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Research article

A hybrid hesitant fuzzy N-soft set framework for algebraic modeling of corporate bond evaluation under parameter-induced uncertainty

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Abstract: The evaluation of corporate bond yields is a multi-criteria decision-making problem involving parameter-based uncertainty and heterogeneity in expert judgments. Approaches, including N-soft sets, hesitant N-soft sets, and hesitant fuzzy N-soft sets, remain limited in their ability to simultaneously represent uncertainty at the level of discrete grade assignment and fuzzy membership degrees. To address these limitations, we proposed a hybrid hesitant fuzzy N-soft set (HHFNSS) framework integrated with a soft yield structure. The proposed model represented each object-parameter pair as a set of possible grade combinations along with corresponding sets of membership values, thereby accommodating two-layer hesitancy within a unified algebraic structure. A formal reduction mechanism was developed to transform the set-based representation into a reduced fuzzy N-soft set, which was subsequently processed using a soft yield-generating function to produce a scalar evaluative index. Based on this framework, a structured decision-making algorithm was constructed to rank alternatives. An application to corporate bond evaluation demonstrated that the proposed model yields stable and discriminative rankings that are consistent with the fundamental financial characteristics of the evaluated firms. Comparative analysis indicated that the framework provides a more comprehensive representation of uncertainty than existing models. In contrast,sensitivity analysis confirmed the robustness of the ranking results against variations in input parameters. Overall, the proposed HHFNSS-based soft yield framework offers a flexible and effective approach for multicriteria evaluation under complex uncertainty conditions. This research aligns with Decent Work and Economic Growth (SDG) 8 , specifically Target 8.10, by contributing to the strengthening of domestic financial institutions' capacity for transparent and replicable corporate bond evaluation. The proposed framework supports more efficient capital allocation, which is foundational to sustained economic growth and the creation of decent work opportunities.

Keywords: hybrid hesitant fuzzy N-soft set; soft yield; hesitant fuzzy soft set; soft set; N-soft set; uncertainty modeling; corporate bond evaluation; SDG's 8
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1. Introduction

The corporate bond market is a cornerstone of the modern financial system, providing medium- and long-term financing for corporations and serving as a primary investment vehicle for institutional investors focused on cash flow stability and risk management [1–3]. Bond yield, defined as the effective annual return from holding a bond, represents more than a nominal rate of return. It aggregates multiple factors, including credit risk, liquidity, inflation expectations, macroeconomic conditions, and the contractual characteristics of the instrument. As a result, bond yield evaluation constitutes a multidimensional problem characterized by parameter interdependencies and inherent uncertainty.

In practice, investment decision-making rarely benefits from complete or precise information when evaluating yield-determining factors. As such, analysts must contend with imperfect data, conflicting indicators, and subjective judgments, such as assessments of business prospects and industry stability. Therefore, evaluation cannot be reduced to a single deterministic value but must account for epistemic and cognitive uncertainty. While classical probabilistic approaches are theoretically robust, they rely on distributional assumptions and sufficient historical data [4–6]. These requirements are frequently unmet in the heterogeneous and dynamic corporate bond market, limiting the effectiveness of such methods in addressing uncertainty from subjective assessments and ambiguous information.

Fuzzy set theory has emerged as an alternative for representing cognitive and linguistic uncertainty [7–12]. In a fuzzy set, each object is assigned a degree of membership ranging continuously from 0 to 1, in contrast to the binary membership of classical set theory. However, classical fuzzy sets require decision-makers to assign a single precise membership value to each element, which is often unrealistic in situations with ambiguous information and varying evaluator expertise. To address this, Torra and Narukawa introduced hesitant fuzzy sets (HFSs) in 2009, enabling an element to be associated with a set of possible membership values [13,14]. This approach explicitly models hesitation, making it more suitable for decision-making environments characterized by conflicting evaluator information.

Since their introduction, HFSs have been extended in several important directions. Probabilistic HFSs incorporate probability distributions over membership values, enabling the representation of hesitation and statistical information simultaneously [15]. Developments have also integrated HFSs with cloud models and divergence measures, leading to sophisticated multi-criteria sorting methodologies for complex real-world problems, such as medical insurance fraud detection [16]. Additionally, case-based reasoning approaches have been combined with hesitant fuzzy frameworks to enhance classification and identification capabilities in fraud detection contexts [17]. These advancements demonstrate the continued evolution and practical utility of hesitant fuzzy modeling across application domains.

Soft set theory, introduced by Dmitri Molodtsov in 1999, offers a flexible parametric framework for managing uncertainty without the need for probabilistic distributions or rigid preference functions [18,19]. The integration of HFSs with soft sets, as developed by Zeshui Xu in 2014, resulted in hesitant fuzzy soft sets, which enable simultaneous representation of parameter uncertainty and evaluative hesitation [20,21]. However, this framework typically operates within a continuous domain and does not explicitly accommodate the discrete granularity often required in practical financial evaluations. Advances in hesitant fuzzy soft set theory have explored connections with other uncertainty frameworks, including probabilistic hesitant fuzzy information and cloud-based representation models [15–17]. Yet, the fundamental challenge of integrating discrete categorical assessments with continuous hesitant membership remains an open problem.

In bond markets, many key parameters, such as credit ratings, risk levels, and liquidity quality, are evaluated for discrete or categorical scales. N-soft sets, an extension of soft sets, facilitate evaluation across discrete levels [22,23]. In this framework, a soft set provides a flexible approach to parameter evaluation, while an N-soft set assigns each parameter to a specific level within the discrete domain. However, the original N-soft set formulation assumes a deterministic mapping to a single level, which does not capture uncertainty in categorical assignment. Hesitant N-soft sets (HNSS) address this limitation by allowing each object-parameter pair to be associated with multiple possible grades [24]. Although HNSS effectively captures uncertainty at the grade level, it does not reflect the intensity of membership within each category.

Hesitant fuzzy N-soft sets (HFNSS) extend the N-soft set framework by incorporating hesitant fuzzy membership degrees, allowing an object-parameter pair to receive several possible membership values that reflect evaluators' hesitation. However, HFNSS still requires each pair to be assigned a single discrete grade, capturing hesitation only in intensity and not in categorical classification [25,26]. This model effectively presumes consensus among evaluators regarding the grade for each pair. The multi-fuzzy N-soft set (MFNSS) approach [27] permits multiple membership values in parallel but lacks a defined process for integrating a grade-based discrete structure or reconciling systematic uncertainty across evaluators. In the model proposed by Fatimah and Alcantud, hesitant structures are absent; each object-parameter pair receives only one membership degree from each evaluator. Even when multiple evaluators contribute, their membership degrees, which indicate intensity or confidence in the grade assignment, do not form an HFS.

The limitations of models are particularly significant in corporate bond evaluation, where differences in evaluator perceptions and information ambiguity are inherent. For example, two evaluators may assign different risk categories to the same bond and express varying levels of confidence in their assessments. Models that capture only one dimension of uncertainty risk omitting critical information that influences decision-making. Therefore, a mathematical framework is needed that can simultaneously integrate uncertainty in categorical classification and membership intensity within a coherent structure.

Extensions of N-soft sets improve uncertainty representation by enriching either the discrete grade dimension or the fuzzy membership dimension, but they continue to treat these two sources of uncertainty independently. From a cognitive perspective, however, these uncertainties arise from different stages of the evaluation process. As such, makers must first determine the most appropriate qualitative category (e.g., risk level or credit quality), after which they assess the confidence or degree to which the evaluated object belongs to that category. In many practical settings, particularly in corporate bond assessment involving multiple experts, disagreement may occur simultaneously at both stages. Existing HNSS models preserve disagreement in categorical classification but assume complete certainty regarding membership intensity, whereas HFNSS models preserve hesitation in membership intensity only after a unique grade has been assigned. Consequently, both frameworks inevitably discard part of the original evaluative information before the decision-making process begins. This information loss may reduce the ability of the resulting model to faithfully represent heterogeneous expert judgments.

Motivated by this representational limitation, we propose the hybrid hesitant fuzzy N-soft set (HHFNSS), not merely as a hybridization of existing fuzzy models, but as a mathematical framework that simultaneously preserves two fundamentally different uncertainty dimensions within a unified algebraic structure. The proposed framework represents each object-parameter pair as a finite collection of ordered pairs consisting of a discrete grade and its associated hesitant fuzzy membership set. This construction enables categorical ambiguity and membership uncertainty to coexist until the reduction stage, thereby avoiding premature information aggregation. Consequently, the proposed model provides a richer representation of expert knowledge while remaining compatible with existing N-soft set theory through rigorous reduction properties.

Additionally, a formal reduction mechanism is introduced to map the set-valued representation to a single pair, resulting in a reduced fuzzy N-soft set. A soft yield-generating function further transforms this information into a parameter-based continuous evaluative index. In this framework, yield is conceptualized as an evaluative construct representing the relative strength of a bond with respect to its yield-determining factors, rather than as a financial quantity to be predicted. Beyond its technical contributions, this research also aligns with United Nations Sustainable Development Goal 8 (Decent Work and Economic Growth), specifically Target 8.10, which calls for strengthening the capacity of domestic financial institutions to encourage and expand access to banking, insurance, and financial services for all (indicator 8.10.2). By developing an advanced mathematical decision-making framework for corporate bond assessment under uncertainty, the proposed model supports more efficient capital allocation and contributes to the stability and sustainability of financial markets. Efficient capital markets are essential for fostering productive investment, job creation, and sustained economic growth (SDG) Target 8.5, thereby providing a direct channel through which the proposed framework contributes to the SDG 8 agenda.

2. Preliminaries

2.1. Soft sets (SS)

Let U be the universal set of objects and E the set of parameters. The notation $P(U)$ denotes the power set of U, while $A, B \subseteq E$. The concept of a soft set is defined as follows:

Definition 1. [18] *An pair (F, A) is called a soft set over U if F is a mapping from A to the set of all subsets of U, that is:*

$$F: A \to P(U).$$

Formally, the soft set (F, A) can be expressed as a set of sorted pairs between each parameter $e \in A$ with the functional value of $F(e)$, shown in Eq (1)

$$(F, A) = \{F(e) \in P(U), e \in A\}. \tag{1}$$

Example 1. Suppose there are six investment alternatives in the universe U given by $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Let the parameter set be $A = \{e_1, e_2, e_3\}$, where e_1 = 'Consistent historical return', e_2 = 'Low probability of loss', and e_3 = 'Consistent historical growth'. An investor aims to describe investment choices in terms of alternatives that exhibit consistent historical returns, low probability of loss, and consistent historical growth. Define:

$$F(e_1) = \{u_2, u_4\}, \qquad F(e_2) = \{u_1\}, \qquad F(e_3) = \{u_2, u_3, u_5, u_6\}.$$

Then, (F, A) constitutes a parameterized family of subsets of U, providing an approximate description of the objects under consideration.

Next, the definition of the soft subset operation is introduced as follows:

Definition 2. [28] *Let (F, A) and (L, B) be two soft sets over the universe U. The soft set (F, A) is said to be a soft subset of (L, B), denoted by $(F, A) \tilde{\subseteq} (L, B)$, if $A \subseteq B$ and $F(e) \subseteq L(e)$ for every $e \in A$.*

This definition extends the classical subset relation from ordinary sets to soft sets. It requires not only that every parameter in A appears in B, but also that the approximate value set under each parameter in A is contained within the corresponding set of B. Thus, $(F, A) \tilde{\subseteq} (L, B)$ formalizes the idea that the soft set (F, A) provides a less or equally specific approximation compared to (L, B).

In addition, the AND and OR operations are defined as follows.

Definition 3. [19] *The AND operation of (F, A) and (L, B), denoted by $(F, A) \wedge (L, B)$, results in a soft set $(K, A \times B)$, where for each $(a, b) \in A \times B$, $K(a, b) = F(a) \cap L(b)$.*

The AND operation combines two soft sets by taking all pairs of parameters from the first and second soft sets. The resulting approximation set under the combined parameter (a, b) is the intersection of $F(a)$ and $L(b)$. Intuitively, this represents objects that satisfy the condition described by a in the first soft set and the condition described by b in the second soft set.

Definition 4. [19] *The OR operation of the soft sets (F, A) and (L, B), denoted by $(F, A) \vee (L, B)$, results in a soft set $(K, A \times B)$, where for each $(a, b) \in A \times B$, $K(a, b) = F(a) \cup L(b)$.*

Unlike the AND operation, the OR operation uses union instead of intersection. Hence, for each parameter pair (a, b), $K(a, b)$ contains elements that satisfy at least one of the two conditions; either the description given by a in (F, A) or the description given by b in (L, B). Together, Definitions 3 and 4 provide the soft set analogues of logical conjuction and disjunction.

2.2. Fuzzy set (FS)

In 1965, Lotfi A. Zadeh introduced a mathematical framework to represent imprecise or vague phenomena, which later became known as fuzzy set theory. Unlike classical (crisp) sets that require an element either to belong fully or not at all, fuzzy sets enable partial membership, making them suitable for modeling real-world ambiguity such as tall height or high temperature. The formal definition of a fuzzy set is given as follows:

Definition 5. [7] *Let U be a universal set of objects. A fuzzy set μ on U is defined by a membership function $\mu: U \to [0,1]$. For each $u \in U$, $\mu(u)$ represents the degree of membership of u in the fuzzy set. The fuzzy set μ can be expressed as*

$$\mu(u) = \{(u, \mu(u)), u \in U\}. \tag{2}$$

2.3. Hesitant fuzzy set (HFS)

HFSs are introduced to enable multiple membership values to be assigned to a single element u in the universal set U. An individual evaluator or a committee may provide more than one membership degree due to uncertainty or hesitation in determining a single precise value. This concept is formally defined as follows:

Definition 6. [13,14] *Let U be a universal set. A hesitant fuzzy set (HFS) on U is characterized by a function h that, when applied to U, returns a subset of the interval* $[0, 1]$.

Let μ be a membership function of a parameter $e \in A$ with respect to an element $u \in U$, and let $M = \{\mu_1, \dots, \mu_I\}$ be a set consisting of I membership functions. Then, the HFS associated with M, denoted by h_M, can be expressed as shown in Eq (3)

$$h_M(u) = \bigcup_{\mu \in M} \{\mu(u)\}. \tag{3}$$

HFS are characterized by the property that $h(u)$ is finite for each $u \in U$. Several illustrative examples of HFS are provided below.

Example 2. Let U be a universal set. The following cases describe particular instances of HFSs:

- Empty set: $h(u) = \{0\}$ for all $u \in U$.
- Full set: $h(u) = \{1\}$ for all $u \in U$.
- Complete ignorance for $u \in U$ (all values possible): $h(u) = [0,1]$.
- Inconsistent or ill-defined set for u: $h(u) = \emptyset$.

It is important to note that, in classical fuzzy set theory, the empty set and the full set are defined, respectively, by $\mu(u) = 0$ and $\mu(u) = 1$. However, such representations are not capable of expressing complete ignorance or inconsistency.

2.4. N-soft set (NSS)

Since its introduction by Dmitri Molodtsov in 1999, soft set theory has undergone numerous extensions. One significant variant is the N-soft set, first proposed by Fatimah et al. [22]. This concept enables evaluation on scales with more than two levels, thereby extending beyond the classical binary framework. The formal definition of an N-soft set is given as follows:

Definition 7. [22] *Let $G = \{0,1, \dots, N-1\}$ be a set of grades, where $N \in N$ and $N \geq 2$. A triple (F, A, N) is called an N-soft set over U if F is a mapping defined by*

$$F: A \to P(U \times G),$$

with the property that for each $e \in A$, there exists exactly one pair $(u, g_{u,e})$, where $u \in U$ and $g_{u,e} \in G$, such that $(u, g_{u,e}) \in F(e)$. Furthermore, the notation $(u, g_{u,e})$ can be equivalently expressed as in Eq (4)

$$F(e) = g_{u,e}. \tag{4}$$

Let $U = \{u_1, u_2, u_3, \dots, u_m\}$ and $A = \{e_1, e_2, e_3, \dots, e_n\}$ be finite sets. Then, the *N-soft set* (F, A, N) can be represented in tabular form, as shown in Table 1, where each entry g_{mn} denotes the value of $F(u_m)(e_n)$ [22].

Table 1. Tabular representation of the N-soft set.

(F, A, N)	e_1	e_2	e_3	$\cdots$	e_n
u_1	g_{11}	g_{12}	g_{13}	$\cdots$	g_{1n}
u_2	g_{21}	g_{22}	g_{23}	$\cdots$	g_{2n}
u_3	g_{31}	g_{32}	g_{33}	$\cdots$	g_{3n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	g_{m1}	g_{m2}	g_{m3}	$\cdots$	g_{mn}

In Table 1, g_{mn} means $F(u_m)(e_n)$, for all m and n.

Example 3. A prospective investor intends to select stocks of companies newly listed on the Indonesia Stock Exchange. There are four companies under consideration, $U = (u_1, u_2, u_3, u_4)$. The evaluation is conducted by three senior investors, $A = (e_1, e_2, e_3)$, using a rating scale from 0 to 4, where 0 denotes very poor and 4 denotes very good. Accordingly, a 5-soft set is constructed as follows:

$$F(e_1) = \{(u_1, 3), (u_2, 1), (u_3, 4), (u_4, 2)\},$$

$$F(e_2) = \{(u_1, 2), (u_2, 2), (u_3, 3), (u_4, 4)\},$$

$$F(e_3) = \{(u_1, 1), (u_2, 3), (u_3, 2), (u_4, 4)\}.$$

The tabular representation of $(F, A, 5)$ is shown in Table 2 [19].

Table 2. Tabular representation of Example 3.

$(F, A, 5)$	e_1	e_2	e_3
u_1	3	2	1
u_2	1	2	3
u_3	4	3	2
u_4	2	4	4

2.5. Fuzzy N-soft set (FNSS)

A further development of the N-soft set is the fuzzy N-soft set, also referred to as the (F, N)-soft set [23], which integrates the advantages of N-soft sets and fuzzy sets. In this model, each pair $(u, g_{u,e})$ is augmented with a corresponding fuzzy membership degree. Based on Definition 5 of fuzzy sets, the fuzzy N-soft set can be formally defined as follows:

Definition 8. [23] *An ordered pair* (μ, K) *is called a fuzzy N-soft set if* $K = (F, A, N)$ *is an N-soft set over* U*, and* μ *is a function that maps each* $e \in A$ *into the fuzzy set* $[0, 1]^{F(e)}$*; that is, for every* $e \in A$*, it holds that* $\mu(e) \in [0,1]^{F(e)}$.

More specifically, for each $e \in A$ and $u \in U$, there exists a unique pair $(u, g_{u,e})$, where $g_{u,e} \in G$, such that $\left((u, g_{u,e}), \mu_e(u)\right) \in \mu(e)$, which is denoted by $\mu_e(u) = \mu(e)(u, g_e)$.

Let $U = \{u_1, u_2, u_3, \dots, u_m\}$ and $A = \{e_1, e_2, e_3, \dots, e_n\}$ be finite sets. Suppose $K = (F, A, N)$ is an N-soft set over U. Then, fuzzy N-soft set (μ, K) can be represented in tabular form, as illustrated in Table 3 [23].

Table 3. Tabular representation of the fuzzy N-soft set.

(μ, K)	e_1	e_2	e_3	...	e_n
u_1	(g_{11}, μ_{11})	(g_{12}, μ_{12})	(g_{13}, μ_{13})	...	(g_{1n}, μ_{1n})
u_2	(g_{21}, μ_{21})	(g_{22}, μ_{22})	(g_{23}, μ_{23})	...	(g_{2n}, μ_{2n})
u_3	(g_{31}, μ_{31})	(g_{32}, μ_{32})	(g_{33}, μ_{33})	...	(g_{3n}, μ_{3n})
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	(g_{m1}, μ_{m1})	(g_{m2}, μ_{m2})	(g_{m3}, μ_{m3})	...	(g_{mn}, μ_{mn})

Example 4. An IT consulting firm intends to recruit a project manager. There are three candidates under consideration, $U = \{u_1, u_2, u_3\}$. The evaluation criteria consist of four aspects:

e_1 = technical skills,

e_2 = team leadership experience,
e_3 = negotiation ability, and
e_4 = professional certification.

The company establishes a grade scale $G = \{0, 1, 2, 3\}$, with a corresponding mapping to fuzzy membership degrees defined as follows:

$$0 \leq \mu(e, u) \leq 0.2 \text{ if } g_e = 0,$$

$$0.2 \leq \mu(e, u) \leq 0.4 \text{ if } g_e = 1,$$

$$0.4 \leq \mu(e, u) \leq 0.7 \text{ if } g_e = 2,$$

$$0.7 \leq \mu(e, u) \leq 1 \text{ if } g_e = 3.$$

Subsequently, the fuzzy 4-soft set is defined as follows:

$$\mu_{e_1} = \{((u_1, 2), 0.62), ((u_2, 1), 0.28), ((u_3, 0), 0.15)\},$$

$$\mu_{e_2} = \{((u_1, 1), 0.30), ((u_2, 3), 0.80), ((u_3, 2), 0.5)\},$$

$$\mu_{e_3} = \{((u_1, 2), 0.60), ((u_2, 2), 0.45), ((u_3, 3), 0.85)\},$$

$$\mu_{e_4} = \{((u_1, 3), 0.75), ((u_2, 2), 0.42), ((u_3, 1), 0.32)\}.$$

Tabular representation of the fuzzy 4-soft set is given in Table 4 [23].

Table 4. Fuzzy 4-soft set in Example 4.

(μ, K)	e_1	e_2	e_3	e_4
u_1	$(2, 0.62)$	$(1, 0.30)$	$(2, 0.60)$	$(3, 0.75)$
u_2	$(1, 0.28)$	$(3, 0.80)$	$(2, 0.45)$	$(2, 0.42)$
u_3	$(0, 0.15)$	$(2, 0.5)$	$(3, 0.85)$	$(1, 0.32)$

2.6. Hesitant N-soft set (HNSS)

Akram et al. [24] first introduced a novel model arising from the hybridization of N-soft sets and hesitancy, which constitutes a key contribution of HFS theory [13,26]. Let $F(U)$ denote the set of all fuzzy sets on U, and $P(U)$ the set of all subsets of U, and $P^*(U)$ the set of all non-empty subsets of U, i.e., $P^*(U) = P(U) \setminus \{\emptyset\}$. This model captures the possibility that evaluators may hold differing opinions when assigning grades to an object under a given parameter, reflecting a multi-granular assessment structure. The formal definition of a hesitant N-soft set is given as follows:

Definition 9. [24] *Let U be a universal set of objects and $A \subseteq E$ a set of parameters. Let $G = \{0,1,2, \dots, N-1\}$ be an ordered set of grades, where $N \in \{2,3, \dots\}$. A triple (H, A, N) is called a hesitant N-soft set (HNSS) over U if H is a mapping*

$$H: A \to P(U \times G),$$

such that for each $e \in A$ and each $u \in U$, there exists at least one $g_{u,e} \in G$ with $(u, g_{u,e}) \in H(e)$.

Equivalently, H can be represented by a function

$$\eta: U \times A \to P^*(G),$$

defined by $\eta(u, e) = \{g_{u,e} \in G \mid (u, g_{u,e}) \in H(e)\}$. Furthermore, for each $e \in A$, the set $H(e)$ can be expressed as shown in Eq (5)

$$H(e) = \{(u, g_{u,e}) \in U \times G \mid g \in \eta(u, e)\}. \tag{5}$$

Example 5. In a company, employee selection is based on star ratings and evaluations provided by a selection panel composed of the director, manager, and HR representatives. For the sake of illustration, let $U = \{u_1, u_2\}$ be the set of candidates participating in the interview process, and let E denote the set of evaluation parameters assessed by the selection panel. Consider a subset $A \subseteq E$ such that $A = \{e_1, e_2, e_3\}$ in this case. A hesitant 4-soft set can then be constructed as given in Eq (6)

$$H = [\{1,2\}\,\{0,1\}\,\{2,3\}\,\{3\}\,\{1,2\}\,\{1,2,3\}\,]. \tag{6}$$

The matrix H can also be expressed in tabular form, as presented in Table 5 [24].

Table 5. Tabular representation of hesitant 4-soft set for Example 5.

$(H, A, 4)$	e_1	e_2	e_3
u_1	{1, 2}	{0, 1}	{2, 3}
u_2	{3}	{1, 2}	{1, 2, 3}

For each star rating $g_{u,e} \in G = \{0,1,2,3\}$, the following interpretations are assigned:

- 3 represents "very good",
- 2 represents "good",
- 1 represents "average",
- 0 represents "poor".

Accordingly, the H4SS can be expressed based on Definition 11 as follows.

$$H(e_1) = \{(u_1, 1), (u_1, 2), (u_2, 3)\} \in P(U \times G),$$

$$H(e_2) = \{(u_1, 0), (u_1, 1), (u_2, 1), (u_2, 2)\} \in P(U \times G),$$

$$H(e_3) = \{(u_1, 2), (u_1, 3), (u_2, 1), (u_2, 2), (u_2, 3)\} \in P(U \times G).$$

2.7. Hesitant fuzzy N-soft set (HFNSS)

In 2019, Akram et al., as cited in Ashraf et al. [26], first introduced the concept of the HNFSS as a model and framework for representing hesitation on the part of an evaluator or a panel of evaluators when assigning the membership degree of a parameter to an object in the form of a HFS set, combined with grade-based assessments according to categories or evaluation classes through the N-soft set structure. In this model, hesitation is represented solely through uncertainty in determining membership degrees, whereas the granular grade N assigned by each evaluator is assumed to be unique. The formal definition of a HFSS is given below.

Definition 10. [26] *Let U be a universal set of objects and $A \subseteq E$ be a set of parameters. Let $G = \{0, 1, 2, \dots, N-1\}$ be an ordered set of grades, where $N \in \{2,3, \dots\}$. A Triple (H', A, N) is called a HFNSS over U if H' is a mapping*

$$H': U \times A \to G \times P^*([0,1]),$$

such that $H'(u, e) = (g_{u,e}, h_{u,e})$, where $g_{u,e} \in G$ is a unique grade assigned to the pair (u, e), and $h_{u,e} \in [0,1]$. $h_{u,e} \neq \emptyset$ represents the hesitancy in the membership degree.

Furthermore, for each $e \in A$, the HNFSS can be expressed as shown in Eq (7)

$$H'(u,e) = \{(g_{u,e}, h_{u,e}) | g_{u,e} \in G, h_{u,e} \in [0,1]\}. \tag{7}$$

Let $U = \{u_1, u_2, u_3, \ldots, u_m\}$ and $A = \{e_1, e_2, e_3, \ldots, e_n\}$ be finite sets, with $g_{u,e} \in G = \{0, 1, 2, 3, \ldots, N-1\}$ and $H_{u,e} \subseteq [0, 1]$. Then, the HFNSS (H', A, N) can be expressed in matrix form as shown in Eq (8)

$$\mathcal{H}' = \begin{bmatrix} (g_{11}, h_{11}) & \cdots & (g_{1n}, h_{1n}) \\ \vdots & \ddots & \vdots \\ (g_{i1}, h_{i1}) & \cdots & (g_{in}, h_{in}) \\ \vdots & \ddots & \vdots \\ (g_{m1}, h_{m1}) & \cdots & (g_{mn}, h_{mn}) \end{bmatrix}. \tag{8}$$

Example 6. Using the same setting as in Example 5, the 4-soft set can be presented in Table 6 [26].

Table 6. Tabular representation of the 4-soft set for Example 6.

$(H', A, 4)$	e_1	e_2	e_3
u_1	1	3	2
u_2	2	2	0

Subsequently, the director, manager, and HR representatives determine the membership degrees for each object with respect to each parameter. The hesitant fuzzy 4-soft set is presented in Table 7.

Table 7. Tabular representation of hesitant fuzzy 4-soft set for Example 6.

$(H', A, 4)$	e_1	e_2	e_3
u_1	$(1, \{0.6, 0.9\})$	$(3, \{0.3, 0.5, 0.8\})$	$(2, \{0.4, 0.5\})$
u_2	$(2, \{0.35, 0.48\})$	$(2, \{0.8\})$	$(0, \{0.2, 0.3, 0.35\})$

2.8. The proposed hybrid hesitant fuzzy N-soft sets

The proposed model integrates the grade hesitancy framework derived from the HFSS [24] into the structure of the HFNSS [26] in order to establish a more robust decision-making framework. Within this framework, hesitancy in the grade reflects disagreement or ambiguity among evaluators when classifying an object into a particular discrete category, whereas hesitancy in the membership degree reflects uncertainty regarding the intensity or strength of an object's affiliation with the evaluated parameter. It should be emphasized that the granularity N is global and fixed; therefore, all evaluators operate under the same categorical scheme. Evaluator heterogeneity is represented internally through the hesitancy-set structure than through differences in granularity.

Formally, the model is defined as follows.

Definition 11. *Let U be the universal set of objects, $A \subseteq E$ be the set of parameters, and $G = \{0,1, \ldots, N-1\}$, where $N \in \{2,3, \ldots\}$ is the ordered grade set (global granularity). The triple (H^*, A, N) is called a HHFNSS over U if H^* is a mapping*

$$H^*: U \times A \to P^*(G \times P^*([0, 1])),$$

such that for each $(u, e) \in U \times A, H^*(u, e)$ is a finite nonempty set of ordered pairs $(g_{u,e}, h_{u,e})$, where $g_{u,e} \in G$ and $h_{u,e} \subseteq [0, 1]$ with $h_{u,e} \neq \emptyset$. Component $g_{u,e}$ represents hesitancy in the grade, while component $h_{u,e}$ represents hesitancy in the membership degree.

Throughout this paper, unless explicitly stated otherwise, we assume that $u \in U, e \in A, g \in G$, and $h \subseteq [0,1]$. Furthermore, define the projections $G_{u,e} = \{g_{u,e} \mid (g_{u,e}, h_{u,e}) \in H^*(u,e)\} \subseteq G$ and $H_{u,e} = \bigcup\{h_{u,e} \mid (g_{u,e}, h_{u,e}) \in \mathcal{H}^*(u,e)\} \subseteq [0,1]$, so that $H^*(u,e)$ can be expressed as Eq (9)

$$H^*(u,e) = \left\{\left(g_{u,e}^{(k)}, h_{u,e}^{(k)}\right)\right\}_{k=1}^{m_{u,e}}, \tag{9}$$

where $m_{u,e}$ denotes the number of pairs in the set $H^*(u,e)$ for object u under parameter e. The properties of the HHFNSS are given as follows:

1. $G_{\{u,e\}} \subseteq G$ is a nonempty hesitant grade set obtained as the projection of $H^*(u,e)$, representing the evaluator's hesitation in assigning a discrete category to object $u \in U$ with respect to parameter $e \in A$.
2. $H_{\{u,e\}} \subseteq [0,1]$ is a nonempty hesitant fuzzy membership degree set obtained from the union of the elements $h_{u,e}$ in $H^*(u,e)$, representing the evaluator's uncertainty in assigning a membership degree to object $u \in U$ with respect to parameter $e \in A$.

Let $U = \{u_1, u_2, \ldots, u_m\}$ and $A = \{e_1, e_2, \ldots, e_n\}$ be finite sets. Then, the HHFNSS (H^*, A, N) can be represented in matrix form as Eq (10)

$$\mathcal{H}^* = \begin{bmatrix} \left\{\left(g_{11}^{(k)}, h_{11}^{(k)}\right)\right\}_{k=1}^{m_{u,e}} & \left\{\left(g_{12}^{(k)}, h_{12}^{(k)}\right)\right\}_{k=1}^{m_{u,e}} & \cdots & \left\{\left(g_{1n}^{(k)}, h_{1n}^{(k)}\right)\right\}_{k=1}^{m_{u,e}} \\ \left\{\left(g_{21}^{(k)}, h_{21}^{(k)}\right)\right\}_{k=1}^{m_{u,e}} & \left\{\left(g_{22}^{(k)}, h_{22}^{(k)}\right)\right\}_{k=1}^{m_{u,e}} & \cdots & \left\{\left(g_{2n}^{(k)}, h_{2n}^{(k)}\right)\right\}_{k=1}^{m_{u,e}} \\ \vdots & \vdots & \ddots & \vdots \\ \left\{\left(g_{m1}^{(k)}, h_{m1}^{(k)}\right)\right\}_{k=1}^{m_{u,e}} & \left\{\left(g_{m2}^{(k)}, h_{m2}^{(k)}\right)\right\}_{k=1}^{m_{u,e}} & \cdots & \left\{\left(g_{mn}^{(k)}, h_{mn}^{(k)}\right)\right\}_{k=1}^{m_{u,e}} \end{bmatrix}. \tag{10}$$

Example 7. Consider the evaluation of an investment based on two parameters, namely credit risk (e_1) and liquidity (e_2). The grade set is $G = \{0,1,2,3,4\}$ with $N = 5$, where the categories are defined as follows: 0 = very low, 1 = low, 2 = moderate, 3 = high, and 4 = very high. Two investments, u_1 and u_2, are assessed. Two evaluators provide judgments regarding this investment. The investment evaluation results are presented in tabular form in Table 8.

Table 8. Tabular representation of hybrid hesitant fuzzy 5-soft set for Example 7.

$(H^*, A, 5)$	e_1	e_2
u_1	$\{(2,\{0.8,0.69\}),(1,\{0.3,0.36,0.43\})\}$	$\{(0,\{0.8,0.67\}),(2,\{0.8,0.93\})\}$
u_2	$\{(4,\{0.6,0.7,0.8\}),(2,\{0.4\})\}$	$\{(3,\{0.5\}),(3,\{0.4,0.48\})\}$

Proposition 1. If, for every $u \in U$ and $e \in A$, it holds that $|G_{u,e}| = 1$ and $H_{u,e} = \{1\}$, then the HHFNSS reduces to the classical N-soft set.

Proof. Let U be the universal set of objects, E the set of parameters with $A \subseteq E$, and $G = \{0,1,2,\ldots,N-1\}$ the ordered grade set, where $N \geq 2$.

Suppose (H^*, A, N) is a HHFNSS over U. According to Definition 13, $G_{u,e} \subseteq G$ and $H_{u,e} \subseteq [0,1]$ are nonempty sets.

Assume that for every $u \in U$ and $e \in A$:

1. $|G_{u,e}| = 1$, so, there exists some $g_{u,e} \in G$ such that

$$G_{u,e} = \{g_{u,e}\} \tag{11}$$

2. $H_{u,e} = \{1\}$.

Hence, each element of the mapping in Eq (9) becomes

$$H^*(u,e) = (\{g_{u,e}\},\{1\}). \tag{12}$$

Since $G_{u,e}$ is a singleton set, there is no discrete hesitation in the assigned grade. Morover, since $H_u^e = \{1\}$, there is no fuzzy uncertainty, and the membership degree is complete.

Therefore, the mapping in Definition 13 simplifies to $H^*: U \times A \to G \times \{1\}$, which is equivalently written as $H^*: U \times A \to G$ with the form

$$H^*(u,e) = (g_{u,e}). \tag{13}$$

Based on Definition 7, the above mapping satisfies the structure of a triple (F, A, N) with mapping $F: U \times A \to G$. Since every HHFNSS satisfying conditions (1) and (2) can be represented by a mapping F that fulfills Definition 7, it follows that (H^*, A, N) reduces to (F, A, N), namely, a classical N-soft set. Thus, the proposition is proved.

2.9. Reduction of fuzzy N-soft set (RFNSS)

In the HHFNSS model, the evaluation information for each pair $(u,e) \in U \times A$ is set-valued, reflecting hesitation in the discrete grade and the membership degree. This hesitation in the discrete grade is introduced by the multi-evaluator setting, where every object u with respect to parameter e receives discrete-grade assessments from several evaluators. Notably, unlike the hesitant fuzzy component represented as a multiset, hesitation occurs at two levels: The multi-evaluator condition generates the hesitant fuzzy component, and each evaluator can assign multiple membership degrees to each u with respect to e.

Building on this structure, the HHFNSS model requires a reduction mechanism to facilitate its practical use in constructing soft yield models and decision-making procedures. This mechanism maps each set-valued element to a single ordered pair comprising a representative grade and a representative membership degree. As a result, the reduction produces a fuzzy N-soft set free of internal hesitancy, while retaining essential information from the original evaluation process. This approach aligns with established reduction concepts in hesitant fuzzy models, transforming multivalued representations into informative single-valued forms [24,25, 27–30].

Definition 12. *(Grade average score) Let (H^*, A, N) be an HHFNSS defined on the universal set U and the parameter set $A \subseteq E$, with global grade set $G = \{0,1,2,\ldots,N-1\}, N \geq 2$. For each $u \in U$ and $e \in A$, let the hesitant grade set obtained from the HHFNSS projection be denoted by $G_{u,e}$. The grade average score of the pair (u,e) is defined as*

$$\bar{g}_{u,e} = \left(\frac{1}{m_{u,e}} \sum_{k=1}^{m_{u,e}} g_{u,e}^{(k)}\right) \in \mathbb{R}, \tag{14}$$

where $m_{u,e} = |G_{u,e}|$ denotes the number of grade values in the hesitant grade set.

Since each $g_{u,e}^{(k)}$ belongs to G, it follows directly that $\underline{g}_{u,e} \in [0, N-1]$, where $\underline{g}_{u,e}$ represents the average form of evaluator hesitation.

Example 8. Using the case in Example 7, the grade average scores are obtained as follows:

- $\bar{g}_{u_1,e_1} = \left(\frac{1}{2}\sum_{k=1}^{2} g_{u_1,e_1}^{(k)}\right) = \frac{1}{2}(2+1) = 1.5$
- $\bar{g}_{u_1,e_2} = \left(\frac{1}{2}\sum_{k=1}^{2} g_{u_1,e_2}^{(k)}\right) = \frac{1}{2}(0+2) = 1$
- $\bar{g}_{u_2,e_1} = \left(\frac{1}{2}\sum_{k=1}^{2} g_{u_2,e_1}^{(k)}\right) = \frac{1}{2}(4+2) = 3$

- $\bar{g}_{u_2,e_2} = \left(\frac{1}{2}\sum_{k=1}^{2} g_{u_2,e_2}^{(k)}\right) = \frac{1}{2}(3+3) = 3$

Definition 13. *(Reduced grade function) Let* $\underline{g}_{u,e}$ *be the grade average score for the pair* (u,e). *Define the projection operator onto the discrete grade set as the mapping*

$$Q: [0, N-1] \to G.$$

The nearest-neighbour selection $Q\left(\underline{g}_{u,e}\right)$ *is defined as*

$$\begin{aligned} Q\left(\underline{g}_{u,e}\right) &= arg\ arg\ \left(\underline{g}_{u,e} - j\right)^2 \\ &= arg\ arg\ \left|\underline{g}_{u,e} - j\right| . \end{aligned} \tag{15}$$

Subsequently, the reduced grade is given by

$$GA(u,e) = Q\left(\underline{g}_{u,e}\right). \tag{16}$$

Operator $Q\left(\underline{g}_{u,e}\right)$ satisfies the nearest-neighbour quantization rule, which is optimal under the mean squared error criterion [31]. For each $\underline{g}_{u,e} \in [0, N-1]$, $Q\left(\underline{g}_{u,e}\right)$ is defined as the element $j \in G$ that minimizes $\left(\underline{g}_{u,e} - j\right)^2$. If more than one minimizer exists, the smaller grade is selected. In other words, $Q\left(\underline{g}_{u,e}\right)$ denotes the grade $j \in G$ closest to $\underline{g}_{u,e}$. Equation (16) gives the unique reduced discrete grade by selecting the smallest $j \in G$ whenever multiple grades have equal distance from $\left|\underline{g}_{u,e} - j\right|$.

Remark 1. The arithmetic mean is employed as the aggregation operator for the hesitant grade values for several reasons. First, it is the most widely used and easily interpretable aggregation operator in multi-criteria decision-making, facilitating transparency and practitioner adoption. Second, while the grade scale $G = \{0, 1, 2, \ldots, N-1\}$ is formally ordinal, the use of a five-level scale $(N = 5)$ with linguistically defined categories (very low, low, medium, high, very high) approximates an interval scale for practical purposes, particularly when fuzzy memberhsip intervals are used in conjunction with the grade values. Third, the nearest-neighbour quantization operator introduced in Definition 13 and the conservative tie-breaking rule (selecting the smaller grade when multiple minimizers exist) mitigate potential distortions arising from the interval assumption by ensuring that the final reduced grade belongs to G. Fourth, the arithmetic mean is equivalent to minimizing the sum of squared deviations, which is the optimal aggregation rule under the mean squared error criterion [31]. Finally, the use of the arithmetic means as a first-stage aggregation step is standard practice in hesitant fuzzy decision-making [24,25, 27–30].

Example 9. Using the case in Example 7 and the grade average scores obtained in Example 8, the reduced grade function is obtained as follows.

- $Q\left(\underline{g}_{u_1,e_1}\right) = arg\ arg\ \left|\underline{g}_{u_1,e_1} - j\right| = arg\ arg\ |1.5 - j| = arg\ arg\ |0.5| = \{1, 2\}$
- $GA(u_1, e_1) = \left(Q\left(\underline{g}_{u_1,e_1}\right)\right) = (\{1, 2\}) = 1$
- $Q\left(\underline{g}_{u_1,e_2}\right) = arg\ arg\ \left|\underline{g}_{u_1,e_2} - j\right| = arg\ arg\ |1 - j| = arg\ arg\ |0| = \{1\}$
- $GA(u_1, e_2) = \left(Q\left(\underline{g}_{u_1,e_2}\right)\right) = (\{1\}) = 1$
- $Q\left(\underline{g}_{u_2,e_1}\right) = arg\ arg\ \left|\underline{g}_{u_2,e_1} - j\right| = arg\ arg\ |3 - j| = arg\ arg\ |0| = \{3\}$

- $GA(u_2, e_1) = \left(Q\left(\underline{g}_{u_1,e_2}\right)\right) = (\{3\}) = 3$
- $Q\left(\underline{g}_{u_2,e_2}\right) = arg\ arg\ \left|\underline{g}_{u_2,e_2} - j\right| = arg\ arg\ |3 - j| = arg\ arg\ |0| = \{3\}$
- $GA(u_2, e_2) = \left(Q\left(\underline{g}_{u_2,e_2}\right)\right) = (\{3\}) = 3.$

Definition 14. *(fuzzy reduction) For every $u \in U$ and $e \in A$, let (H^*, A, N) be an HHFNSS defined on U, with $e \in A$. For every $(u, e) \in U \times A$, an internal reduction is first applied to the hesitant fuzzy element $h_{u,e}^{(k)}$, yielding a single value via Eq (17)*

$$\phi\left(h_{u,e}^{(k)}\right) = \left(\frac{1}{\left|h_{u,e}^{(k)}\right|} \sum_{\mu \in h_{u,e}^{(k)}} \mu\right) \in [0, 1]. \tag{17}$$

Subsequently, the fuzzy average function is defined as

$$FA(u, e) = \left(\frac{1}{m_{u,e}} \sum_{k=1}^{m_{u,e}} \phi\left(h_{u,e}^{(k)}\right)\right) \in [0, 1]. \tag{18}$$

Example 10. Using the case presented in Example 7, the fuzzy average values are obtained as follows.

- $$\begin{aligned} FA(u_1, e_1) &= \tfrac{1}{m_{u_1,e_1}} \textstyle\sum_{k=1}^{m_{u_1,e_1}} \phi\left(h_{u_1,e_1}^{(k)}\right) \\ &= \tfrac{1}{m_{u_1,e_1}} \textstyle\sum_{k=1}^{m_{u_1,e_1}} \left(\tfrac{1}{\left|h_{u_1,e_1}^{(k)}\right|} \textstyle\sum_{\mu \in h_{u_1,e_1}^{(k)}} \mu\right) \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}(0.8 + 0.69) + \tfrac{1}{3}(0.3 + 0.36 + 0.43)\right) \\ &= 0.5541 \\ &\approx 0.55 \end{aligned}$$
- $$\begin{aligned} FA(u_1, e_2) &= \tfrac{1}{m_{u_1,e_2}} \textstyle\sum_{k=1}^{m_{u_1,e_2}} \phi\left(h_{u_1,e_2}^{(k)}\right) \\ &= \tfrac{1}{m_{u_1,e_2}} \textstyle\sum_{k=1}^{m_{u_1,e_2}} \left(\tfrac{1}{\left|h_{u_1,e_2}^{(k)}\right|} \textstyle\sum_{\mu \in h_{u_1,e_2}^{(k)}} \mu\right) \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}(0.8 + 0.67) + \tfrac{1}{2}(0.8 + 0.93)\right) \\ &= 0.8 \end{aligned}$$
- $$\begin{aligned} FA(u_2, e_1) &= \tfrac{1}{m_{u_2,e_1}} \textstyle\sum_{k=1}^{m_{u_2,e_1}} \phi\left(h_{u_2,e_1}^{(k)}\right) \\ &= \tfrac{1}{m_{u_2,e_1}} \textstyle\sum_{k=1}^{m_{u_2,e_1}} \left(\tfrac{1}{\left|h_{u_2,e_1}^{(k)}\right|} \textstyle\sum_{\mu \in h_{u_2,e_1}^{(k)}} \mu\right) \\ &= \tfrac{1}{2}\left(\tfrac{1}{3}(0.6 + 0.7 + 0.8) + \tfrac{1}{1}(0.4)\right) \\ &= 0.55 \end{aligned}$$
- $$\begin{aligned} FA(u_2, e_2) &= \tfrac{1}{m_{u_2,e_2}} \textstyle\sum_{k=1}^{m_{u_2,e_2}} \phi\left(h_{u_2,e_2}^{(k)}\right) \\ &= \tfrac{1}{m_{u_2,e_2}} \textstyle\sum_{k=1}^{m_{u_2,e_2}} \left(\tfrac{1}{\left|h_{u_2,e_2}^{(k)}\right|} \textstyle\sum_{\mu \in h_{u_2,e_2}^{(k)}} \mu\right) \\ &= \tfrac{1}{2}\left(\tfrac{1}{1}(0.5) + \tfrac{1}{2}(0.4 + 0.48)\right) \\ &= 0.47. \end{aligned}$$

Definition 15. *(Reduced fuzzy N-soft set) Let* (H^*, A, N) *be an HHFNSS defined on* U*. Let* $GA(u, e)$ *and* $FA(u, e)$ *denote the aggregation functions introduced in Definitions 12 and 13, respectively. The triple* (R, A, N) *is called a reduced fuzzy N-soft set (RFNSS) induced by* H^**, defined as the mapping*

$$R: U \times A \rightarrow G \times [0, 1],$$

such that

$$R(u, e) = \big(GA(u, e), FA(u, e)\big). \tag{19}$$

Each object-parameter pair in the HHFNSS is thus reduced to a single ordered pair consisting of a representative grade and a representative fuzzy membership degree. This structure serves as a formal bridge from the HHFNSS framework to the construction of the soft yield.

Example 11. Using the case presented in Example 7, together with the single-grade results obtained in Example 9 and the single fuzzy membership degrees obtained in Example 10, the corresponding reduced fuzzy N-soft set is derived as follows.

- $R(u_1, e_1) = (1, 0.55) \in G \times [0, 1]$.
- $R(u_1, e_2) = (3, 0.55) \in G \times [0, 1]$.
- $R(u_2, e_1) = (1, 0.8) \in G \times [0, 1]$.
- $R(u_2, e_2) = (3, 0.47) \in G \times [0, 1]$.

2.10. The proposed soft yield function

The soft yield-generating function is a mapping that integrates discrete grade information and fuzzy membership degrees into a single continuous evaluative score. The contribution of a parameter to the yield strength of an object becomes high only when both evaluation components, namely the discrete grade and the fuzzy membership intensity, are simultaneously high. In the reduced fuzzy N-soft set, each object-parameter pair has been reduced into a single ordered pair consisting of a representative discrete grade and a representative fuzzy membership value. In order to utilize this pair as the basis for constructing an evaluative yield structure, a transformation function is required to map the pair into a scalar score within the unit interval. This score is not interpreted as a prediction of the bond's actual yield level, but rather as parametric evaluative indices representing the relative strength of a corporate bond with respect to yield-related criteria.

Let H^* be a HHFNSS over the universe of objects U and the parameter set $A \subseteq E$, with global granularity $G = \{0, 1, 2, \dots, N-1\}$, where $N \geq 2$. Let R denote the reduced fuzzy N-soft set induced by H^*, such that for every $u \in U$ and $e \in A$ $R(u, e) = \big(GA(u, e), FA(u, e)\big)$, where $GA(u, e) \in \{0, 1, 2, \dots, N-1\}$ and $FA(u, e) \in [0, 1]$. To unify the domains of both components, the grade normalization function is first introduced.

Definition 16. *(Soft yield generating function) Let* $G = \{0, 1, 2, \dots, N-1\}$ *be the global discrete set with* $N \geq 2$*. Let* $GA(u, e) \in G$ *and* $FA(u, e) \in [0, 1]$ *denote the representative grade and representative fuzzy membership value in the RFNSS, respectively. The grade normalization function is a mapping*

$$\eta: G \rightarrow [0, 1],$$

defined as Eq (20)

$$\eta\big(GA(u,e)\big) = \frac{GA(u,e)}{N-1}, \quad \forall GA(u,e) \in G. \tag{20}$$

Furthermore, the soft yield generating function is defined as a mapping

$$\psi: G \times [0,1] \to [0,1]$$

satisfies

$$\begin{aligned}\psi\big(GA(u,e), FA(u,e)\big) &= \eta\big(GA(u,e)\big).FA(u,e)\\ &= \frac{GA(u,e)}{N-1}.FA(u,e).\end{aligned} \tag{21}$$

The function ψ is referred as the soft yield generating function, which combines discrete grade information and fuzzy membership degrees into a single continuous evaluative score. The contribution of a parameter to the yield strength of an object becomes high only when evaluation components, namely the discrete grade value and the intensity of fuzzy membership, are simultaneously high.

Example 12. Using the RFNSS structure obtained in Example 11 based on the case presented in Example 7, the resulting soft yield generating function is given as follows:

- $$\begin{aligned}\psi\big(GA(u_1,e_1), FA(u_1,e_1)\big) &= \eta\big(GA(u_1,e_1)\big)\cdot FA(u_1,e_1)\\ &= \frac{GA(u_1,e_1)}{N-1}\cdot FA(u_1,e_1)\\ &= \frac{1}{5-1}\cdot(0.55)\\ &= 0.1375\end{aligned}$$
- $$\begin{aligned}\psi\big(GA(u_1,e_2), FA(u_1,e_2)\big) &= \eta\big(GA(u_1,e_2)\big)\cdot FA(u_1,e_2)\\ &= \frac{GA(u_1,e_2)}{N-1}\cdot FA(u_1,e_2)\\ &= \frac{1}{5-1}\cdot(0.8)\\ &= 0.2\end{aligned}$$
- $$\begin{aligned}\psi\big(GA(u_2,e_1), FA(u_2,e_1)\big) &= \eta\big(GA(u_2,e_1)\big)\cdot FA(u_2,e_1)\\ &= \frac{GA(u_2,e_1)}{N-1}\cdot FA(u_2,e_1)\\ &= \frac{3}{5-1}\cdot(0.55)\\ &= 0.4125\end{aligned}$$
- $$\begin{aligned}\psi\big(GA(u_2,e_2), FA(u_2,e_2)\big) &= \eta\big(GA(u_2,e_2)\big)\cdot FA(u_2,e_2)\\ &= \frac{GA(u_2,e_2)}{N-1}\cdot FA(u_2,e_2)\\ &= \frac{3}{5-1}\cdot(0.47)\\ &= 0.3525.\end{aligned}$$

Definition 17. (Soft yield structure) *Let $R(e)$ be a reduced fuzzy N-soft set induced by a HHFNSS as defined in Definition 17. For each parameter $e \in A$, where $A \subseteq E$ and for each $u \in U$, let $R(e) = \big(GA(u,e), FA(u,e)\big)$, where $GA(u,e) \in G$ and $FA(u,e) \in [0,1]$. The soft yield function is defined as the mapping*

$$Y: U \times A \to [0,1]$$

given by

$$Y(u,e)=\psi\big(GA(u,e),FA(u,e)\big). \tag{22}$$

The pair $Y=(Y,A)$ *is called a soft yield structure over the universe of objects* U, *where*

$$Y: A \to [0,1]^U$$

satisfies

$$Y(e)=\big\{\big(u,Y(u,e)\big)\mid u\in U\big\} \tag{23}$$

for all $u\in U$ *and* $e\in A$.

Proposition 2. Let U be a universe of objects and let $A\subseteq E$ be a set of parameters. If a mapping

$$Y: U\times A\to[0,1]$$

is given, then it induces a parametric mapping

$$Y: E\to[0,1]$$

defined by $Y(e)(u)=Y(u,e)$ for all $u\in U$ and $e\in A$.

Proof. Take an arbitrary parameter $e\in A$. Since Y is defined on the domain $U\times A$, for every $u\in U$, the value $Y(u,e)$ is well-defined and belongs to the interval $[0,1]$. Therefore, for each $e\in A$ a mapping

$$Y_e: U\to[0,1]$$

can be defined by

$$Y_e(u)=Y(u,e).$$

Since Y_e maps each element $u\in U$ to an element in $[0,1]$, it follows that Y_e is an element of the function space $[0,1]^U$. Hence, a mapping

$$Y: A\to[0,1]^U$$

can be defined by

$$Y(e)=Y_e.$$

Explicitly, for every pair $(u,e)\in U\times A$, we have

$$Y(e)(u)=Y_e(u)=Y(u,e).$$

Thus, every function $Y: U\times A\to[0,1]$ naturally induces a family of parametric functions over the universe of objects, represented by the mapping $Y: A\to[0,1]^U$.

Remark 2. The above result shows that a two-variable evaluation function $Y: U\times A\to[0,1]$ may equivalently be viewed as a family of functions parameterized by A. In the context of the soft yield structure, the mapping Y represents the fundamental evaluation function on object-parameter pairs, whereas the induced mapping $Y: A\to[0,1]^U$ represents the parametric structure generated by that function, consistent with functional representations in soft set theory.

Example 13. Using the computed results of the soft yield generating function in Example 12, the resulting soft yield structure is obtained as follows.

- $Y(e_1)=\big\{\big(u_1,Y(u_1,e_1)\big),\big(u_2,Y(u_2,e_1)\big)\big\}=\{(u_1,0.1375),(u_2,0.4125)\}\in[0,1]^U.$
- $Y(e_2)=\big\{\big(u_1,Y(u_1,e_2)\big),\big(u_2,Y(u_2,e_2)\big)\big\}=\{(u_1,0.2),(u_2,0.3525)\}\in[0,1]^U.$

2.11. Decision-making algorithm based on the hybrid hesitant fuzzy N-soft set−soft yield framework

The HHFNSS–soft yield framework constructs a criterion-based yield evaluation structure under uncertainty arising from multi-evaluator assessments, without performing term-structure forecasting. This procedure transforms the set-valued information contained in HHFNSS into: (i) A single grade–membership pair within the reduced fuzzy N-soft set, and (ii) a scalar soft yield index in the interval [0,1] for each alternative–criterion pair, which is subsequently used to generate final scores and decision outcomes.

Algorithm Multi-Criteria Decision-Making Algorithm Based on HHFNSS and Soft Yield Structure

1. Select N according to the category partition adopted for the HHFNSS evaluation framework.
2. Input $U = \{u_1, u_2, u_3, \dots, u_m\}$ as the universal set of objects, $A = \{e_1, e_2, e_3, \dots, e_n\}$ as the set of parameters, and $D = \{d_1, d_2, d_3, \dots, d_s\}$ as the set of evaluators.
3. Input $G = \{0,1,2, \dots, N-1\}, N \geq 2,$ for all $u_i \in U, e_j \in A,$ assign $g_{u,e} \in G$ according to each evaluator $d_k \in D$.
4. For each evaluator v_k, alternative u_i, and parameter e_j, provide an assessment consisting fo a discrete grade $g_{i,j}^{(k)} \in G$ and a hesitant fuzzy element $h_{i,j}^{(k)} \subseteq [0, 1]$.
5. Construct the HHFNSS based on all evaluator assessments according to Eq (9).
6. Compute the representative grade for each pair (u_i, e_j) using Eqs (14)–(16).
7. Compute the representative fuzzy membership value for each pair (u_i, e_j) using Eqs (17) and (18).
8. Construct the reduced fuzzy N-soft set according to Eq (19).
9. Compute the soft yield value for each alternative u_i and parameter e_j using Eq (22), subject to Eq (21).
10. If e_j is a benefit criterion, use $Y(u_i, e_j)$;
a. If e_j is a cost criterion, compute

$$Y'(u_i, e_j) = 1 - Y(u_i, e_j). \tag{24}$$

11. Compute the final score of each alternative:

$$S(u_i) = \sum_{j=1}^{n} Y'(u_i, e_j). \tag{25}$$

12. Determine all indices i such that:

$$S(u_i) = S(u_i)\,. \tag{26}$$

13. The alternative with the maximum value is selected as the optimal decision.

3. Results

3.1. Application of the HHFNSS-soft yield model

In this section, the HHFNSS–Soft Yield model developed in the previous section is applied to a corporate bond investment decision-making problem. Our primary objective of this case study is to evaluate bond yield quality based on multiple yield-determining factors (yield drivers), incorporating multi-criteria considerations and accounting for uncertainty arising from assessments by several evaluators. In the bond market, the yield of a financial instrument is not determined by a single factor; rather, it emerges from the complex interaction of credit risk, market conditions, and the issuing company's fundamental performance. Therefore, a multi-criteria approach that explicitly incorporates evaluative uncertainty is highly relevant for producing a more comprehensive assessment.

Step 1–3: In this study, five corporate bonds were selected as decision alternatives, namely bonds issued by:

- PT Bank Mandiri (Persero) Tbk;
- PT Telekomunikasi Indonesia (Persero) Tbk;
- PT Astra Internasional Tbk;
- PT Indofood Sukses Makmur Tbk; and
- PT Adaro Energi Indonesia Tbk.

The five companies were selected to represent sectoral diversity and risk profiles relevant to the corporate bond market. First, these firms are large-scale entities with established track records of bond issuance in the capital market, making them appropriate representatives of corporate bond investment instruments. Goldstein et al. [32], De Jong and Driessen [33], and Li et al. [34] indicated that corporate bonds issued by firms with established market access tend to exhibit more stable spread and liquidity characteristics. Second, the five companies originate from different industrial sectors, namely banking, telecommunications, automotive, conglomerates, food manufacturing, and energy. Such sectoral diversity is important for capturing variations in risk and return characteristics that directly influence bond yield structures [35]. Third, these firms possess different risk profiles and levels of financial stability, thereby enabling a richer analysis of the model's sensitivity in distinguishing yield quality. Accordingly, the set of alternatives is defined as:

$$U = \{u_1, u_2, u_3, u_4, u_5\}$$

where each u_i represents the bond issued by the corresponding company.

The bond evaluation in this study is based on several criteria that, in theory, serve as determinants of yield. These criteria do not represent yield directly but rather the factors that influence whether a bond yield is high or low. The set of criteria is defined as:

$$A = \{e_1, e_2, e_3, \dots, e_8\}$$

consisting of:

- e_1*: credit rating (benefit)*
- e_2*: default risk (cost)*
- e_3*: interest rate sensitivity (cost)*
- e_4*: liquidity (benefit)*
- e_5*: debt-t-equity ratio (cost)*
- e_6*: interest coverage ratio (benefit)*
- e_7*: yield spread (benefit)*
- e_8*: cash flow stability (benefit)*

The selection of these criteria is grounded in the financial literature, which suggests that bond yields are influenced by a combination of credit risk factors, market conditions, and the firm's underlying financial strength [36–38]. Therefore, the adopted criteria structure reflects a comprehensive approach to evaluating yield quality.

In the proposed decision-making algorithm, the final score of each alternative is computed using additive aggregation, as shown in Eq (25). This formulation implicitly assigns equal weights to all eight criteria. The equal-weighting assumption is adopted for the following reasons:

1. In the context of this case study, there is no established consensus or regulatory framework specifying differential importance among the eight yield-determining criteria for the Indonesia corporate bond market. The criteria are drawn from financial theory as established determinants of expert interpretation.
2. The proposed framework is designed to support investment decisions without imposing any particular preference structure on the part of the decision-maker. Equal weighting represents a neutral baseline that enables the evaluation to reflect the intrinsic characteristics of each alternative across all criteria.
3. Equal weighting ensures that no single criterion dominates the evaluation, enabling a balanced assessment of each bond's overall quality. This is particularly important given the diverse nature of the criteria, which span credit risk, market conditions, and firm fundamentals.
4. Equal weighting is transparent and easily understood by decision-makers, facilitating the interpretation and adoption of the model in practice.

The set of evaluators is defined as:

$$D = \{d_1, d_2, d_3\}.$$

Evaluators were asked to independently assess each alternative-criterion pair by providing:

1. A discrete grade $g_{i,j}^{(k)} \in G = \{0,1,2,3,4\}$ representing the categorical quality level;
2. A hesitant fuzzy element $h_{i,j}^{(k)} \subseteq [0,1]$ representing their uncertainty or hesitation regarding the membership degree.

The evaluation data presented in Tables 10–14 were obtained through a structured expert elicitation process conducted between January and February 2026. Three evaluators with specialized expertise in corporate bond analysis were recruited to participate in the study:

- Evaluator d_1: A credit analyst with 12 years of experience in corporate credit assessment at a major Indonesian commercial bank, specializing in credit risk modeling and rating assignment.
- Evaluator d_2: A bond market analyst with 8 years of experience in fixed-income research at a securities firm, focusing on corporate bond yield analysis and market trends.
- Evaluator d_3: An investment manager with 15 years of experience in portfolio management, overseeing fixed-income investments and bond selection strategies.

Each evaluator was provided with a comprehensive information package containing:

1. The latest financial reports (annual reports, quarterly statements) for the five companies under consideration;
2. Credit rating information from major rating agencies (PEFINDO and Fitch Ratings Indonesia);
3. Recent bond issuance data and secondary market trading information;
4. The grade scale interpretation (Table 9) and the fuzzy membership interval mapping;
5. Guidelines for assigning hesitant assessments, including examples and best practices.

The selection of the global granularity level $N = 5$ in this study is based on several considerations. First, a five-level scale is among the most commonly used in multi-criteria decision-making, as it provides a balance between discriminatory precision and ease of interpretation. A scale that is too coarse (e.g., $N = 3$) tends to lack sensitivity in distinguishing alternatives, whereas a scale that is too fine (e.g., $N \geq 7$) may increase inconsistency in evaluator judgments. Second, in the context of bond evaluation, the quality of financial instruments is typically classified into a limited number of principal categories, such as very low to very high. Therefore, using five categories provides sufficient representation to capture variations in quality without imposing an excessive cognitive burden on evaluators.

Based on these considerations, each grade value is interpreted as presented in Table 9.

Table 9. Grade interpretation.

Grade	Interpretation
0	Very Low
1	Low
2	Medium
3	High
4	Very High

This interpretation applies to all criteria while taking into account the criterion type (benefit or cost).

Furthermore, to maintain consistency between the discrete and fuzzy representations, each grade $g_{i,j} \in G$ is associated with a corresponding fuzzy membership interval. This is intended to ensure that the fuzzy values assigned by evaluators remain within the range consistent with the quality level represented by that grade. Formally, for each pair (u, e), the fuzzy membership value $\mu(e, u)$ satisfies:

$$0 \leq \mu(e, u) < 0.2 \text{ if } g = 0,$$

$$0.2 \leq \mu(e, u) \leq 0.4 \text{ if } g = 1,$$

$$0.4 \leq \mu(e, u) \leq 0.6 \text{ if } g = 2,$$

$$0.6 \leq \mu(e, u) \leq 0.8 \text{ if } g = 3,$$

$$0.8 \leq \mu(e, u) \leq 1 \text{ if } g = 4.$$

The interval allocation is derived from a uniform partition of the unit interval $[0,1]$, as follows:

$$[0, 1] = \cup_{g=0}^{4} \left[\frac{g}{5}, \frac{g+1}{5}\right].$$

This partition guarantees that each grade maintains a consistent fuzzy representation with minimal overlap.

Step 4: Evaluator assessments for each alternative–criterion pair were compiled into a HHFNSS structure. This framework facilitates the simultaneous representation of discrete information as grades and uncertainty as hesitant fuzzy elements. All numerical computations in this study were performed using Microsoft Excel 2021. The assessment results for each evaluator and each alternative–criterion combination are presented below.

Table 10. Evaluation assessments provided by the evaluators for the bond alternative u_1.

e_j	d_1	d_2	d_3
e_1	$(4, \{0.85, 0.9\})$	$(4, \{0.88\})$	$(3, \{0.75, 0.79\})$
e_2	$(1, \{0.25, 0.3\})$	$(1, \{0.28\})$	$(2, \{0.45\})$
e_3	$(2, \{0.45, 0.5\})$	$(2, \{0.48\})$	$(3, \{0.65\})$
e_4	$(4, \{0.8, 0.95\})$	$(4, \{0.82\})$	$(4, \{0.83\})$
e_5	$(2, \{0.45, 0.5\})$	$(2, \{0.48\})$	$(3, \{0.65\})$
e_6	$(4, \{0.8, 0.85\})$	$(4, \{0.82\})$	$(4, \{0.83\})$
e_7	$(3, \{0.65, 0.7\})$	$(3, \{0.68\})$	$(3, \{0.7\})$
e_8	$(4, \{0.85, 0.9\})$	$(4, \{0.88\})$	$(4, \{0.87\})$

Table 11. Evaluation assessments provided by the evaluators for the bond alternative u_2.

e_j	d_1	d_2	d_3
e_1	$(4,\{0.9\})$	$(4,\{0.88\})$	$(4,\{0.8,0.97\})$
e_2	$(1,\{0.25\})$	$(1,\{0.22\})$	$(2,\{0.45,0.5\})$
e_3	$(2,\{0.42,0.5\})$	$(2,\{0.45\})$	$(2,\{0.5\})$
e_4	$(4,\{0.81,0.9\})$	$(4,\{0.85\})$	$(4,\{0.9\})$
e_5	$(2,\{0.42,0.48\})$	$(3,\{0.65\})$	$(2,\{0.48\})$
e_6	$(4,\{0.85,0.9\})$	$(4,\{0.88\})$	$(3,\{0.63,0.78\})$
e_7	$(3,\{0.7,0.75\})$	$(3,\{0.75\})$	$(3,\{0.72\})$
e_8	$(4,\{0.85,0.95\})$	$(4,\{0.88\})$	$(4,\{0.92\})$

Table 12. Evaluation assessments provided by the evaluators for the bond alternative u_3.

e_j	d_1	d_2	d_3
e_1	$(3,\{0.7\})$	$(3,\{0.75\})$	$(2,\{0.5\})$
e_2	$(2,\{0.45\})$	$(2,\{0.5\})$	$(3,\{0.65\})$
e_3	$(3,\{0.6,0.65\})$	$(2,\{0.5,0.58\})$	$(3,\{0.7,0.78\})$
e_4	$(3,\{0.7\})$	$(3,\{0.75\})$	$(3,\{0.72\})$
e_5	$(3,\{0.6\})$	$(3,\{0.65\})$	$(2,\{0.45\})$
e_6	$(3,\{0.6,0.65\})$	$(3,\{0.75\})$	$(3,\{0.72\})$
e_7	$(3,\{0.65\})$	$(2,\{0.5\})$	$(3,\{0.68,0.75\})$
e_8	$(3,\{0.7,0.76\})$	$(3,\{0.78\})$	$(2,\{0.5\})$

Table 13. Evaluation assessments provided by the evaluators for the bond alternative u_4.

e_j	d_1	d_2	d_3
e_1	$(3,\{0.68\})$	$(2,\{0.5,0.55\})$	$(3,\{0.7\})$
e_2	$(2,\{0.47\})$	$(3,\{0.63\})$	$(2,\{0.4,0.48\})$
e_3	$(2,\{0.45\})$	$(2,\{0.48,0.54\})$	$(3,\{0.65\})$
e_4	$(3,\{0.68,0.74\})$	$(3,\{0.68\})$	$(2,\{0.5\})$
e_5	$(3,\{0.7\})$	$(2,\{0.47\})$	$(3,\{0.65\})$
e_6	$(3,\{0.65\})$	$(3,\{0.69\})$	$(2,\{0.5\})$
e_7	$(3,\{0.6\})$	$(4,\{0.82,0.88\})$	$(2,\{0.5\})$
e_8	$(4,\{0.82\})$	$(2,\{0.46\})$	$(3,\{0.68\})$

Table 14. Evaluation assessments provided by the evaluators for the bond alternative u_5.

e_j	d_1	d_2	d_3
e_1	$(2,\{0.4,0.45\})$	$(3,\{0.63\})$	$(2,\{0.46\})$
e_2	$(3,\{0.66\})$	$(3,\{0.7\})$	$(4,\{0.88\})$
e_3	$(3,\{0.6\})$	$(3,\{0.7\})$	$(2,\{0.5,0.56\})$
e_4	$(2,\{0.43\})$	$(2,\{0.5\})$	$(3,\{0.64\})$
e_5	$(3,\{0.65\})$	$(4,\{0.85,0.9\})$	$(3,\{0.7\})$
e_6	$(2,\{0.45\})$	$(2,\{0.55\})$	$(3,\{0.72\})$
e_7	$(4,\{0.82,0.88\})$	$(4,\{0.9\})$	$(3,\{0.6,0.68\})$
e_8	$(2,\{0.44\})$	$(3,\{0.6,0.7\})$	$(2,\{0.5\})$

Step 5: All data presented in Tables 10 through 14 constitute the HHFNSS structure in Tables 15a and 15b based on Definition 11, which contains multi-evaluator assessment information in the form of discrete grades and hesitant fuzzy elements.

Table 15a. Hybrid hesitant fuzzy 5-soft set of u_i.

e_j	u_1	u_2	u_3
e_1	$\{(4,\{0.85,0.9\}),(4,\{0.88\}),$ $(3,\{0.75,0.79\})\}$	$\{(4,\{0.9\}),(4,\{0.88\}),$ $(4,\{0.8,0.97\})\}$	$\{(3,\{0.7\}),(3,\{0.75\}),$ $(2,\{0.5\})\}$
e_2	$\{(1,\{0.25,0.3\}),$ $(1,\{0.28\}),(2,\{0.45\})\}$	$\{(1,\{0.25\}),(1,\{0.22\}),$ $(2,\{0.45,0.5\})\}$	$\{(2,\{0.45\}),(2,\{0.5\}),$ $(3,\{0.65\})\}$
e_3	$\{(2,\{0.45,0.5\}),$ $(2,\{0.48\}),(3,\{0.65\})\}$	$\{(2,\{0.42,0.5\}),$ $(2,\{0.45\}),(2,\{0.5\})\}$	$\{(3,\{0.6,0.65\}),(2,\{0.5,0.58\}),$ $(3,\{0.7,0.78\})\}$
e_4	$\{(4,\{0.8,0.95\}),$ $(4,\{0.82\}),(4,\{0.83\})\}$	$\{(4,\{0.81,0.9\}),$ $(4,\{0.85\}),(4,\{0.9\})\}$	$\{(3,\{0.7\}),(3,\{0.75\}),$ $(3,\{0.72\})\}$
e_5	$\{(2,\{0.45,0.5\}),$ $(2,\{0.48\}),(3,\{0.65\})\}$	$\{(2,\{0.42,0.48\}),$ $(3,\{0.65\}),(2,\{0.48\})\}$	$\{(3,\{0.6\}),(3,\{0.65\}),$ $(2,\{0.45\})\}$
e_6	$\{(4,\{0.8,0.85\}),$ $(4,\{0.82\}),(4,\{0.83\})\}$	$\{(4,\{0.85,0.9\}),$ $(4,\{0.88\}),(3,\{0.63,0.78\})\}$	$\{(3,\{0.6,0.65\}),$ $(3,\{0.75\}),(3,\{0.72\})\}$
e_7	$\{(3,\{0.65,0.7\}),$ $(3,\{0.68\}),(3,\{0.7\})\}$	$\{(3,\{0.7,0.75\}),$ $(3,\{0.75\}),(3,\{0.72\})\}$	$\{(3,\{0.65\}),(2,\{0.5\}),$ $(3,\{0.68,0.75\})\}$
e_8	$\{(4,\{0.85,0.9\}),$ $(4,\{0.88\}),(4,\{0.87\})\}$	$\{(4,\{0.85,0.95\}),$ $(4,\{0.88\}),(4,\{0.92\})\}$	$\{(3,\{0.7,0.76\}),$ $(3,\{0.78\}),(2,\{0.5\})\}$

Table 15b. Hybrid hesitant fuzzy 5-soft set of u_i.

e_j	u_4	u_5
e_1	$\{(3,\{0.68\}),(2,\{0.5,0.55\}),(3,\{0.7\})\}$	$\{(2,\{0.4,0.45\}),(3,\{0.63\}),(2,\{0.46\})\}$
e_2	$\{(2,\{0.47\}),(3,\{0.63\}),(2,\{0.4,0.48\})\}$	$\{(3,\{0.66\}),(3,\{0.7\}),(4,\{0.88\})\}$
e_3	$\{(2,\{0.45\}),(2,\{0.48,0.54\}),(3,\{0.65\})\}$	$\{(3,\{0.6\}),(3,\{0.7\}),(2,\{0.5,0.56\})\}$
e_4	$\{(3,\{0.68,0.74\}),(3,\{0.68\}),(2,\{0.5\})\}$	$\{(2,\{0.43\}),(2,\{0.5\}),(3,\{0.64\})\}$
e_5	$\{(3,\{0.7\}),(2,\{0.47\}),(3,\{0.65\})\}$	$\{(3,\{0.65\}),(4,\{0.85,0.9\}),(3,\{0.7\})\}$
e_6	$\{(3,\{0.65\}),(3,\{0.69\}),(2,\{0.5\})\}$	$\{(2,\{0.45\}),(2,\{0.55\}),(3,\{0.72\})\}$
e_7	$\{(3,\{0.6\}),(4,\{0.82,0.88\}),(2,\{0.5\})\}$	$\{(4,\{0.82,0.88\}),(4,\{0.9\}),(3,\{0.6,0.68\})\}$
e_8	$\{(4,\{0.82\}),(2,\{0.46\}),(3,\{0.68\})\}$	$\{(2,\{0.44\}),(3,\{0.6,0.7\}),(2,\{0.5\})\}$

Step 6: The HHFNSS in Tables 15a and 15b is reduced based on Eqs (14)–(16) to get a representative grade. An example of determining grade average score based on Definition 12 and reduced grade based on Definition 13 for HHFNSS (u_1,e_1) with respect to $H^*(u_1,e_1)=\{(4,\{0.85,0.9\}),(4,\{0.88\}),(3,\{0.75,0.79\})\}$ is given as follows:

$$
\begin{aligned}
\bar{g}_{u_1,e_1} &= \left(\frac{1}{m_{u_1,e_1}}\sum_{k=1}^{m_{u_1,e_1}} g_{u_1,e_1}^{(k)}\right)\\
&= \frac{1}{3}\sum_{k=1}^{3} g_{u_1,e_1}^{(k)}\\
&= \frac{1}{3}\left(g_{u_1,e_1}^{(1)}+g_{u_1,e_1}^{(2)}+g_{u_1,e_1}^{(3)}\right)
\end{aligned}
$$

$$= \frac{1}{3}(4+4+3) = 3.667$$

$$Q(\bar{g}_{u_1,e_1}) = arg \min_{j\in G}\left|\bar{g}_{u_1,e_1} - j\right|$$

$$= arg \min_{j\in G}|3.667 - j|$$

$$= arg \min_{j\in G}|0.333| = \{4\}$$

$$GA(u_1, e_1) = \min Q(\bar{g}_{u_1,e_1})$$

$$= \min(\{4\}) = 4.$$

Step 7: After aggregating the discrete grade of multi-evaluator assessment, the HHFNSS in Tables 15a and 15b is reduced again using Eqs (17) and (18) to get a representative fuzzy. An example of determining fuzzy reduction based on Definition 14 for HHFNSS (u_1, e_1) with respect to $H^*(u_1, e_1) = \{(4, \{0.85, 0.9\}), (4, \{0.88\}), (3, \{0.75, 0.79\})\}$ is given as follows:

$$\phi\left(h^{(1)}_{u_1,e_1}\right) = \left(\frac{1}{\left|h^{(1)}_{u_1,e_1}\right|}\sum_{\mu\in h^{(1)}_{u_1,e_1}}\mu\right)$$

$$= \frac{1}{|2|}(0.85 + 0.9) = 0.875$$

$$\phi\left(h^{(2)}_{u_1,e_1}\right) = \left(\frac{1}{\left|h^{(2)}_{u_1,e_1}\right|}\sum_{\mu\in h^{(2)}_{u_1,e_1}}\mu\right)$$

$$= \frac{1}{|1|}(0.88) = 0.88$$

$$\phi\left(h^{(3)}_{u_1,e_1}\right) = \left(\frac{1}{\left|h^{(3)}_{u_1,e_1}\right|}\sum_{\mu\in h^{(3)}_{u_1,e_1}}\mu\right)$$

$$= \frac{1}{|2|}(0.75 + 0.79) = 0.77$$

$$FA(u_1, e_1) = \left(\frac{1}{m_{u_1,e_1}}\sum_{k=1}^{m_{u_1,e_1}}\phi\left(h^{(k)}_{u_1,e_1}\right)\right)$$

$$= \frac{1}{2}\left(\phi\left(h^{(1)}_{u_1,e_1}\right) + \phi\left(h^{(2)}_{u_1,e_1}\right) + \phi\left(h^{(3)}_{u_1,e_1}\right)\right)$$

$$= \frac{1}{2}(0.875 + 0.88 + 0.77) = 0.842 \approx 0.84.$$

Step 8: Based on Definition 15, the outcomes of the RFNSS calculation are summarized in Table 16.

Table 16. Reduced fuzzy 5-soft set for each pair (u_i, e_j).

e_j	$R(u_1, e_j)$	$R(u_2, e_j)$	$R(u_3, e_j)$	$R(u_4, e_j)$	$R(u_5, e_j)$
e_1	$(4, 0.84)$	$(4, 0.89)$	$(3, 0.65)$	$(3, 0.64)$	$(2, 0.25)$
e_2	$(1, 0.34)$	$(1, 0.32)$	$(2, 0.53)$	$(2, 0.51)$	$(3, 0.37)$
e_3	$(2, 0.54)$	$(2, 0.47)$	$(3, 0.64)$	$(2, 0.54)$	$(3, 0.31)$
e_4	$(4, 0.84)$	$(4, 0.87)$	$(3, 0.72)$	$(3, 0.63)$	$(2, 0.26)$
e_5	$(2, 0.54)$	$(2, 0.53)$	$(3, 0.57)$	$(3, 0.61)$	$(3, 0.37)$
e_6	$(4, 0.83)$	$(4, 0.82)$	$(3, 0.70)$	$(3, 0.61)$	$(2, 0.29)$
e_7	$(3, 0.69)$	$(3, 0.74)$	$(3, 0.62)$	$(3, 0.65)$	$(4, 0.40)$
e_8	$(4, 0.88)$	$(4, 0.90)$	$(3, 0.76)$	$(3, 0.65)$	$(2, 0.27)$

Step 9: After obtaining the reduced fuzzy N-soft set, the soft yield generating function was obtained using Eq (22) subject to Eq (21) based on Definitions 16 and 17 for RNFSS (u_1, e_1) with respect to $R(u_1, e_1) = (4, 0.84)$ given as follows:

$$\begin{aligned} Y(u_1, e_1) &= \psi\big(GA(u_1, e_1), FA(u_1, e_1)\big) \\ &= \tfrac{GA(u_1, e_1)}{5-1} \cdot FA(u_1, e_1) \\ &= \tfrac{4}{4}\, .(0.84) = 0.84. \end{aligned}$$

The resulting soft yield structure based on Definition 17 is presented in Table 17.

Table 17. Soft yield structure for each alternative u_i.

e_j	$Y(u_1, e_j)$	$Y(u_2, e_j)$	$Y(u_3, e_j)$	$Y(u_4, e_j)$	$Y(u_5, e_j)$
e_1	0.842	0.888	0.488	0.476	0.126
e_2	0.084	0.079	0.267	0.257	0.280
e_3	0.268	0.235	0.476	0.268	0.229
e_4	0.842	0.868	0.543	0.473	0.131
e_5	0.268	0.263	0.428	0.455	0.278
e_6	0.825	0.820	0.524	0.460	0.143
e_7	0.514	0.556	0.466	0.488	0.398
e_8	0.875	0.900	0.566	0.490	0.133

Step 10: To enable the calculation of each alternative's final score, the criteria were first classified into benefit and cost categories. Accordingly, the criteria belonging to the cost category, namely e_2, e_3, and e_5, require the soft yield values to be determined using Eq (24). The adjusted soft yield structure for the cost criteria is presented in Table 18.

Table 18. Adjustment of the soft yield structure for cost criteria.

e_j	$Y(u_1, e_j)$	$Y(u_2, e_j)$	$Y(u_3, e_j)$	$Y(u_4, e_j)$	$Y(u_5, e_j)$
e_1	0.842	0.888	0.488	0.476	0.126
e_2	0.916	0.921	0.733	0.743	0.720
e_3	0.733	0.765	0.524	0.732	0.771
e_4	0.842	0.868	0.543	0.473	0.131
e_5	0.733	0.737	0.572	0.545	0.722
e_6	0.825	0.820	0.524	0.460	0.143
e_7	0.514	0.556	0.466	0.488	0.398
e_8	0.875	0.900	0.566	0.490	0.133

Step 11: An example of computing the final score using Eq (25) for $S(u_1)$ is given as follows:

$$\begin{aligned} S(u_1) &= \textstyle\sum_{j=1}^{n} Y(u_1, e_j) \\ &= \big(Y(u_1, e_1) + Y(u_1, e_2) + Y(u_1, e_3) + Y(u_1, e_4) + Y(u_1, e_5)\big) \\ &= 0.842 + 0.916 + 0.733 + 0.842 + 0.733 + 0.825 + 0.514 + 0.875 \\ &= 6.3. \end{aligned}$$

The final scores of each alternative, together with their corresponding rankings, are presented in Table 19.

Table 19. Final scores and ranking of each alternative.

u_i	$S(u_i)$	Rank
u_1	6.3	2
u_2	6.5	1
u_3	4.4	3
u_4	4.4	4
u_5	3.1	5

Based on Step 12 and Table 18, PT Telekomunikasi Indonesia (Persero) Tbk (u_2) has the highest score among corporate bond options with respect to $S(u_i) = \{6.3, 6.5, 4.4, 4.4, 3.1\} = 6.5$, while PT Adaro Energi Indonesia Tbk (u_5) has the lowest. Accordingly, the best corporate bond investment decision, based on this scoring process, which incorporated evaluations by credit analysts, bond market analysts, and investment managers using credit risk, market conditions, and the fundamental strength of the firms, is PT Telekomunikasi Indonesia (Persero) Tbk.

3.2. Comparative analysis

To assess the effectiveness and advantages of the proposed HHFNSS model, a comparative analysis was conducted with two frameworks: The hesitant fuzzy N-soft set (HFNSS) and the hesitant N-soft set (HNSS). In this analysis, we aim to evaluate each mode's ability to represent uncertainty and generate consistent decisions in the context of corporate bond evaluation. As a fair basis for comparison, all models used the same assessment data: Evaluation results from multiple evaluators for bond alternatives, based on predetermined criteria. However, due to differences in the mathematical structures of the frameworks, adjustments were made to the data-processing procedures to ensure compatibility with each approach's characteristics.

Within the HHFNSS model, all informational components are utilized simultaneously, namely, discrete grade values and hesitant fuzzy membership values. Consequently, this framework can capture two dimensions of uncertainty: Selection of evaluation categories and the intensity of evaluator preferences. In contrast, the HFNSS model considers hesitation only in fuzzy membership values. Accordingly, the data were processed by separating assessments according to individual evaluators. Each evaluator independently generated alternative scores, which were subsequently aggregated using the arithmetic mean to obtain the final score. This approach assumes homogeneous contributions across evaluators. The HNSS model uses only discrete grade information, ignoring the fuzzy component. Thus, its evaluation process depends fully on differences in grade values given by evaluators. It does not consider confidence levels or preference intensity.

The comparative results of scores and rankings generated by the HHFNSS, HFNSS, and HNSS models are presented in Table 20.

Table 20. Comparison of corporate bond alternative rankings across models.

u_i	HHFNSS		HFNSS		HNSS	
	Score	Rank	Score	Rank	Score	Rank
u_1	6.3	2	4.5	1	24.5	1
u_2	6.5	1	4.4	2	24.2	2
u_3	4.4	3	4.0	3	21.4	4
u_4	4.4	4	3.2	5	20.8	5
u_5	3.1	5	3.6	4	22.0	3

Table 20 shows that the three models produce distinct rankings of the alternatives, especially for those with medium-quality profiles. The HHFNSS model ranks alternative u_2 first, followed by u_1, u_4, u_3, and u_5. This outcome demonstrates that the HHFNSS model consistently identifies the optimal alternative by integrating all relevant evaluation dimensions. In contrast, the HFNSS model ranks alternative u_1 first, followed by u_2. This difference suggests that when only fuzzy information is considered, without explicit connection to grade values, the preference order among alternatives may change. This finding further implies that discrete information is important for distinguishing the quality of alternatives that fuzzy information alone cannot fully capture. The HNSS model produces a more divergent ranking, with alternative u_5 placed higher than in the other models. This result indicates that omitting the fuzzy component may result in the loss of information regarding preference intensity, which can lead to less representative decision outcomes.

The observed differences in ranking outcomes demonstrate that each model displays distinct sensitivity to the underlying data structure. The HNSS model generates coarser decisions due to its reliance solely on discrete values. The HFNSS model yields smoother results but remains limited because it does not consider the relationship between grade values and fuzzy membership levels. In contrast, the HHFNSS model integrates both types of information, resulting in more balanced and comprehensive decisions.

To facilitate visual comparison of the three models' performance, Figure 1 presents a bar chart of the final scores for each alternative under HHFNSS, HFNSS, and HNSS. The bar chart shows the score distribution across models and alternatives, enabling readers to quickly identify patterns in the comparative results.

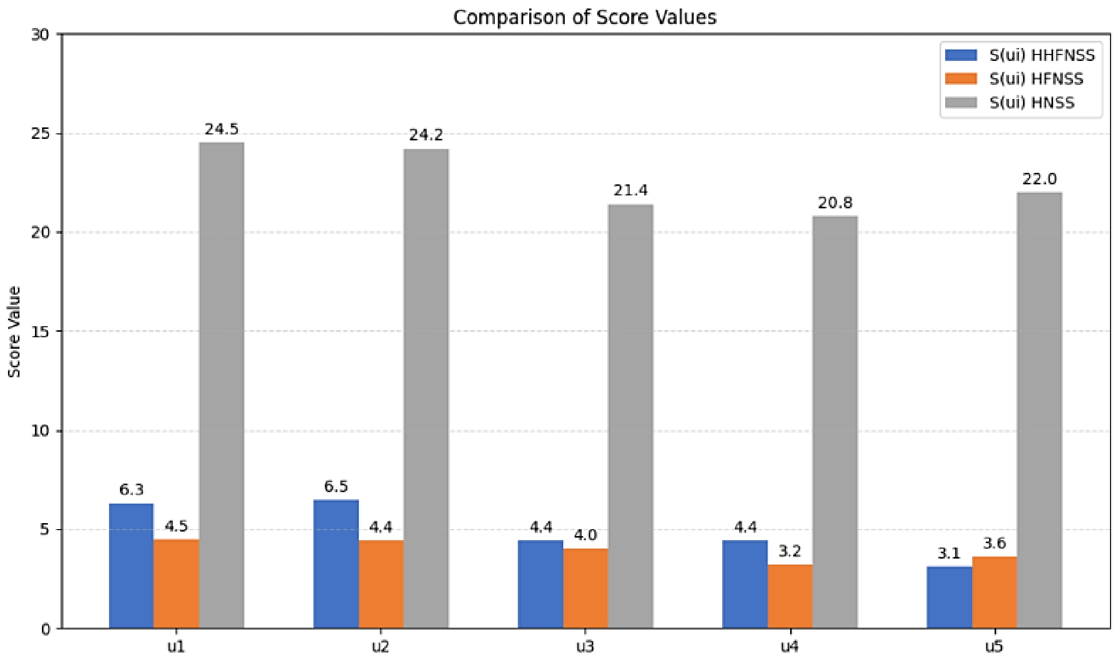

Figure 1. Comparison of alternative scores under HHFNSS, HFNSS, and HNSS models.

Figure 1 reveals several important patterns. First, the HHFNSS model produces scores that are more evenly distributed across alternatives, with a separation between the top three alternatives (u_1, u_2) and the bottom three (u_3, u_4, u_5). Second, the HFNSS model produces compressed scores that are closer together, making it more difficult to distinguish alternatives. Third, the HNSS model produces scores on a different scale, approximately 20–25, reflecting its use of grade-level information only, without the fuzzy intensity component.

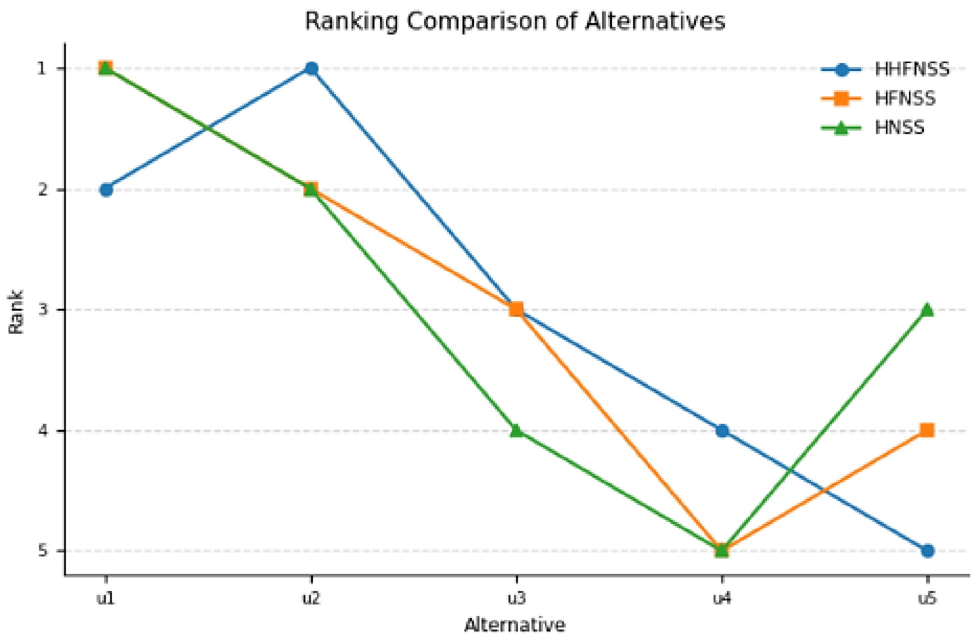

Figure 2. Ranking comparison across HHFNSS, HFNSS, and HNSS models.

Figure 2 illustrates how the ranking order changes across the three models. The ranking of u_2 (PT

Telekomunikasi Indonesia) remains stable at first or second position across all models. However, the rankings of u_5 (PT Adaro Energi Indonesia) and u_4 (PT Indofood Sukses Makmur) exhibit notable variation. Under HNSS, u_5 ranks third, while under HHFNSS and HFNSS, it ranks fifth. This discrepancy highlights the importance of incorporating fuzzy intensity information, which HNSS neglects, and the importance of grade-level information, which HFNSS partially overlooks.

To further validate the model results, an additional analysis was performed by comparing the rankings with the companies' quantitative financial indicators obtained from Yahoo Finance (www.yahoo.finance.com accessed on February, 10th). Financial indicators were selected according to sector: For non-financial firms, ratios such as debt-to-equity, Earnings Before Interest, Taxes, Depreciation, and Amortization (EBITDA) margin, and coverage ratio were used, while for the banking sector, Return on Equity (ROE), profit margin, and price-to-book ratio were applied. The quantitative financial indicators for the companies analyzed in this study are presented in Table 21.

Table 21. Comparison of quantitative financial indicators.

Company Name	**Sector**	**Financial Metric 1**	**Financial Metric 2**	**Financial Metric 3**
Non-Financial Firms		**Debt-to-Equity**	**EBITDA Margin**	**Coverage Ratio**
PT Telekomunikasi Indonesia (Persero) Tbk (u_2)	Telecommunications	50.11%	50.74%	7.67
PT Astra International Tbk (u_3)	Automotive / Conglomerate	37.93%	16.40%	14.20
PT Indofood Sukses Makmur Tbk (u_4)	Food Manufacturing	1.25%	18.83%	732.65
PT Adaro Energi Indonesia Tbk (u_5)	Energy/Mining	16.41%	33.54%	52.84
Banking Firm		**ROE**	**Profit Margin**	**Price-to-Book**
PT Bank Mandiri (Persero) Tbk (u_1)	Banking	19.14%	38.87%	1.50

Note: Financial indicators are selected based on sector-specific relevance. For non-financial firms, debt-to-equity ratio, EBITDA margin, and interest coverage ratio are standard measures of financial health and debt-serving capacity. For banking institutions, Return on Equity (ROE), profit margin, and price-to-book ratio are more appropriate indicators of profitability and valuation. Data sourced from Yahoo Finance (accessed February 10th, 2026), reflecting the most recent available fiscal year-end data for each company as of that date.

Alternative u_2, namely PT Telekomunikasi Indonesia (Persero) Tbk, ranks first under the HHFNSS model. This result is consistent with its financial indicators, which show an EBITDA margin of 50.74% and a coverage ratio of 7.67, reflecting the company's strong ability to generate operating cash flows and meet its interest obligations. Although its debt-to-equity ratio is relatively high (50.11%), its capacity to generate operating earnings continues to support its top-ranked position. Alternative u_1, namely PT Bank Mandiri (Persero) Tbk, is placed second. This is aligned with financial indicators showing a Return on Equity (ROE) of 19.14%, a profit margin of 38.87%, and a price-to-book ratio of 1.50. These values indicate strong profitability and relatively stable valuation. However, the banking sector's sensitivity to interest rates and macroeconomic conditions places it below sectors characterized by more stable cash flows.

Alternative u_3, namely PT Astra International Tbk, has a debt-to-equity ratio of 37.93% and an EBITDA margin of 16.40%. Its coverage ratio of 14.20 indicates adequate debt-servicing capacity, yet its exposure to economic cycles places it below alternatives with stronger stability characteristics. Alternative u_4, namely PT Indofood Sukses Makmur Tbk, exhibits a very low debt-to-equity ratio (1.25%) and an exceptionally high coverage ratio (732.65), indicating a highly robust capital structure and excellent capacity to service interest payments. Nevertheless, its more moderate EBITDA margin (18.83%) positions it in the middle tier of the ranking. Alternative u_5, namely PT Adaro Energi Indonesia Tbk, records a relatively high EBITDA margin (33.54%) and a coverage ratio of 52.84. However, the energy sector's strong dependence on commodity price fluctuations results in elevated cash-flow volatility. This explains why the HHFNSS model ranks this alternative last despite its strong profitability indicators.

Overall, the analysis indicates that the HHFNSS model produces rankings consistent with firms' underlying fundamentals. Companies with stable cash flows and lower risk tend to receive higher rankings, whereas firms exposed to greater volatility are generally placed lower. Moreover, compared with the HFNSS and HNSS models, HHFNSS demonstrates a stronger ability to capture the balance between risk and return. The HFNSS model tends to overlook the discrete structural dimension, whereas HNSS neglects fuzzy intensity, leading to rankings that are less consistent with empirical conditions. Accordingly, the HHFNSS model offers not only mathematical advantages but also strong empirical validity in capturing the quality of corporate bonds using relevant financial indicators.

3.3. Sensitivity analysis

To strengthen the sensitivity analysis, numerical simulations were conducted on changes in evaluator assessments within the HHFNSS model. These simulations were intended to examine how small variations in fuzzy membership values and grade values affect the soft yield scores and the rankings of alternatives.

3.3.1. Numerical simulation of fuzzy value changes

In this scenario, each fuzzy membership value was uniformly adjusted by ± 0.05, while maintaining the corresponding interval boundaries associated with each grade. The re-aggregated results produced the scores presented in Table 22.

Table 22. Simulation of score changes due to fuzzy value variation.

u_i	Initial Score	Rank	Score (-0.05)	Rank	Score (+0.05)	Rank
u_1	6.3	2	6.1	2	6.5	2
u_2	6.5	1	6.3	1	6.7	1
u_3	4.4	3	3.8	4	4.2	4
u_4	4.4	4	4.2	3	4.6	3
u_5	3.1	5	2.9	5	3.3	5

Based on the simulation results presented in Table 21, it can be observed that small changes in fuzzy membership values, specifically within the range of ± 0.05, produce score variations that are linear and proportional to the corresponding input changes. Each alternative experiences relatively uniform increases or decreases in score, without causing significant changes in the overall ranking structure. Across all tested scenarios, the ranking order remains consistent, with u_2 identified as the best alternative and u_5 retaining the lowest score.

The observed pattern indicates that the HHFNSS model is highly stable to minor fluctuations in fuzzy membership values. Conceptually, this suggests that the fuzzy component of the model functions as a refinement mechanism within the evaluation process rather than as a dominant factor capable of triggering drastic decision changes. Consequently, small variations that may be interpreted as evaluator noise do not materially affect the final outcomes.

Furthermore, this robustness demonstrates that the model maintains consistency in distinguishing the relative quality of alternatives. The score differentials among alternatives remain sufficiently large, thereby preventing rank reversal from small input perturbations. This is a desirable property in decision-making models, as it indicates that the framework is not overly sensitive to minor disturbances that do not reflect substantive changes in evaluator preferences.

3.3.2. Simulation of grade value changes

In the second scenario, the grade values were adjusted by one level $(g \pm 1)$ for selected evaluators, followed by corresponding modifications to the associated fuzzy intervals. The re-aggregated results produced the scores presented in Table 23.

Table 23. Simulation of score changes due to grade variation.

u_i	Initial Score	Rank	Score (grade-1)	Rank	Score (grade+1)	Rank
u_1	6.3	2	5.9	2	6.7	2
u_2	6.5	1	6.1	1	6.9	1
u_3	4.4	3	3.6	4	4.4	4
u_4	4.4	4	4	3	4.8	3
u_5	3.1	5	2.7	5	3.5	5

The simulation results presented in Table 22 indicate that variations in grade values have a greater impact on the scores and rankings of alternatives. A one-level change in grade produces larger score adjustments than those observed in the previous scenario and, in several cases, leads to positional shifts among alternatives occupying middle ranks. Nevertheless, these changes remain within controlled limits. Alternatives with extreme quality profiles, namely, u_2 as the highest-ranked and u_5 as the lowest-ranked, retain their respective positions across all tested scenarios. This demonstrates that the model can preserve the global ranking when substantial changes to the input are confined primarily to alternatives with relatively similar characteristics, such as u_3 and u_4. Intuitively, this is expected, since alternatives of nearly comparable quality are naturally more susceptible to rank changes when evaluation inputs vary. Accordingly, the sensitivity to grade variation reflects the model's capacity to respond appropriately to more fundamental changes in categorical assessments.

3.4. Sensitivity analysis of criteria weights

To strengthen the sensitivity analysis, numerical simulations were conducted on changes in criteria weights within the HHFNSS model. These simulations were intended to examine how variations in the relative importance of the eight evaluation criteria affect the soft yield scores and the rankings of alternatives. The first scenario is a single criterion. This scenario was assigned a dominant weight of $w_j = 0.5$, while the remaining weight of 0.5 was distributed equally among the other seven criteria, i.e., $w_k = 0.5/7$ for all $k \neq j$. This design tests whether any individual criterion possesses sufficient influence to unilaterally determine the ranking. We modified final score in Eq (25) as the Eq (27)

$$S(u_i) = \sum_{j=1}^{n} w_j . Y^{'} \left(u_i, e_j\right). \tag{27}$$

The re-aggregated results produced the rankings presented in Table 24.

Table 24. Ranking of alternatives under single-criterion dominance scenarios.

e_j Dominant	u_1 Rank	u_2 Rank	u_3 Rank	u_4 Rank	u_5 Rank
e_1	2	1	4	3	5
e_2	2	1	4	3	5
e_3	2	1	5	3	4
e_4	2	1	3	4	5
e_5	2	1	5	3	4
e_6	2	1	4	3	5
e_7	2	1	4	3	5
e_8	2	1	3	4	5

Based on the simulation results presented in Table 24, it can be observed that u_2 consistently secures the top rank across all single-criterion dominance scenarios. This persistence indicates that the preeminence of u_2 is not contingent upon any particular criterion but rather reflects a consistently strong performance across the full spectrum of yield-determining attributes. In parallel, u_5 consistently occupies the lowest rank, confirming the robustness of its inferior position. The only variations observed involve the relative ordering of alternatives u_3 and u_4, which occupy the middle ranks. Specifically, when e_3 (interest rate sensitivity) or e_5 (debt-to-equity) dominates, these two alternatives exchange positions, with u_5 advancing from rank 5 to rank 4. This sensitivity is expected, as alternatives with comparable overall quality profiles are inherently more susceptible to rank reversals when the evaluation weights are subtantially altered. This finding is consistent with the underlying data, that is u_3 (PT Astra International) demonstrates moderate performance on financial leverage and yield spread, whereas u_4 (PT Indofood Sukses Makmur) exhibits a stronger financial structure yet comparatively weaker yield spread characteristics.

The second scenario is two extreme weighting allocations. These scenarios were examined: (a) Credit-related criteria $(e_1, e_2, e_5, e_6, e_8)$ assigned 80% of the total weight, with the remaining 20% distributed equally among the market-related criteria (e_3, e_4, e_7); and (b) market-related criteria assigned 80% of the total weight, with the remaining 20% distributed equally among the credit-related criteria. These scenarios probe the sensitivity of the ranking to fundamentally divergent perspectives on the relative importance of credit fundamentals versus market conditions. The re-aggregated results produced the rankings presented in Table 25.

Table 25. Ranking of alternatives under extreme weighting scenarios.

Weighting Scenario	u_1 Rank	u_2 Rank	u_3 Rank	u_4 Rank	u_5 Rank
Equal weights	2	1	4	3	5
Credit criteria dominant (80%)	2	1	4	3	5
Market criteria dominant (80%)	2	1	4	3	5

Based on the simulation results presented in Table 25, it can be observed that the global ranking remains invariant even under the most extreme group-weighting allocations. The absence of rank reversals under such extreme assumptions confirms that the decision outcomes are not artefacts of the particular weighting scheme employed but rather reflect fundamental, multidimensional differences among the evaluated alternatives. The observed pattern indicates that the HHFNSS-soft yield framework exhibits a high degree of robustness to variations in criteria weights. Conceptually, this suggests that the weighting component of the model functions as a refinement mechanism within the evaluation process rather than as a dominant factor capable of triggering drastic decision changes. Consequently, variations in stakeholder preferences regarding criteria importance do not materially affect the identification of the best overall alternative.

4. Discussion

4.1. Comparative positioning with related hesitancy models

The HHFNSS framework occupies a unique position among N-soft set-based uncertainty models, particularly in comparison to three established frameworks: HFNSS, HNSS, and MFNSS. While each model addresses a specific aspect of uncertainty, none offers a comprehensive representation of dual-layer hesitation, which is the defining feature of the HHFNSS framework.

The hesitant N-soft set, introduced by Muhammad Akram et al. [24], represents the first systematic attempt to incorporate hesitation at the grade level into the N-Soft Set paradigm. In this model, for each object-parameter pair $(u, e) \in U \times A$, the evaluation $H(e)$ yields a set of possible grades $G_{u,e} \subseteq G$, where $|G_{u,e}| \geq 1$. This structure effectively captures multigranular hesitation, namely situations in which multiple evaluators disagree on the appropriate discrete category, or a single evaluator simultaneously considers several plausible classifications. Nevertheless, HNSS operates under the implicit assumption that once the hesitant grade set has been determined, the evaluation process is complete. It does not provide a mechanism for representing uncertainty about the strength of association between an object and a parameter. In the context of corporate bond evaluation, this limitation is substantial. Two evaluators who assign grade 3 (high) to a bond's creditworthiness may have very different levels of confidence: One may be highly certain, while the other may believe the bond meets the grade 3 threshold only marginally. HNSS treats these two situations identically, thereby discarding valuable epistemic information.

In contrast, the hesitant fuzzy N-soft set framework, developed by Muhammad Akram et al. in Ashraf et al. [26] and Adeel et al. [25], addresses this limitation by incorporating hesitation in fuzzy membership values. Within HFNSS, each object–parameter pair receives a unique grade $g_{u,e} \in G$ together with an HFS $h_{u,e} \subseteq [0,1]$. This structure enables evaluators to express uncertainty regarding membership intensity, for example, by indicating that the liquidity of a bond lies around 0.8 with plausible values $\{0.78, 0.82, 0.85\}$. However, HFNSS retains the classical assumption that the grade is certain and unambiguous. This assumption becomes problematic in multi-evaluator settings or cognitively complex environments, where HNSS performs better. If three analysts disagree on whether the default risk of a bond should be classified as grade 2 (moderate) or grade 3 (high), forcing a single grade through voting or averaging eliminates precisely the uncertainty that hesitant modeling seeks to preserve. Thus, HFNSS captures only one dimension of hesitation (fuzzy intensity) while ignoring the other (grade ambiguity).

The MFNSS, proposed by Fatimah and Alcantud [27], offers a third approach in which each object–parameter pair is associated with a vector of fuzzy membership values, each reflecting a distinct evaluation dimension or evaluator perspective. While this framework provides considerable representational richness, it lacks the parsimony of grade-based categorization and does not inherently address the discrete-continuous interface central to practical financial decision-making. Additionally, MFNSS does not offer a clear mechanism for reconciling conflicting evaluations beyond basic aggregation.

The HHFNSS framework synthesizes all three preceding models as special cases under specific reduction conditions, as formally defined in Definition 15 of Subsection 2.9. The cardinality $m_{u,e} \geq 1$ captures hesitation at the grade level, while the associated HFS for each grade represents hesitation at the intensity level. This dual-layer structure enables the modeling of scenarios where evaluators differ in discrete categorization and continuous intensity within each category. The theoretical advantages of HHFNSS are demonstrated by its reduction properties: When $m_{u,e} = 1$ for all (u, e), the model reduces to HFNSS; when $h_{u,e}^{(k)} = 1$, it reduces to HNSS; and when both conditions are met, the classical N-soft set is obtained. Additionally, if each $h_{u,e}^{(k)}$ is a singleton, the model reduces to the weighted N-soft set [39].

From the perspective of decision theory, HHFNSS demonstrates advantages that can be empirically substantiated relative to its predecessors. As shown in the comparative analysis in Subsection 3.2, Table 20, the three models generate different rankings for corporate bond alternatives. HNSS places PT Adaro Energi Indonesia Tbk u_5 in third position, whereas HFNSS and HHFNSS rank it fifth. This discrepancy arises because HNSS, which ignores fuzzy intensity, cannot distinguish a grade-3 assessment accompanied by high confidence $(\mu \approx 0.8)$, and one accompanied by lower confidence $(\mu \approx 0.6)$. The Adaro bond receives relatively high grades but shows substantial intensity hesitancy across evaluators, an indicator of underlying volatility that HNSS fails to capture. Conversely, although HFNSS can model intensity, it cannot capture disagreement among evaluators over whether the yield spread of u_4 (criterion e_7) should be classified as grade 3 or grade 4. The imposition of a single grade (the mode of $\{3,4\}$) suppresses an ambiguity signal that HHFNSS explicitly preserves and propagates through its reduction mechanism.

This comparative positioning reveals a hierarchy of representational capacity: HHFNSS ⊃ HFNSS ⊃ N-soft set and HHFNSS ⊃ HNSS ⊃ N-soft set, with the multi-fuzzy variant occupying an orthogonal direction of generalization. This hierarchy is not solely taxonomic but carries substantive implications for model selection in applied decision-making. When evaluator consensus exists regarding categorical classification but uncertainty persists about intensity, HFNSS may suffice. Conversely, when grade-level ambiguity is predominant and intensity judgments are precise, HNSS may be appropriate. In the corporate bond market, which involves multiple analysts, diverse information sources, and inherently imprecise risk categories, both dimensions of hesitation frequently occur. In such contexts, the comprehensive HHFNSS framework is essential for accurately and thoroughly representing uncertainty.

4.2. Positioning within the broader hesitant fuzzy literature

The proposed HHFNSS framework is within a broader landscape of hesitant fuzzy modeling that has seen significant recent development. While the foundational HFS model of Torra and Narukawa [14] established the paradigm of representing hesitation through sets of membership values, subsequent extensions have explored various directions of generalization. Probabilistic HFSs, for instance, incorporate probability distributions to reflect the relative likelihood of different membership values. This approach, while powerful, assumes that probability information is available or can be reliably estimated. In contrast, the HHFNSS framework does not require probabilistic information but instead captures uncertainty through the structure of grade-membership pairs, making it more suitable for contexts where frequency-based information is unavailable or unreliable.

Application of hesitant fuzzy modeling to complex real-world problems, such as the medical insurance fraud detection frameworks developed in [16,17], demonstrates the practical value of sophisticated hesitant fuzzy methodologies. These works employ advanced techniques, including f-divergence measures and case-based reasoning within hesitant fuzzy environments, thereby addressing specific application needs in the domain of fraud identification and classification. While these approaches focus on sorting and classification methodologies, the HHFNSS framework offers a complementary contribution by providing a unified algebraic structure for representing dual-layer hesitation in parametric decision-making contexts. Cloud-based hesitant fuzzy models represent another recent direction of development, integrating the representational power of cloud theory, which captures fuzziness and randomness, with the hesitation representation of HFSs [17]. These models are particularly suited for environments where uncertainty exhibits fuzzy and stochastic characteristics. The HHFNSS framework, by contrast, addresses a different type of complexity, that is the simultaneously occurrence of categorical ambiguity and intensity uncertainty in multi-evaluator assessment settings.

4.3. Theoretical advancements and model integration

The integration of dual-layer uncertainty, namely, grade-level hesitation and fuzzy membership hesitation, into the N-soft set framework constitutes the principal theoretical contribution of this study. It bridges three separate evolutionary streams of soft set theory: (i) The N-soft set, which permits multilevel discrete evaluation but assumes a unique grade; (ii) the hesitant N-soft set, which allows multiple grades but ignores fuzzy intensity; and (iii) the hesitant fuzzy N-soft set, which captures fuzzy hesitation while preserving a single grade assumption. The HHFNSS framework synthesizes these three models through a paired structure of the form $\left\{\left(g^{(k)}, h^{(k)}\right)\right\}$, where each uncertain grade is associated with an HFS conditioned on that grade. Consequently, both dimensions of uncertainty can be represented simultaneously, while the framework rigorously reduces to prior models as special cases.

From a theoretical standpoint, the reduction mechanisms defined in Definitions 12 through 15 provide a measurable transformation from set-valued structures into a single grade-membership pair through mean-based aggregation and nearest-neighbour quantization. This is followed by the multiplicative soft yield generating function defined in Eq (21), which embodies a non-compensatory logic: The yield index becomes high only when both components, the discrete grade and the fuzzy membership degree, are substantively strong.

From the perspective of category theory, HHFNSS may be interpreted as a fibre product between the grade-hesitant mapping $G(u, e)$ and the conditional fuzzy-hesitant mapping $F(u, e, g)$. Thus, it is not merely a disjoint combination of existing structures, but rather a genuine interaction in which the hesitant fuzzy space is indexed by the grade-hesitancy space.

4.4. Methodological workflow and rationale

The methodological workflow of the HHFNSS framework is a structured sequence of transformations that convert layered uncertainty into a quantitatively justifiable decision outcome. The process begins with selecting the granularity parameter $N = 5$, which balances evaluators' cognitive limitations with the need for sufficient discrimination in financial analysis. This five-level scale enables a consistent mapping between discrete categories and fuzzy membership intervals over the domain $[0,1]$. In the subsequent stage, multi-evaluator assessments are constructed within the HHFNSS formalism by explicitly separating grade assignment from fuzzy membership valuation. This distinction reflects the cognitive difference between categorical classification and intensity judgment, while simultaneously enabling uncertainty to be represented without imposing artificial consistency constraints.

The reduction stage then consolidates the information using the arithmetic mean. Grade values are summarized through averaging and subsequently quantized using a nearest-neighbour approach, which implicitly assumes interval-scale properties and introduces a conservative bias under ambiguous conditions. In parallel, fuzzy reduction is performed via a two-level averaging procedure, both within hesitant sets and across evaluators, thereby preserving the full range of uncertainty. The final values are then computed using a multiplicative soft yield function, obtained by multiplying normalized grades and fuzzy membership values. This creates a non-compensatory mechanism in which weakness in one dimension cannot be fully offset by strength in another. For cost-type criteria, a complementary transformation is applied to ensure consistency in evaluation orientation.

The ranking process is carried out by additive aggregation across criteria, followed by selecting the maximum score. Although this approach assumes criterion independence and equal weighting, its principal advantages lie in transparency and ease of interpretation. Nevertheless, for more complex settings with interdependencies among criteria, future extensions using non-additive aggregation methods, such as the Choquet integral, may provide a more realistic representation of decision-making dynamics.

Regarding the weighting scheme employed in the case study, the equal-weight assumption provides a neutral and transparent baseline for evaluation. The sensitivity analysis conducted in Section 3.4 confirms that the main decision outcomes are robust to variations in criteria weights, with u_2 maintaining its top position across all tested scenarios. This robustness is an important property of the HHFNSS-soft yield framework, as it suggests that the model can support reliable decisions even when precise weight information is unavailable or contested among stakeholders.

The sensitivity analysis yields several insights for practical application of the proposed framework. First, for investment decisions aimed at identifying the best overall bond, the HHFNSS-soft yield model produces reliable and stable recommendations that are largely insensitive to variations in criteria weights. Second, for more nuanced decisions involving mid-ranked alternatives, explicit weight elicitation from decision-makers, potentially through structured methods such as the Analytic Hierarchy Process (AHP), may be beneficial for resolving preference-dependent ambiguities. Third, the stability of extreme alternatives across all tested scenarios provides confidence in the model's ability to distinguish fundamentally strong and weak performers, irrespective of the preferential weighting structure imposed.

4.5. Computational considerations

The computational implementation of the HHFNSS-soft yield framework requires careful attention to several operational aspects that influence the feasibility and reliability of the decision-making process, particularly data quality, numerical stability, consistency verification, and scalability. The most fundamental aspect concerns the consistency between grades and fuzzy membership degrees. Each grade $g \in G$ is mapped to a corresponding membership interval, as specified in Section 3, for example, grade $0 \rightarrow [0, 0.2]$ through grade $4 \rightarrow [0.8, 1]$. This mapping ensures conceptual coherence while simultaneously providing a mechanism for computational validation. In implementation, a tolerance level of $\varepsilon = 0.01$ is introduced to accommodate minor deviations caused by rounding errors without compromising mapping integrity.

Furthermore, HFSs must be reduced into finite operational forms. In practical settings, evaluators typically provide 1 to 3 discrete membership values, allowing aggregation using the arithmetic mean. For continuous interval cases, approximation techniques such as interval-bound averaging may be employed, although more precise methods, such as centroid-based estimation, remain feasible. The grade quantization process based on Euclidean distance entails very low computational cost; however, it requires special handling of numerical equivalence arising from floating-point precision limitations, including the application of tolerance thresholds and consistent tie-breaking rules.

The soft yield function generates an index within the interval $[0,1]$ by combining grade normalization with multiplication by the membership value, thereby providing sufficient granularity for ranking. From an implementation perspective, the proposed framework is computationally efficient and can be readily integrated into standard computing environments. Accordingly, the principal challenge does not lie in computation, but rather in the accurate collection, preparation, and validation of evaluation data, all of which demand a high level of precision.

4.6. Analysis of computational complexity

The computational complexity analysis shows that the HHFNSS-soft yield framework is highly efficient, with total time complexity linear in the number of alternatives (m) and the number of criteria (n), specifically $O(m \cdot n \cdot s \cdot \kappa)$, assuming the number of evaluators (s), the cardinality of the HFS (κ), and the granularity level (N) are fixed. Each procedure stage, including construction of the HHFNSS structure, grade reduction, fuzzy reduction, and calculation of soft yield values, uses basic aggregation operations such as summation and averaging, imposing minimal computational overhead. Consequently, the model achieves robust scalability across problem sizes.

A major strength of this framework is the elimination of pairwise comparisons among alternatives, which are the main source of quadratic complexity in many traditional Multi-Criteria Decision-Making (MCDM) methods. Methods such as Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [40], Elimination and Choice Expressing Reality (ELECTRE) [41], Preference Ranking Organization METhod for Enrichment of Evaluations (PROMETHEE) [42], and Combinative Distance-based ASsessment (CODAS) [43] often require distance computations or dominance evaluations for each alternative pair, resulting in $O(m^2 \cdot n)$ or higher complexity. In contrast, the HHFNSS-soft yield framework integrates multi-evaluator information into a unified format before the final aggregation, enabling the ranking process to be completed with a linear scan. This approach enhances efficiency and streamlines implementation while maintaining sufficient representational capacity to handle uncertainty.

In terms of space complexity, storage requirements are also linear ($O(m \cdot n \cdot s \cdot \kappa)$), primarily driven by the HHFNSS data structure, which stores multi-evaluator assessments with HFSs. Supplementary structures, including the reduced fuzzy N-soft set and the resulting soft yield values, require only $O(m \cdot n)$ additional space and thus do not significantly increase overall complexity. Thus, the model remains memory-efficient as alternatives and criteria increase, provided that the evaluator counts and hesitant fuzzy remain limited.

For large-scale applications, the number of evaluators s should be determined by the desired level of confidence and perspective diversity rather than by the scale of m and n. Based on [44,45], $s = 3 - 5$ evaluators suffice for small-scale applications, while $s = 5 - 7$ are recommended for medium-scale applications, with diminishing returns beyond approximately $s = 7$. The practical upper bound on m and n is primarily determined by evaluator cognitive capacity rather than computational constraints. For applications with $m > 50$ or $n > 20$, we recommend two-stage screening to reduce the alternative set hierarchical criteria structures, automated data extraction for quantitative indicators, or parallel evaluation with specialized evaluators panels. From a computational perspective, even for $m = 500$ and $n = 20$ with $s = 7$ and $\kappa = 3$, the framework requires storing approximately 210,000 data points and executing a few thousand arithmetic operations, well within standard computational limits. Thus, the computational upper bound is practically unlimited; the binding constraint is the cognitive load on evaluators.

Despite its computational advantages, the main challenge in practical implementation is data acquisition. The total number of required assessments increases multiplicatively $(m \cdot n \cdot s \cdot \kappa)$, which may create significant cognitive load for evaluators, especially when the number of alternatives and criteria is extensive.

4.7. Limitations and future research directions

Despite the significant theoretical advances and empirical validation presented in this study, the HHFNSS-soft yield framework exhibits several limitations that restrict its scope of application and present opportunities for future research. These limitations pertain to underlying assumptions, operational constraints, and empirical coverage.

The grade reduction mechanism formulated in Definition 12 employs the arithmetic mean to aggregate multiple grade values into an average score $\underline{g}_{u,e}$. This approach implicitly assumes that the grade scale possesses interval properties, meaning that the distance between adjacent grade levels is considered equal. Although such an assumption is common in multicriteria decision analysis, its validity becomes questionable when grade labels carry only an ordinal meaning. Researchers may therefore explore alternative aggregation operators that preserve ordinal properties, such as the median or mode. Empirically calibrated approaches based on psychometric techniques, including Thurstone's Law of Comparative Judgment, could also be employed to estimate the actual perceptual distances between grade levels.

The HHFNSS framework treats each evaluator's assessment as a valid input to the hesitant set structure $H^*(u, e)$ without imposing probabilistic distributional assumptions. It does not assume statistical independence among evaluators. Nevertheless, in practical applications such as the corporate bond evaluation presented in Section 3, the three evaluators, a credit analyst, a bond market analyst, and an investment manager, may share common information sources, including financial reports, credit rating agency data, and macroeconomic indicators. Such shared information may induce correlation in their judgments. This is particularly likely when evaluators rely on the same publicly available data or are influenced by prevailing market sentiment.

In the HHFNSS framework, correlated evaluator biases would manifest as reduced dispersion in the hesitant grade set $G_{u,e}$ and the hesitant fuzzy membership set $H_{u,e}$, leading to smaller values of $m_{u,e}$ in Eq (16) and consequently a narrower range of possible grade-membership pairs. This effect is naturally captured by the reduction mechanism. When assessments are highly correlated, the reduction process yields representative values closer to the consensus assessment, whereas when assessments are more diverse, the reduction preserves the range of uncertainty.

The HHFNSS framework does not require independence for its mathematical validity. The hesitant set structure is purely representational. It collects all observed assessments without assuming that they are drawn from independent or identically distributed sources. Consequently, the model remains applicable even when evaluator judgments exhibit correlation. However, users of the framework should be aware that correlated biases may lead to underestimation of the true epistemic uncertainty in the evaluation process. To mitigate this risk, future applications may consider diversifying the evaluator panel to include perspectives from different institutional backgrounds, regulatory jurisdictions, or analytical frameworks. Additionally, inter-rater reliability measures such as Fleiss' Kappa 47 or Krippendorff's alpha 48 could be employed to quantify the degree of evaluator agreement and to inform sensitivity analyses regarding the impact of correlated biases.

The HHFNSS model does not explicitly capture interactions among criteria. The additive aggregation mechanism assumes criterion independence, whereas real-world financial evaluation often involves complex interdependencies. For instance, a high debt-to-equity ratio may be offset by strong interest coverage capacity. Additive models are unable to adequately reflect such relationships and may therefore introduce aggregation bias. Future extensions may consider non-additive aggregation approaches, including geometric scoring with conjunctive logic, the Choquet integral for modeling substitution and complementarity effects under fuzzy measures, or the Sugeno integral for ordinal-scale contexts.

Based on the limitations identified above, three principal directions for future development of the HHFNSS-soft yield framework are proposed. First, regarding the interval-scale assumption, future studies may investigate alternative grade-aggregation operators that preserve ordinal characteristics, such as the median and mode, along with empirical calibration approaches based on Thurstone's Law of Comparative Judgment to estimate the actual distances between grades. Second, concerning evaluator dependence, differential evaluator-weighting schemes may be integrated using criteria such as professional experience, historical validity, or inter-rater agreement statistics such as Fleiss' Kappa. Such schemes would enable more competent or more consistent evaluators to exert proportionate influence on relevant criteria. Third, with respect to criterion interactions, non-additive aggregation mechanisms may be developed to model complex interdependencies. Promising approaches include geometric scoring with conjunctive logic, the Choquet Integral for substitution and complementarity effects under fuzzy measures, and the Sugeno integral for ordinal-scale contexts. Future applications may also benefit from employing structured weight-elicitation methods, such as Analytic Hierarchy Process (AHP) [12] or the entropy weight method [46], to derive criteria weights from stakeholder input or objective data characteristics. Collectively, these research directions aim to enhance representational richness, decision accuracy, and robustness against assumption-driven bias, while preserving the fundamental HHFNSS structure that has proven effective in representing dual-layer uncertainty.

5. Conclusions

In this study, we have developed and validated an HHFNSS (HHFNSS) framework, integrated with a soft yield structure, for modeling corporate bond evaluation under parametric uncertainty. Several key findings emerged. First, the proposed framework effectively integrates two dimensions of uncertainty, hesitation in determining discrete grades and hesitation in assigning fuzzy membership degrees, into a unified mathematical structure. This integration is formally verified through strict reduction properties, whereby HHFNSS reduces to a hesitant fuzzy N-soft set when the grade-level hesitation is absent, to a hesitant N-soft set when fuzzy membership hesitation is absent, and to the classical N-soft set when both conditions are met.

Second, application of the model to five Indonesian corporate bonds using eight yield-determining criteria demonstrates that the HHFNSS-soft yield approach produces decisions consistent with the firms' underlying fundamentals. PT Telekomunikasi Indonesia (Persero) Tbk was identified as the best alternative with a final score of 6.5, supported by an EBITDA margin of 50.74% and a coverage ratio of 7.67, both indicating strong cash-flow stability. Comparative analysis with HFNSS and HNSS shows that HHFNSS can simultaneously capture both dimensions of uncertainty, whereas HFNSS neglects the discrete grade structure and HNSS ignores fuzzy intensity. This is further evidenced by the ranking divergence of PT Adaro Energi Indonesia Tbk, which was ranked third under HNSS due to high grade values but fifth under HHFNSS because of substantial hesitation in membership intensity, reflecting the firm's underlying volatility.

Third, the proposed framework demonstrates notable robustness and reliability. Sensitivity analysis indicates that small variations in fuzzy membership values result in linear and proportional score changes without significantly affecting the ranking structure, suggesting the model is not overly sensitive to evaluator noise. In contrast, a one-level change in grade produces a more pronounced but still controlled effect, with the highest and lowest alternatives maintaining their positions across all scenarios. This stability implies that the model consistently distinguishes alternatives in terms of relative quality and is resistant to rank reversals due to minor input disturbances.

The principal theoretical contribution of this study is the integration of two-layer hesitation into the N-soft set framework, formalized through a product construction between a grade-hesitant mapping and a conditional fuzzy-hesitant mapping. The proposed multiplicative soft yield generating function embodies a non-compensatory logic, where the yield index becomes high only when both evaluation components are substantively strong. This approach contrasts with conventional additive aggregation methods that assume full compensability across components. Practically, the proposed model is more representative, stable, and transparent than competing approaches, while maintaining linear computational complexity, making it suitable for broader implementation.

Several limitations warrant acknowledgement, including the interval-scale assumption underpinning grade aggregation, the uniform treatment of evaluator expertise, and the additive aggregation mechanism that presumes criterion independence. To address these, researchers may explore ordinal-preserving aggregation operators, differential evaluator weighting informed by expertise or inter-rater reliability, non-additive integration frameworks such as the Choquet and Sugeno integrals to accommodate criterion interactions, and structured weight elicitation via the Analytic Hierarchy Process or entropy-based methods. These directions collectively aim to enhance the framework's representational capacity, decision-making accuracy, and robustness while retaining its foundational structure for dual-layer uncertainty modeling.

Appendix: List of symbols

The Table 26 summarizes the principal notation used throughout this paper.

Table 26. List of symbols.

Symbols	Description
U	Universal set of objects (alternatives)
E	Universal set of parameters
A	Subset of parameters under consideration, $A \subseteq E$
G	Ordered grade set, $G = \{0, 1, \dots, N-1\}, N \geq 2$
$P(U)$	Power set of U (set of all subsets of U)
$P^*(U)$	Set of all non-empty subsets U
$[0, 1]$	Unit interval of real numbers
$[0, 1]^U$	Set of all functions from U to $[0, 1]$
N	Granularity parameter, $N \in N, N \geq 2$
(F, A)	Soft set over U with parameter set A
(F, A, N)	N-soft set over U with grade set G
(μ, K)	Fuzzy N-soft set
(H, A, N)	Hesitant N-soft set (HNSS)
(H', A, N)	Hesitant fuzzy N-soft set (HNSS)
(H^*, A, N)	Hybrid hesitant fuzzy N-soft set (HHFNSS)
(R, A, N)	Reduced fuzzy N-soft set (RFNSS) induced by H^*
(Y, A)	Soft yield structure over U
u	Object (alternative), $u \in U$
e	Parameter (criterion), $e \in E$
d	Evaluator (decision maker), $d \in D$
$g_{u,e}$	Grade assigned to pair $(u, e), g_{u,e} \in G$
$h_{u,e}$	Hesitant fuzzy element for pair $(u, e), h_{u,e} \subseteq [0, 1]$
$H^*(u, e)$	Set of grade-membership pairs for (u, e) in HHFNSS
$m_{u,e}$	Cardinality of $H^*(u, e)$
$G_{u,e}$	Projection of $H^*(u, e)$ onto G
$H_{u,e}$	Projection of $H^*(u, e)$ onto hesitant fuzzy sets
$\underline{g}_{u,e}$	Grade average score for pair (u, e)
$Q\left(\underline{g}_{u,e}\right)$	Nearest-neighbour quantization of $\underline{g}_{u,e}$
$GA(u, e)$	Reduced representative grade for pair (u, e)
$\phi\left(h_{u,e}^{(k)}\right)$	Internal reduction of hesitant fuzzy element
$FA(u, e)$	Reduced representative fuzzy membership value for pair (u, e)
$R(u, e)$	Reduced pair $\big(GA(u, e), FA(u, e)\big)$ in RFNSS
$\eta\big(GA(u, e)\big)$	Grade normalization function, $\eta: G \to [0, 1]$
$\psi\big(GA(u, e), FA(u, e)\big)$	Soft yield generating function
$Y(u, e)$	Soft yield value for pair (u, e)
$Y(e)$	Parametric soft yield mapping for parameter e
$S(u_i)$	Final score of alternatives u_i

Author contributions

E. C., R. and S.: Conceptualization; E. C., R., Z. A. Z. and S.: Methodology; A. A. S. HS., Z. A. Z. and M. P. A. S.: Software; E. C., S., and M. P. A. S.: Validation; E. C., R., Z. A. Z. and A. A. S. HS.: Formal analysis; M. L., A. S. A. and E. C.: Investigation; S., M. P. A. S. and A. S. A.: Resources;

R., A. S. A., M. L. and M. P. A. S.: Data curation; A. A. S. HS., E. C. and R.: Writing—original draft preparation; Z. A. Z., E. C. and S.: Writing—review and editing; A. A. S. HS., M. P. A. S. and A. S. A.: Visualization; E. C., R., S. and M. L.: Supervision; S., E. C. and A. S. A.: Project administration; S. and E. C.: Funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

The original contributions presented in this study are included in the article/supplementary material. Further inquiries can be directed to the corresponding author.

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Conflicts of interest

The authors declare no conflicts of interest.

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