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Research article

A neural network-based Newton–Cotes method for solving neutrosophic fuzzy integro-differential equations

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Abstract: In this article, a modified computational approach based on Newton-Cotes methods and an artificial neural network with positive coefficients is reformulated, presented, and implemented for the first time in the literature to solve the neutrosophic fuzzy integro-differential equations (NFFIDEs), where the variables and parameters are considered as fuzzy neutrosophic numbers. The neutrosophic triangular number (TNFN) is used for expressing neutrosophic fuzzy aspects. Applying fuzzification to the deterministic α-cut, β-cut, and γ-cut solutions results in a neutrosophic numerical solution based on artificial neural networks. The main aim of using neural networks was to address the neutrosophic-fuzzy aspects of NFFIDEs. A numerical experiment is provided to demonstrate the developed approach. The findings are consistent with theoretical predictions and highlight the improved accuracy and efficiency achieved by combining artificial neural networks with the Newton–Cotes method for solving NFFIDEs, supporting its applicability in medicine, engineering, and physics.

Keywords: nutrosophic fuzzy numbers; fuzzy neutrosophic integro-differential equations; Newton-Cotes methods; artificial neural network
Mathematics Subject Classification: 30E10, 34K05

1. Introduction

The Integro-differential equations (IDEs) combine the main features of integral and differential equations were considered as an important tool for modeling various real-life problems across several fields, including engineering, economy, and physics [1]. IDEs are classified based on the limits of integration into two types, that are Fredholm integro-differential equations (F-IDEs) and

Volterra integro-differential equations (V-IDEs). In particular, the classification of IDEs is based on whether the limits of integration are constant or variable. IDEs with constant limits are known as the F-IDEs, where the name of F-IDEs refers to the mathematician Erik Fredholm [2]. F-IDEs are used to model complex real-life systems in numerous areas, including medicine, physics, engineering, and biology [3]. F-IDEs usually relate to boundary value problems. The unknown function relies on values under fixed integration, which often makes solving the F-IDEs more complicated. Hafez et al. (2026) formulated a spectral Bernoulli–Gauss collocation approach for solving pantograph-type V-IDEs. The suggested approach achieved well-approximately with exponential convergence and high effectiveness by using a finite number of collocation points [4]. Youssri et al. (2026) presented a Delannoy polynomial-based collocation model for solving non-linear IDEs. The presented method transforms the problem into an algebraic system and offers highly accurate solutions with fast convergence and computational stability [5]. A novel spectral collocation approach built on modified shifted Chebyshev polynomials was developed for solving pantograph IDEs with weakly singular kernels and delay terms, the presented approach transforms the problem into an algebraic system and achieves rapid convergence, high accuracy, and efficient numerical performance [6]. By merging the fuzzy logic into F-IDEs, the systems can model where uncertainty and imprecision exist. Fuzzy sets aim to represent and handle the vague or incomplete information that commonly appear in real-world problems, such as in medicine, physics, and engineering [7–9]. The use of fuzzy numbers in F-IDEs improves their ability to handle uncertainty by providing more accurate and realistic solutions in the modeling of complex systems where data is often unavailable. Therefore, fuzzy integro-differential equations (F-IDEs) and uncertainty modeling constitute fundamental tools for addressing mathematical problems. These lead to the fuzzy Fredholm integro-differential equations (FFIDEs).

Artificial neural networks (ANNs) are mathematical models inspired by the human brain and implemented through computer software. ANNs act as parallel distributed processors made up of interconnected elements known as neurons. Information is processed by these neurons by passing signals from one neuron to another through connection links. Every link contains a weight that affects the passing signal. The nonlinear activation function of each neuron generates an output from its input. There are feedback connections in ANNs in which outputs are passed back to previous layers. Similarly, there are feed-forward connections in which signals flow in one direction. The backpropagation algorithm is generally involved in the learning of ANNs. It identifies the change in weights that resulted in differences between actual and predicted outputs in various cycles of learning. The performance of ANN is then calculated by using error metrics such as mean squared error (MSE) or root mean squared error (RMSE) [10,11]. On the other hand, fuzzy neural networks (FNNs) have attracted considerable interest because they manage complex, nonlinear systems that include uncertain or imprecise data. By merging the flexibility of neural networks with fuzzy logic, FNNs model and approximate a wide variety of real-world problems where traditional methods may not succeed [12]. Therefore, FNNs can learn from data and respond to uncertainties to make them effective for solving problems in engineering, medicine, and science. Recently, some researchers examined using FNNs for solving fuzzy integral and integro-differential equations.

Fadravi et al. (2014) [13] presented a new method for solving linear second-kind Fredholm fuzzy integral equations using ANNs. The presented method transforms the fuzzy integral equation into two parametric linear Fredholm integral equations. Then, the Fredholm integral equations are solved numerically with neural networks based on using optimizing weights through an unconstrained optimization problem. Numerical examples are given to show the proposed approach's reliability and accuracy by highlighting its ability to handle complex fuzzy equations well in various real-life applications. In the same year, Mosleh (2014) [14] introduced a modified method for solving second-kind Fredholm integro-differential equations with fuzzy initial values by combining FNNs with Newton-Cotes methods. The FNNs are used as universal approximators by adjusting fuzzy weights with a learning algorithm based on a cost function. Numerical examples are presented to indicate that the proposed method provides accurate solutions and is effective in solving complex fuzzy integro-differential equations in different scientific fields. Also, Jafarian and Nia (2013) [12] presented a feedback neural network approach for approximating solutions of linear Fredholm integral equations of the second kind. The presented method involves substituting the n-th truncation of the Taylor series into the integral equations and employing a two-layer feedback neural network to adjust the coefficients with a learning algorithm based on gradient descent. The results show that the proposed FNN method outperforms traditional numerical methods, such as the trapezoidal quadrature rule and provides accurate approximations with high efficiency. After that, a novel hybrid approach combining FNNs and Newton-Cotes methods is presented by Mosleh and Otadi (2017) [15] for solving systems of fuzzy linear Fredholm integro-differential equations. The hybrid approach uses FNNs as universal approximators to calculate fuzzy parameters in the system, with a learning algorithm designed to adjust fuzzy weights. Numerical simulations validate the effectiveness of the presented approach and demonstrate their ability to provide accurate approximations of fuzzy solutions in complex integral equations. Berná and Falcó (2022) formulated a mathematical structure for solving non-linear convex variational problems in Banach spaces reflexive by using greedy optimization and radial dictionaries approaches. Their approach Included neural networks together with fixed architectures and bounded parameters as approximation utilities. The presented method reached a convergence rate of the ordering $O(m^{-1})$, Is comparable to the Steepest Descent method [16]. A recent study Proposed two robust techniques for generating competitive examples in the deep neural networks through a modified MILP structure. The study showed that competitive examples generated from the training-data-related information exhibit higher reliability against network retraining contrasted with parameter-based techniques[17].

After the introduction of fuzzy sets and fuzzy numbers by Zadeh in 1965 [18]. After that, several extensions of fuzzy set theory have emerged, one of which is the neutrosophic fuzzy set (NFS), first proposed by the mathematician Florentin Smarandache in 1998 [19]. A neutrosophic Set extends classical and fuzzy set theories by characterizing each element with three independent membership degrees: truth (T), indeterminacy (I), and falsity (F), each ranging independently within the standard or non-standard unit interval. This richer structure allows for a more flexible and comprehensive representation of information, accommodating contradictions and incomplete knowledge naturally. Neutrosophic Sets have since found applications in fields such as decision-making, artificial intelligence, engineering, image processing, and numerical analysis, especially where traditional models fall short due to the presence of ambiguous or incomplete data [19–21].

FNNs have gained significant attention due to their ability to handle complex nonlinear systems with uncertain or imprecise data. By combining the flexibility of neural networks with the power of fuzzy logic, these networks can model and approximate a wide range of real-world problems, where traditional deterministic approaches may fail. However, when FNNs are combined with neutrosophic fuzzy sets, which capture all truth-membership, indeterminacy-membership and falsity-membership degrees of uncertainty, the model becomes even more powerful due to providing a more comprehensive representation of uncertainty by incorporating a degree of hesitation or doubt, enhancing decision-making processes. Many researchers have considered fuzzy integro-differential equations and proposed various methods to analyze and solve them, such as numerical and approximation methods. These equations are significant in modeling the real-world phenomena that involve uncertainty. However, the merger of Neutrosophic set theory in integro-differential equations remains limited. Neutrosophic theory enables a more flexible way to handle uncertainty by regarding truth, indeterminacy, and falsity apart. Although some recent studies have addressed neutrosophic differential and integral equations, the study of neutrosophic fuzzy integro-differential equations is still very limited and remains in its early stages. This gap encourages the development of effective numerical frameworks for these equations. In many real-world applications, such as engineering systems, physical processes, and medical diagnosis, the data are often inconsistent, incomplete, or indeterminate. Standard fuzzy models cannot fully represent these distinct types of vagueness, by contrast neutrosophic theory offers a more extensive mathematical framework.

In recent years, the neural-network built on methods have interest notable attention for solving differential and integro-differential equations because of their significant approximation ability and flexibility in dealing with complex linear and nonlinear systems [22–24]. Particularly, Physics-based on Neural Networks and related deep-learning techniques have exhibited reliable results in engineering and scientific applications. These innovations motivate the use of neural-network approaches for solving fuzzy integro-differential equations in neutrosophic environment, where indeterminacy and uncertainty play a crucial role.

By combining fuzzy neural networks with neutrosophic fuzzy sets, we can model and solve more complex systems where both the data and the relationships between variables are inherently uncertain, thus enhancing the decision-making processes across various areas and advancing computational intelligence techniques. Therefore, the main aim of this paper is to extend and implement the FNNs to solve the neutrosophic fuzzy integro-differential equations (NFFIDEs) where the parameters and variables are considered as neutrosophic fuzzy numbers. In particular, a novel modified method based on FNNs and Newton-Cotes methods is presented and applied to solve the NFFIDEs.

In conclusion, this paper presents a novel hybrid computational framework for solving neutrosophic fuzzy integro-differential equations (NFFIDEs) by combining artificial neural networks (ANNs) with the Newton–Cotes quadrature method. To the best of our knowledge, this is the first study that integrates these techniques within a neutrosophic fuzzy environment, enabling the simultaneous handling of truth, indeterminacy, and falsity membership functions. The proposed approach employs the (α, β, γ)-cut representation to transform the original problem into a system of tractable parametric equations, which are then efficiently approximated using feed-forward neural networks. By leveraging the learning capability of ANNs alongside the numerical stability of the Newton–Cotes method, the developed model achieves high accuracy and robustness. The effectiveness of the method is validated through numerical experiments, demonstrating its potential applicability in modeling complex systems with uncertainty across various scientific and engineering fields.

2. Preliminaries

Definition 2.1. [19] Let X be a universe of discourse. A neutrosophic set M in X is defined as:

$$M = \{\langle x, T_M(x), I_M(x), F_M(x)\rangle : x \in X\},$$

where the mappings are defined as:

$$T_M, I_M, F_M : X \to [0,1] \text{ (or)}]^{-}0, 1^{+}[\text{ in the extended form)},$$

and represent respectively truth-membership ($T_M(x)$), indeterminacy-membership (($I_M(x)$) and falsity-membership $F_M(x)$) with the constraint:

$$.^{-}0 \le T_M(x) + I_M(x) + F_M(x) \le 3^{+},$$

where.$^{-}$ 0 and3^{+}denote infinitesimal extensions of 0 and 3. In the general neutrosophic framework, $T_M(x), I_M(x), F_M(x)$may also be set-valued (multi-valued).

Now, the non-standard interval $]^{-}0, 1^{+}[$ in Definition 2.1 is used to represent the degrees of membership of truth (T), indeterminacy (I), and falsity (F) components in neutrosophic sets. This interval reflects the philosophical basis of neutrosophic theory, which allows for the representation of uncertainty beyond classical limits.

However, in practical scientific and engineering applications, it is not possible to directly use non-standard intervals in computational models and real data analysis. To address this limitation, Wang et al. (2010) [17] introduced the concept of single-valued neutrosophic sets (SVNS) by restricting the membership degrees to the standard unit interval [0,1], making the framework more suitable for numerical computation and real-world applications.

Definition 2.2. (Single-Valued Neutrosophic Set (SVNS) [25]) Let M be any single-valued Neutrosophic Set which is defined as $M = \{(T_M(x), I_M(x), F_M(x)) : x \epsilon X\}$, where $T_M(x), I_M(x), F_M(x) : X \to [0,1]$ represents the truth membership $T_M(X)$, indeterminacy membership $I_M(x)$, and false membership $F_M(x)$. of the element $x \in X$, with the condition $0 \le T_M(x) + I_M(x) + F_M(x) \le 3$.

Definition 2.3. (Neutrosophic number [26]) A neutrosophic number M is a neutrosophic set defined on the set of real numbers $\mathbb{R}$ and satisfies the following properties:

i. M is normal: if $x_a \in R, , T_M(x_a) = 1, I_M(x_a) = F_M(x_a) = 0$.

ii. M is a convex set on the truth function, $T_M(x)$ where $T_M(\lambda x_1 + (1-\lambda)x_2) \ge min(T_M(x_1), T_M(x_2)),\ \forall x_1, x_2 \in R, \lambda \in [0,1]$.

iii. M is a concave set on indeterministic and fuzzy functions of falsity, $I_M(x), F_M(x)$ such that:

$$I_M(\lambda x_1 + (1-\lambda)x_2) \ge max(I_M(x_1), I_M(x_2)), \qquad \forall x_1, x_2 \in R, \lambda \in [0,1].$$

$$F_M(\lambda x_1 + (1-\lambda)x_2) \ge max(F_M(x_1), F_M(x_2)), \qquad \forall x_1, x_2 \in R, \lambda \in [0,1].$$

Definition 2.4. (Triangular neutrosophic number [27]) Let M be a (SVNS) having truth $T_M(x)$, indeterminacy $I_M(x)$ and false $F_M(x)$ membership function over the universal set X, then the triangular neutrosophic number is defined as:

$$T_M(x) = \begin{cases} \left(\dfrac{x - a_1}{b_1 - a_1}\right) & for\ a_1 \le x \le b_1 \\ 1 & for\ x = b_1 \\ \left(\dfrac{c_1 - x}{c_1 - b_1}\right) & for\ b_1 < x \le c_1 \\ 0 & o.w \end{cases}$$

$$I_M(x) = \begin{cases} \left(\dfrac{b_1 - x}{b_1 - a_1}\right) & for\ a_1 \le x \le b_1 \\ 0 & for\ x = b_2 \\ \left(\dfrac{x - b_1}{c_1 - b_1}\right) & for\ b_1 < x \le c_1 \\ 1 & o.w \end{cases}$$

$$T_M(x) = \begin{cases} \left(\dfrac{b_1 - x}{b_1 - a}\right) & for\ a_3 \le x \le b_3 \\ 0 & for\ x = b_3 \\ \left(\dfrac{x - c_1}{c_1 - b_1}\right) & for\ b_1 < x \le c_1 \\ 1 & o.w \end{cases}$$

where $a_1 \le b_1 \le c_1$ and $a_1, b_1, c_1 \in R$. Triangular neutrosophic numbers are denoted as $M_T\langle(a_1, b_1, c_1)\rangle$ where the truth membership function $T_M(x)$ increases in a linear way for $x \in [a_1, b_1]$ and a decrease in a linear form for $x \in [b_1, c_1]$ for $I_M(x)$ and $F_M(x)$ inverse behavior is seen from the truth membership for $x \in [a_1, b_1]$ and for $x \in [b_1, c_1]$, which is depicted in Figure 1.

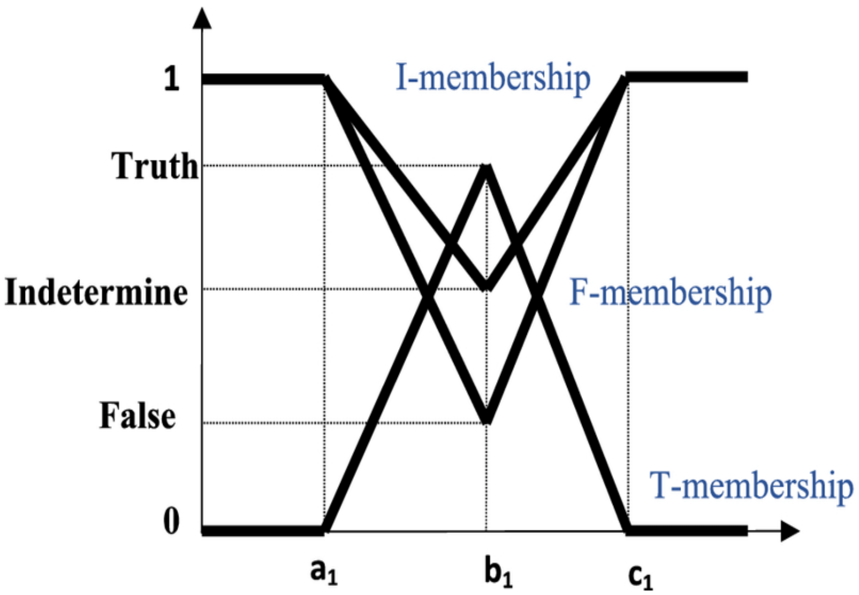

Figure 1. Neutrosophic triangular number shape.

Definition 2.5. ((α, β, γ)-cut of a triangular neutrosophic number [25]) A Triangular neutrosophic set with (α, β, γ)-cut is denoted by $A_{TM(\alpha,\beta,\gamma)}$, where $\alpha, \beta, \gamma \in [0,1]$, and is defined as:

$$A_{TM(\alpha,\beta,\gamma)} = \{T_M(x), I_M(x), F_M(x)\}: T_M(x) \geq \alpha, I_M(x) \leq \beta, F_M(x) \leq \gamma, x \in X\}.$$

Here $0 \leq \alpha + \beta + \gamma \leq 3$, for each $\alpha, \beta, \gamma \in [0,1]$ and

$A_{TM(\alpha,\beta,\gamma)} = [(a_1 + \alpha\,(b_1 - a_1)), (c_1 - \alpha(c_1 - b_1))]$,

$[(b_1 - \beta\,(b_1 - a_1)), (b_1 + \beta(c_1 - b_1))]$,

$[(b_1 - \gamma\,(b_1 - a_1)), (b_1 + \gamma(c_1 - b_1))]$.

3. Fredholm integro differential equation in neutrosophic fuzzy environment

In this section, the general form of the NFFIDE of the second kind is defined as follows:

$$\tilde{z}^{(n)}(s) = \tilde{f}(s) + \tilde{\lambda} \int_c^d k(s,t)\tilde{z}(t)dt\,, 0 \leq s \leq 1, \tag{1}$$

$$\tilde{z}(0) = \tilde{g}_{(1)}, \tilde{z}'(0) = \tilde{g}_{(2)}, \ldots, \tilde{z}^{(n-1)}(0) = \tilde{g}_{(n-1)}.$$

such that $\tilde{z}^{(n)}(s)$ is n^{th} order derivative of the fuzzy neutrosophic function $\tilde{z}(s)$ i.e $\tilde{z}^{(n)}(s) = \frac{d^n\tilde{z}(s)}{ds^n}$, $\tilde{f}(s)$ is a known fuzzy neutrosophic function, $k(s,t)$ is a known function of the variables s and t, calling the kernel of NFFIDE and $k(s,t) \geq 0$. The fuzzy neutrosophic initial conditions are donated by

$$\tilde{z}(0) = \tilde{g}_{(1)}, \tilde{z}'(0) = \tilde{g}_{(2)}, \ldots, \tilde{z}^{(n-1)}(0) = \tilde{g}_{(n-1)} \tag{2}$$

where c and d are real constants.

Currently, the NFFIDE in Eq (1) is then defuzzified based on using the singular parametric formula under α, β, γ-cut framework explained in section 2 at $\alpha, \beta, \gamma \in [0,1]$ as follows:

$$\left[\tilde{z}^{(n)}(s)\right]_{\alpha,\beta,\gamma} = \left\{\left[\underline{z}^{(n)}(s;\alpha), \overline{z}^{(n)}(s;\alpha)\right], \left[\underline{z}^{(n)}(s;\beta), \overline{z}^{(n)}(s;\beta)\right], \left[\underline{z}^{(n)}(s;\gamma), \overline{z}^{(n)}(s;\gamma)\right]\right\}, \tag{3}$$

$$\left[\tilde{f}(s)\right]_{\alpha,\beta,\gamma} = \left\{\left[\underline{f}(s;\alpha), \overline{f}(s;\alpha)\right], \left[\underline{f}(s;\beta), \overline{f}(s;\beta)\right], \left[\underline{f}(s;\gamma), \overline{f}(s;\gamma)\right]\right\}, \tag{4}$$

$$[\tilde{z}(t)]_{\alpha,\beta,\gamma} = \{[\underline{z}(t;\alpha), \overline{z}(t;\alpha)], [\underline{z}(t;\beta), \overline{z}(t;\beta)], [\underline{z}(t;\gamma), \overline{z}(t;\gamma)]\}, \tag{5}$$

$$[\tilde{g}(0)]_{\alpha,\beta,\gamma} = \left\{\left[\underline{g}(0;\alpha), \overline{g}(0;\alpha)\right], \left[\underline{g}(0;\beta), \overline{g}(0;\beta)\right], \left[\underline{g}(0;\gamma), \overline{g}(0;\gamma)\right]\right\}, \tag{6}$$

$$[\tilde{\lambda}]_{\alpha,\beta,\gamma} = \{[\underline{\lambda}_\alpha \overline{\lambda}_\alpha], [\underline{\lambda}_\beta, \overline{\lambda}_\beta], [\underline{\lambda}_\gamma \overline{\lambda}_\gamma]\}. \tag{7}$$

Now, by applying the (α, β, γ)-cut representations defined in Eqs (3)–(7) to the neutrosophic fuzzy integro-differential equation in Eq (1) and using interval arithmetic under the assumption of order preservation, the original problem is decomposed into equivalent lower and upper parametric deterministic systems. Accordingly, the general form of the NFFIDE can be expressed as follows:

$$\begin{cases} \underline{z^{(n)}}(s,\alpha) = \underline{f}(s,\alpha) + \underline{\lambda}_\alpha \int_c^d k(s,t)\underline{z}(t,\alpha)dt \\ \overline{z^{(n)}}(s,\alpha) = \overline{f}(s,\alpha) + \overline{\lambda}_\alpha \int_c^d k(s,t)\overline{z}(t,\alpha)dt \end{cases} \tag{8}$$

$$\begin{cases} \underline{z^{(n)}}(s,\beta) = \underline{f}(s,\beta) + \underline{\lambda}_\beta \int_c^d k(s,t)\underline{z}(t,\beta)dt \\ \overline{z^{(n)}}(s,\beta) = \overline{f}(s,\beta) + \overline{\lambda}_\beta \int_c^d k(s,t)\overline{z}(t,\beta)dt \end{cases} \tag{9}$$

$$\begin{cases} \underline{z^{(n)}}(s,\gamma) = \underline{f}(s,\gamma) + \underline{\lambda}_\gamma \int_c^d k(s,t)\underline{z}(t,\gamma)dt \\ \overline{z^{(n)}}(s,\gamma) = \overline{f}(s,\gamma) + \overline{\lambda}_\gamma \int_c^d k(s,t)\overline{z}(t,\gamma)dt \end{cases} \tag{10}$$

The Eq (8) represents the lower neutrosophic bounds and upper neutrosophic bounds of the truth-membership function for the general formula of NFFIDE. Although, the Eq (9) represents the lower neutrosophic bounds and upper neutrosophic bounds of the indeterminacy-membership function of the general expression of NFFIDE. Finally, Eq (10) represents the lower neutrosophic bounds and upper neutrosophic bounds of the falsity-membership function of the general expression of NFFIDE.

4. Functions approximation using neutrosophic fuzzy neural networks

The use of fuzzy neural networks offers mathematical solutions for certain real-life problems with outstanding generalizability. On the other hand, a significant characteristic of multi-layer perceptrons is their ability to approximate complex functions, making them useful for solving a wide range of problems and applicable to a wide range of real-life phenomena.

In this section, the ability of the feed-forward fuzzy neural network to approximate functions is implemented to express a trial solution for the NFFIDE defined in Eqs (8)–(10) as the sum of two parts. The first part satisfies the initial conditions and has no adjustable parameters for the truth-membership, indeterminacy-membership, and falsity-membership functions. The second part uses three feed-forward fuzzy neural networks: the first one for the fuzzy truth-membership (T), the second one for the fuzzy indeterminacy-membership (I) and the third for the fuzzy falsity-membership functions (F), all trained to meet the NFFIDE. This type of fuzzy network was chosen for the architecture due to the ability of a multilayer perceptron with one hidden layer to approximate arbitrary functions closely. Now, we define the lower and upper trial approximate solutions of the neutrosophic truth-membership component as follows:

$$\left[\underline{z}_T^\eta(s; p_\eta), \overline{z}_T^\eta(s; p_\eta)\right],$$

where $\eta \in \{\alpha, \beta, \gamma\}$ and p_η represents the adjustable neural network parameters.

Specifically, for Eqs (8)–(10) the trial solutions are given respectively as follows:

$$\left[\underline{z}_T^{\alpha}(s;p_\alpha),\ \bar{z}_T^{\alpha}(s;p_\alpha)\right],$$

$$\left[\underline{z}_T^{\beta}(s;p_\beta),\ \bar{z}_T^{\beta}(s;p_\beta)\right],$$

$$\left[\underline{z}_T^{\gamma}(s;p_\gamma),\ \bar{z}_T^{\gamma}(s;p_\gamma)\right].$$

These trial functions approximate the exact neutrosophic solutions $\underline{z}(s;\eta)$ and $\bar{z}(s;\eta)$, respectively. Therefore, solving Eqs (8)–(10) over the interval $[c,d]$ is equivalent to determining the optimal parameter sets $p_\alpha, p_\beta, p_\gamma$ that satisfy the optimization problem outlined in [19] as follows:

$$\begin{cases} \min_{\breve{p}} \sum_{i=1}^{M} \left\{ \left(\underline{z}'_{T_\alpha}\left(s;\alpha;\underline{p}_\alpha\right) - \underline{f}(s_i;\alpha) - \underline{F}(s_i;\alpha;\breve{p}_\alpha) \right)^2 + \left(\overline{z}'_{T_\alpha}\left(s;\alpha;\overline{p}_\alpha\right) - \overline{f}(s_i;\alpha) - \overline{F}(s_i;\alpha;\breve{p}_\alpha) \right)^2 \right\} \\ \underline{z}_{T_\alpha}\left(s_0;\alpha;\underline{p}_\alpha\right) = \underline{z}_0(\alpha),\ \overline{z}_{T_\alpha}\left(s_0;\alpha;\overline{p}_\alpha\right) = \overline{z}_0(\alpha) \end{cases} \tag{11}$$

Such that, $\tilde{p}_\alpha = \left(\underline{p}_\alpha,\ \overline{p}_\alpha\right)$ include all adjustable fuzzy parameters of the neural network (weights and biases associated with the input and output layer) as follows:

$$\begin{cases} \underline{F}(s;\alpha;\tilde{p}_\alpha) = \underline{\lambda} \int_c^d k(s,t)\underline{z}_{T_\alpha}(t;\alpha;\tilde{p}_\alpha)dt \\ \overline{F}(s;\alpha;\tilde{p}_\alpha) = \overline{\lambda} \int_c^d k(s,t)\overline{z}_{T_\alpha}(t;\alpha;\tilde{p}_\alpha)dt \end{cases} \tag{12}$$

and for the fuzzy fuzzy indeterminacy-membership (I),

$$\begin{cases} \min_{\breve{p}`} \sum_{i=1}^{M} \left\{ \left( \underline{z}'_{T_\beta}\left(s;\beta;\underline{p}_\beta\right) - \underline{f}(s_i;\beta) - \underline{F}`\left(s_i;\beta;\breve{p}_\beta\right) \right)^2 + \left(\overline{z}'_{T}`\left(s;\beta;\overline{p}_\beta\right) - \overline{f}(s_i;\beta) - \overline{F}`\left(s_i;\beta;\breve{p}_\beta\right) \right)^2 \right\} \\ \underline{z}_{T_\beta}\left(s_0;\beta;\underline{p}_\beta\right) = \underline{z}_0(\beta),\ \overline{z}_{T_\beta}\left(s_0;\beta;\overline{p}_\beta\right) = \overline{z}_0(\beta) \end{cases} \tag{13}$$

where, $\tilde{p}_\beta = \left(\underline{p}_\beta,\ \overline{p}_\beta\right)$ include all adjustable fuzzy parameters of the neural network (weights and biases associated with the input and output layer) as follows:

$$\begin{cases} \underline{F}\left(s;\beta;\tilde{p}_\beta\right) = \underline{\lambda} \int_c^d k(s,t)\underline{z}_{T_\beta}\left(t;\beta;\tilde{p}_\beta\right)dt \\ \overline{F}\left(s;\beta;\tilde{p}_\beta\right) = \overline{\lambda} \int_c^d k(s,t)\overline{z}_{T_\beta}\left(t;\beta;\tilde{p}_\beta\right)dt \end{cases} \tag{14}$$

and for the fuzzy falsity-membership functions (F),

$$\begin{cases} \min_{\breve{p}`} \sum_{i=1}^{M} \left\{ \left( \underline{z}'_{T_\gamma}\left(s;\gamma;\underline{p}_\gamma\right) - \underline{f}(s_i;\gamma) - \underline{F}`\left(s_i;\gamma;\breve{p}_\gamma\right) \right)^2 + \left(\overline{z}'_{T}`\left(s;\gamma;\overline{p}_\gamma\right) - \overline{f}(s_i;\gamma) - \overline{F}`\left(s_i;\gamma;\breve{p}_\gamma\right) \right)^2 \right\} \\ \underline{z}_{T_\gamma}\left(s_0;\gamma;\underline{p}_\gamma\right) = \underline{z}_0(\gamma),\ \overline{z}_{T_\gamma}\left(s_0;\gamma;\overline{p}_\gamma\right) = \overline{z}_0(\gamma) \end{cases} \tag{15}$$

where, $\tilde{p}_\gamma = \left(\underline{p}_\gamma, \overline{p}_\gamma\right)$ include all adjustable fuzzy parameters of the neural network (weights and biases associated with the input and output layer) as follows:

$$\begin{cases} \underline{F}(s;\gamma;\tilde{p}_\gamma) = \underline{\lambda}\int_c^d k(s,t)\underline{z}_{T_\gamma}(t;\gamma;\tilde{p}_\gamma)dt \\ \overline{F}(s;\gamma;\tilde{p}_\gamma) = \overline{\lambda}\int_c^d k(s,t)\overline{z}_{T_\gamma}(t;\beta;\tilde{p}_\gamma)dt \end{cases} \tag{16}$$

In this hybrid approach, the neural network approximates the unknown solution, while the Newton–Cotes method accurately evaluates the integral term, combining learning flexibility with numerical stability. Generally, the integration in Eq (16) cannot be obtained analytically. Therefore, in this situation, we naturally use numerical methods. Thus, to approximate the integral term, the Newton–Cotes quadrature rule is used through its compatibility and simplicity with uniformly spaced quantisation nodes. In contrast to Gaussian quadrature, it does not need special non-regular points, which is easy to directly within the neutrosophic fuzzy frame. Specifically, the uniform system enables concerted numerical dealing of the truth-membership, indeterminacy-membership and falsity-membership functions in the suggested dual-network structure. A moderating-order scheme is implemented to confirm numerical stability. In this study, the numerical quadrature method R for the interval $[c;d]$ is used under fuzzy conditions with non-negative fuzzy weights g_j and N fuzzy nodes t_j, represented as follows:

$$Rf = \sum_{j=1}^{N} g_j\, f(t_i) = If - Ef = \int_c^d f(t)dt - Ef. \tag{17}$$

It is supposed that the quadrature rule is selected with a positive weight, i.e., $g_j > 0, j = 1, \dots, N$. Now, for every collocation point $\mathrm{s} = \mathrm{s_i}$, the integral with respect to the variable t is represented by using the Newton–Cotes quadrature method on the quadrature nodes $\mathrm{t_j}$.

Such that, Ef this is the error. Now, at first, we ignore this rule-of-square error, and then the Eqs (11), (13), and (15) are replaced by the following fuzzy approximate equations, respectively.

$$\begin{cases} \min_{\breve{p}} \sum_{i=1}^{M} \begin{Bmatrix} \left(\underline{z}'_{T_\alpha}\left(s_i;\alpha;\underline{p}_\alpha\right) - \underline{f}(s_i;\alpha) - \underline{\lambda}\sum_{j=1}^{N} g_j\, k(s_i,t_j)\underline{z}_{T_r}(t_i;\alpha;\breve{p}_\alpha)\right)^2 \\ + \left(\overline{z}'_T(s_i;\alpha;\overline{p}_\alpha) - \overline{f}(s_i;\alpha) - \overline{\lambda}\sum_{j=1}^{N} g_j\, k(s_i,t_j)\overline{z}_T(t_i;\alpha;\breve{p}_\alpha)\right)^2 \end{Bmatrix} \\ \underline{z}_{T_\alpha}\left(s_0;\alpha;\underline{p}_\alpha\right) = \underline{z}_0(\alpha),\ \overline{z}_{T_\alpha}(s_0;\alpha;\overline{p}_\alpha) = \overline{z}_0(\alpha) \end{cases} \tag{18}$$

$$\begin{cases} \min_{\breve{p}`} \sum_{i=1}^{M} \begin{Bmatrix} \left(\underline{z}'_{T_\beta}\left(s_i;\beta;\underline{p}`\right) - \underline{f}(s_i;\beta) - \underline{\lambda}\sum_{j=1}^{N} g_j\, k(s_i,t_j)\underline{z}_{T_\beta}(t_i;\beta;\breve{p}_\beta)\right)^2 \\ + \left(\overline{z}'_{T_\beta}\left(s_i;\beta;\overline{p}_\beta\right) - \overline{f}(s_i;r) - \overline{\lambda}\sum_{j=1}^{N} g_j\, k(s_i,t_j)\overline{z}_{T_\beta}(t_i;\beta;\breve{p}_\beta)\right)^2 \end{Bmatrix} \\ \underline{z}_{T_\beta}\left(s_0;\beta;\underline{p}_\beta\right) = \underline{z}_0(\beta),\ \overline{z}_{T_\beta}\left(s_0;\beta;\overline{p}_\beta\right) = \overline{z}_0(\beta) \end{cases} \tag{19}$$

and

$$\begin{cases} \min_{\breve{p}} \sum_{i=1}^{M} \left\{ \begin{array}{l} \left(\underline{z}'_{T_\gamma}\left(s_i;\gamma;\underline{p}_\gamma\right) - \underline{f}(s_i;\gamma) - \underline{\lambda} \sum_{j=1}^{N} g_j\, k(s_i,t_j) \underline{z}_{T_\gamma}(t_i;\gamma;\breve{p}_\gamma) \right)^2 \\ + \left(\overline{z}'_T\left(s_i;\gamma;\overline{p}_\gamma\right) - \overline{f}(s_i;\gamma) - \overline{\lambda} \sum_{j=1}^{N} g_j\, k(s_i;\gamma;t_j) \overline{z}_T(t_i;\gamma;\breve{p}_\gamma) \right)^2 \end{array} \right\} \\ \underline{z}_{T_\gamma}\left(s_0;\gamma;\underline{p}_\gamma\right) = \underline{z}_0(\gamma), \qquad \overline{z}_{T_\gamma}\left(s_0;\gamma;\overline{p}_\gamma\right) = \overline{z}_0(\gamma) \end{cases}$$

Each trial solution of $\underline{z}_T(\alpha)$ and $\overline{z}_T(\alpha)$ for each truth-membership function, it employs a single feed-forward fuzzy neural network, where the corresponding fuzzy networks are defined by $\underline{N}(\alpha)$ and $\overline{N}(\alpha)$, with fuzzy truth-membership adjustable parameters $\underline{p}(\alpha)$ and $\overline{p}(\alpha)$, respectively. While for each indeterminacy, a membership function employs a single feed-forward fuzzy neural network, where the corresponding fuzzy networks are defined by $\underline{N}(\beta)$ and $\overline{N}(\beta)$, with fuzzy indeterminacy-membership adjustable parameters $\underline{p}(\beta)$ and $\overline{p}(\beta)$, respectively. Also, for each falsity-membership function, a single feed-forward fuzzy neural network, where the corresponding fuzzy networks are defined by $\underline{N}(\gamma)$ and $\overline{N}(\gamma)$, with fuzzy falsity-membership functions adjustable parameters $\underline{p}(\gamma)$ and $\overline{p}(\gamma)$, respectively. So, the related fuzzy trial functions are presented in the following formula [28]:

For truth-membership function:

$$\underline{z}_{T_\alpha}\left(s_i;\alpha;\underline{p}_\alpha\right) = \underline{z}(s_0,\alpha) + (s-s_0)\,\underline{N}_\alpha\left(s;\alpha;\underline{p}_\alpha\right),$$

$$\overline{z}_{T_\alpha}\left(s_i;\alpha;\overline{p}_\alpha\right) = \overline{z}. \tag{20}$$

For indeterminacy-membership components:

$$\underline{z}_{T_\beta}\left(s_i;\beta;\underline{p}_\beta\right) = \underline{z}`(s_0,\beta) + (s-s_0)\,\underline{N}_\beta\left(s;\beta;\underline{p}_\beta\right),$$

$$\overline{z}_{T_\beta}\left(s_i;\beta;\overline{p}_\beta\right) = \overline{z}`(s_0,\beta) + (s-s_0)\,\overline{N}_\beta\left(s;\beta;\overline{p}_\beta\right). \tag{21}$$

And, for falsity-membership components:

$$\underline{z}_{T_\gamma}\left(s_i;\gamma;\underline{p}_\gamma\right) = \underline{z}`(s_0,\gamma) + (s-s_0)\underline{N}_\gamma\left(s;\gamma;\underline{p}_\gamma\right),$$

$$\overline{z}_{T_\gamma}\left(s_i;\gamma;\overline{p}_\gamma\right) = \overline{z}`(s_0,\gamma) + (s-s_0)\,\overline{N}_\gamma\left(s;\gamma;\overline{p}_\gamma\right). \tag{22}$$

Such that $\underline{N}(\alpha), \overline{N}(\alpha)$, $\underline{N}(\beta)$, $\overline{N}(\beta)$, $\underline{N}(\gamma)$, and $\overline{N}(\gamma)$ are the single-output truth-membership, indeterminacy-membership and falsity-membership of feed-forward fuzzy neural networks with neutrosophic adjustable parameters $\underline{p}(\alpha)$, $\overline{p}(\alpha)$, $\underline{p}(\beta)$, $\overline{p}(\beta)$, $\underline{p}(\gamma)$, and $\overline{p}(\gamma)$ respectively, for the lower and upper bounds of truth-membership, indeterminacy-membership, and falsity-membership functions. This neutrosophic solution satisfies the neutrosophic fuzzy initial conditions in Eqs (20)–(22).

Since the neural network trial solutions are defined in Eqs (20)–(22) using sigmoid activation functions in the hidden layer and a linear output layer, they are differentiable with respect to the independent variable s. Therefore, their derivatives can be obtained analytically using the chain rule, which leads to the expressions given Eqs (23)–(26).

For truth-membership components:

$$\underline{z}'_{T_\alpha}\left(s;\alpha;\underline{p}_\alpha\right)=\underline{N}_\alpha\left(s;\alpha;\underline{p}_\alpha\right)+(s-s_0)\frac{\partial \underline{N}_\alpha}{\partial s}, \tag{23}$$

$$\overline{z}'_{T_\alpha}\left(s;\alpha;\overline{p}_\alpha\right)=\overline{N}_\alpha\left(s;\alpha;\overline{p}_\alpha\right)+(s-s_0)\frac{\partial \overline{N}_\alpha}{\partial s}. \tag{24}$$

For indeterminacy-membership components:

$$\underline{z}'_{T_\beta}\left(s;\beta;\underline{p}_\beta\right)=\underline{N}_\beta\left(s;\beta;\underline{p}_\beta\right)+(s-s_0)\frac{\partial \underline{N}_\beta}{\partial s},$$

$$\overline{z}'_{T_\beta}\left(s;\beta;\overline{p}_\beta\right)=\overline{N}_\beta\left(s;\beta;\overline{p}_\beta\right)+(s-s_0)\frac{\partial \overline{N}_\beta}{\partial s}. \tag{25}$$

And for falsity-membership components:

$$\underline{z}'_{T_\gamma}\left(s;\gamma;\underline{p}_\gamma\right)=\underline{N}_\gamma\left(s;\gamma;\underline{p}_\gamma\right)+(s-s_0)\frac{\partial \underline{N}_\gamma}{\partial s},$$

$$\overline{z}'_{T_\gamma}\left(s;\gamma;\overline{p}_\gamma\right)=\overline{N}_\gamma\left(s;\gamma;\overline{p}_\gamma\right)+(s-s_0)\frac{\partial \overline{N}_\gamma}{\partial s}. \tag{26}$$

Let's now assume a multilayer neutrosophic perceptron having a single hidden layer involving $H\ sigmoid$ units and a neutrosophic fuzzy linear output unit for truth-membership, indeterminacy-membership, and falsity-membership components. As shown in Figures 2–7.

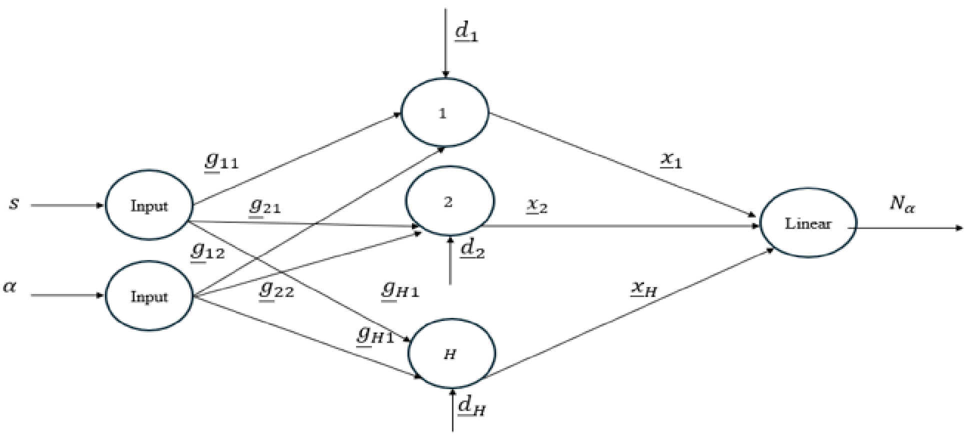

Figure 2. A three-layer sensory neuron featuring two inputs and a single $\underline{N}_\alpha$ output.

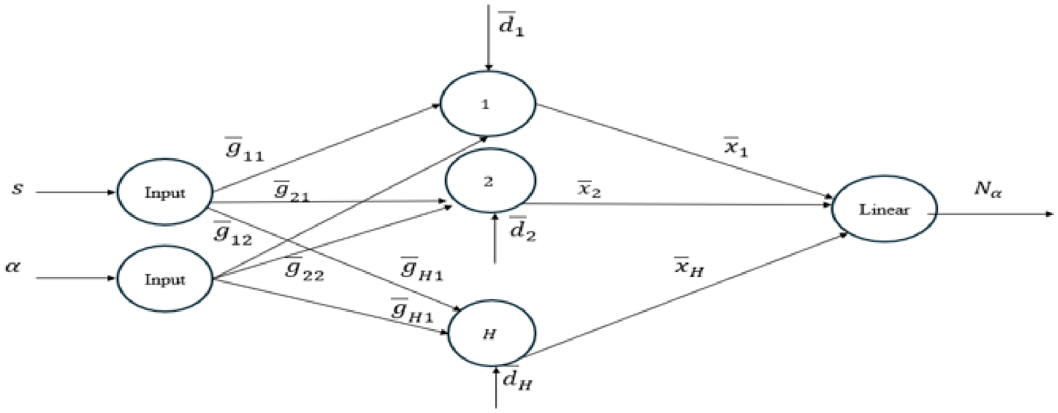

Figure 3. A three-layer neural network featuring two input nodes and a single $\bar{N}_\alpha$ output.

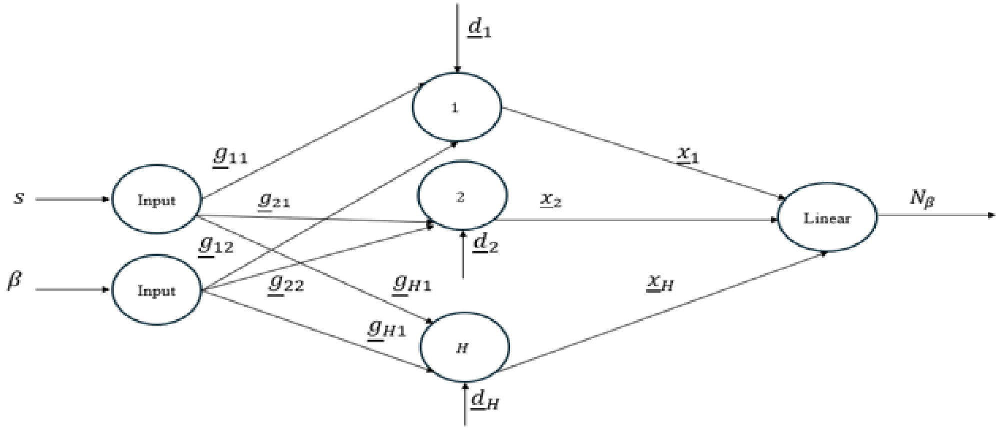

Figure 4. A three-layer feedforward neural network with two inputs and one $\underline{N}_\beta$ output.

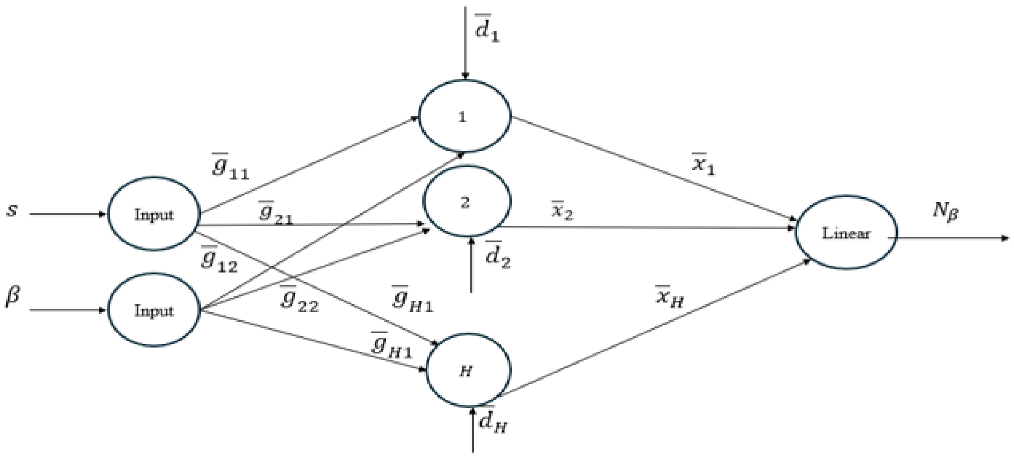

Figure 5. A three-layer sensory network with two input nodes and one $\bar{N}_\beta$ output.

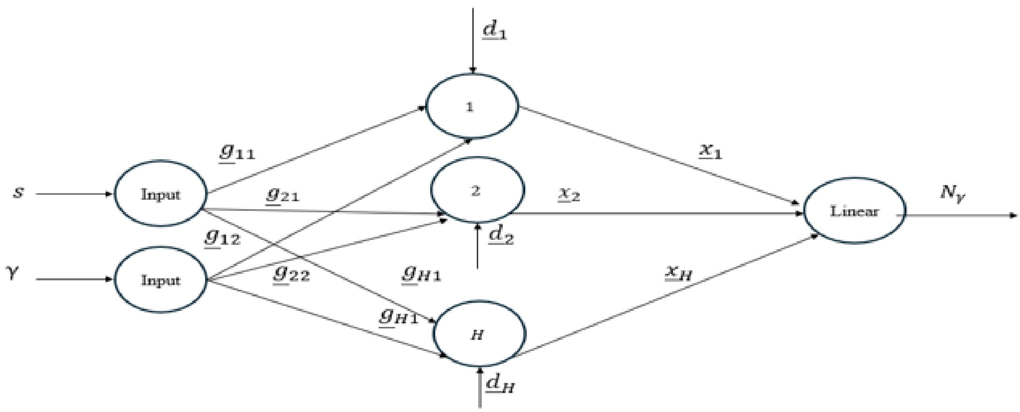

Figure 6. A three-layer feedforward neural network with two input nodes and one $\underline{N}_\gamma$ output node.

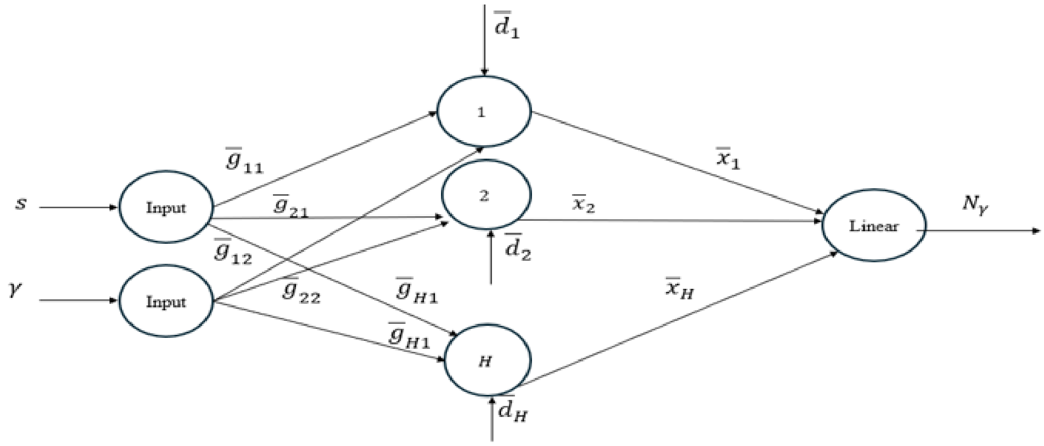

Figure 7. A three-layer feedforward neural network with two input nodes and one $\overline{N}_\gamma$ output node.

As seen in the above four figures, the figures illustrate the use of a three-layer FNNs to approximate the lower and upper bounds of fuzzy truth-membership, indeterminacy-membership and falsity-membership functions in a system described by an IFFIDE. Figures 2 and 3 focus on the truth-membership function, with $\underline{N}_\alpha$ and $\overline{N}_\alpha$ representing the lower and upper neutrosophic fuzzy solutions for truth-membership, respectively. Also, Figures 4 and 5 focus on the indeterminacy-membership function, where $\underline{N}_\beta$ and $\underline{N}_\beta$ represent the lower and upper fuzzy solutions for indeterminacy-membership. And Figures 6 and 7 focus on the falsity-membership function, where $\underline{N}_\gamma$ and $\underline{N}_\gamma$ represent the lower and upper fuzzy solutions for falsity-membership. Each figure represents the FNNs processing two inputs to generate these neutrosophic fuzzy outputs, which are then used to solve the NFFIDE.

Now, we have:

For truth-membership components:

$$\underline{N}_\alpha = \sum_{i=1}^{H} \underline{u}_{i_\alpha} \sigma_\alpha\left(\underline{z}_{i_\alpha}\right), \qquad \underline{z}_{i_\alpha} = \underline{g}_{i1} s + \underline{g}_{i2}\alpha + \underline{d}_i,$$

$$\overline{N}_\alpha = \sum_{i=1}^{H} \overline{u}_{i_\alpha} \sigma_\alpha\left(\overline{z}_{i_\alpha}\right), \qquad \overline{z}_{i_\alpha} = \overline{g}_{i1} s + \overline{g}_{i2}\alpha + \overline{d}_i. \tag{27}$$

Note that: d_i is denotes bias of the *i-th* neuron and is handled as a learnable parameter that is adapted Along with the network weights throughout training.

Where $\sigma_\alpha\left(\underline{z_i}_\alpha\right)$ and $\sigma_\alpha\left(\overline{z_i}_\alpha\right)$ are the neutrosophic lower and neutrosophic upper bounds of the sigmoid transfer neural function, respectively. Then, the following are evaluated:

$$\frac{\partial \underline{N}_\alpha}{\partial s} = \sum_{i=1}^{H} \underline{u}_{i_\alpha}\, \underline{g}_{i1} \sigma_\alpha{}'\left(\underline{z_i}_\alpha\right),$$

$$\frac{\partial \overline{N}_\alpha}{\partial s} = \sum_{i=1}^{H} \overline{u}_{i_\alpha} \overline{g}_{i1}\, \sigma_\alpha{}'\left(\overline{z_i}_\alpha\right), \tag{28}$$

where $\sigma_r{}'(\tilde{z})$ is the neutrosophic fuzzy first derivative of the truth-membership function of the sigmoid function. Furthermore, among the many possible sigmoid functions, we select the following fuzzy $\sigma_\alpha(\tilde{z}) = \frac{1}{(1+e^{-z})}$ since all derivatives can be obtained from the fuzzy $\sigma_\alpha(\tilde{z})$ in terms of the fuzzy sigmoid function itself. i.e.,

$$\sigma_\alpha{}'(\tilde{z}) = \sigma_\alpha(\tilde{z})\big(1 - \sigma_\alpha(\tilde{z})\big). \tag{29}$$

And for indeterminacy-membership components:

$$\underline{N}_\beta = \sum_{i=1}^{H} \underline{u}_{i_\beta}\, \sigma_\beta\left(\underline{z_i}_\beta\right), \qquad \underline{z}_{i_\beta} = \underline{g}_{i1}s + \underline{g}_{i2}\beta + \underline{d}_i,$$

$$\overline{N}_\beta = \sum_{i=1}^{H} \overline{u}_{i_\beta}\, \sigma_\beta\left(\overline{z}_{i_\beta}\right), \qquad \overline{z}_{i_\beta} = \overline{g}_{i1}s + \overline{g}_{i2}\beta + \overline{d}_i, \tag{30}$$

where $\sigma_\beta(\tilde{z})$ is a neutrosophic fuzzy sigmoid transfer function. Then, the following are evaluated:

$$\frac{\partial \underline{N}_\beta}{\partial s} = \sum_{i=1}^{H} \underline{u}_{i_\beta}\, \underline{g}_{i1} \sigma'_\beta\left(\underline{z_i}_\beta\right), \tag{31}$$

$$\frac{\partial \overline{N}_\beta}{\partial s} = \sum_{i=1}^{H} \overline{u}_{i_\beta} \overline{g}_{i1}\, \sigma'_\beta\left(\overline{z}_{i_\beta}\right), \tag{32}$$

where $\sigma'_\beta\left(\overline{z}_{i_\beta}\right)$ is the neutrosophic first derivative of the indeterminacy-membership function of the sigmoid function. Furthermore, among the many possible sigmoid functions, we select the following fuzzy $\sigma_\beta(\tilde{z}) = \frac{1}{(1+e^{-z})}$ since all derivatives can be obtained from the fuzzy $\sigma_\beta`(z)$ in terms of the fuzzy sigmoid function itself. i.e.,

$$\sigma_\beta{}'(\tilde{z}) = \sigma_\beta(\tilde{z})\left(1 - \sigma_\beta(\tilde{z})\right). \tag{33}$$

Also, for falsity-membership components:

$$\underline{N}_\gamma = \sum_{i=1}^{H} \underline{u}_{i_\gamma}\, \sigma_\gamma\left(\underline{z_i}_\gamma\right), \qquad \underline{z}_{i_\gamma} = \underline{g}_{i1}s + \underline{g}_{i2}\gamma + \underline{d}_i,$$

$$\overline{N}_\gamma = \sum_{i=1}^{H} \overline{u}_{i_\gamma}\, \sigma_\gamma\left(\overline{z_{i}}_\gamma\right), \qquad \overline{z_{i}}_\gamma = \overline{g}_{i1}s + \overline{g}_{i2}\gamma + \overline{d}_i, \tag{34}$$

where $\sigma_\gamma(\tilde{z})$ is a neutrosophic fuzzy sigmoid transfer function. Then, the following are evaluated:

$$\frac{\partial \underline{N}_\gamma}{\partial s} = \sum_{i=1}^{H} \underline{u}_{i_\gamma}\, \underline{g}_{i1} \sigma'_\gamma\left(\underline{z_{i}}_\gamma\right),$$

$$\frac{\partial \overline{N}_\gamma}{\partial s} = \sum_{i=1}^{H} \overline{u}_{i_\gamma} \overline{g}_{i1}\, \sigma'_\gamma\left(\overline{z_{i}}_\gamma\right), \tag{35}$$

where $\sigma'_\gamma\left(\overline{z_{i}}_\gamma\right)$ is the neutrosophic first derivative of the falsity-membership function of the sigmoid function. Furthermore, among the many possible sigmoid functions, we select the following fuzzy $\sigma_\gamma(\tilde{z}) = \frac{1}{(1+e^{-z})}$ since all derivatives can be obtained from the fuzzy $\sigma_\gamma`(z)$ in terms of the fuzzy sigmoid function itself. i.e.,

$$\sigma_\gamma{}'(\tilde{z}) = \sigma_\gamma(\tilde{z})\left(1 - \sigma_\gamma(\tilde{z})\right). \tag{36}$$

For reproducibility, the proposed ANN–Newton–Cotes method is summarised as follows: The integral term in the NFFIDE is approximated using a Newton–Cotes quadrature rule. The optimization process was carried out using the Adam optimizer with a learning rate of 0.001. The optimization is conducted only on biases and the neural network weights, whereas α, β, and $\gamma$are preassigned admissible cut levels. The neutrosophic fuzzy solution bounds are represented by ANN-based trial functions. The ANN weights and biases are initialized with small random values and then updated iteratively using backpropagation by minimizing the mean squared error (MSE) of the equation residuals at selected collocation points until convergence the training process was executed for 5000 epochs. Training is stopped when the MSE reaches a prescribed tolerance or when the maximum number of epochs is achieved. In addition to MSE, RMSE may be used to evaluate approximation accuracy. In this study, a feed-forward neural network with one hidden layer is employed, where the hidden layer uses a sigmoid activation function, and the output layer is linear. The number of hidden neurons H and the number of Newton–Cotes nodes N= 12 are selected to balance accuracy and computational cost (e.g., $H = 10$ in the numerical example), and a small positive learning rate is used to ensure stable convergence.

5. Numerical example

Example 1. Consider the following linear NFFIDE of the second kind:

$$\tilde{z}'(s) = \tilde{f}(s) + \int_0^1 \tilde{z}(t,\alpha)dt \tag{37}$$

with

$$\tilde{z}(0,\alpha;\beta;\gamma) = \left\{\left[\underline{z}(\alpha);\overline{z}(\alpha)\right], \left[\underline{z}(\beta);\overline{z}(\beta)\right], \left[\underline{z}(\gamma);\overline{z}(\gamma)\right]\right\},$$

$$\tilde{z}(0,\alpha;\beta;\gamma)=\{[0.75+0.25\alpha,1.25-0.25\alpha];[1-0.5\beta,1+0.5\beta];[1-0.25\gamma,1+0.25\gamma]\},$$

where $0\le t,s\le 1$, $0\le \alpha,\beta,\gamma\le 1$, $k(s,t)=1$, and

$$\tilde{f}(s)=\left[\underline{f}(s,\alpha),\overline{f}(s,\alpha);\underline{f}(s,\beta),\overline{f}(s,\beta);\underline{f}(s,\gamma),\overline{f}(s,\gamma)\right].$$

i.e., $\tilde{f}(s)=\{(e^s-s)[0.75+0.25\alpha,1.25-0.25\alpha],(e^s-s)[1-0.5\beta,1+0.5\beta],(e^s-s)[1-0.25\gamma,1+0.25\gamma]\}$.

The exact solution is provided as:

$$\tilde{z}(s)=\{e^s[0.75+0.25\alpha,1.25-0.25\alpha],e^s[1-0.5\beta,1+0.5\beta],e^s[1-0.25\gamma,1+0.25\gamma]\}.$$

The functions of the trial for this problem are:

$$\underline{z}_{T_\alpha}(s;\alpha)=(0.75+0.25\alpha)+s\sum_{i=1}^{H}\frac{\underline{u}_{i_\alpha}}{1+e^{\left(-\underline{w}_{i1_\alpha}\right)s-\left(\underline{w}_{i2_\alpha}\right)\alpha-\underline{b}_i}},$$

$$\overline{z}_{T_\alpha}(s;\alpha)=(1.25-0.25\alpha)+s\sum_{i=1}^{H}\frac{\overline{u}_{i_r}}{1+e^{\left(-\overline{w}_{i1_\alpha}\right)s-\left(\overline{w}_{i2_\alpha}\right)\alpha-\overline{b}_i}},\tag{38}$$

$$\underline{z}_{T_\beta}(s;\beta)=(1-0.5\beta)+s\sum_{i=1}^{H}\frac{\underline{u}_{i_\beta}}{1+e^{\left(-\underline{w}_{i1_\beta}\right)s-\left(\underline{w}_{i2_\beta}\right)\beta-\underline{b}_i}},$$

$$\overline{z}_{T_\beta}(s;\beta)=(1+0.5\beta)+s\sum_{i=1}^{H}\frac{\overline{u}_{i_\beta}}{1+e^{\left(-\overline{w}_{i1_\beta}\right)s-\left(\overline{w}_{i2_\beta}\right)\beta-\overline{b}_i}},\tag{39}$$

and,

$$\underline{z}_{T_\gamma}(s;\gamma)=(1-0.25\gamma)+s\sum_{i=1}^{H}\frac{\underline{u}_{i_\gamma}}{1+e^{\left(-\underline{w}_{i1_\gamma}\right)s-\left(\underline{w}_{i2_\gamma}\right)\gamma-\underline{b}_i}},$$

$$\overline{z}_{T_\gamma}(s;\gamma)=(1+0.25\gamma)+s\sum_{i=1}^{H}\frac{\overline{u}_{i_\gamma}}{1+e^{\left(-\overline{w}_{i1_\gamma}\right)s-\left(\overline{w}_{i2_\gamma}\right)\gamma-\overline{b}_i}}.\tag{40}$$

Upon examining the results presented in Figure 10, we can find many important remarks and conclusions regarding the performance and reliability of our proposed method for solving Eq (37) at $s=0.2$. In general, it can be noted from Tables 1–3, and Figures 8–10 that the results obtained by the artificial neural networks are in good agreement with the exact solution for Eq (37) at $s=0.2$ and all $\alpha,\beta,\gamma\in[0,1]$, attaining the neutrosophic triangular fuzzy number shape described in Section 2 for the truth-membership functions $\alpha-cut$, indeterminacy -membership functions $\beta-cut$ and falsity-membership $\gamma-cut$ functions, thereby satisfying the properties of fuzzy neutrosophic numbers. Furthermore, the results demonstrate that the FNN method accurately approximates the neutrosophic fuzzy truth-membership, indeterminacy-membership and falsity-membership solutions, with the outputs closely aligning with the exact solutions. The FNN's ability to compute these fuzzy bounds effectively handles the imprecision inherent in neutrosophic fuzzy systems. This method's results are crucial for solving the NFFIDE, as the FNNs provide reliable approximations truth-membership, indeterminacy-membership and falsity-membership functions, as seen in the low absolute errors across the computed outputs.

Table 1. Neutrosophic exact and neutrosophic approximation lower and upper solution of truth-membership of Eq (37) by artificial neural networks at $s = 0.2$ for all $\alpha \in [0,1]$.

	Lower fuzzy solution of truth-membership			Upper fuzzy solution of truth-membership		
(α)	Exact solution	Approximation solution	Absolute error	Exact solution	Approximation solution	Absolute error
0	0.916052	0.916052	6.862×10^{-8}	1.526753	1.526753	1.143×10^{-7}
0.2	0.977122	0.977122	7.319×10^{-8}	1.465683	1.465683	1.097×10^{-7}
0.4	1.038192	1.038192	7.776×10^{-8}	1.404613	1.404613	1.052×10^{-7}
0.6	1.099262	1.099262	8.234×10^{-8}	1.343543	1.343542	1.006×10^{-7}
0.8	1.160332	1.160332	8.691×10^{-8}	1.282472	1.282472	9.606×10^{-8}
1	1.221402	1.221402	9.149×10^{-8}	1.221402	1.221402	9.149×10^{-8}

Table 2. Neutrosophic exact and neutrosophic approximation lower and upper solution of indeterminacy-membership of Eq (37) by artificial neural networks at $s = 0.2$ for all $\beta \in [0,1]$.

	Lower fuzzy solution of indeterminacy-membership			Upper fuzzy solution of indeterminacy -membership		
(β)	Exact solution	Approximation solution	Absolute error	Exact solution	Approximation solution	Absolute error
0	1.221402	1.221402	9.149×10^{-8}	1.221402	1.221402	9.149×10^{-8}
0.2	1.099262	1.099262	8.234×10^{-8}	1.343543	1.343542	1.006×10^{-7}
0.4	0.977122	0.977122	7.319×10^{-8}	1.465683	1.465683	1.097×10^{-7}
0.6	0.854981	0.854981	6.404×10^{-8}	1.587823	1.587823	1.189×10^{-7}
0.8	0.732841	0.732841	5.489×10^{-8}	1.709963	1.709963	1.280×10^{-7}
1	0.610701	0.610701	4.574×10^{-8}	1.8321045	1.832104	1.372×10^{-7}

Table 3. Neutrosophic exact and neutrosophic approximation lower and upper solution of falsity-membership of Eq (37) by artificial neural networks at $s = 0.2$ for all $\gamma \in [0,1]$.

	Lower fuzzy solution of falsity-membership			Upper fuzzy solution of falsity-membership		
(γ)	Exact solution	Approximation solution	Absolute error	Exact solution	Approximation solution	Absolute error
0	1.221402	1.221402	9.149×10^{-8}	1.221402	1.221402	9.149×10^{-8}
0.2	1.160332	1.160332	8.691×10^{-8}	1.282472	1.282472	9.606×10^{-8}
0.4	1.099262	1.099262	8.234×10^{-8}	1.343543	1.343542	1.006×10^{-7}
0.6	1.038192	1.038192	7.776×10^{-8}	1.404613	1.404613	1.052×10^{-7}
0.8	0.977122	0.977122	7.319×10^{-8}	1.465683	1.465683	1.097×10^{-7}
1	0.916052	0.916052	6.862×10^{-8}	1.526753	1.526753	1.143×10^{-7}

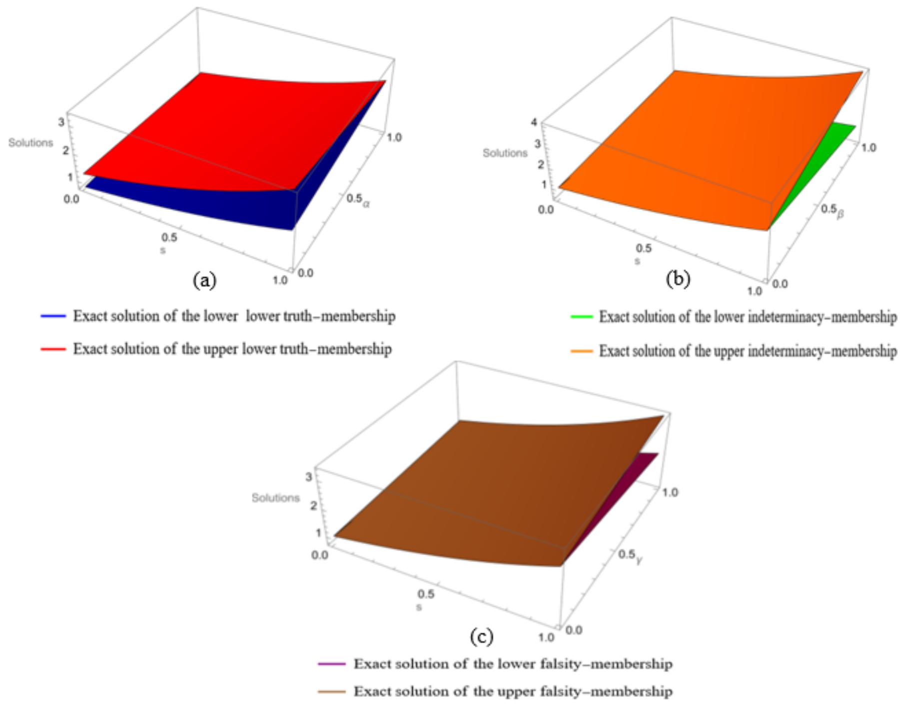

Figure 8. Neutrosophic fuzzy lower and upper exact solution for Eq (37) (a) of truth-membership, (b) indeterminacy-membership, and (c) falsity -membership for all $s \in [0,1]$ and $r, \beta \in [0,1]$.

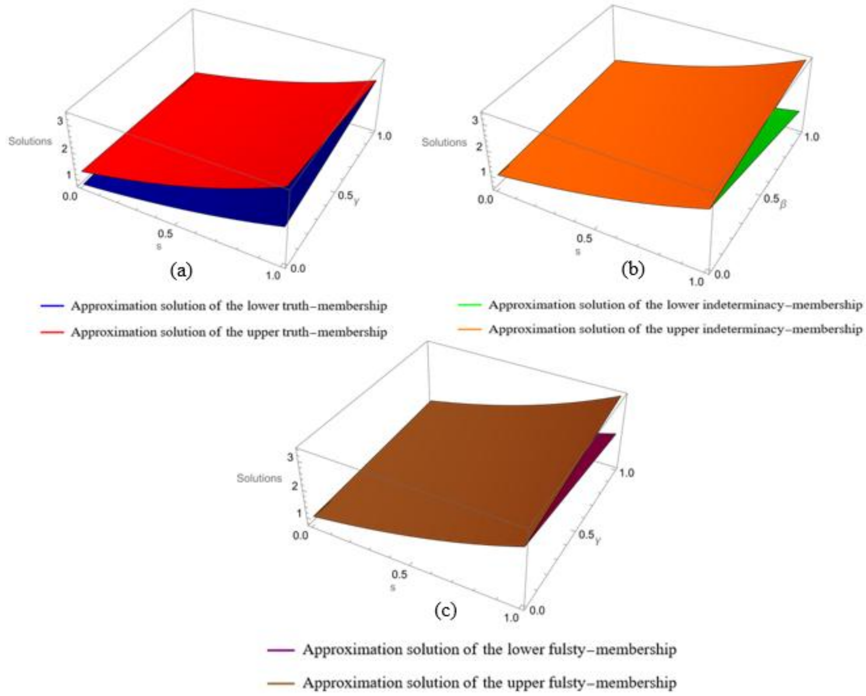

Figure 9. Neutrosophic fuzzy lower and upper approximations solution for Eq (37) (a) of truth-membership, (b) indeterminacy-membership, and (c) falsity -membership for all $s \in [0,1]$ and $r, \beta \in [0,1]$.

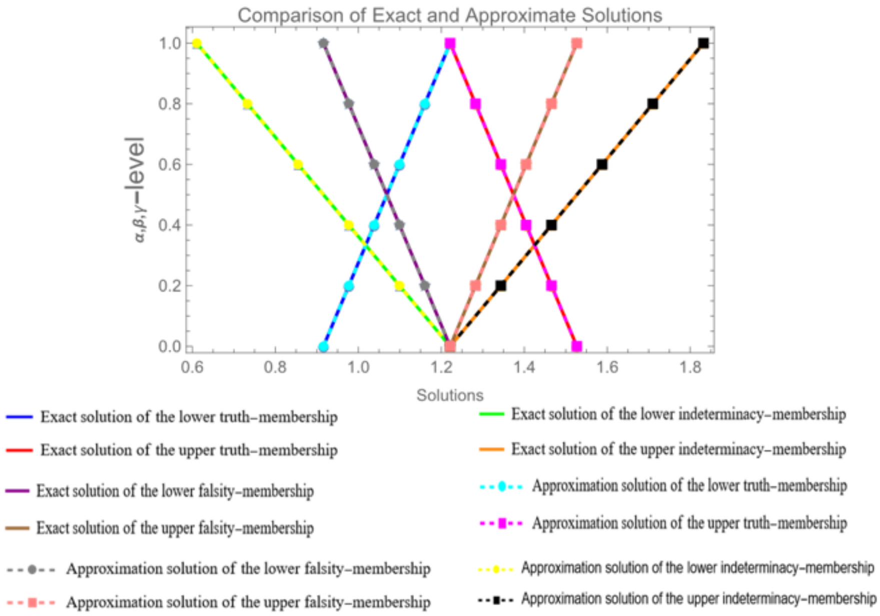

Figure 10. 2D plot comparing the neutrosophic exact and neutrosophic approximation solutions of Eq (37) using artificial neural networks for $\alpha, \beta, \gamma \in [0,1]$ at $s = 0.2$.

6. Conclusions

In this paper, we solved the NFFIDEs based on developed, and applied a modified method based on artificial neural network and Newton-Cotes methods with positive coefficients. The parameters and variables of NFFIDEs are considered as neutrosophic fuzzy numbers. TNFN is used to represent the fuzzy neutrosophic parameters and variables. The fuzzification to the deterministic α-cut, β-cut, and γ-cut solutions results in a neutrosophic numerical solution based on artificial neural networks. The main aim for using neural networks is their feasibility in handling the neutrosophic fuzzy parameters and variables of NFFIDEs. A numerical experiment is provided to illustrate the proposed method. The study shows that the results align with theoretical predictions, emphasizing how the integration of Newton-Cotes methods and artificial neural networks improves both the accuracy and efficiency in solving NFFIDEs.

Author contributions

Hamzeh Zureigat: Conceptualization, methodology, software, validation, formal analysis, investigation, writing—original draft preparation, writing-review and editing; Belal Batiha: Formal analysis, resources, writing—original draft preparation, funding acquisition. All authors have read and agreed to the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no conflicts of interest.

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