma$ is contained in the kernel of $(d\theta_{p_i})_e$, the universal property of quotient vector spaces gives the following: a unique linear map

$$\overline{d\theta}_{p_i} : \mathfrak{g}/\mathfrak{c}_\sigma \to T_{p_i}(\mathcal{A})$$

such that

$$(d\theta_{p_i})_e = \overline{d\theta}_{p_i} \circ \pi_\sigma.$$

This factorization means that all infinitesimal directions belonging to $\mathfrak{c}_\sigma$ are annihilated before the orbit tangent space is generated. Thus, the tangent space to the orbit is only determined by the quotient directions in $\mathfrak{g}/\mathfrak{c}_\sigma$. Indeed,

$$T_{p_i}(\mathcal{O}_{a_i}) = \mathrm{Im}((d\theta_{p_i})_e).$$

Using the factorization above, we obtain the following:

$$\mathrm{Im}((d\theta_{p_i})_e) = \mathrm{Im}(\overline{d\theta}_{p_i} \circ \pi_\sigma) = \mathrm{Im}(\overline{d\theta}_{p_i}).$$

Therefore,

$$T_{p_i}(O_{a_i}) = \mathrm{Im}(\overline{d\theta}_{p_i}) \subseteq T_{p_i}(\mathcal{A}).$$

Taking dimensions gives the following:

$$\rho(O_{a_i}) = \dim T_{p_i}(O_{a_i}) = \dim \mathrm{Im}(\overline{d\theta}_{p_i}).$$

Since the image of a linear map cannot have a dimension larger than the dimension of its domain,

$$\dim \mathrm{Im}(\overline{d\theta}_{p_i}) \leq \dim(\mathfrak{g}/\mathfrak{c}_\sigma).$$

Hence,

$$\rho(O_{a_i}) \leq \dim(\mathfrak{g}/\mathfrak{c}_\sigma).$$

Finally,

$$\dim(\mathfrak{g}/\mathfrak{c}_\sigma) = \dim(\mathfrak{g}) - \dim(\mathfrak{c}_\sigma) = \dim(\mathfrak{g}) - r.$$

Thus,

$$\rho(O_{a_i}) \leq \dim(\mathfrak{g}) - r.$$

This completes the proof. □

Corollary 3.9. Let $\sigma = \{O_{a_0}, \ldots, O_{a_k}\} \in \Delta(\mathcal{G}, \mathcal{A})$ and suppose that $\Gamma(\sigma) = r > 0$. Then $r \leq \dim(\mathfrak{g}_{p_i})$ for every representative $p_i \in O_{a_i}$ realizing $\Gamma(\sigma)$. Consequently,

$$\rho(O_{a_i}) \leq \dim(\mathfrak{g}) - r, \quad i = 0, \ldots, k.$$

Proof. By Theorem 3.8, there exists an r -dimensional common stabilizer Lie subalgebra $\mathfrak{c}_\sigma$ such that $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_{p_i}$ for every representative p_i realizing $\Gamma(\sigma)$. Hence,

$$r = \dim(\mathfrak{c}_\sigma) \leq \dim(\mathfrak{g}_{p_i}).$$

Using the rank–nullity formula for the orbit map,

$$\rho(O_{a_i}) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_{p_i}),$$

we obtain the following: $\rho(O_{a_i}) \leq \dim(\mathfrak{g}) - r.$ □

The factorization obtained in Theorem 3.8 naturally induces a rank filtration on the orbit-rank simplicial complex. For each integer $m \geq 1$, define the following: $\Delta_{\leq m}(\mathcal{G}, \mathcal{A})$ to be the collection of all simplices $\sigma \in \Delta(\mathcal{G}, \mathcal{A})$ whose vertices satisfy $\rho(O_a) \leq m$.

Theorem 3.10. For every integer $m \geq 1$, the family $\Delta_{\leq m}(\mathcal{G}, \mathcal{A})$ is a simplicial subcomplex of $\Delta(\mathcal{G}, \mathcal{A})$. Furthermore,

$$\Delta_{\leq 1}(\mathcal{G}, \mathcal{A}) \subseteq \Delta_{\leq 2}(\mathcal{G}, \mathcal{A}) \subseteq \cdots \subseteq \Delta_{\leq \dim(\mathfrak{g})}(\mathcal{G}, \mathcal{A}) = \Delta(\mathcal{G}, \mathcal{A}).$$

Moreover, if $\sigma \in \Delta(\mathcal{G}, \mathcal{A})$ satisfies $\Gamma(\sigma) = r > 0$, then

$$\sigma \in \Delta_{\leq \dim(\mathfrak{g}) - r}(\mathcal{G}, \mathcal{A}).$$

Proof. Fix $m \geq 1$. Let
$$\sigma = \{O_{a_0}, \ldots, O_{a_k}\} \in \Delta_{\leq m}(\mathcal{G}, \mathcal{A}).$$

By definition, $\rho(O_{a_i}) \leq m$, $i = 0, \ldots, k$. Let $\tau \subseteq \sigma$ be a nonempty face. Then $\tau = \{O_{a_{i_0}}, \ldots, O_{a_{i_s}}\}$ for suitable indices $0 \leq i_0 < \cdots < i_s \leq k$. Since every vertex of τ already belongs to σ, we still have $\rho(O_{a_{i_j}}) \leq m$, $j = 0, \ldots, s$. Since $\Delta(\mathcal{G}, \mathcal{A})$ is a simplicial complex, every face of σ belongs to $\Delta(\mathcal{G}, \mathcal{A})$. Therefore, $\tau \in \Delta_{\leq m}(\mathcal{G}, \mathcal{A})$. Hence, $\Delta_{\leq m}(\mathcal{G}, \mathcal{A})$ is closed under taking faces and Therefore, forms a simplicial subcomplex.

Now, let $m_1 \leq m_2$. Suppose that $\sigma \in \Delta_{\leq m_1}(\mathcal{G}, \mathcal{A})$. Then, every vertex O_a of σ satisfies $\rho(O_a) \leq m_1 \leq m_2$. Hence, $\sigma \in \Delta_{\leq m_2}(\mathcal{G}, \mathcal{A})$. Therefore

$$\Delta_{\leq m_1}(\mathcal{G}, \mathcal{A}) \subseteq \Delta_{\leq m_2}(\mathcal{G}, \mathcal{A}).$$

This proves the filtration property.

Next, we show that the final term of the filtration coincides with the whole complex. Let O_a be any non-trivial orbit. Since $(d\theta_a)_e \colon \mathfrak{g} \longrightarrow T_a(\mathcal{A})$ is a linear transformation, we have the following:

$$\rho(O_a) = \operatorname{rank}((d\theta_a)_e) \leq \dim(\mathfrak{g}).$$

Equivalently, using $\ker((d\theta_a)_e) = \mathfrak{g}_a$, the rank–nullity theorem yields

$$\rho(O_a) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_a) \leq \dim(\mathfrak{g}).$$

Thus, every vertex of $\Delta(\mathcal{G}, \mathcal{A})$ belongs to the rank level $\leq \dim(\mathfrak{g})$. Consequently,

$$\Delta_{\leq \dim(\mathfrak{g})}(\mathcal{G}, \mathcal{A}) = \Delta(\mathcal{G}, \mathcal{A}).$$

Finally, let
$$\sigma = \{O_{a_0}, \ldots, O_{a_k}\} \in \Delta(\mathcal{G}, \mathcal{A})$$

with $\Gamma(\sigma) = r > 0$. By Theorem 3.8, there exists an r-dimensional common stabilizer Lie subalgebra $\mathfrak{c}_\sigma \subseteq \mathfrak{g}$ such that $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_{p_i}$ for suitable representatives $p_i \in O_{a_i}$, $i = 0, \ldots, k$. Furthermore, Theorem 3.8 shows that the differential $(d\theta_{p_i})_e$ factors through the quotient space $\mathfrak{g}/\mathfrak{c}_\sigma$. Hence,

$$\rho(O_{a_i}) = \dim \operatorname{Im}((d\theta_{p_i})_e) \leq \dim(\mathfrak{g}/\mathfrak{c}_\sigma).$$

Since $\dim(\mathfrak{c}_\sigma) = r$, we obtain the following:

$$\dim(\mathfrak{g}/\mathfrak{c}_\sigma) = \dim(\mathfrak{g}) - r.$$

Therefore,
$$\rho(O_{a_i}) \leq \dim(\mathfrak{g}) - r, \quad i = 0, \ldots, k.$$

Consequently, $\sigma \in \Delta_{\leq \dim(\mathfrak{g}) - r}(\mathcal{G}, \mathcal{A})$. $\square$

4. Facets and dimension bounds

The purpose of this section is to investigate maximal simplices and dimensional properties of the orbit-rank simplicial complex. Special attention is devoted to facets and the influence of common stabilizer directions on the combinatorial structure of the complex.

A simplex $\sigma \in \Delta(\mathcal{G}, \mathcal{A})$ is called a facet whenever there exists no simplex $\tau \in \Delta(\mathcal{G}, \mathcal{A})$ such that $\sigma \subsetneq \tau$. For every simplex $\sigma \in \Delta(\mathcal{G}, \mathcal{A})$, we fix a common stabilizer Lie subalgebra $\mathfrak{c}_\sigma$ realizing $\Gamma(\sigma)$, that is, $\dim(\mathfrak{c}_\sigma) = \Gamma(\sigma)$.

Lemma 4.1. Let σ be a facet of $\Delta(\mathcal{G}, \mathcal{A})$, and let $\mathfrak{c}_\sigma$ be the fixed common stabilizer Lie subalgebra realizing $\Gamma(\sigma)$. Then, there exists no orbit $\mathcal{O}_b \notin \sigma$ such that $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_q$ for some $q \in \mathcal{O}_b$.

Proof. Contrarily, assume that there exist an orbit $\mathcal{O}_b \notin \sigma$ and a point $q \in \mathcal{O}_b$ such that $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_q$. Write $\sigma = \{\mathcal{O}_{a_0}, \dots, \mathcal{O}_{a_k}\}$. Since $\mathfrak{c}_\sigma$ realizes $\Gamma(\sigma)$, there exist representatives $p_i \in \mathcal{O}_{a_i}$ for $i = 0, \dots, k$ such that

$$\mathfrak{c}_\sigma = \bigcap_{i=0}^{k} \mathfrak{g}_{p_i}.$$

By combining this identity with $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_q$, we obtain the following:

$$\mathfrak{c}_\sigma \subseteq \left(\bigcap_{i=0}^{k} \mathfrak{g}_{p_i} \right) \cap \mathfrak{g}_q.$$

Hence,

$$\dim \left(\left(\bigcap_{i=0}^{k} \mathfrak{g}_{p_i} \right) \cap \mathfrak{g}_q \right) \geq \dim(\mathfrak{c}_\sigma) = \Gamma(\sigma) > 0.$$

Therefore, $\Gamma(\sigma \cup \{\mathcal{O}_b\}) > 0$. By the definition of $\Delta(\mathcal{G}, \mathcal{A})$, the family $\sigma \cup \{\mathcal{O}_b\}$ is a simplex. Since $\mathcal{O}_b \notin \sigma$, we have the following: $\sigma \subsetneq \sigma \cup \{\mathcal{O}_b\}$, which contradicts the assumption that σ is a facet. Therefore, no such orbit $\mathcal{O}_b$ exists. □

Corollary 4.2. Let σ be a facet of $\Delta(\mathcal{G}, \mathcal{A})$, and let $\mathfrak{c}_\sigma$ be the fixed common stabilizer Lie subalgebra realizing $\Gamma(\sigma)$. Then, every orbit $\mathcal{O}_b$ satisfying $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_q$ for some $q \in \mathcal{O}_b$ already belongs to σ.

Proof. Assume that $\mathcal{O}_b$ does not belong to σ. Then, $\mathcal{O}_b \notin \sigma$ and there exists $q \in \mathcal{O}_b$ such that $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_q$. This contradicts Lemma 4.1. Therefore, $\mathcal{O}_b$ must belong to σ. □

The following result gives a geometric characterization of facets through the common zero set of their stabilizer directions.

Theorem 4.3. Let σ be a facet of $\Delta(\mathcal{G}, \mathcal{A})$, and let $\mathfrak{c}_\sigma$ be the fixed common stabilizer Lie subalgebra realizing $\Gamma(\sigma)$. Define the following

$$\mathcal{Z}(\mathfrak{c}_\sigma) = \{x \in \mathcal{A} : X_{\mathcal{A}}(x) = 0 \text{ for all } X \in \mathfrak{c}_\sigma\}.$$

Then, every non-trivial orbit intersecting $\mathcal{Z}(\mathfrak{c}_\sigma)$ is a vertex of σ. Consequently, σ coincides with the family of all non-trivial orbit classes meeting $\mathcal{Z}(\mathfrak{c}_\sigma)$. Moreover, if $X_1, \dots, X_r$ is a basis of $\mathfrak{c}_\sigma$ and the smooth section $\Phi_\sigma \colon \mathcal{A} \to T\mathcal{A} \oplus \cdots \oplus T\mathcal{A}$ defined by $\Phi_\sigma(x) = (X_{1,\mathcal{A}}(x), \dots, X_{r,\mathcal{A}}(x))$ is transverse to the zero section, then $\mathcal{Z}(\mathfrak{c}_\sigma)$ is an embedded submanifold of $\mathcal{A}$, and for each $x \in \mathcal{Z}(\mathfrak{c}_\sigma)$, one has the following:

$$T_x\mathcal{Z}(\mathfrak{c}_\sigma) = \bigcap_{\ell=1}^{r} \ker(dX_{\ell,\mathcal{A}})_x.$$

Proof. Let $\sigma = \{O_{a_0}, \ldots, O_{a_k}\}$. Since $\mathfrak{c}_\sigma$ realizes $\Gamma(\sigma)$, there exist representatives $p_i \in O_{a_i}$ such that

$$\mathfrak{c}_\sigma = \bigcap_{i=0}^{k} \mathfrak{g}_{p_i}.$$

Hence, for every $X \in \mathfrak{c}_\sigma$ and every $i = 0, \ldots, k$, we have the following: $X \in \mathfrak{g}_{p_i}$. By the definition of the stabilizer Lie algebra, this is equivalent to $X_{\mathcal{A}}(p_i) = 0$. Thus $p_i \in \mathcal{Z}(\mathfrak{c}_\sigma)$ for all i. Therefore, every orbit appearing in σ intersects $\mathcal{Z}(\mathfrak{c}_\sigma)$.

Conversely, let O_b be a non-trivial orbit such that $O_b \cap \mathcal{Z}(\mathfrak{c}_\sigma) \neq \emptyset$. Choose $q \in O_b \cap \mathcal{Z}(\mathfrak{c}_\sigma)$. Then $X_{\mathcal{A}}(q) = 0$ for every $X \in \mathfrak{c}_\sigma$. Equivalently, $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_q$. By Corollary 4.2, every orbit which satisfies this condition must already belong to σ. Hence, $O_b \in \sigma$. This proves that σ is precisely the collection of all non-trivial orbits meeting $\mathcal{Z}(\mathfrak{c}_\sigma)$.

It remains to prove the smooth submanifold statement. Let $X_1, \ldots, X_r$ be a basis of $\mathfrak{c}_\sigma$. Each $X_{\ell,\mathcal{A}}$ is a smooth vector field on $\mathcal{A}$ because it is generated by the smooth action of the Lie group $\mathcal{G}$. Hence $\Phi_\sigma(x) = (X_{1,\mathcal{A}}(x), \ldots, X_{r,\mathcal{A}}(x))$ is a smooth section of the Whitney sum $T\mathcal{A} \oplus \cdots \oplus T\mathcal{A}$.

By construction, $x \in \mathcal{Z}(\mathfrak{c}_\sigma)$ if and only if $X_{\ell,\mathcal{A}}(x) = 0$ for every $\ell = 1, \ldots, r$. Therefore, $\mathcal{Z}(\mathfrak{c}_\sigma) = \Phi_\sigma^{-1}(0)$, where 0 denotes the zero section of $T\mathcal{A} \oplus \cdots \oplus T\mathcal{A}$.

If Φ_σ is transverse to the zero section, then the preimage theorem implies that $\mathcal{Z}(\mathfrak{c}_\sigma)$ is an embedded submanifold of $\mathcal{A}$. Moreover, the tangent space to this zero set at $x \in \mathcal{Z}(\mathfrak{c}_\sigma)$ is the kernel of the differential of Φ_σ at x. Since Φ_σ has components $X_{1,\mathcal{A}}, \ldots, X_{r,\mathcal{A}}$, we obtain the following:

$$T_x\mathcal{Z}(\mathfrak{c}_\sigma) = \ker(d\Phi_\sigma)_x = \bigcap_{\ell=1}^{r} \ker(dX_{\ell,\mathcal{A}})_x.$$

This completes the proof. □

The next result describes how two facets interact through the intersection of their common zero sets.

Theorem 4.4. Let σ and τ be two facets of $\Delta(\mathcal{G}, \mathcal{A})$ with fixed common stabilizer Lie subalgebras $\mathfrak{c}_\sigma$ and $\mathfrak{c}_\tau$ realizing $\Gamma(\sigma)$ and $\Gamma(\tau)$, respectively. Set $\mathfrak{c}_{\sigma,\tau} = \mathfrak{c}_\sigma + \mathfrak{c}_\tau$. Then,

$$\mathcal{Z}(\mathfrak{c}_{\sigma,\tau}) = \mathcal{Z}(\mathfrak{c}_\sigma) \cap \mathcal{Z}(\mathfrak{c}_\tau).$$

Moreover, every non-trivial orbit intersecting $\mathcal{Z}(\mathfrak{c}_{\sigma,\tau})$ belongs to $\sigma \cap \tau$. Consequently, if $\sigma \cap \tau = \emptyset$, then no non-trivial orbit intersects $\mathcal{Z}(\mathfrak{c}_{\sigma,\tau})$.

Proof. By definition, $\mathfrak{c}_{\sigma,\tau} = \mathfrak{c}_\sigma + \mathfrak{c}_\tau$ is the vector subspace of $\mathfrak{g}$ consisting of all elements of the form $X + Y$, where $X \in \mathfrak{c}_\sigma$ and $Y \in \mathfrak{c}_\tau$.

First, we prove the equality of zero sets. Let $x \in \mathcal{Z}(\mathfrak{c}_{\sigma,\tau})$. Then, by definition, $U_{\mathcal{A}}(x) = 0$ for every $U \in \mathfrak{c}_{\sigma,\tau}$. Since $\mathfrak{c}_\sigma \subseteq \mathfrak{c}_{\sigma,\tau}$ and $\mathfrak{c}_\tau \subseteq \mathfrak{c}_{\sigma,\tau}$, we have the following: $X_{\mathcal{A}}(x) = 0$ for every $X \in \mathfrak{c}_\sigma$ and $Y_{\mathcal{A}}(x) = 0$ for every $Y \in \mathfrak{c}_\tau$. Hence, $x \in \mathcal{Z}(\mathfrak{c}_\sigma)$ and $x \in \mathcal{Z}(\mathfrak{c}_\tau)$. Thus, $x \in \mathcal{Z}(\mathfrak{c}_\sigma) \cap \mathcal{Z}(\mathfrak{c}_\tau)$.

Conversely, let $x \in \mathcal{Z}(\mathfrak{c}_\sigma) \cap \mathcal{Z}(\mathfrak{c}_\tau)$. Then $X_{\mathcal{A}}(x) = 0$ for every $X \in \mathfrak{c}_\sigma$ and $Y_{\mathcal{A}}(x) = 0$ for every $Y \in \mathfrak{c}_\tau$. Take any $U \in \mathfrak{c}_{\sigma,\tau}$. Then $U = X + Y$ for some $X \in \mathfrak{c}_\sigma$ and $Y \in \mathfrak{c}_\tau$. Since the assignment $X \mapsto X_{\mathcal{A}}$ is linear, we have the following: $U_{\mathcal{A}} = X_{\mathcal{A}} + Y_{\mathcal{A}}$. Therefore,

$$U_{\mathcal{A}}(x) = X_{\mathcal{A}}(x) + Y_{\mathcal{A}}(x) = 0.$$

This holds for every $U \in \mathfrak{c}_{\sigma,\tau}$, so $x \in \mathcal{Z}(\mathfrak{c}_{\sigma,\tau})$. Hence,

$$\mathcal{Z}(\mathfrak{c}_{\sigma,\tau}) = \mathcal{Z}(\mathfrak{c}_\sigma) \cap \mathcal{Z}(\mathfrak{c}_\tau).$$

Now, let $\mathcal{O}_b$ be a non-trivial orbit intersecting $\mathcal{Z}(\mathfrak{c}_{\sigma,\tau})$. Choose $q \in \mathcal{O}_b \cap \mathcal{Z}(\mathfrak{c}_{\sigma,\tau})$. By the equality just proved, $q \in \mathcal{Z}(\mathfrak{c}_\sigma)$ and $q \in \mathcal{Z}(\mathfrak{c}_\tau)$. Thus, $X_{\mathcal{A}}(q) = 0$ for every $X \in \mathfrak{c}_\sigma$ and $Y_{\mathcal{A}}(q) = 0$ for every $Y \in \mathfrak{c}_\tau$.

Equivalently, $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_q$ and $\mathfrak{c}_\tau \subseteq \mathfrak{g}_q$. Since σ is a facet, Corollary 4.2 implies the following: $\mathcal{O}_b \in \sigma$. Similarly, since τ is a facet, Corollary 4.2 implies the following: $\mathcal{O}_b \in \tau$. Therefore $\mathcal{O}_b \in \sigma \cap \tau$.

Consequently, every non-trivial orbit intersecting $\mathcal{Z}(\mathfrak{c}_{\sigma,\tau})$ belongs to $\sigma \cap \tau$. If $\sigma \cap \tau = \emptyset$, then there is no non-trivial orbit belonging to both facets. Hence, no non-trivial orbit can intersect $\mathcal{Z}(\mathfrak{c}_{\sigma,\tau})$. □

The next result refines Theorem 4.4 by relating the intersection of two facets to quotient orbit differentials.

Theorem 4.5. Let σ and τ be two facets of $\Delta(\mathcal{G}, \mathcal{A})$ with fixed common stabilizer Lie subalgebras $\mathfrak{c}_\sigma$ and $\mathfrak{c}_\tau$ realizing $\Gamma(\sigma)$ and $\Gamma(\tau)$, respectively. Put $\eta = \sigma \cap \tau$ and $\mathfrak{c}_{\sigma,\tau} = \mathfrak{c}_\sigma + \mathfrak{c}_\tau$. If $\eta \neq \emptyset$, then η is a face of both σ and τ. Moreover, for every orbit $\mathcal{O}_a \in \eta$ and every point $p \in \mathcal{O}_a \cap \mathcal{Z}(\mathfrak{c}_{\sigma,\tau})$, the differential $(d\theta_p)_e \colon \mathfrak{g} \to T_p(\mathcal{A})$ factors through the quotient space $\mathfrak{g}/\mathfrak{c}_{\sigma,\tau}$. In particular,

$$\rho(\mathcal{O}_a) \leq \dim(\mathfrak{g}) - \dim(\mathfrak{c}_{\sigma,\tau}).$$

Proof. Since σ and τ are simplices of the simplicial complex $\Delta(\mathcal{G}, \mathcal{A})$, every nonempty subset of each of them is again a simplex. Hence, if $\eta = \sigma \cap \tau$ is nonempty, then $\eta \subseteq \sigma$ and $\eta \subseteq \tau$; therefore, η is a face of both σ and τ.

Now, let $\mathcal{O}_a \in \eta$. Since $\eta \subseteq \sigma$ and $\eta \subseteq \tau$, the orbit $\mathcal{O}_a$ belongs to both facets. Let $p \in \mathcal{O}_a \cap \mathcal{Z}(\mathfrak{c}_{\sigma,\tau})$. By definition of the zero set, for every $U \in \mathfrak{c}_{\sigma,\tau}$, one has $U_{\mathcal{A}}(p) = 0$. Since $\mathfrak{c}_{\sigma,\tau} = \mathfrak{c}_\sigma + \mathfrak{c}_\tau$, each $U \in \mathfrak{c}_{\sigma,\tau}$ can be written as $U = X + Y$ with $X \in \mathfrak{c}_\sigma$ and $Y \in \mathfrak{c}_\tau$.

The assignment $X \mapsto X_{\mathcal{A}}$ is linear because it is obtained by differentiating the smooth action of $\mathcal{G}$ on $\mathcal{A}$. Hence, $U_{\mathcal{A}} = X_{\mathcal{A}} + Y_{\mathcal{A}}$. Since $p \in \mathcal{Z}(\mathfrak{c}_{\sigma,\tau})$, we have the following: $U_{\mathcal{A}}(p) = 0$ for every such U. Equivalently, $U \in \mathfrak{g}_p$ for every $U \in \mathfrak{c}_{\sigma,\tau}$. Thus, $\mathfrak{c}_{\sigma,\tau} \subseteq \mathfrak{g}_p$.

Recall that $\mathfrak{g}_p = \ker((d\theta_p)_e)$. Therefore, $\mathfrak{c}_{\sigma,\tau} \subseteq \ker((d\theta_p)_e)$. Let $\pi_{\sigma,\tau} \colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{c}_{\sigma,\tau}$ be the canonical projection. Since the subspace $\mathfrak{c}_{\sigma,\tau}$ is contained in the kernel of $(d\theta_p)_e$, the universal property of quotient spaces gives a unique linear map

$$\overline{d\theta}_p : \mathfrak{g}/\mathfrak{c}_{\sigma,\tau} \to T_p(\mathcal{A})$$

such that

$$(d\theta_p)_e = \overline{d\theta}_p \circ \pi_{\sigma,\tau}.$$

Consequently, the tangent space to the orbit at p satisfies

$$T_p(\mathcal{O}_a) = \mathrm{Im}((d\theta_p)_e) = \mathrm{Im}(\overline{d\theta}_p).$$

Taking dimensions gives the following:

$$\rho(O_a) = \dim T_p(O_a) \leq \dim(\mathfrak{g}/\mathfrak{c}_{\sigma,\tau}).$$

Since

$$\dim(\mathfrak{g}/\mathfrak{c}_{\sigma,\tau}) = \dim(\mathfrak{g}) - \dim(\mathfrak{c}_{\sigma,\tau}),$$

we obtain the following:

$$\rho(O_a) \leq \dim(\mathfrak{g}) - \dim(\mathfrak{c}_{\sigma,\tau}).$$

This completes the proof. □

The next result gives a geometric obstruction to the link of a facet. For a simplex $\sigma \in \Delta(\mathcal{G}, \mathcal{A})$, the link of σ is the family

$$\mathrm{Lk}_\Delta(\sigma) = \{\tau \in \Delta(\mathcal{G}, \mathcal{A}) : \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta(\mathcal{G}, \mathcal{A})\}.$$

Theorem 4.6. Let σ be a facet of $\Delta(\mathcal{G}, \mathcal{A})$, and let $\mathfrak{c}_\sigma$ be the fixed common stabilizer Lie subalgebra realizing $\Gamma(\sigma)$. Then $\mathrm{Lk}_\Delta(\sigma) = \emptyset$. Moreover, for every non-trivial orbit $O_b \notin \sigma$ and every $q \in O_b$, there exists $X \in \mathfrak{c}_\sigma$ such that $(d\theta_q)_e(X) \neq 0$. Equivalently, $X_{\mathcal{A}}(q) \neq 0$.

Proof. Since σ is a facet, there exists no simplex $\mu \in \Delta(\mathcal{G}, \mathcal{A})$ such that $\sigma \subsetneq \mu$. Assume that $\mathrm{Lk}_\Delta(\sigma)$ is not empty. Then, there exists a simplex $\tau \in \mathrm{Lk}_\Delta(\sigma)$. By the definition of the link, we have the following: $\tau \cap \sigma = \emptyset$ and $\tau \cup \sigma \in \Delta(\mathcal{G}, \mathcal{A})$. Since τ is nonempty and disjoint from σ, we get $\sigma \subsetneq \sigma \cup \tau$. This contradicts the fact that σ is a facet. Hence, $\mathrm{Lk}_\Delta(\sigma) = \emptyset$.

Now, let $O_b \notin \sigma$ be a non-trivial orbit, and let $q \in O_b$. Contrarily, suppose that $(d\theta_q)_e(X) = 0$ for every $X \in \mathfrak{c}_\sigma$. Since $(d\theta_q)_e(X) = X_{\mathcal{A}}(q)$, this means that $X_{\mathcal{A}}(q) = 0$ for every $X \in \mathfrak{c}_\sigma$. Therefore, $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_q$. However, this contradicts Lemma 4.1, because $O_b \notin \sigma$. Thus, there must exist $X \in \mathfrak{c}_\sigma$ such that $(d\theta_q)_e(X) \neq 0$. Equivalently, $X_{\mathcal{A}}(q) \neq 0$. The proof is complete. □

The previous theorem shows that every orbit outside a facet is detected by a non-vanishing infinitesimal direction. This observation can be reformulated through a smooth evaluation map associated with the common stabilizer directions.

For a facet $\sigma \in \Delta(\mathcal{G}, \mathcal{A})$ with a fixed common stabilizer Lie subalgebra $\mathfrak{c}_\sigma$ realizing $\Gamma(\sigma)$, define the smooth map $\Psi_\sigma \colon \mathcal{A} \to \mathrm{Hom}(\mathfrak{c}_\sigma, T\mathcal{A})$ by $\Psi_\sigma(x)(X) = X_{\mathcal{A}}(x)$ for every $x \in \mathcal{A}$ and $X \in \mathfrak{c}_\sigma$.

Theorem 4.7. Let σ be a facet of $\Delta(\mathcal{G}, \mathcal{A})$, and let $\mathfrak{c}_\sigma$ be the fixed common stabilizer Lie subalgebra realizing $\Gamma(\sigma)$. Then $\mathcal{Z}(\mathfrak{c}_\sigma) = \Psi_\sigma^{-1}(0)$. Moreover, if $O_b \notin \sigma$ is a non-trivial orbit, then $\Psi_\sigma(q) \neq 0$ for every $q \in O_b$. Equivalently, $\Psi_\sigma|_{O_b}$ is nowhere zero. Furthermore, for every $q \in O_b$, the restriction $(d\theta_q)_e|_{\mathfrak{c}_\sigma} \colon \mathfrak{c}_\sigma \longrightarrow T_q(\mathcal{A})$ has a positive rank.

Proof. Since the action of $\mathcal{G}$ on $\mathcal{A}$ is smooth, every element $X \in \mathfrak{g}$ induces a smooth fundamental vector field $X_{\mathcal{A}}$ on $\mathcal{A}$. Consequently, every element of the finite-dimensional subspace $\mathfrak{c}_\sigma$ determines a smooth vector field. Therefore, the assignment $x \mapsto \Psi_\sigma(x)$ is a smooth map from $\mathcal{A}$ into the vector bundle $\mathrm{Hom}(\mathfrak{c}_\sigma, T\mathcal{A})$. Let $x \in \mathcal{A}$. By definition, $x \in \mathcal{Z}(\mathfrak{c}_\sigma)$ if and only if $X_{\mathcal{A}}(x) = 0$ for every $X \in \mathfrak{c}_\sigma$.

On the other hand, $\Psi_\sigma(x) = 0$ means that the linear transformation $\Psi_\sigma(x) \colon \mathfrak{c}_\sigma \longrightarrow T_x(\mathcal{A})$ is identically zero. Since $\Psi_\sigma(x)(X) = X_{\mathcal{A}}(x)$, the two conditions are equivalent. Hence

$$\mathcal{Z}(\mathfrak{c}_\sigma) = \Psi_\sigma^{-1}(0).$$

Now, let O_b be a non-trivial orbit outside σ and choose $q \in O_b$. Assume that $\Psi_\sigma(q) = 0$. Then $X_{\mathcal{A}}(q) = 0$ for every $X \in \mathfrak{c}_\sigma$. Therefore $\mathfrak{c}_\sigma \subseteq \mathfrak{g}_q$. By Lemma 4.1, this is impossible because O_b does not belong to σ. Hence, $\Psi_\sigma(q) \neq 0$ for every $q \in O_b$. Consequently, the restriction $\Psi_\sigma|_{O_b}$ never vanishes. Finally, recall that the orbit map at q is as follows:

$$\theta_q : \mathcal{G} \longrightarrow \mathcal{A}, \;\; \theta_q(g) = g \cdot q.$$

Its differential at the identity satisfies $(d\theta_q)_e(X) = X_{\mathcal{A}}(q)$ for every $X \in \mathfrak{g}$. Restricting this identity to $\mathfrak{c}_\sigma$ gives the following:

$$(d\theta_q)_e|_{\mathfrak{c}_\sigma}(X) = \Psi_\sigma(q)(X).$$

Since $\Psi_\sigma(q) \neq 0$, there exists $X \in \mathfrak{c}_\sigma$ such that

$$(d\theta_q)_e(X) = \Psi_\sigma(q)(X) \neq 0.$$

Therefore, the restriction $(d\theta_q)_e|_{\mathfrak{c}_\sigma}$ cannot be the zero transformation. Since a linear transformation is nonzero if and only if its image has positive dimension, we conclude that the rank $\left((d\theta_q)_e|_{\mathfrak{c}_\sigma}\right) > 0$. □

5. Topological and combinatorial invariants

The purpose of this section is to investigate topological and combinatorial invariants associated with the orbit-rank simplicial complex. These invariants provide quantitative information describing the global organization of orbit classes arising from smooth Lie group actions. We establish several properties relating the simplicial structure to the geometry of the underlying manifold through orbit-map differentials, stabilizer Lie algebras, and common infinitesimal symmetries. The results obtained in this section complement the structural theory developed in the previous sections and provide computable invariants for orbit-rank simplicial complexes.

The first objective is to show that the common stabilizer dimension naturally induces a nested family of simplicial subcomplexes of the orbit-rank simplicial complex.

For every integer $r \geq 1$, define the following:

$$\Delta_r(\mathcal{G}, \mathcal{A}) = \{\sigma \in \Delta(\mathcal{G}, \mathcal{A}) : \Gamma(\sigma) \geq r\}.$$

Theorem 5.1. For every integer $r \geq 1$, the family $\Delta_r(\mathcal{G}, \mathcal{A})$ is a simplicial subcomplex of $\Delta(\mathcal{G}, \mathcal{A})$. Moreover,

$$\Delta_1(\mathcal{G}, \mathcal{A}) \supseteq \Delta_2(\mathcal{G}, \mathcal{A}) \supseteq \cdots \supseteq \Delta_{\dim(\mathfrak{g})}(\mathcal{G}, \mathcal{A}).$$

Proof. Fix an integer $r \geq 1$. Let $\sigma \in \Delta_r(\mathcal{G}, \mathcal{A})$. Then, $\sigma \in \Delta(\mathcal{G}, \mathcal{A})$ and $\Gamma(\sigma) \geq r$. Write $\sigma = \{O_{a_0}, \ldots, O_{a_k}\}$. By the definition of $\Gamma(\sigma)$, there exist representatives $p_i \in O_{a_i}$ such that

$$\dim\Big(\bigcap_{i=0}^{k} \mathfrak{g}_{p_i}\Big) = \Gamma(\sigma).$$

Let $\tau \subseteq \sigma$ be a nonempty face. Then, τ has the form $\tau = \{O_{a_{i_0}}, \ldots, O_{a_{i_s}}\}$. Since $\Delta(\mathcal{G}, \mathcal{A})$ is a simplicial complex, $\tau \in \Delta(\mathcal{G}, \mathcal{A})$.

Now, removing stabilizer Lie algebras from an intersection can only enlarge the resulting vector subspace. Hence

$$\bigcap_{i=0}^{k} \mathfrak{g}_{p_i} \subseteq \bigcap_{j=0}^{s} \mathfrak{g}_{p_{i_j}}.$$

Taking dimensions gives the following:

$$\dim\Big(\bigcap_{j=0}^{s} \mathfrak{g}_{p_{i_j}}\Big) \geq \dim\Big(\bigcap_{i=0}^{k} \mathfrak{g}_{p_i}\Big) = \Gamma(\sigma) \geq r.$$

Since $\Gamma(\tau)$ is the maximum of such dimensions over all representatives of the orbits in τ, we obtain the following: $\Gamma(\tau) \geq r$. Thus, $\tau \in \Delta_r(\mathcal{G}, \mathcal{A})$. Therefore, $\Delta_r(\mathcal{G}, \mathcal{A})$ is closed under taking faces; hence, it is a simplicial subcomplex of $\Delta(\mathcal{G}, \mathcal{A})$.

It remains to prove the nested property. Let $r_1 \leq r_2$. If $\sigma \in \Delta_{r_2}(\mathcal{G}, \mathcal{A})$, then $\Gamma(\sigma) \geq r_2$. Since $r_2 \geq r_1$, we have the following: $\Gamma(\sigma) \geq r_1$. Hence, $\sigma \in \Delta_{r_1}(\mathcal{G}, \mathcal{A})$. Therefore, $\Delta_{r_2}(\mathcal{G}, \mathcal{A}) \subseteq \Delta_{r_1}(\mathcal{G}, \mathcal{A})$. Consequently,

$$\Delta_1(\mathcal{G}, \mathcal{A}) \supseteq \Delta_2(\mathcal{G}, \mathcal{A}) \supseteq \cdots \supseteq \Delta_{\dim(\mathfrak{g})}(\mathcal{G}, \mathcal{A}).$$

This completes the proof. □

The filtration obtained in Theorem 5.1 allows us to measure how simplices disappear when stronger common stabilizer directions are required. The following result gives a differential characterization of this loss.

Theorem 5.2. Let $r \geq 1$, and let $\sigma = \{\mathcal{O}_{a_0}, \ldots, \mathcal{O}_{a_k}\} \in \Delta_r(\mathcal{G}, \mathcal{A})$. Then, $\sigma \notin \Delta_{r+1}(\mathcal{G}, \mathcal{A})$ if and only if $\Gamma(\sigma) = r$. Equivalently, the maximum possible dimension of the common kernel $\bigcap_{i=0}^{k} \ker((d\theta_{p_i})_e)$ over all choices of representatives $p_i \in \mathcal{O}_{a_i}$ is equal to r. In this case, if $\mathfrak{c}_\sigma$ is the fixed common stabilizer Lie subalgebra realizing $\Gamma(\sigma) = r$, then the orbit-map differentials associated with σ factor through $\mathfrak{g}/\mathfrak{c}_\sigma$, but no larger subspace $\mathfrak{d} \subseteq \mathfrak{g}$ with $\dim(\mathfrak{d}) > r$ can be simultaneously annihilated by all these differentials.

Proof. Let $\sigma = \{\mathcal{O}_{a_0}, \ldots, \mathcal{O}_{a_k}\} \in \Delta_r(\mathcal{G}, \mathcal{A})$. By the definition of $\Delta_r(\mathcal{G}, \mathcal{A})$, we have the following: $\Gamma(\sigma) \geq r$. Since $\Gamma(\sigma)$ is defined by

$$\Gamma(\sigma) = \max_{p_i \in \mathcal{O}_{a_i}} \dim\left(\bigcap_{i=0}^{k} \mathfrak{g}_{p_i}\right),$$

and since $\mathfrak{g}_{p_i} = \ker((d\theta_{p_i})_e)$, we may rewrite this quantity as

$$\Gamma(\sigma) = \max_{p_i \in \mathcal{O}_{a_i}} \dim\left(\bigcap_{i=0}^{k} \ker((d\theta_{p_i})_e)\right).$$

First, assume that $\sigma \notin \Delta_{r+1}(\mathcal{G}, \mathcal{A})$. Then, by definition of the filtration level Δ_{r+1}, we must have $\Gamma(\sigma) < r + 1$. On the other hand, $\sigma \in \Delta_r$ gives the following: $\Gamma(\sigma) \geq r$. Since $\Gamma(\sigma)$ is an integer, these two inequalities imply $\Gamma(\sigma) = r$.

For every choice of representatives $p_i \in \mathcal{O}_{a_i}$, one has the following:

$$\dim\left(\bigcap_{i=0}^{k} \ker((d\theta_{p_i})_e)\right) \leq \Gamma(\sigma) = r.$$

Moreover, since $\Gamma(\sigma) = r$ is the largest possible common kernel dimension, there exist representatives $p_i^* \in \mathcal{O}_{a_i}$ for which

$$\dim\left(\bigcap_{i=0}^{k} \ker((d\theta_{p_i^*})_e)\right) = r.$$

Thus, the simplex appears at level r but does not survive to level $r+1$. More precisely, the maximal simultaneous infinitesimal vanishing space associated with σ has a dimension exactly r.

Conversely, suppose that the largest possible common kernel dimension of the orbit-map differentials associated with σ is exactly r, that is, $\Gamma(\sigma) = r$. Then, $\Gamma(\sigma) \geq r$, so $\sigma \in \Delta_r(\mathcal{G}, \mathcal{A})$. However, $\Gamma(\sigma) = r < r+1$, so $\sigma \notin \Delta_{r+1}(\mathcal{G}, \mathcal{A})$. Therefore, $\sigma \in \Delta_r(\mathcal{G}, \mathcal{A}) \setminus \Delta_{r+1}(\mathcal{G}, \mathcal{A})$.

It remains to prove the final assertion. Let $\mathfrak{c}_\sigma$ be the fixed common stabilizer Lie subalgebra realizing $\Gamma(\sigma) = r$. Then, there exist representatives $p_i \in \mathcal{O}_{a_i}$ such that

$$\mathfrak{c}_\sigma = \bigcap_{i=0}^{k} \mathfrak{g}_{p_i} = \bigcap_{i=0}^{k} \ker((d\theta_{p_i})_e).$$

In particular, $\mathfrak{c}_\sigma \subseteq \ker((d\theta_{p_i})_e)$ for each i. Therefore, as in Theorem 3.8, each differential $(d\theta_{p_i})_e\colon \mathfrak{g} \to T_{p_i}(\mathcal{A})$ factors through the quotient $\mathfrak{g}/\mathfrak{c}_\sigma$. Indeed, if $\pi_\sigma\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{c}_\sigma$ denotes the quotient projection, then there exists a linear map $\overline{d\theta}_{p_i}\colon \mathfrak{g}/\mathfrak{c}_\sigma \to T_{p_i}(\mathcal{A})$ such that

$$(d\theta_{p_i})_e = \overline{d\theta}_{p_i} \circ \pi_\sigma.$$

Now, suppose that there is a subspace $\mathfrak{d} \subseteq \mathfrak{g}$ with $\dim(\mathfrak{d}) > r$ such that $\mathfrak{d} \subseteq \ker((d\theta_{p_i})_e)$ for every $i = 0, \ldots, k$ for some choice of representatives. Then $\mathfrak{d}$ would be contained in $\bigcap_{i=0}^{k} \ker((d\theta_{p_i})_e)$, and so

$$\dim\left(\bigcap_{i=0}^{k} \ker((d\theta_{p_i})_e)\right) \geq \dim(\mathfrak{d}) > r.$$

This would imply $\Gamma(\sigma) > r$, which contradicts $\Gamma(\sigma) = r$. Therefore, no such larger common annihilated subspace exists.

Thus, the layer $\Delta_r \setminus \Delta_{r+1}$ is precisely the collection of simplices whose maximal simultaneous kernel of orbit-map differentials has a dimension of exactly r. □

The previous result identifies the layer $\Delta_r(\mathcal{G}, \mathcal{A}) \setminus \Delta_{r+1}(\mathcal{G}, \mathcal{A})$ as the family of simplices whose maximal common kernel dimension is exactly r. This motivates the following invariant.

For each $j \geq 0$ and $r \geq 1$, let $f_j^{[r]}(\Delta)$ denote the number of j-dimensional simplices $\sigma \in \Delta(\mathcal{G}, \mathcal{A})$ satisfying $\Gamma(\sigma) = r$.

Theorem 5.3. For each $j \geq 0$ and $r \geq 1$, the number $f_j^{[r]}(\Delta)$ coincides with the number of j-dimensional simplices contained in $\Delta_r(\mathcal{G}, \mathcal{A})$ but not contained in $\Delta_{r+1}(\mathcal{G}, \mathcal{A})$. In particular, if $f_j(\Delta)$ denotes the total number of j-dimensional simplices of $\Delta(\mathcal{G}, \mathcal{A})$, then

$$f_j(\Delta) = \sum_{r \geq 1} f_j^{[r]}(\Delta).$$

Proof. Fix $j \geq 0$ and $r \geq 1$. Let σ be a j-dimensional simplex of $\Delta(\mathcal{G}, \mathcal{A})$. Then σ has the form $\sigma = \{O_{a_0}, \ldots, O_{a_j}\}$. By the definition of $\Gamma(\sigma)$, we have the following:

$$\Gamma(\sigma) = \max_{p_i \in O_{a_i}} \dim \left(\bigcap_{i=0}^{j} \mathfrak{g}_{p_i} \right).$$

Using the identity $\mathfrak{g}_{p_i} = \ker((d\theta_{p_i})_e)$, this can be rewritten as follows:

$$\Gamma(\sigma) = \max_{p_i \in O_{a_i}} \dim \left(\bigcap_{i=0}^{j} \ker((d\theta_{p_i})_e) \right).$$

Thus, $\Gamma(\sigma)$ measures the largest dimension of a subspace of $\mathfrak{g}$ simultaneously annihilated by all orbit-map differentials attached to representatives of the orbit vertices of σ.

Now, by the definition of the filtration, $\sigma \in \Delta_r(\mathcal{G}, \mathcal{A})$ if and only if $\Gamma(\sigma) \geq r$. Similarly, $\sigma \in \Delta_{r+1}(\mathcal{G}, \mathcal{A})$ if and only if $\Gamma(\sigma) \geq r+1$. Therefore, $\sigma \in \Delta_r(\mathcal{G}, \mathcal{A}) \setminus \Delta_{r+1}(\mathcal{G}, \mathcal{A})$ if and only if $\Gamma(\sigma) \geq r$ and $\Gamma(\sigma) < r+1$. Since $\Gamma(\sigma)$ is an integer, this is equivalent to $\Gamma(\sigma) = r$. Hence, the j -dimensional simplices counted by $f_j^{[r]}(\Delta)$ are exactly the j-dimensional simplices lying in the layer $\Delta_r(\mathcal{G}, \mathcal{A}) \setminus \Delta_{r+1}(\mathcal{G}, \mathcal{A})$.

It remains to prove the decomposition of $f_j(\Delta)$. Every j-dimensional simplex σ of $\Delta(\mathcal{G}, \mathcal{A})$ has $\Gamma(\sigma) > 0$ by the definition of orbit-rank simplices. Moreover, $\Gamma(\sigma)$ is a positive integer not exceeding $\dim(\mathfrak{g})$. Indeed, for every choice of representatives, $\bigcap_i \mathfrak{g}_{p_i}$ is a vector subspace of $\mathfrak{g}$, and hence, its dimension is at most $\dim(\mathfrak{g})$. Thus, each j-dimensional simplex belongs to exactly one layer $\Delta_r \setminus \Delta_{r+1}$, namely the layer indexed by $r = \Gamma(\sigma)$.

Consequently, the collection of all j-dimensional simplices of $\Delta(\mathcal{G}, \mathcal{A})$ is the disjoint union of the collections

$$\{\sigma \in \Delta(\mathcal{G}, \mathcal{A}) : \dim(\sigma) = j,\ \Gamma(\sigma) = r\},\ \ r \geq 1.$$

Counting the elements in this disjoint union gives the following:

$$f_j(\Delta) = \sum_{r \geq 1} f_j^{[r]}(\Delta).$$

This completes the proof. □

The decomposition in Theorem 5.3 allows the Euler characteristic to be refined according to the common kernel dimensions of the orbit-map differentials.

For each $r \geq 1$, define the r-th differential Euler contribution of $\Delta(\mathcal{G}, \mathcal{A})$ by the following:

$$\chi^{[r]}(\Delta) = \sum_{j \geq 0} (-1)^j f_j^{[r]}(\Delta).$$

Theorem 5.4. Assume that $\Delta(\mathcal{G}, \mathcal{A})$ is finite. Then, the Euler characteristic of the orbit-rank simplicial complex admits the decomposition

$$\chi(\Delta) = \sum_{r=1}^{\dim(\mathfrak{g})} \chi^{[r]}(\Delta).$$

Proof. Since $\Delta(\mathcal{G}, \mathcal{A})$ is finite, there are only finitely many simplices in each dimension and only finitely many nonzero face numbers. Let $d = \dim(\Delta)$. Therefore, the Euler characteristic is given by the following:

$$\chi(\Delta) = \sum_{j=0}^{d} (-1)^j f_j(\Delta).$$

By Theorem 5.3, for every $j = 0, \ldots, d$, the number $f_j(\Delta)$ decomposes as

$$f_j(\Delta) = \sum_{r=1}^{\dim(\mathfrak{g})} f_j^{[r]}(\Delta).$$

Indeed, each j-simplex σ has a well-defined integer $\Gamma(\sigma)$ satisfying $1 \leq \Gamma(\sigma) \leq \dim(\mathfrak{g})$. The lower bound follows from the defining condition of an orbit-rank simplex, while the upper bound follows because every common stabilizer intersection is a subspace of $\mathfrak{g}$.

Substituting the decomposition of $f_j(\Delta)$ into the Euler characteristic gives the following:

$$\chi(\Delta) = \sum_{j=0}^{d} (-1)^j \sum_{r=1}^{\dim(\mathfrak{g})} f_j^{[r]}(\Delta).$$

Since all sums are finite, we may interchange the order of summation. Hence,

$$\chi(\Delta) = \sum_{r=1}^{\dim(\mathfrak{g})} \sum_{j=0}^{d} (-1)^j f_j^{[r]}(\Delta).$$

By the definition of $\chi^{[r]}(\Delta)$, this becomes the following:

$$\chi(\Delta) = \sum_{r=1}^{\dim(\mathfrak{g})} \chi^{[r]}(\Delta).$$

It remains to explain why this decomposition is differential in nature and not merely combinatorial. Let $\sigma = \{O_{a_0}, \ldots, O_{a_j}\}$ be a j-simplex. Then, σ contributes to $f_j^{[r]}(\Delta)$ if and only if $\Gamma(\sigma) = r$.

By definition,

$$\Gamma(\sigma) = \max_{p_i \in O_{a_i}} \dim \left(\bigcap_{i=0}^{j} \mathfrak{g}_{p_i} \right).$$

Since $\mathfrak{g}_{p_i} = \ker((d\theta_{p_i})_e)$, we have the following:

$$\Gamma(\sigma) = \max_{p_i \in O_{a_i}} \dim \left(\bigcap_{i=0}^{j} \ker((d\theta_{p_i})_e) \right).$$

Thus, the layer $\Gamma(\sigma) = r$ is determined by the largest possible dimension of the common kernel of the orbit-map differentials associated with suitable representatives of the orbit vertices of σ.

Therefore $\chi^{[r]}(\Delta)$ is precisely the alternating contribution of all simplices whose representing orbit-map differentials possess a maximal common kernel dimension r. This proves the stated decomposition and its geometric interpretation. $\square$

The preceding decomposition can be localized along the filtration introduced in Theorem 5.1. The next result shows that the change of Euler characteristic between two consecutive filtration levels is exactly the contribution of simplices whose common differential kernel has the corresponding dimension.

Theorem 5.5. Assume that $\Delta(\mathcal{G}, \mathcal{A})$ is finite. For every $r \geq 1$,

$$\chi(\Delta_r) - \chi(\Delta_{r+1}) = \chi^{[r]}(\Delta).$$

Equivalently, the loss of the Euler characteristic from Δ_r to Δ_{r+1} is precisely determined by those orbit-rank simplices σ satisfying $\Gamma(\sigma) = r$.

Proof. By Theorem 5.1, each $\Delta_r(\mathcal{G}, \mathcal{A})$ is a simplicial subcomplex of $\Delta(\mathcal{G}, \mathcal{A})$, and the sequence $\Delta_1 \supseteq \Delta_2 \supseteq \ldots$ is decreasing. Hence, Δ_{r+1} is a subcomplex of Δ_r.

For each $j \geq 0$, let $f_j(\Delta_r)$ denote the number of j-dimensional simplices in Δ_r. By the definition of Δ_r, a simplex σ belongs to Δ_r if and only if $\Gamma(\sigma) \geq r$. Similarly, σ belongs to Δ_{r+1} if and only if $\Gamma(\sigma) \geq r + 1$. Therefore, the simplices contained in Δ_r but not in Δ_{r+1} are exactly those satisfying $\Gamma(\sigma) = r$.

This is precisely the layer described in Theorem 5.7. In differential terms, such a simplex is one for which the largest possible common kernel $\bigcap_i \ker((d\theta_{p_i})_e)$ has a dimension of exactly r. Thus, at the level of j-simplices, Theorem 5.3 gives the following:

$$f_j(\Delta_r) - f_j(\Delta_{r+1}) = f_j^{[r]}(\Delta).$$

Since the complex is finite, Euler characteristics are finite alternating sums. Hence,

$$\chi(\Delta_r) = \sum_{j \geq 0} (-1)^j f_j(\Delta_r) \quad \text{and} \quad \chi(\Delta_{r+1}) = \sum_{j \geq 0} (-1)^j f_j(\Delta_{r+1}).$$

Subtracting these two equalities gives the following:

$$\chi(\Delta_r) - \chi(\Delta_{r+1}) = \sum_{j \geq 0} (-1)^j (f_j(\Delta_r) - f_j(\Delta_{r+1})).$$

Using

$$f_j(\Delta_r) - f_j(\Delta_{r+1}) = f_j^{[r]}(\Delta),$$

we obtain the following:

$$\chi(\Delta_r) - \chi(\Delta_{r+1}) = \sum_{j \geq 0} (-1)^j f_j^{[r]}(\Delta) = \chi^{[r]}(\Delta).$$

This also agrees with the global decomposition in Theorem 5.4, since summing the above identity over all r recovers the following:

$$\chi(\Delta) = \sum_r \chi^{[r]}(\Delta).$$

Finally, by Theorem 3.8, whenever $\Gamma(\sigma) = r$, the orbit-map differentials associated with representatives of the vertices of σ factor through the quotient $\mathfrak{g}/\mathfrak{c}_\sigma$ for the fixed r-dimensional common stabilizer Lie subalgebra $\mathfrak{c}_\sigma$ realizing $\Gamma(\sigma)$. Thus, the difference $\chi(\Delta_r) - \chi(\Delta_{r+1})$ does not merely count abstract simplices; it measures the alternating contribution of those simplices whose common infinitesimal stabilizer kernel has exactly dimension r. □

For $r \geq 1$, two vertices O_a and O_b of $\Delta_r(\mathcal{G}, \mathcal{A})$ are said to be r-connected if there exists a finite sequence of vertices $O_{a_0}, \ldots, O_{a_m}$ such that $O_{a_0} = O_a$ and $O_{a_m} = O_b$, and for every $\ell = 0, \ldots, m-1$ there exists a simplex $\sigma_\ell \in \Delta_r(\mathcal{G}, \mathcal{A})$ containing both O_{a_ℓ} and $O_{a_{\ell+1}}$.

Theorem 5.6. Let $r \geq 1$. The connected components of $\Delta_r(\mathcal{G}, \mathcal{A})$ are precisely the r-connected classes of orbit vertices. Moreover, if O_a and O_b lie in the same connected component of $\Delta_r(\mathcal{G}, \mathcal{A})$, then there exist a finite chain of orbit vertices $O_{a_0}, \ldots, O_{a_m}$ and, for each ℓ, representatives $p_\ell \in O_{a_\ell}$, $p_{\ell+1} \in O_{a_{\ell+1}}$, together with an r-dimensional Lie subalgebra $\mathfrak{c}_\ell \subseteq \mathfrak{g}$, such that

$$\mathfrak{c}_\ell \subseteq \ker((d\theta_{p_\ell})_e) \cap \ker((d\theta_{p_{\ell+1}})_e).$$

Consequently, the orbit-map differentials along each link of the chain factor through the quotient $\mathfrak{g}/\mathfrak{c}_\ell$. In particular, every orbit vertex in a connected component of $\Delta_r(\mathcal{G}, \mathcal{A})$ satisfies $\rho(O_a) \leq \dim(\mathfrak{g}) - r$.

Proof. By Theorem 5.1, $\Delta_r(\mathcal{G}, \mathcal{A})$ is a simplicial subcomplex of $\Delta(\mathcal{G}, \mathcal{A})$. Hence, its connected components are determined by its 1-skeleton. Thus, two vertices lie in the same connected component if and only if they can be joined by a finite edge path in the 1-skeleton of $\Delta_r(\mathcal{G}, \mathcal{A})$.

First, assume that O_a and O_b lie in the same connected component of $\Delta_r(\mathcal{G}, \mathcal{A})$. Then, there exists a finite edge path $O_{a_0}, \ldots, O_{a_m}$ with $O_{a_0} = O_a$ and $O_{a_m} = O_b$. Since each edge is a 1-simplex of $\Delta_r(\mathcal{G}, \mathcal{A})$, for every ℓ the pair $\{O_{a_\ell}, O_{a_{\ell+1}}\}$ is contained in some simplex $\sigma_\ell \in \Delta_r(\mathcal{G}, \mathcal{A})$. Hence, the vertices are r-connected.

Now, fix ℓ. Since $\sigma_\ell \in \Delta_r(\mathcal{G}, \mathcal{A})$, the definition of $\Delta_r(\mathcal{G}, \mathcal{A})$ implies that $\Gamma(\sigma_\ell) \geq r$. Hence, there exist representatives of the orbit vertices of σ_ℓ whose common stabilizer intersection has dimension at least r. In particular, for the two vertices O_{a_ℓ} and $O_{a_{\ell+1}}$, we may choose representatives $p_\ell \in O_{a_\ell}$ and $p_{\ell+1} \in O_{a_{\ell+1}}$ such that $\mathfrak{g}_{p_\ell} \cap \mathfrak{g}_{p_{\ell+1}}$ contains a fixed r-dimensional Lie subalgebra $\mathfrak{c}_\ell$.

Since

$$\mathfrak{g}_{p_\ell} = \ker((d\theta_{p_\ell})_e) \quad \text{and} \quad \mathfrak{g}_{p_{\ell+1}} = \ker((d\theta_{p_{\ell+1}})_e),$$

we obtain the following:

$$\mathfrak{c}_\ell \subseteq \ker((d\theta_{p_\ell})_e) \cap \ker((d\theta_{p_{\ell+1}})_e).$$

Thus, every $X \in \mathfrak{c}_\ell$ satisfies the following:

$$(d\theta_{p_\ell})_e(X) = 0 \quad \text{and} \quad (d\theta_{p_{\ell+1}})_e(X) = 0.$$

Using the identity $(d\theta_p)_e(X) = X_{\mathcal{A}}(p)$, this means that the fundamental vector field $X_{\mathcal{A}}$ simultaneously vanishes at p_ℓ and $p_{\ell+1}$.

Let π_ℓ: $\mathfrak{g} \to \mathfrak{g}/\mathfrak{c}_\ell$ be the quotient projection. Since $\mathfrak{c}_\ell$ is contained in the kernel of both orbit-map differentials, the universal property of quotient spaces gives linear maps $\overline{d\theta}_{p_\ell}$ and $\overline{d\theta}_{p_{\ell+1}}$ such that

$$(d\theta_{p_\ell})_e = \overline{d\theta}_{p_\ell} \circ \pi_\ell \quad \text{and} \quad (d\theta_{p_{\ell+1}})_e = \overline{d\theta}_{p_{\ell+1}} \circ \pi_\ell.$$

Hence, the infinitesimal orbit geometries along each link of the chain are controlled by the quotient $\mathfrak{g}/\mathfrak{c}_\ell$.

Conversely, suppose that two vertices are connected by such a finite chain. Since each consecutive pair admits an r-dimensional common stabilizer direction, each pair belongs to $\Delta_r(\mathcal{G}, \mathcal{A})$. Thus, the chain gives an edge path in the 1-skeleton of $\Delta_r(\mathcal{G}, \mathcal{A})$, so the two vertices lie in the same connected component.

Finally, let O_a be any vertex in a connected component of $\Delta_r(\mathcal{G}, \mathcal{A})$. Since $\{O_a\} \in \Delta_r(\mathcal{G}, \mathcal{A})$, the defining condition gives the following: $\Gamma(\{O_a\}) \geq r$. By the rank-nullity formula for the orbit map,

$$\rho(O_a) = \dim(\mathfrak{g}) - \dim(\mathfrak{g}_a).$$

Since the stabilizer dimension is at least r, we obtain the following: $\rho(O_a) \leq \dim(\mathfrak{g}) - r$. This is consistent with the rank estimate established in Theorem 3.8. □

Example 5.7. Let $\mathcal{G} = \mathbb{T}^4 = (S^1)^4$ with Lie algebra $\mathfrak{g} = \mathbb{R}^4$ and basis e_1–e_4. Consider the smooth manifold $\mathcal{A} = \{1, 2, 3, 4, 5, 6\} \times \mathbb{T}^2$. For $g = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathcal{G}$ and $u = (u_1, u_2) \in \mathbb{T}^2$, define the following: $g \cdot (i, u) = (i, \beta_i(g)u)$, where

$$\begin{aligned} &\beta_1(g) = (\lambda_3, \lambda_4), \quad \beta_2(g) = (\lambda_2, \lambda_4), \quad \beta_3(g) = (\lambda_2, \lambda_3), \\ &\beta_4(g) = (\lambda_1, \lambda_4), \quad \beta_5(g) = (\lambda_1, \lambda_3), \quad \beta_6(g) = (\lambda_1, \lambda_2). \end{aligned}$$

Each β_i: $\mathbb{T}^4 \to \mathbb{T}^2$ is surjective. Hence, each component $\{i\} \times \mathbb{T}^2$ is one orbit. Therefore, the action has exactly six non-trivial orbits $O_i = \{i\} \times \mathbb{T}^2$, for $i = 1, \ldots, 6$. The differentials $d\beta_i$: $\mathbb{R}^4 \to \mathbb{R}^2$ are represented by

$$B_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Thus, $\mathfrak{g}_{O_i} = \ker(B_i)$. Consequently,

$$\begin{aligned} &\mathfrak{g}_{O_1} = \operatorname{span}\{e_1, e_2\}, \quad \mathfrak{g}_{O_2} = \operatorname{span}\{e_1, e_3\}, \quad \mathfrak{g}_{O_3} = \operatorname{span}\{e_1, e_4\}, \\ &\mathfrak{g}_{O_4} = \operatorname{span}\{e_2, e_3\}, \quad \mathfrak{g}_{O_5} = \operatorname{span}\{e_2, e_4\}, \quad \mathfrak{g}_{O_6} = \operatorname{span}\{e_3, e_4\}. \end{aligned}$$

Hence, $\dim(\mathfrak{g}_{O_i}) = 2$ and $\rho(O_i) = 4 - 2 = 2$ for all i. This agrees with Theorem 3.8, since the orbit-map differentials associated with each orbit factor through quotient spaces determined by the corresponding stabilizer Lie algebras. Consequently, every orbit satisfies the rank estimate predicted by the general theory. The only zero pairwise intersections are $\mathfrak{g}_{O_1} \cap \mathfrak{g}_{O_6} = \{0\}$, $\mathfrak{g}_{O_2} \cap \mathfrak{g}_{O_5} = \{0\}$, and $\mathfrak{g}_{O_3} \cap \mathfrak{g}_{O_4} = \{0\}$. Thus, $f_0 = 6$ and $f_1 = \binom{6}{2} - 3 = 12$. The nonzero triple intersections are exactly

$$\mathfrak{g}_{O_1} \cap \mathfrak{g}_{O_2} \cap \mathfrak{g}_{O_3} = \operatorname{span}\{e_1\},$$

$$\mathfrak{g}_{O_1} \cap \mathfrak{g}_{O_4} \cap \mathfrak{g}_{O_5} = \mathrm{span}\{e_2\},$$
$$\mathfrak{g}_{O_2} \cap \mathfrak{g}_{O_4} \cap \mathfrak{g}_{O_6} = \mathrm{span}\{e_3\},$$

and

$$\mathfrak{g}_{O_3} \cap \mathfrak{g}_{O_5} \cap \mathfrak{g}_{O_6} = \mathrm{span}\{e_4\}.$$

Hence, the 2-simplices are as follows:

$$\{O_1, O_2, O_3\}, \quad \{O_1, O_4, O_5\}, \quad \{O_2, O_4, O_6\}, \quad \{O_3, O_5, O_6\}.$$

There is no 3-simplex. Therefore, $f(\Delta) = (6, 12, 4)$ and $\chi(\Delta) = 6 - 12 + 4 = -2$. Thus, the simplicial complex is two-dimensional, illustrating the facet and dimension bounds established in the previous section. Moreover, every maximal simplex has a dimension 2, which is in agreement with the common stabilizer dimensions computed above. With the vertex order $O_1, \ldots, O_6$, the adjacency matrix of the 1-skeleton is as follows:

$$A_\Delta = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Moreover, $\Delta_1 = \Delta$, Δ_2 only consists of the six vertices and $\Delta_3 = \emptyset$. Hence, the filtration $\Delta_1 \supseteq \Delta_2 \supseteq \Delta_3$ is strictly decreasing, exactly as predicted by Theorem 5.1. The disappearance of edges and higher-dimensional simplices reflects the stronger requirement on the common stabilizer dimension. Thus $f^{[1]} = (0, 12, 4)$ and $f^{[2]} = (6, 0, 0)$. Hence, $\chi^{[1]} = -8$, $\chi^{[2]} = 6$, and $\chi(\Delta) = \chi^{[1]} + \chi^{[2]} = -2$. Therefore, this example explicitly verifies Theorems 5.3–5.5. The simplices are partitioned according to the values of $\Gamma(\sigma)$, the filtration levels determine the corresponding differential Euler contributions, and the total Euler characteristic is recovered from the sum of these contributions. Furthermore, the 1-skeleton is connected, so all orbit vertices belong to a single connected component of $\Delta_1(\mathcal{G}, \mathcal{A})$, thus illustrating Theorem 5.6.

6. Conclusions

In this paper, we introduced a new simplicial framework associated with smooth Lie group actions through orbit ranks and common stabilizer Lie algebras. The proposed orbit-rank simplicial complex provides a natural higher-dimensional model that combines infinitesimal geometric information arising from orbit maps with the combinatorial structure of simplicial complexes, extending graph-based approaches to higher-order orbit interactions. We established the fundamental properties of the proposed construction, proving that it is intrinsic and independent of the choice of orbit representatives. We investigated facets, dimension bounds and rank filtrations, characterized facets through common stabilizer Lie subalgebras, and their zero sets, and established explicit connections between orbit-map differentials, quotient Lie algebras, and the simplicial structure. In addition, refined f-vectors and Euler characteristic decompositions were derived, providing new topological invariants for orbit-rank simplicial complexes. The presented examples demonstrate the applicability of the proposed framework to explicit smooth Lie group actions and illustrate how it captures both geometric and infinitesimal symmetry information. Future work will focus on equivariant topological invariants, computational aspects of orbit-rank simplicial complexes, and applications to homogeneous spaces, transformation groups, and geometric mechanics.

Author contributions

Hadba Alqahtani: conceptualization, formal analysis, investigation, resources, writing-original draft; Faizah Alharbi: conceptualization, formal analysis, investigation, resources, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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