Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 26308–26336.
DOI:10.3934/math.20261056
Received: 23 June 2026
Revised: 05 August 2026
Accepted: 12 August 2026
Published: 24 August 2026

Research article

Asymptotic behavior of bootstrapped extreme m-generalized order statistics with application to left endpoint estimation

M. E. Sobh[1], H. M. Barakat[2], Magdy E. El-Adll[3], Amany E. Aly[3] and Asamh Saleh M. Al Luhayb[4,*]

[1] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt

[2] Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt

[3] Department of Mathematics, Faculty of Science, Capital University (Formerly Helwan University), Cairo, Egypt

[4] Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia

* **Correspondence:** Email: a.alluhayb@qu.edu.sa.

Abstract: This paper develops an asymptotic theory for the bootstrapped distributions of normalized upper and lower extreme m-generalized order statistics (m-GOSes). Sufficient conditions are established for their weak convergence when the linear normalizing constants are unknown and must be estimated. The proposed asymptotic framework is developed under suitable consistency conditions for the estimators of the normalizing constants together with the asymptotic requirement that the bootstrap sample size satisfies $M = o(n)$. As an application, asymptotic bootstrap confidence intervals for the left endpoint of distributions in the Weibull domain of attraction are constructed using a pivotal quantity based on lower extreme m-GOSes. The proposed procedure is applicable to both original and sequential order statistics models. A Monte Carlo simulation study is conducted to evaluate the performance of constructing bootstrap confidence intervals for the left endpoints of both exponential and gamma distributions. The results reveal that the bootstrap method provides robust inference whenever the theoretical assumptions are met, which supports the use of bootstrap procedures for reliable estimation of the lower endpoint in finite samples.

Keywords: m-GOSes; bootstrap; weak convergence; Weibull domain of attraction; pivotal quantity; left endpoint estimation; Monte Carlo simulation
Mathematics Subject Classification: Primary 62G32; Secondary 60F05, 62G09, 62G20, 62G30

1. Introduction

Extreme value theory (EVT) is regarded as a fundamental tool for modeling rare events. It is particularly relevant in reliability engineering and survival analysis, where the primary focus is on ordered data, specifically concerning the distribution of lower or upper extremes. Although classical EVT provides a robust framework for analyzing ordinary order statistics (OOSes), real-world applications frequently involve complex sampling designs that deviate from the standard independent and identically distributed (IID) assumption. Examples include progressive censoring schemes, adaptive or sequential sampling procedures, multistage experimental designs, and record-based data structures. Each of these produces ordering patterns that traditional OOSes approaches cannot adequately describe. To address these challenges within a single coherent framework, Kamps [15–17] and Kamps and Cramer [18] developed the concept of GOSes, which provides a versatile and unified model for representing a broad class of ordered random variables (RVs) through a common parametric formulation.

One of the most significant subclasses of GOSes is the class of m-GOSes. This subclass arises when the parameter vector is restricted such that $m_1 = \cdots = m_{n-1} = m$. Under this condition, the model becomes both flexible and analytically manageable, encompassing several well-known models as special cases, including OOSes, record values, Type-II censored samples, and sequential order statistics (SOSes). Moreover, the m-GOS framework extends the classical theory of ordered samples while retaining key analytical features, such as explicit forms of joint distributions, closed-form marginal distributions, and stable transformation properties. These characteristics make it highly valuable for both theoretical development and practical applications.

The mathematical framework of m-GOSes has attracted considerable attention in the literature due to its unified treatment of OOSes, record values, SOSes, Pfeifer records, and several other ordered random structures. Their marginal distributions admit convenient representations through powers of the underlying survival function, which considerably facilitates both theoretical investigations and practical applications. Moreover, the asymptotic behavior of their extreme values closely parallels the classical theory of extreme value distributions, allowing many results from EVT to be extended naturally to the generalized setting. Early contributions by Nasri-Roudsari [20, 21] established fundamental limit theorems for extreme GOSes under both linear and power normalizations, thereby identifying the corresponding domains of convergence and characterizing the limiting extreme value distributions. These pioneering results were substantially generalized by Barakat [2], who developed a comprehensive limit theory for m-GOSes, and by Barakat and El-Adll [3], who investigated the asymptotic behavior of extreme dual GOSes and established analogous convergence results for lower extremes.

Subsequent developments have further enriched the asymptotic theory of m-GOSes from several perspectives. Schmiedt [24] provided a detailed characterization of the domains of attraction associated with asymptotic distributions of extreme GOSes, clarifying the relationship between GOSes and the classical Fisher-Tippett-Gnedenko framework. More recently, Bieniek and Rychlik [7] derived general conditions for the finiteness of moments and established sharp moment bounds for GOSes, yielding important tools for studying their asymptotic behavior, tail properties, and probabilistic inequalities. These advances, together with earlier investigations on convergence rates, tail characteristics, and increasingly flexible parameterizations, have significantly broadened the theoretical foundations of

GOSes and strengthened their role in reliability analysis, survival studies, risk theory, and extreme value modeling. Owing to their analytical tractability and remarkable ability to unify diverse ordered RVs within a single mathematical framework, m-GOSes continue to represent a central topic in the theory of ordered RVs. The next two theorems summarize the principal asymptotic results concerning the upper and lower extremes of m-GOSes, which serve as the theoretical foundation for the developments presented in this work.

We adopt the notations $\underset{n}{\overset{w}{\to}}$, $\underset{n}{\overset{p}{\to}}$, $\underset{n}{\overset{\text{w.p.1}}{\longrightarrow}}$, and $\underset{n}{\longrightarrow}$ to denote weak convergence (convergence in distribution), convergence in probability, convergence with probability 1 (almost sure convergence), and convergence as $n \to \infty$, respectively.

Theorem 1.1 (Nasri-Roudsari [20]). *Let $m_1 = m_2 = \cdots = m_{n-1} = m > -1$, and let $r < n$ be an integer independent on n. Then, there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that*

$$\Psi^{(m,k)}_{n-r+1:n}(a_n x + b_n) = P\left(X^{(m,k)}_{n-r+1:n} \le a_n x + b_n\right) \underset{n}{\overset{w}{\to}} \Psi^{(m,k)}_{r;t}(x),\ t \in \{1, 2, 3\}, \tag{1.1}$$

where $X^{(m,k)}_{n-r+1:n}$ denotes the $(n-r+1)$th m-GOS based on a DF F, and $\Psi^{(m,k)}_{j;t}(x)$ is a nondegenerate DF. Moreover, this convergence holds if and only if there exist sequences of constants $a^*_n > 0$ and $b^*_n \in \mathbb{R}$ for which

$$\Psi^{(0,1)}_{n-r+1:n}(a^*_n x + b^*_n) \underset{n}{\overset{w}{\to}} \Psi^{(0,1)}_{r;t}(x) = 1 - \Gamma_r(w_{t,\delta}(x)), \qquad t = 1, 2, 3,$$

where $\Gamma_r(.)$ denotes the incomplete gamma function

$$\Gamma_r(t) := \frac{1}{(r-1)!}\int_t^\infty u^{r-1} e^{-u}\, du = 1 - \sum_{\ell=0}^{r-1} \frac{t^\ell}{\ell!} e^{-t},$$

and the function $w_{t,\delta}(x)$ is defined as follows:

$$\begin{aligned}
&\textit{Type I:} \quad && w_{1,\delta}(x) = \begin{cases} x^{-\delta}, & x > 0, \\ \infty, & x \le 0, \end{cases} \\
&\textit{Type II:} \quad && w_{2,\delta}(x) = \begin{cases} (-x)^{\delta}, & x \le 0, \\ 0, & x > 0, \end{cases} \\
&\textit{Type III:} \quad && w_{3,\delta}(x) = e^{-x}, \qquad \forall x.
\end{aligned}$$

Furthermore, the limit distribution of the upper extreme m-GOS is

$$\Psi^{(m,k)}_{r;t}(x) = 1 - \Gamma_J\Big((w_{t,\delta}(x))^{m+1}\Big), \quad J = r + \left\lfloor \frac{k}{m+1} \right\rfloor - 1.$$

The normalizing constants can be chosen in the form $a_n = a^*_{M(n)}$, $b_n = b^*_{M(n)}$, with $M(n) = n^{1/(m+1)}$.

Theorem 1.2 (Barakat and El-Adll [5]). *Assume that $m_1 = \cdots = m_{N-1} = m > -1$ and that $r < n$ is an integer independent on n. Then, there exist constants $c_n > 0$ and $d_n \in \mathbb{R}$ such that*

$$\tilde{\Psi}^{(m,k)}_{r:n}(c_n x + d_n) \underset{n}{\overset{w}{\to}} \tilde{\Psi}^{(m,k)}_{r,t}(x),\ t = 1, 2, 3, \tag{1.2}$$

where the limit distribution $\tilde{\Psi}^{(m,k)}_{r,t}(x)$ is nondegenerate. This convergence holds if and only if there exist constants $c^*_n > 0$ and $d^*_n \in \mathbb{R}$ satisfying

$$\tilde{\Psi}^{(0,1)}_{r:n}(c^*_n x + d^*_n) \xrightarrow[n]{w} \tilde{\Psi}^{(0,1)}_{r,t}(x) = \Gamma_r\Big(w^*_{t,\delta}(x)\Big), \qquad t = 1, 2, 3,$$

where the function $w^*_{t,\delta}(x)$ is defined according to the following three cases:

$$\text{Type I:} \quad w^*_{1,\delta}(x) = \begin{cases} (-x)^{-\delta}, & x \leq 0, \\ \infty, & x > 0, \end{cases}$$
$$\text{Type II:} \quad w^*_{2,\delta}(x) = \begin{cases} x^{\delta}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$
$$\text{Type III:} \quad w^*_{3,\delta}(x) = e^x, \qquad \forall x.$$

In this case, the limiting distribution can be expressed as

$$\tilde{\Psi}^{(m,k)}_{r,t}(x) = \Gamma_r(w^*_{t,\delta}(x)) = 1 - \sum_{\ell=0}^{r-1} \frac{(w^*_{t,\delta}(x))^{\ell}}{\ell!} e^{-w^*_{t,\delta}(x)}.$$

The normalizing sequences may be chosen in the form $c_n = c^*_{R(n)}$ and $d_n = d^*_{R(n)}$, where $R(n) = (m+1)n$.

The bootstrap method, originally proposed by Efron [11], has become a cornerstone of modern statistical inference, particularly in situations where the underlying DF $F(x)$ is unknown. Due to its ease of use and effectiveness in producing positive findings, the approach has been extensively employed in the literature for a range of statistical problems; see [12, 19]. The bootstrap approach may be quite useful in situations where little is known about an appropriate distribution [10]. The bootstrap technique has been employed in a number of practical applications; for further information, see [9, 27].

For an IID sample $\mathbf{X}_n = (X_1, \ldots, X_n)$ from a DF $F(x)$, the bootstrap approach approximates $F(x)$ by its empirical counterpart

$$F_n(x) = \frac{1}{n} \sum_{j=1}^{n} \mathbb{I}(X_j \leq x),$$

where $\mathbb{I}(X_j \leq x)$ denotes the indicator function. Bootstrap samples $\mathbf{Y} = (Y_1, \ldots, Y_N)$ are then constructed by sampling with replacement from $F_n(x)$, where the bootstrap sample size $N = N(n)$ increase such that $N \xrightarrow[n]{} \infty$. Conditionally on the observed sample, each resampled observation satisfies

$$P(Y_i = X_j \mid \mathbf{X}_n) = \frac{1}{n}, \quad \forall\, i = 1, \ldots, N.$$

Moreover, let $Y_{1:N}, Y_{2:N}, \ldots, Y_{N:N}$ denote the OOSes of the bootstrap sample $Y_1, \ldots, Y_N$.

The rth $m-$GOS $Y^{(m,k)}_{r:N}$ based on the empirical DF F_n may be easily obtained by considering the independent RVs with respective beta distribution $\beta(\gamma_j, 1)$; that is B_j follows a power DF with exponent $\gamma_j = k + (N - j)(m + 1)$, $j = 1, ..., N$, where $B_j := F^{\frac{1}{\gamma_j}}(Y_{r:N})$ and $Y_{r:N} := Y^{(0,1)}_{r:N}$ is the rth order statistic. Therefore, in view of the results of Cramer [8]

$$Y^{(m,k)}_{r:N} = F_n^{-1}\left(1 - \prod_{j=1}^{r} B_j\right), \; r = 1, 2, ..., N,$$

is the rth $m-$GOS based on the empirical DF F_n. Furthermore, let

$$H_{r,n,N}^{(m,k)}(x) = P\left(\frac{Y_{r:N}^{(m,k)} - b_N}{a_N} \leq x|\,\mathbf{X_n}\right) = \mathrm{I}_{\Psi_{n,m:N}(x)}(N^\star - r^\star + 1, r^\star) \tag{1.3}$$

be the bootstrap distribution of $(X_{r:N}^{(m,k)} - b_n)/a_n$, where $r^\star = r + k/(m+1) - 1$, $N^\star = N + k/(m+1) - 1$, $\mathrm{I}_x(a,b)$, $0 \leq a, b \leq 1$ is the usual incomplete beta function, and $\Psi_{n,m:N}(x) = 1 - (1 - F_n(a_N x + b_N))^{m+1}$ is a DF. Define

$$\Omega_{N-r+1,N}^{(m,k)}(x) = P\left(a_N^{-1}(Y_{N-r+1:N}^{(m,k)} - b_N) \leq x|\mathbf{X}_n\right)$$

as the bootstrap distribution of the normalized upper m-GOS extreme, $a_n^{-1}(X_{n-r+1:n}^{(m,k)} - b_n)$. Similarly, for any integer $1 \leq r < n$, let

$$\Phi_{r,N}^{(m,k)}(x) = P\left(c_N^{-1}(Y_{r:N}^{(m,k)} - d_N) \leq x|\mathbf{X}_n\right)$$

represent the bootstrap distribution of the normalized lower m-GOS extreme, $c_n^{-1}(X_{r:n}^{(m,k)} - d_n)$, where n and N are the sample size and resample size, respectively.

Despite the extensive use of bootstrap techniques in the study of OOSes and m-GOSes, several theoretical challenges, particularly those related to exact distributional characterizations and asymptotic behavior, continue to attract attention. Early theoretical groundwork was laid by Athreya and Fukuchi [1], Fukuchi [14], and Davison and Hinkley [10], while more recent developments were addressed by Nigm [22], Schendel and Thongwichian [23], and Barakat et al. [4]. Further advances, including extensions to related distributional settings, were provided by Barakat et al. [6] and Sobh et al. [25, 26].

In this paper, we further develop this line of research by investigating the asymptotic properties of bootstrap procedures for both lower and upper m-GOSes under unknown linear normalization. The results presented here contribute to a deeper understanding of their probabilistic structure and extend the theoretical framework of GOSes. The remainder of the paper is organized as follows. Section 2 presents the theoretical results on the bootstrap behavior of both upper and lower extreme m-GOSes under unknown linear normalizing constants. Section 3 provides a Monte Carlo simulation study to construct confidence intervals for the left endpoint of a distribution based on the first r lower extreme m-GOSes. Finally, Section 4 concludes the simulation with a summary of the main findings and their practical implications.

2. Bootstrapping extreme m-GOSes under unknown normalizing constants

Suppose that the normalizing constants a_N and b_N are unknown and must therefore be estimated from the sample data $\mathbf{X}_n$. Let $\hat{a}_N$ and $\hat{b}_N$ denote suitable estimators of a_N and b_N, respectively. We then define the bootstrap distribution of the normalized m-GOS $a_n^{-1}(X_{n-r+1:n}^{(m,k)} - b_n)$ with estimated normalizing constants by

$$\hat{\Omega}_{N-r+1,N}^{(m,k)}(x) = P\left(\hat{a}_N^{-1}(Y_{N-r+1:N}^{(m,k)} - \hat{b}_N) \leq x|\mathbf{X}_n\right).$$

Similarly, let $\hat{c}_N$ and $\hat{d}_N$ be estimators of the normalizing constants c_N and d_N based on the sample $\mathbf{X}_n$. The corresponding bootstrap distribution of $c_n^{-1}(X_{r:n}^{(m,k)} - c_n)$ is then given by

$$\hat{\Phi}_{r,N}^{(m,k)}(x) = P\left(\hat{c}_N^{-1}(Y_{r:N}^{(m,k)} - \hat{d}_N) \leq x|\mathbf{X}_n\right).$$

To investigate the asymptotic behavior of the bootstrap distribution of the upper and lower extreme m-GOSes, in the case where the normalizing constants are unknown, we begin by the following fundamental theorem.

Theorem 2.1. *Let $(\hat{a}_N, \hat{b}_N)$ and $(\hat{c}_N, \hat{d}_N)$ be sequences of estimators of the normalizing constants (a_N, b_N) and (c_N, d_N), respectively, where $N = N(n)$. Assume that the following conditions hold:*

(i) $\Omega^{(m,k)}_{N-r+1,N}(x) \xrightarrow[n]{\text{w.p.1}} \Psi^{(m,k)}_{r,t}(x) \qquad \left(\Phi^{(m,k)}_{r,N}(x) \xrightarrow[n]{\text{w.p.1}} \tilde{\Psi}^{(m,k)}_{r,t}(x)\right);$

(ii) $\dfrac{\hat{a}_N}{a_N} \xrightarrow[n]{\text{w.p.1}} 1 \qquad \left(\dfrac{\hat{c}_N}{c_N} \xrightarrow[n]{\text{w.p.1}} 1\right);$

(iii) $\dfrac{\hat{b}_N - b_N}{a_N} \xrightarrow[n]{\text{w.p.1}} 0 \qquad \left(\dfrac{\hat{d}_N - d_N}{c_N} \xrightarrow[n]{\text{w.p.1}} 0\right).$

Then,

$$\sup_{x\in\mathbb{R}} \left|\hat{\Omega}^{(m,k)}_{N-r+1,N}(x) - \Psi^{(m,k)}_{r,t}(x)\right| \xrightarrow[n]{\text{w.p.1}} 0 \tag{2.1}$$

for upper m-GOSes, and

$$\sup_{x\in\mathbb{R}} \left|\hat{\Phi}^{(m,k)}_{r,N}(x) - \tilde{\Psi}^{(m,k)}_{r,t}(x)\right| \xrightarrow[n]{\text{w.p.1}} 0 \tag{2.2}$$

for lower m-GOSes.

Moreover, the conclusions (2.1) *and* (2.2) *remain valid if the almost sure convergence in* (i)–(iii) *is replaced by convergence in probability.*

Proof. The proof follows the same general approach as that of Theorem 3.3 in Barakat et al. [6], with only the necessary modifications to accommodate the present setting. □

Theorem 2.2. *Let $m_1 = m_2 = \cdots = m_{n-1} = m > -1$, and let $r < n$ be an integer independent on n. Assume that there exist normalizing constants $a_N > 0$ and b_N such that (1.1) holds, where a_N and b_N are unknown. Let $\hat{a}_N$ and $\hat{b}_N$ denote estimators of a_N and b_N based on the sample $\mathbf{X}_n$. Further, define $\nu = \lfloor n/M(N)\rfloor$, $\nu' = \lfloor n/eM(N)\rfloor$, and let ω_F be the upper endpoint of the DF F. Suppose that the estimators $\hat{a}_N$ and $\hat{b}_N$ are chosen as follows:*

(i) For $\Psi^{(m,k)}_{r,1}(x) = 1 - \Gamma_J\Big((w_{1,\delta}(x))^{m+1}\Big)$,

$$\hat{a}_N = F_n^{-1}\left(1 - \frac{1}{M(N)}\right) = X^{(m,k)}_{n-\nu:n}, \quad \hat{b}_N = 0;$$

(ii) For $\Psi^{(m,k)}_{r,2}(x) = 1 - \Gamma_J\Big((w_{2,\delta}(x))^{m+1}\Big)$,

$$\hat{a}_N = \omega_{F_n} - F_n^{-1}\left(1 - \frac{1}{M(N)}\right) = X^{(m,k)}_{n:n} - X^{(m,k)}_{n-\nu:n}, \quad \hat{b}_N = \omega_{F_n} = X^{(m,k)}_{n:n};$$

(iii) For $\Psi^{(m,k)}_{r,3}(x) = 1 - \Gamma_J\Big((w_{3,\delta}(x))^{m+1}\Big)$,

$$\begin{aligned}\hat{a}_N &= F_n^{-1}\left(1 - \frac{1}{eM(N)}\right) - F_n^{-1}\left(1 - \frac{1}{M(N)}\right) = X^{(m,k)}_{n-\nu':n} - X^{(m,k)}_{n-\nu:n},\\ \hat{b}_N &= F_n^{-1}\left(1 - \frac{1}{M(N)}\right) = X^{(m,k)}_{n-\nu:n}.\end{aligned}$$

If $M(N) = o(n)$, then

$$\sup_{x\in\mathbb{R}} \left|\hat{\Omega}^{(m,k)}_{N-r+1:N}(x) - \Psi^{(m,k)}_{r,t}(x)\right| \xrightarrow[n]{p} 0. \tag{2.3}$$

Furthermore, if $\sum_{n=1}^{\infty} \eta^{n/M(N)} < \infty$ for every $\eta \in (0,1)$, then (2.3) holds almost surely.

Proof. We begin with $\Psi^{(m,k)}_{r,1}(x) = 1 - \Gamma_J\Big((w_{1,\eta}(x))^{m+1}\Big)$. Because $\hat{b}_N = 0$, Condition (ii) of Theorem 2.1 is satisfied, and no further verification is required. Therefore, it suffices to establish the condition (iii), which we address below. Specifically, we aim to show that

$$\frac{\hat{a}_N}{a_N} = \frac{X^{(m,k)}_{n-\nu:n}}{a_N} \xrightarrow[n]{p} 1, \tag{2.4}$$

either in probability or almost surely. To verify (2.4), let, $\epsilon \in (0,1)$ be arbitrary, and define $\mathcal{V}_N(\epsilon) = (\epsilon+1)a_N$. Denote by $U_{n,m}(x)$ the number of observations X_j that exceed x for $1 \le r \le n$. Then,

$$\begin{aligned}
P\left(\frac{X^{(m,k)}_{n-\nu:n}}{a_N} - 1 > \epsilon\right) &= P\left(X^{(m,k)}_{n-\nu:n} > (\epsilon+1)a_N\right) = P\left(X^{(m,k)}_{n-\nu:n} > \mathcal{V}_N(\epsilon)\right) \\
&= P(U_{n,N}(x)(\mathcal{V}_N(\epsilon)) > k) \sim P\left(\frac{M(N)}{n} U_{n,N}(\mathcal{V}_N(\epsilon)) > 1\right) \\
&= P\left(\frac{M(N)}{n} \sum_{i=1}^{n} I(X_i > \mathcal{V}_N(\epsilon)) > 1\right) \\
&= P\left(M(N)(1 - F_n(\mathcal{V}_N(\epsilon))) > 1\right). \qquad (2.5)
\end{aligned}$$

Moreover,

$$M(N)(1-F_n(\mathcal{V}_N(\epsilon))) = M(N)(1-F_n((\epsilon+1)a_N) = M(N)(1-F_n((\epsilon+1)a^*_{M(N)}) \xrightarrow[n]{p} \log\Big(e^{(\epsilon+1)^{-\delta}}\Big) = (\epsilon+1)^{-\delta} < 1.$$

Hence,

$$P\left(\frac{X^{(m,k)}_{n-\nu:n}}{a_N} - 1 > \epsilon\right) \xrightarrow[n]{} 0. \tag{2.6}$$

Following an analogous argument, we take $\tilde{\mathcal{V}}_N(\epsilon) = (1-\epsilon)a_N$. Because

$$M(N)(1 - F_n(\tilde{\mathcal{V}}_N(\epsilon))) = M(N)(1 - F_n((1-\epsilon)a_N)) = M(N)(1 - F_n((1-\epsilon)a^*_{M(N)})),$$

it follows that

$$M(N)(1 - F_n((1-\epsilon)a^*_{M(N)})) \xrightarrow[n]{p} \log\left(e^{(1-\epsilon)^{-\delta}}\right) = (1-\epsilon)^{-\delta} > 1.$$

Thus, we have

$$\begin{aligned}
P\left(\frac{X^{(m,k)}_{n-\nu:n}}{a_N} - 1 < -\epsilon\right) &= P\left(X^{(m,k)}_{n-\nu:n} < (1-\epsilon)a_N\right) \\
&= P\left(X^{(m,k)}_{n-\nu:n} < \tilde{\nu}_N(\epsilon)\right) = P\left(U_{n,N}(x)(\tilde{\mathcal{V}}_N(\epsilon))) < k\right) \\
&\sim P\left(\frac{M(N)}{n} U_{n,N}(\tilde{\mathcal{V}}_N(\epsilon)) < 1\right) = P\left(\frac{M(N)}{n} \sum_{i=1}^{n} I(X_i > \tilde{\mathcal{V}}_N(\epsilon)) < 1\right)
\end{aligned}$$

$$= P\Big(M(N)(1 - F_n(\tilde{\mathcal{V}}_N(\epsilon))) < 1\Big) \underset{n}{\longrightarrow} 0. \tag{2.7}$$

Combining (2.6) and (2.7), we obtain $P\left(\left|X^{(m,k)}_{n-\nu:n}/a_N - 1\right| > \epsilon\right) \underset{n}{\longrightarrow} 0$. Hence, Relation (2.4) holds in probability. To strengthen this result to almost sure convergence, assume that

$$\sum_{n=1}^{\infty} \eta^{\frac{n}{M(N)}} < \infty \quad \text{for all } 0 < \eta < 1.$$

Let $\mathcal{G}_N(\tau)$ denote the moment-generating function of a Bernoulli random variable with parameter $1 - F_n(\mathcal{V}_N(\epsilon))$. Because

$$M(N)(1 - F_n(\mathcal{V}_N(\epsilon))) \xrightarrow[n]{p} (\epsilon + 1)^{-\delta},$$

we deduce, for any fixed $\tau > 0$, that

$$\begin{aligned}
\frac{M(N)}{n} \log P\left(M(N)(1 - F_n(\mathcal{V}_N(\epsilon))) > 1\right) &= \frac{M(N)}{n} \log P\left(n(1 - F_n(\mathcal{V}_N(\epsilon))) > \frac{n}{M(N)}\right) \\
&= \frac{M(N)}{n} \log P\left(e^{\tau n(1 - F_n(\mathcal{V}_N(\epsilon)))} > e^{\tau n/M(N)}\right) \\
&\leq \frac{M(N)}{n} \log\left(e^{-\tau n/M(N)} E\left[e^{\tau n(1 - F_n(\mathcal{V}_N(\epsilon)))}\right]\right) \\
&\sim -t + \frac{M(N)}{n} \log \mathcal{G}^n_N(\tau) = -\tau + \log \mathcal{G}^{M(N)}_N(\tau) \\
&\longrightarrow -\tau + \log e^{(\epsilon+1)^{-\delta}(e^{\tau}-1)} \\
&= -\tau + (\epsilon + 1)^{-\delta}(e^{\tau} - 1) =: g(\tau, \epsilon).
\end{aligned} \tag{2.8}$$

Differentiating $g(\tau, \epsilon)$ with respect to τ shows that the minimizer is $\tau_0 = \delta \log(\epsilon + 1)$. Now, define

$$z(\epsilon) = g(\tau_0, \epsilon) = -\delta \log(\epsilon + 1) - (\epsilon + 1)^{-\delta} + 1.$$

Clearly, $z(0) = 0$. Moreover,

$$z'(\varepsilon) = -\frac{\delta}{\varepsilon + 1} + \frac{\delta}{(\varepsilon + 1)^{\delta+1}} < 0;$$

hence, $z(\epsilon)$ is strictly negative for all $\epsilon > 0$. Next, define

$$z_n(\epsilon) = \frac{M(N)}{n} \log\left(e^{-\tau n/M(N)} E\left[e^{\tau n(1 - F_n(\mathcal{V}_N(\epsilon)))}\right]\right).$$

Then, $z_n(\epsilon) \underset{n}{\longrightarrow} z(\epsilon)$. Substituting $\tau = \tau_0(\epsilon)$ into (2.8) yields

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\frac{X^{(m,k)}_{n-\nu:n}}{a_N} - 1 > \epsilon\right) &\sim \sum_{n=1}^{\infty} P\left\{M(N)(1 - F_n(\mathcal{V}_N(\epsilon))) > 1\right\} \\
&= \sum_{n=1}^{\infty} e^{\frac{n}{M(N)}\left(\frac{M(N)}{n} \log P\{M(N)(1 - F_n(\mathcal{V}_N(\epsilon)))>1\}\right)} \leq \sum_{n=1}^{\infty} e^{\frac{n}{M(N)} z_n(\epsilon)}.
\end{aligned}$$

Given $\epsilon > 0$, choose $\delta_\epsilon > 0$ such that $z(\epsilon) + \delta_\epsilon < 0$, and select N_ϵ large enough that $g_n(\epsilon) < z(\epsilon) + \delta_\epsilon$ for all $n \geq N_\epsilon$. Then,

$$\sum_{n=1}^{\infty} e^{\frac{n}{M(N)} z_n(\epsilon)} = \sum_{n=1}^{N_\epsilon - 1} e^{\frac{n}{M(N)} z_n(\epsilon)} + \sum_{n=N_\epsilon}^{\infty} e^{\frac{n}{M(N)} z_n(\epsilon)} = \sum_{n=1}^{N_\epsilon - 1} e^{\frac{n}{M(N)} z_n(\epsilon)} + \sum_{n=N_\epsilon}^{\infty} e^{\frac{n}{M(N)} (z(\epsilon)+\delta_\epsilon)}.$$

Because $n/M(N) \xrightarrow[n]{} 0$, and $z(\epsilon) + \delta_\epsilon < 0$, the tail sum converges 0, and thus,

$$\sum_{n=1}^{\infty} P\left(\frac{X_{n-\nu:n}^{(m,k)}}{a_N} - 1 > \epsilon\right) \sim \sum_{n=1}^{\infty} P\left(M(N)(1 - F_n(\mathcal{V}_N(\epsilon))) > 1\right) < \infty.$$

A symmetric argument for $\tilde{\mathcal{V}}_N(\epsilon)$ yields

$$\sum_{n=1}^{\infty} P\left(\frac{X_{n-\nu:n}^{(m,k)}}{a_N} - 1 < -\epsilon\right) \sim \sum_{n=1}^{\infty} P\left(M(N)(1 - F_n(\tilde{\mathcal{V}}_N(\epsilon))) < 1\right) < \infty.$$

Therefore, by the Borel-Cantelli lemma,

$$\frac{\hat{a}_N}{a_N} = \frac{X_{n-\nu:n}^{(m,k)}}{a_N} \xrightarrow[n]{w.p.1} 1,$$

which establishes the first point of our theorem.

Next, assume $\Psi_{r,2}^{(m,k)}(x) = 1 - \Gamma_J\Big((w_{2,\eta}(x))^{m+1}\Big)$. By Theorem 2.1, verifying Conditions (ii) and (iii) reduces to showing that

$$\frac{\hat{a}_N}{a_N} = \frac{X_{n:n}^{(m,k)} - X_{n-\nu:n}^{(m,k)}}{a_N} \xrightarrow[n]{p} 1, \tag{2.9}$$

and

$$\frac{\hat{b}_N - b_N}{a_N} = \frac{X_{n:n}^{(m,k)} - b_N}{a_N} \xrightarrow[n]{p} 0. \tag{2.10}$$

Noting that

$$\frac{\hat{a}_N}{a_N} = \frac{X_{n:n}^{(m,k)} - b_N}{a_N} - \frac{X_{n-\nu:n}^{(m,k)} - b_N}{a_N},$$

to establish Eqs (2.9) and (2.10), it suffices to show that

$$\frac{X_{n:n}^{(m,k)} - b_N}{a_N} \xrightarrow[n]{p} 0, \tag{2.11}$$

and

$$\frac{X_{n-\nu:n}^{(m,k)} - b_N}{a_N} \xrightarrow[n]{p} -1. \tag{2.12}$$

To prove (2.11), let $0 < \epsilon < 1$, and define $\mathcal{V}_N(\epsilon) = a_N\epsilon + b_N$. Then,

$$M(N)\left(1 - F_n(\mathcal{V}_N(\epsilon))\right) = M(N)\left(1 - F_n(\epsilon a_{M(N)}^* + b_{M(N)}^*\right) \xrightarrow[n]{p} 0,$$

implying

$$\begin{aligned}
P\left(\frac{X_{n:n}^{(m,k)}-b_N}{a_N}>\epsilon\right)&=P\left(X_{n:n}^{(m,k)}>\epsilon a_N+b_N\right)\\
&=P(X_{n:n}^{(m,k)}>\mathcal{V}_N(\epsilon))=P(U_{n,N}(\mathcal{V}_N(\epsilon))>0)\\
&\sim P\left(\frac{M(N)}{n}U_{n,N}(\mathcal{V}_N(\epsilon))>0\right)=P\left(\frac{M(N)}{n}\sum_{i=1}^{n}I(X_i>\mathcal{V}_N(\epsilon))>0\right)\\
&=P\left(M(N)(1-F_n(\mathcal{V}_N(\epsilon)))>0\right)\xrightarrow[n]{p}0. \qquad (2.13)
\end{aligned}$$

Similarly, for $\hat{\mathcal{V}}_N(\epsilon)=-\epsilon a_N+b_N$, we have

$$M(N)(1-F_n(\hat{\mathcal{V}}_N(\epsilon)))=M(N)(1-F_n(-\epsilon a^*_{M(N)}+b^*_{M(N)})\xrightarrow[n]{p}\epsilon^{\delta}\neq 0,$$

which leads to

$$\begin{aligned}
P\left(\frac{X_{n:n}^{(m,k)}-b_N}{a_N}<-\epsilon\right)&=P\left(X_{n:n}^{(m,k)}<-\epsilon a_N+b_N\right)\\
&=P\left(X_{n:n}^{(m,k)}<\hat{\mathcal{V}}_N(\epsilon)\right)=P(U_{n,N}(\hat{\mathcal{V}}_N(\epsilon))=0)\\
&\sim P\left(\frac{M(N)}{n}U_{n,N}(\hat{\mathcal{V}}_N(\epsilon))=0\right)=P\left(\frac{M(N)}{n}\sum_{i=1}^{n}I(X_i>\hat{\mathcal{V}}_N(\epsilon))=0\right)\\
&=P\left(M(N)(1-F_n(\hat{\mathcal{V}}_N(\epsilon)))=0\right)\xrightarrow[n]{p}0. \qquad (2.14)
\end{aligned}$$

Combining (2.13) and (2.14) yields $P\left(\left|(X_{n:n}^{(m,k)}-b_N)/a_N\right|>\epsilon\right)\xrightarrow[n]{}0$, establishing (2.11) in probability. Following the same argument as in the first part of the theorem, this also holds almost surely. We now proceed to prove Eq (2.12). For any $0<\epsilon<1$, let $\mathcal{V}_N(\epsilon)=(\epsilon-1)a_N+b_N$. Then,

$$\begin{aligned}
P\left(\frac{X_{n-\nu:n}^{(m,k)}-b_N}{a_N}-(-1)>\epsilon\right)&=P\left(X_{n-\nu:n}^{(m,k)}>(\epsilon-1)a_N+b_N\right)\\
&=P\left(X_{n-\nu:n}^{(m,k)}>\mathcal{V}_N(\epsilon)\right)=P(U_{n,N}(\mathcal{V}_N(\epsilon))>k)\\
&\sim P\left(\frac{M(N)}{n}U_{n,N}(\mathcal{V}_N(\epsilon))>1\right)\\
&=P\left(\frac{M(N)}{n}\sum_{i=1}^{n}I(X_i>\mathcal{V}_N(\epsilon))>1\right)\\
&=P\left(M(N)(1-F_n(\mathcal{V}_N(\epsilon)))>1\right). \qquad (2.15)
\end{aligned}$$

It is straightforward to show that

$$M(N)(1-F_n(\mathcal{V}_N(\epsilon)))=M(N)(1-F_n((\epsilon-1)a^*_{M(N)}+b^*_{M(N)})\xrightarrow[n]{p}(1-\epsilon)^{\delta}<1,$$

which implies $P\left((X_{n-\nu:n}^{(m,k)}-b_N)/a_N-(-1)>\epsilon\right)\xrightarrow[n]{}0$. By following an analogous procedure, we obtain $P\left((X_{n-\nu:n}^{(m,k)}-b_N)/a_N-(-1)<-\epsilon\right)\xrightarrow[n]{}0$. Thus, Eq (2.12) is established in probability. Applying the arguments from the first part of the theorem yields almost sure convergence, which completes the proof of the second part.

Finally, for $\Psi_{r,3}^{(m,k)}(x) = 1 - \Gamma_J\Big((w_{3,\eta}(x))^{m+1}\Big)$, verifying the conditions of Theorem 2.1 requires showing

$$\frac{\hat{a}_N}{a_N} = \frac{X_{n-\nu':n}^{(m,k)} - X_{n-\nu:n}^{(m,k)}}{a_N} \xrightarrow[n]{p} 1 \tag{2.16}$$

and

$$\frac{\hat{b}_N - b_N}{a_N} = \frac{X_{n-\nu:n}^{(m,k)} - b_N}{a_N} \xrightarrow[n]{p} 0, \tag{2.17}$$

both in probability or almost surely. Because

$$\frac{\hat{a}_N}{a_N} = \frac{X_{n-\nu':n}^{(m,k)} - b_N}{a_N} - \frac{X_{n-\nu:n}^{(m,k)} - b_N}{a_N},$$

to prove (2.16) and (2.17), it is enough to show

$$\frac{X_{n-\nu':n}^{(m,k)} - b_N}{a_N} \xrightarrow[n]{p} 1 \tag{2.18}$$

and

$$\frac{X_{n-\nu:n}^{(m,k)} - b_N}{a_N} \xrightarrow[n]{p} 0. \tag{2.19}$$

To prove (2.18), for arbitrary $0 < \epsilon < 1$, let $\hat{\mathcal{V}}_N(\epsilon) = (\epsilon + 1)a_N + b_N$. Then,

$$\begin{aligned} P\left(\frac{X_{n-\nu':n}^{(m,k)} - b_N}{a_N} - 1 > \epsilon\right) &= P\left(X_{n-\nu':n}^{(m,k)} > (\epsilon+1)a_N + b_N\right) \\ &= P(X_{n-\nu':n}^{(m,k)} > \mathcal{V}_N(\epsilon)) = P(U_{n,N}(\mathcal{V}_N(\epsilon)) > k') \\ &\sim P\left(\frac{M(N)}{n} U_{n,N}(\mathcal{V}_N(\epsilon)) > e^{-1}\right) \\ &= P\left(\frac{M(N)}{n} \sum_{i=1}^{n} I(X_i > \mathcal{V}_N(\epsilon)) > e^{-1}\right) \\ &= P\left(M(N)(1 - F_n(\mathcal{V}_N(\epsilon))) > \frac{1}{e}\right). \end{aligned}$$

It follows directly that

$$M(N)(1 - F_n(\mathcal{V}_N(\epsilon))) = M(N)(1 - F_n((\epsilon+1)a^*_{M(N)} + b^*_{M(N)}) \xrightarrow[n]{p} e^{-(\epsilon+1)} < e^{-1},$$

which implies

$$P\left(\frac{X_{n-\nu':n}^{(m,k)} - b_N}{a_N} - 1 > \epsilon\right) \xrightarrow[n]{} 0. \tag{2.20}$$

Analogous to Eq (2.20), it can be shown that

$$P\left(\frac{X_{n-\nu':n}^{(m,k)}-b_N}{a_N}-1<-\epsilon\right)\underset{n}{\longrightarrow}0, \tag{2.21}$$

which establishes (2.18). Almost sure convergence follows by the same reasoning as before. Furthermore, Eq (2.19) can be established using the same approach as in Eq (2.10), thereby completing the proof of the theorem. □

Theorem 2.3. *Assume that* $m_1=m_2=\cdots=m_{n-1}=m>-1$, *and fix* $1\le r<n$. *Let* $c_N>0$ *and* d_N *be constants satisfying the condition in (1.2), whose exact values are unknown. Define the indices*

$$\xi=\left\lfloor\frac{n}{R(N)}\right\rfloor \quad and \quad \xi'=\left\lfloor\frac{n}{eR(N)}\right\rfloor,$$

and let θ_F *denote the lower endpoint of the DF* F. *Let* $\hat{c}_N$ *and* $\hat{d}_N$ *be the estimators of* c_N *and* d_N *derived from the sample* $\mathbf{X}_n$, *defined according to the target limiting DFs* $\tilde{\Psi}_{r,t}^{(m,k)}(x)$ *as follows:*

(i) For $\tilde{\Psi}_{1,t}^{(m,k)}(x)=\Gamma_r(z_{1,\delta}(x))$,

$$\hat{c}_N=-F_n^{-1}\left(\frac{1}{R(N)}\right)=-X_{\xi:n}^{(m,k)}, \qquad \hat{d}_N=0;$$

(ii) For $\tilde{\Psi}_{2,t}^{(m,k)}(x)=\Gamma_r(z_{2,\delta}(x))$,

$$\hat{c}_N=F_n^{-1}\left(\frac{1}{R(N)}\right)-\theta_{F_n}=X_{\xi:n}^{(m,k)}-X_{1:n}^{(m,k)}, \qquad \hat{d}_N=\theta_{F_n}=X_{1:n}^{(m,k)};$$

(iii) For $\tilde{\Psi}_{3,t}^{(m,k)}(x)=\Gamma_r(z_{3,\delta}(x))$,

$$\hat{c}_N=F_n^{-1}\left(\frac{1}{R(N)}\right)-F_n^{-1}\left(\frac{1}{eR(N)}\right)=X_{\xi:n}^{(m,k)}-X_{\xi':n}^{(m,k)},\ \hat{d}_N=F_n^{-1}\left(\frac{1}{R(N)}\right)=X_{\xi:n}^{(m,k)}.$$

If $R(N)=o(n)$, *then*

$$\sup_{x\in\mathbb{R}}\left|\hat{\Phi}_{r:N}^{(m,k)}(a_Nx+b_N)-\tilde{\Psi}_{r,t}^{(m,k)}(x)\right|\xrightarrow[n]{w.p.1}0. \tag{2.22}$$

Furthermore, if $\sum_{n=1}^{\infty}\eta^{n/R(N)}<\infty$ *for every* $\eta\in(0,1)$, *then the convergence in (2.22) holds almost surely.*

Proof. We begin with the first case, where $\tilde{\Psi}_{1,t}^{(m,k)}(x)=\Gamma_r(z_{1,\delta}(x))$. Because $\hat{d}_N=0$, Condition (ii) of Theorem 2.1 is automatically satisfied, so it suffices to verify Condition (iii). Specifically, we show that

$$\frac{\hat{c}_N}{c_N}=\frac{-X_{\xi:n}^{(m,k)}}{c_N}\xrightarrow[n]{p}1, \tag{2.23}$$

either in probability or almost surely. To prove (2.23) in probability, let $\epsilon\in(0,1)$ be an arbitrary constant, and define $\mathcal{V}_N(\epsilon)=-(\epsilon+1)c_N$. Let $U_{n,m}^{\star}(x)$ denote the number of observations X_j not exceeding x. Then, we have

$$P\left(\frac{-X_{\xi:n}^{(m,k)}}{c_N}-1>\epsilon\right)=P\left(-X_{\xi:n}^{(m,k)}>(\epsilon+1)c_N\right)$$

$$
\begin{aligned}
&= P\left(X_{\xi:n}^{(m,k)} < \mathcal{V}_N(\epsilon)\right) = P\left(U_{n,N}^{\star}(\mathcal{V}_N(\epsilon)) > \xi\right) \\
&\sim P\left(\frac{R(N)}{n} U_{n,N}^{\star}(\mathcal{V}_N(\epsilon)) > 1\right) \\
&= P\left(R(N)F_n(\mathcal{V}_N(\epsilon)) > 1\right).
\end{aligned}
$$

Moreover,

$$
R(N)F_n(\mathcal{V}_N(\epsilon)) = R(N)F_n(-(\epsilon+1)c_N) = R(N)F_n(-(\epsilon+1)c^{*}_{R(N)}) \xrightarrow[n]{p} (\epsilon+1)^{-\delta} < 1,
$$

which implies

$$
P\left(\frac{-X_{\xi:n}^{(m,k)}}{a_N} - 1 > \epsilon\right) \xrightarrow[n]{} 0. \tag{2.24}
$$

A similar reasoning applies to the lower tail. Define $\hat{\mathcal{V}}_N(\epsilon) = (\epsilon - 1)c_N$. Clearly,

$$
R(N)F_n(\hat{\mathcal{V}}_N(\epsilon)) = R(N)F_n((\epsilon-1)c_N) = R(N)F_n((\epsilon-1)c^{*}_{R(N)}) \xrightarrow[n]{p} (1-\epsilon)^{-\delta} > 1,
$$

so it follows that

$$
\begin{aligned}
P\left(\frac{-X_{\xi:n}^{(m,k)}}{c_N} - 1 < -\epsilon\right) &= P\left(-X_{\xi:n}^{(m,k)} < (1-\epsilon)c_N\right) \\
&= P\left(X_{\xi:n}^{(m,k)} > \hat{\mathcal{V}}_N(\epsilon)\right) \\
&= P\left(U_{n,N}^{\star}(\hat{\mathcal{V}}_N(\epsilon)) < \xi\right) \\
&\sim P\left(\frac{R(N)}{n} U_{n,N}^{\star}(\hat{\mathcal{V}}_N(\epsilon)) < 1\right) \\
&= P\left(R(N)F_n(\hat{\mathcal{V}}_N(\epsilon)) < 1\right) \xrightarrow[n]{} 0.
\end{aligned} \tag{2.25}
$$

Combining the results obtained in Eqs (2.24) and (2.25), we conclude that

$$
P\left(\left|-X_{\xi:n}^{(m,k)}/a_N - 1\right| > \epsilon\right) \xrightarrow[n]{} 0,
$$

which establishes Eq (2.23). To extend this result to almost sure convergence, let $\mathcal{G}_N^{\star}(\tau)$ be the moment-generating function of a Bernoulli variable with success probability $F_n(\mathcal{V}_N(\epsilon))$. Because $R(N)F_n(\mathcal{V}_N(\epsilon)) \xrightarrow[n]{p} (\epsilon+1)^{-\delta}$, it follows that for any fixed $\tau > 0$,

$$
\begin{aligned}
\frac{R(N)}{n} \log P\left(R(N)F_n(\mathcal{V}_N(\epsilon)) > 1\right) &= \frac{R(N)}{n} \log P\left(nF_n(\mathcal{V}_N(\epsilon)) > \frac{n}{R(N)}\right) \\
&= \frac{R(N)}{n} \log P\left(e^{\tau n F_n(\mathcal{V}_N(\epsilon))} > e^{\tau n/R(N)}\right) \\
&\leq \frac{R(N)}{n} \log\left(e^{-\tau n/R(N)} E\left[e^{\tau n F_n(\mathcal{V}_N(\epsilon))}\right]\right) \\
&\sim -\tau + \frac{R(N)}{n} \log {\mathcal{G}^{\star}}^{n}_{N}(\tau) = -t + \log {\mathcal{G}^{\star}}^{R(n)}_{N}(\tau)
\end{aligned}
$$

$$\underset{n}{\longrightarrow} -\tau + (\epsilon+1)^{-\delta}(e^{\tau}-1) =: g(\tau,\epsilon). \tag{2.26}$$

Differentiation of $g(\tau,\epsilon)$ with respect to t shows that the unique minimizer is $\tau_0 = \delta\log(\epsilon+1)$. Define

$$z(\epsilon) = g(\tau_0,\epsilon) = -\delta\log(\epsilon+1) - (\epsilon+1)^{-\delta} + 1.$$

Clearly, $z(0) = 0$, and $z'(\epsilon) = -(\delta+1)/\epsilon + \delta/(\epsilon+1)^{\delta+1} < 0$, which implies that $z(\epsilon) < 0$ for all $\epsilon > 0$. Next, define

$$\frac{R(N)}{n}\log\left(e^{-\tau n/R(N)}E\left[e^{\tau n F_n(\mathcal{V}_N(\epsilon))}\right]\right).$$

Then, $z_n(\epsilon) \underset{n}{\longrightarrow} z(\epsilon)$. Substituting $\tau = \tau_0(\epsilon)$ into (2.26) yields

$$\begin{aligned}\sum_{n=1}^{\infty} P\left(\frac{-X_{\xi:n}^{(m,k)}}{c_N} - 1 > \epsilon\right) &\sim \sum_{n=1}^{\infty} P\{R(N)F_n(\mathcal{V}_N(\epsilon)) > 1\}\\ &= \sum_{n=1}^{\infty}\exp\left(\frac{n}{R(N)} \times \frac{R(N)}{n}\log P\{R(N)F_n(\mathcal{V}_N(\epsilon)) > 1\}\right)\\ &\le \sum_{n=1}^{\infty}\exp\left(\frac{n}{R(N)}z_n(\epsilon)\right).\end{aligned}$$

Fix $\epsilon > 0$. Choose $\delta_\epsilon > 0$ such that $z(\epsilon) + \delta_\epsilon < 0$, and take N_ϵ sufficiently large so that

$$z_n(\epsilon) \le z(\epsilon) + \delta_\epsilon \quad \text{for all } n \ge N_\epsilon.$$

Then,

$$\begin{aligned}\sum_{n=1}^{\infty}\exp\left(\frac{n}{R(N)}z_n(\epsilon)\right) &= \sum_{n=1}^{N_\epsilon-1}\exp\left(\frac{n}{R(N)}z_n(\epsilon)\right) + \sum_{n=N_\epsilon}^{\infty}\exp\left(\frac{n}{R(N)}z_n(\epsilon)\right)\\ &\le \sum_{n=1}^{N_\epsilon-1}\exp\left(\frac{n}{R(N)}z_n(\epsilon)\right) + \sum_{n=N_\epsilon}^{\infty}\exp\left(\frac{n}{R(N)}(z(\epsilon)+\delta_\epsilon)\right).\end{aligned}$$

Because $n/R(N) \underset{n}{\longrightarrow} 0$, and $z(\epsilon) + \delta_\epsilon < 0$, the tail series converges to 0. Hence,

$$\sum_{n=1}^{\infty} P\left(\frac{-X_{\xi:n}^{(m,k)}}{c_N} - 1 > \epsilon\right) \sim \sum_{n=1}^{\infty} P(R(N)F_n(\mathcal{V}_N(\epsilon)) > 1) < \infty.$$

A symmetric argument applied to $\hat{\mathcal{V}}_N(\epsilon)$ yields

$$\sum_{n=1}^{\infty} P\left(\frac{-X_{\xi:n}^{(m,k)}}{c_N} - 1 < -\epsilon\right) \sim \sum_{n=1}^{\infty} P\Big(R(N)F_n(\hat{\mathcal{V}}_N(\epsilon)) < 1\Big) < \infty.$$

Therefore, by the Borel-Cantelli lemma,

$$\frac{\hat{c}_N}{c_N} = \frac{-X_{\xi:n}^{(m,k)}}{c_N} \xrightarrow[n]{\text{w.p.1}} 1,$$

which establishes the first part of the theorem.

Next, let $F \in D\left(\tilde{\Psi}_{2,t}^{(m,k)}(x)\right)$. We now verify Conditions (ii) and (iii), namely,

$$\frac{\hat{c}_N}{c_N} = \frac{X_{\xi:n}^{(m,k)} - X_{1:n}^{(m,k)}}{c_N} \xrightarrow[n]{p} 1, \tag{2.27}$$

and

$$\frac{\hat{d}_N - d_N}{c_N} = \frac{X_{1:n}^{(m,k)} - d_N}{c_N} \xrightarrow[n]{p} 0, \tag{2.28}$$

either in probability or almost surely. Evidently,

$$\frac{\hat{c}_N}{c_N} = \frac{X_{\xi:n}^{(m,k)} - X_{1:n}^{(m,k)}}{c_N} = \frac{X_{\xi:n}^{(m,k)} - d_N}{c_N} - \frac{X_{1:n}^{(m,k)} - d_N}{c_N}.$$

Therefore, to prove (2.27) and (2.28), it suffices to show that

$$\frac{X_{\xi:n}^{(m,k)} - d_N}{c_N} \xrightarrow[n]{p} 1, \tag{2.29}$$

and

$$\frac{X_{1:n}^{(m,k)} - d_N}{c_N} \xrightarrow[n]{p} 0, \tag{2.30}$$

either in probability or with probability 1. To establish (2.29) in probability, let ϵ be an arbitrary constant satisfying $0 < \epsilon < 1$. Define $\mathcal{V}_N(\epsilon) = (\epsilon + 1)c_N + d_N$, and let $U_{n,m}^{\star}(x)$ denote the number of observations X_j that do not exceed x for $1 \le j \le n$. Because

$$R(N)F_n(\mathcal{V}_N(\epsilon)) = R(N)F_n((\epsilon + 1)c_N + d_N) = R(N)F_n((\epsilon + 1)c_{R(N)}^{*} + d_{R(N)}^{*}) = (\epsilon + 1)^{\delta} > 1,$$

we obtain

$$\begin{aligned} P\left(\frac{X_{\xi:n}^{(m,k)} - d_N}{c_N} - 1 > \epsilon\right) &= P\left(X_{\xi:n}^{(m,k)} > (\epsilon + 1)c_N + d_N\right) = P\left(X_{\xi:n}^{(m,k)} > \mathcal{V}_N(\epsilon)\right) \\ &= P\left(U_{n,N}^{\star}(\mathcal{V}_N(\epsilon)) < \xi\right) \sim P\left(\frac{R(N)}{n} U_{n,N}^{\star}(\mathcal{V}_N(\epsilon)) < 1\right) \\ &= P\left(R(N)F_n(\mathcal{V}_N(\epsilon)) < 1\right) \xrightarrow[n]{} 0. \end{aligned} \tag{2.31}$$

Similarly, for $\hat{\mathcal{V}}_N(\epsilon) = (1 - \epsilon)c_N + d_N$, we have

$$R(N)F_n(\hat{\mathcal{V}}_N(\epsilon)) = R(N)F_n((1 - \epsilon)c_N + d_N) = R(N)F_n((1 - \epsilon)c_{R(N)}^{*} + d_{R(N)}^{*}) = (1 - \epsilon)^{\delta} < 1,$$

which implies

$$P\left(\frac{X_{\xi:n}^{(m,k)} - d_N}{c_N} - 1 < -\epsilon\right) = P\left(X_{\xi:n}^{(m,k)} < (1 - \epsilon)c_N + d_N\right) = P\left(X_{\xi:n}^{(m,k)} < \hat{\mathcal{V}}_N(\epsilon)\right)$$

$$= P\left(U^{\star}_{n,N}(\hat{\mathcal{V}}_N(\epsilon)) > \xi\right) \sim P\left(\frac{R(N)}{n} U^{\star}_{n,N}(\hat{\mathcal{V}}_N(\epsilon)) > 1\right)$$
$$= P\left(R(N)F_n(\hat{\mathcal{V}}_N(\epsilon)) > 1\right) \underset{n}{\longrightarrow} 0. \tag{2.32}$$

Combining (2.31) and (2.32), we conclude that

$$P\left(\left|\frac{X^{(m,k)}_{\xi:n} - d_N}{c_N} - 1\right| > \epsilon\right) \underset{n}{\longrightarrow} 0, \tag{2.33}$$

and hence, (2.29) holds in probability. Using the same argument as in the first part of the theorem, this convergence can be strengthened to almost sure convergence. We now establish (2.30). Let $0 < \epsilon < 1$, and define $\mathcal{V}_N(\epsilon) = \epsilon c_N + d_N$. Then,

$$\begin{aligned} P\left(\frac{X^{(m,k)}_{1:n} - d_N}{c_N} > \epsilon\right) &= P\left(X^{(m,k)}_{1:n} > \epsilon c_N + d_N\right) \\ &= P\left(X^{(m,k)}_{1:n} > \mathcal{V}_N(\epsilon)\right) = P\left(U^{\star}_{n,N}(\mathcal{V}_N(\epsilon)) = 0\right) \\ &\sim P\left(\frac{R(N)}{n} U^{\star}_{n,N}(\mathcal{V}_N(\epsilon)) = 0\right) = P\left(R(N)F_n(\mathcal{V}_N(\epsilon)) = 0\right). \end{aligned} \tag{2.34}$$

Moreover, $R(N)F_n(\mathcal{V}_N(\epsilon)) = R(N)F_n(\epsilon c_N + d_N) = R(N)F_n(\epsilon c^*_{R(N)} + d^*_{R(N)}) = \epsilon^{\delta} \neq 0$, which yields

$$P\left(\frac{X^{(m,k)}_{1:n} - d_N}{c_N} > \epsilon\right) \underset{n}{\longrightarrow} 0.$$

By a similar argument,

$$P\left(\frac{X^{(m,k)}_{1:n} - d_N}{c_N} < -\epsilon\right) \underset{n}{\longrightarrow} 0.$$

Hence, (2.30) holds in probability. The almost sure convergence follows by the same argument used earlier, completing the proof of second part.

In the third case, assume that $F \in D\left(\tilde{\Psi}^{(m,k)}_{3,t}(x)\right)$. To verify the conditions of Theorem 2.1, it suffices to prove that

$$\frac{\hat{c}_N}{c_N} = \frac{X^{(m,k)}_{\xi:n} - X^{(m,k)}_{\xi:n}}{c_N} \overset{p}{\underset{n}{\rightarrow}} 1, \tag{2.35}$$

and

$$\frac{\hat{d}_N - d_N}{c_N} = \frac{X^{(m,k)}_{\xi:n} - d_N}{c_N} \overset{p}{\underset{n}{\rightarrow}} 0, \tag{2.36}$$

in probability (or almost surely). Noting that

$$\frac{\hat{c}_N}{c_N} = \frac{X^{(m,k)}_{\xi:n} - d_N}{c_N} - \frac{X^{(m,k)}_{\xi:n} - d_N}{c_N},$$

it is sufficient to show that

$$\frac{X^{(m,k)}_{\xi:n} - d_N}{c_N} \overset{p}{\underset{n}{\rightarrow}} 0, \tag{2.37}$$

and

$$\frac{X_{\xi:n}^{(m,k)} - d_N}{c_N} \xrightarrow[n]{p} -1. \tag{2.38}$$

To prove (2.37), let $0 < \epsilon < 1$, and define $\mathcal{V}_N(\epsilon) = \epsilon c_N + d_N$. Then,

$$\begin{aligned} P\left(\frac{X_{\xi:n}^{(m,k)} - d_N}{c_N} > \epsilon\right) &= P\left(X_{\xi:n}^{(m,k)} > \epsilon c_N + d_N\right) \\ &= P\left(X_{\xi:n}^{(m,k)} > \mathcal{V}_N(\epsilon)\right) = P\left(U_{n,N}^{\star}(\mathcal{V}_N(\epsilon)) < \xi\right) \\ &\sim P\left(\frac{R(N)}{n} U_{n,N}^{\star}(\mathcal{V}_N(\epsilon)) < 1\right) = P\left(R(N)F_n(\mathcal{V}_N(\epsilon)) < 1\right). \end{aligned} \tag{2.39}$$

Because $R(N)F_n(\mathcal{V}_N(\epsilon)) = R(N)F_n(\epsilon c_N + d_N) = R(N)F_n(\epsilon c^*_{R(N)} + d^*_{R(N)}) = e^{\epsilon} > 1$, we obtain $P\left((X_{\xi:n}^{(m,k)} - d_N)/c_N > \epsilon\right) \xrightarrow[n]{} 0$. A similar argument shows that $P\left((X_{\xi:n}^{(m,k)} - d_N)/c_N < -\epsilon\right) \xrightarrow[n]{} 0$, and hence, (2.37) holds in probability (and almost surely by the same argument as before). Finally, we turn to prove (2.38). For any $0 < \epsilon < 1$, set $\mathcal{V}_N(\epsilon) = (\epsilon - 1)c_N + d_N$. It can be shown that

$$R(N)F_n(\mathcal{V}_N(\epsilon)) = R(N)F_n((\epsilon-1)c_N + d_N) = R(N)F_n((\epsilon-1)c^*_{R(N)} + d^*_{R(N)}) = e^{\epsilon-1} > e^{-1}.$$

Consequently,

$$\begin{aligned} P\left(\frac{X_{\xi:n}^{(m,k)} - d_N}{c_N} - (-1) > \epsilon\right) &= P\left(X_{\xi:n}^{(m,k)} > (\epsilon-1)c_N + d_N\right) = P\left(X_{\xi:n}^{(m,k)} > \mathcal{V}_N(\epsilon)\right) \\ &= P\left(U_{n,N}^{\star}(\mathcal{V}_N(\epsilon)) < \acute{\xi}\right) \sim P\left(\frac{R(N)}{n} U_{n,N}^{\star}(\mathcal{V}_N(\epsilon)) < e^{-1}\right) \\ &= P\left(R(N)F_n(\mathcal{V}_N(\epsilon)) < e^{-1}\right) \xrightarrow[n]{} 0. \end{aligned} \tag{2.40}$$

An analogous argument shows that

$$P\left(\frac{X_{\xi:n}^{(m,k)} - d_N}{c_N} - (-1) < -\epsilon\right) \xrightarrow[n]{} 0. \tag{2.41}$$

Combining (2.40) and (2.41) implies that (2.38) holds in probability. The corresponding almost sure convergence follows from an analogous argument, which completes the proof of the final part of the theorem. □

Remark 2.1. *The estimators proposed in Theorems 2.2 and 2.3 are intended to provide explicit and computationally convenient choices satisfying the assumptions of Theorem 2.1. However, the asymptotic weak convergence established in Theorem 2.1 is not tied to these particular estimators. Any estimators of the normalizing constants satisfying the consistency conditions of Theorem 2.1, namely*

$$\frac{\widehat{a}_N}{a_N} \xrightarrow[n]{w.p.1} 1, \qquad \frac{\widehat{b}_N - b_N}{a_N} \xrightarrow[n]{w.p.1} 0,$$

together with the analogous conditions for the lower extremes, may be used in place of the empirical quantile estimators. Consequently, the proposed asymptotic framework is applicable to a broad class of estimation procedures, including maximum likelihood estimators and other asymptotically equivalent consistent estimators whenever the required consistency conditions can be verified.

Connection with left endpoint estimation. The inferential objective of the proposed procedure is the unknown left endpoint θ_L of the underlying distribution. Although θ_L is not directly observable, information about this parameter is contained in the first lower extreme m-generalized order statistic, which reduces to the sample minimum in the ordinary order statistics model. The asymptotic results established in Section 2 provide the theoretical justification for constructing a pivotal quantity based on this observable statistic together with consistent estimators of the unknown normalizing constants. Consequently, confidence intervals derived from the limiting bootstrap distribution of the pivotal quantity translate directly into confidence intervals for the unknown left endpoint. The following section develops this construction in detail.

3. Asymptotic bootstrap confidence intervals for the finite left endpoint of a continuous distribution

Estimating the endpoints of continuous distributions is of significant interest across various statistical applications, particularly when the analysis depends on lower extreme OOSes or lower extreme GOSes. This challenge becomes more pronounced with finite sample sizes because the variability of lower OOSes can greatly affect the stability of estimators and the accuracy of confidence intervals.

Although confidence intervals (CIs) are the standard for quantifying uncertainty in this framework, their validity remains contingent upon distributional assumptions and construction methodology. Classical asymptotic approaches, though analytically convenient, utilize limiting approximations that frequently lose precision in moderate sample sizes or under model misspecification. In particular, inference based on the spacing properties of OOSes is highly sensitive to distributional deviations. These constraints necessitate the use of resampling based techniques, specifically the bootstrap, which offers a robust, data-driven alternative. By estimating the sampling distribution directly from the observed data, the bootstrap effectively captures finite sample variability and relaxes the reliance on restrictive parametric assumptions.

This simulation study is conducted to evaluate the finite sample performance of 95% CIs for the left endpoint parameter of positive distributions. The analysis focuses on the standard exponential distribution and gamma distribution with shape parameter of 1.35 and scale parameter of 1, and it considers two data generating frameworks: the OOS and the SOS models.

To investigate the impact of sample size, several configurations are examined, including $n = 50, 100, 1000$, and $10,000$. In the bootstrap framework, resamples of size $M = 50$ and $M = 100$ are drawn from the original sample, with $B = 1000$ bootstrap replications within each Monte Carlo experiment. The empirical coverage probabilities are computed from $\tilde{R} = 10,000$ independent Monte Carlo replications for the asymptotic confidence intervals and from $\tilde{R} = 100$ independent Monte Carlo replications for the bootstrap confidence intervals. In the latter case, each Monte Carlo replication uses $B = 1000$ bootstrap resamples to construct the corresponding bootstrap confidence interval.

It should be emphasized that the bootstrap replications and the Monte Carlo replications play different roles in the simulation study. Specifically, each bootstrap confidence interval is constructed from $B = 1000$ bootstrap resamples, whereas the reported empirical coverage probabilities are obtained by averaging over $\tilde{R} = 100$ independent Monte Carlo experiments. Thus, the value $\tilde{R} = 100$ refers only to the number of repeated simulation runs used for performance evaluation and should not be interpreted as the number of bootstrap replications.

3.1. Choice of the bootstrap sample size

The theoretical results established in Theorems 2.2 and 2.3 require only that the bootstrap sample size satisfies $M = o(n)$, which guarantees the asymptotic validity of the proposed bootstrap procedure. This condition permits a broad class of admissible growth rates, for example,

$$M = \log n, \qquad M = \sqrt{n}, \qquad M = n^{2/3},$$

or, more generally, $M = n^{\alpha}, \quad 0 < \alpha < 1$. In the present simulation study, we adopt $M = 100$ for a sample size of $n = 10,000$, which clearly satisfies the theoretical requirement. This value was selected to provide a computationally efficient implementation while remaining fully consistent with the assumptions of Theorems 2.2 and 2.3. Larger admissible values of M may reduce the Monte Carlo variability of the bootstrap approximation, but they also increase the computational cost. Because all admissible choices satisfying $M = o(n)$ lead to the same asymptotic results established in Theorems 2.2 and 2.3, the choice of M primarily represents a practical tradeoff between computational efficiency and finite-sample precision rather than a theoretical requirement.

In this section, we focus on distributions for which the normalized lower extreme m-GOSes belong to the Weibull domain of attraction. The construction of bootstrap CIs for the left endpoint is based on a modified pivotal quantity adapted from Fukuchi [14], which was originally inspired by the pivotal quantity proposed by Weissman [28]. Our proposed pivotal quantity for m-GOSes is defined as

$$Q^*_{\nu,M,n} := \frac{Y^{(m,k)}_{1:M} - X^{(m,k)}_{1:n}}{Y^{(m,k)}_{\nu:M} - Y^{(m,k)}_{1:M}}, \quad \nu \geq 2, \tag{3.1}$$

where $Y^{(m,k)}_{i:M}$ denotes the ith $m-$GOS of a bootstrap sample of size M, and $X^{(m,k)}_{1:n}$ is the minimum of the original GOS sample of size n.

In view of Theorem 4.8 of Fukuchi [14] and the results of Barakat [2], together with Theorem 2.3 of the present work, it is ensured that the asymptotic distribution of the pivotal quantity $Q^*_{\nu,M,n}$ can be approximated by

$$G(q) = P\left(Q^*_{\nu,M,n} \leq q\right) \underset{n}{\longrightarrow} \left[1 - \left(1 - \left(\frac{q}{1+q}\right)^{\delta}\right)^{\nu-1}\right], \quad q > 0, \quad \nu \geq 2. \tag{3.2}$$

Clearly, the distribution of $Q^*_{\nu,M,n}$ depends on ν and the shape parameter δ, which can be estimated from the data. For a given δ, the pth quantile of the pivotal quantity is given by

$$G^{-1}(p, \nu, \delta) = \frac{\left(1 - (1-p)^{1/(\nu-1)}\right)^{1/\delta}}{1 - \left(1 - (1-p)^{1/(\nu-1)}\right)^{1/\delta}}, \quad 0 < p < 1. \tag{3.3}$$

These quantiles are used to construct CIs by inverting the pivotal relationship.

The shape parameter δ of the Weibull domain of attraction can be estimated using a modified version of the estimator proposed by El-Adll et al. [13], which is based on the first r m-GOSes,

$Y_{1:M}^{(m,k)}, Y_{2:M}^{(m,k)}, \ldots, Y_{r:M}^{(m,k)}$, and is given by

$$\tilde{\delta} = \frac{r-2}{\sum_{i=2}^{r-1}(i-1)\log\left(\frac{Y_{i+1:M}^{(m,k)} - X_{1:n}^{(m,k)}}{Y_{i:M}^{(m,k)} - X_{1:n}^{(m,k)}}\right)}. \tag{3.4}$$

All simulation results reported in Tables 1–12 are obtained through repeated application of the bootstrap procedure described in Algorithm 1. This algorithm integrates resampling, estimation of the shape parameter, and interval construction in a unified framework.

The performance of the CIs is evaluated using two criteria: coverage probability (CP), defined as the proportion of intervals that contain the true parameter value, and average interval width (AIW), which measures the precision of the intervals.

The simulation relies on two significant submodels:

(1) OOSes, with $m = 0$ and $k = 1$.
(2) SOSes, with model parameters defined as $m = 1$ and $k = 1$.

Table 1. Estimated CPs and AIWs for 95% asymptotic CIs for the left endpoint of the EXP(1) distribution, based on 10,000 Monte Carlo replications with sample size $n = 50$ for the OOS model.

ν	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	89.45	1.102	91.33	0.955	92.82	0.857	93.98	0.785	94.33	0.737
4	87.36	0.190	90.37	0.168	91.79	0.154	92.26	0.143	93.03	0.132
8	87.41	0.122	89.43	0.111	90.75	0.103	91.58	0.095	91.32	0.085
16	84.79	0.118	90.54	0.099	90.78	0.091	90.58	0.082	90.45	0.074
32	81.24	0.156	87.65	0.128	90.57	0.109	92.33	0.094	92.57	0.080

CP = Coverage probability (%), AIW = Average interval width.

Table 2. Estimated CPs and AIWs for 95% asymptotic CIs for the left endpoint of the EXP(1) distribution, based on 10,000 Monte Carlo replications with sample size $n = 100$ for the OOS model.

ν	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	89.00	0.573	91.16	0.488	91.84	0.455	93.48	0.433	93.68	0.409
4	86.87	0.097	89.07	0.088	90.83	0.082	91.69	0.078	92.72	0.076
8	86.66	0.061	89.10	0.058	89.94	0.055	90.94	0.052	91.55	0.050
16	84.37	0.056	88.78	0.049	90.43	0.047	90.52	0.045	90.59	0.043
32	80.05	0.064	86.45	0.054	88.90	0.049	90.22	0.045	91.22	0.042

Table 3. Estimated CPs and AIWs for 95% asymptotic bootstrap CIs for the left endpoint of the EXP(1) distribution. Results are based on $\tilde{R} = 100$ Monte Carlo replications, each using $B = 1000$ bootstrap samples of size $M = 50$ drawn from a sample of size $n = 1000$ for the OOS model.

ν	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	90.43	1.378	92.05	1.246	88.51	1.097	91.67	0.867	94.74	0.788
4	89.91	0.227	89.69	0.183	85.26	0.196	94.68	0.165	96.91	0.142
8	90.72	0.141	89.58	0.133	93.81	0.094	91.67	0.101	95.92	0.096
16	83.16	0.159	92.63	0.134	94.62	0.112	96.81	0.095	98.89	0.073
32	70.13	0.265	90.72	0.164	84.38	0.141	96.88	0.122	93.55	0.093

Table 4. Estimated CPs and AIWs for 95% asymptotic bootstrap CIs for the left endpoint of the EXP(1) distribution. Results are based on $\tilde{R} = 100$ Monte Carlo replications, each using $B = 1000$ bootstrap samples of size $M = 100$ drawn from a sample of size $n = 10,000$ for the OOS model.

ν	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	89.55	0.693	91.98	0.577	93.03	0.515	93.85	0.482	94.26	0.454
4	88.87	0.125	91.75	0.106	92.85	0.097	93.64	0.089	94.18	0.085
8	89.54	0.084	92.14	0.073	93.43	0.067	94.14	0.062	94.54	0.058
16	86.51	0.084	92.54	0.067	94.24	0.061	95.03	0.056	95.52	0.052
32	80.49	0.103	89.67	0.079	93.22	0.067	95.05	0.061	96.17	0.054

Table 5. Estimated CPs and AIWs for 95% asymptotic CIs for the left endpoint of the Gamma(1.35, 1) distribution, based on 10,000 Monte Carlo replications with sample size $n = 100$ for the OOS model.

ν	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	89.51	1.755	91.28	1.527	93.05	1.454	93.18	1.357	93.60	1.313
4	86.65	0.270	88.58	0.253	90.04	0.240	91.14	0.227	91.35	0.219
8	84.56	0.160	87.08	0.152	88.47	0.146	89.36	0.139	89.34	0.136
16	81.17	0.137	85.54	0.122	86.81	0.117	87.73	0.115	87.64	0.110
32	76.49	0.140	81.44	0.124	84.96	0.115	86.30	0.109	86.37	0.102

Table 6. Estimated CPs and AIWs for 95% asymptotic bootstrap CIs for the left endpoint of the Gamma(1.35, 1) distribution. Results are based on $\tilde{R} = 100$ Monte Carlo replications, each using $B = 1000$ bootstrap samples of size $M = 50$ drawn from a sample of size $n = 10,000$ for the OOS model.

v	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	90.39	3.694	92.82	3.001	94.04	2.648	94.81	2.371	95.41	2.152
4	90.05	0.619	92.9	0.515	94.04	0.457	94.69	0.41	94.79	0.372
8	90.67	0.403	93.63	0.341	94.75	0.3	95.07	0.267	94.83	0.237
16	87.58	0.392	94.19	0.308	95.9	0.268	96.2	0.236	95.65	0.207
32	78.06	0.523	89.85	0.396	94.61	0.325	96.96	0.271	98.06	0.227

Table 7. Estimated CPs and AIWs for 95% asymptotic bootstrap CIs for the left endpoint of the Gamma(1.35, 1) distribution. Results are based on $\tilde{R} = 100$ Monte Carlo replications, each using $B = 1000$ bootstrap samples of size $M = 100$ drawn from a sample of size $n = 10,000$ for the OOS model.

v	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	89.69	2.246	92.13	1.881	93.19	1.693	93.92	1.568	94.38	1.479
4	89.16	0.373	91.86	0.322	93.09	0.291	93.94	0.271	94.23	0.255
8	89.43	0.238	92.43	0.207	93.38	0.192	94.35	0.175	94.69	0.165
16	86.71	0.220	92.83	0.181	94.37	0.164	95.16	0.152	95.52	0.142
32	80.08	0.251	89.65	0.199	93.44	0.172	95.29	0.154	96.47	0.141

Table 8. Estimated CPs and AIWs for 95% asymptotic CIs for the left endpoint of the EXP(1) distribution, based on 10,000 Monte Carlo replications with sample size $n = 100$ for the SOS model.

v	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	88.92	0.561	91.28	0.486	92.68	0.448	92.91	0.440	93.69	0.422
4	86.68	0.097	89.28	0.088	90.85	0.083	91.68	0.079	92.55	0.076
8	86.50	0.061	89.15	0.056	89.75	0.054	91.03	0.052	92.00	0.050
16	84.50	0.056	89.06	0.050	90.26	0.047	90.46	0.045	91.11	0.043
32	79.72	0.064	86.08	0.054	89.06	0.049	90.39	0.045	91.23	0.042

Table 9. Estimated CPs and AIWs for 95% asymptotic bootstrap CIs for the left endpoint of the EXP(1) distribution. Results are based on $\tilde{R} = 100$ Monte Carlo replications, each using $B = 1000$ bootstrap samples of size $M = 50$ drawn from a sample of size $n = 10,000$ for the SOS model.

v	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	90.29	0.613	92.47	0.502	93.52	0.446	94.21	0.405	94.36	0.371
4	89.83	0.112	92.25	0.094	93.24	0.083	93.92	0.075	94.03	0.068
8	90.69	0.079	93.15	0.067	94.04	0.059	94.34	0.053	94.42	0.047
16	87.61	0.083	93.62	0.064	95.25	0.056	95.65	0.049	95.17	0.043
32	79.59	0.121	90.09	0.088	94.11	0.071	96.11	0.059	97.13	0.049

Table 10. Estimated CPs and AIWs for 95% asymptotic bootstrap CIs for the left endpoint of the EXP(1) distribution. Results are based on $\tilde{R} = 100$ Monte Carlo replications, each using $B = 1000$ bootstrap samples of size $M = 100$ drawn from a sample of size $n = 10,000$ for the SOS model.

v	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	88.43	0.341	90.93	0.273	92.07	0.246	92.58	0.229	93.05	0.218
4	87.61	0.063	91.04	0.051	91.93	0.045	92.34	0.042	92.74	0.04
8	88.27	0.041	91.42	0.034	92.23	0.033	92.72	0.028	92.68	0.027
16	84.88	0.041	91.90	0.032	93.25	0.029	93.66	0.026	93.95	0.024
32	77.88	0.053	89.09	0.038	92.59	0.032	93.99	0.028	94.83	0.025

Table 11. Estimated CPs and AIWs for 95% asymptotic bootstrap CIs for the left endpoint of the Gamma(1.35, 1) distribution. Results are based on $\tilde{R} = 100$ Monte Carlo replications, each using $B = 1000$ bootstrap samples of size $M = 100$ drawn from a sample of size $n = 10,000$ for the SOS model.

v	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	88.21	1.355	90.73	1.092	91.88	0.986	92.38	0.918	92.9	0.871
4	87.38	0.219	90.85	0.183	91.68	0.166	92.14	0.155	92.47	0.146
8	88.07	0.137	91.21	0.116	92.02	0.105	92.5	0.098	92.44	0.092
16	84.59	0.131	91.76	0.103	93.13	0.093	93.52	0.086	93.72	0.081
32	77.21	0.15	88.81	0.113	92.47	0.098	93.91	0.088	94.75	0.08

Table 12. Estimated CPs and AIWs for 95% asymptotic bootstrap CIs for the left endpoint of the Gamma(1.35, 1) distribution. Results are based on $\tilde{R} = 100$ Monte Carlo replications, each using $B = 1000$ bootstrap samples of size $M = 50$ drawn from a sample of size $n = 10,000$ for the SOS model.

v	$r = 10$		$r = 15$		$r = 20$		$r = 25$		$r = 30$	
	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW	CP%	AIW
2	90.33	2.048	92.54	1.683	93.57	1.49	94.21	1.35	94.35	1.23
4	89.9	0.345	92.29	0.288	93.26	0.255	93.8	0.231	93.85	0.21
8	90.81	0.226	93.26	0.192	94.08	0.17	94.2	0.153	93.98	0.137
16	87.63	0.221	93.83	0.174	95.39	0.153	95.59	0.136	94.78	0.12
32	78.93	0.288	90.06	0.22	94.29	0.181	96.3	0.153	97.21	0.13

Algorithm 1: Asymptotic bootstrap confidence intervals for the left endpoint.

Input: F: Known CDF; n: Original sample size; M: Bootstrap sample size; B: Number of bootstrap resamples; $\tilde{R}$: Number of Monte Carlo replications; r, ν: Tuning parameters ($r \geq 3$, $\nu \geq 2$); $\alpha = 0.95$: confidence level.

Output: $\widehat{\mathrm{CP}}$: Estimated coverage probability; $\widehat{\mathrm{AIW}}$: Estimated average interval width

Initialization: Define pivotal quantity $Q^*_{\nu,M,n}$ and its quantile function via (3.1) and (3.3), respectively.

for $rep = 1$ **to** $\tilde{R}$ **do**

  Step 1: Choose CDF F and m-GOS model parameters m and k.

  Step 2: Generate sample $\mathbf{X}_n = (X^{(m,k)}_{1:n}, \ldots, X^{(m,k)}_{n:n})$ via F and extract $X^{(m,k)}_{1:n}$.

  Step 3: Initialize interval storage $C_{rep} = \emptyset$.

  for $b = 1$ **to** B **do**

    Draw $\mathbf{Y}_M$ via bootstrap with replacement; sort to get $Y^{(m,k)}_{1:M} \leq \cdots \leq Y^{(m,k)}_{M:M}$.

    if $\left|Y^{(m,k)}_{1:M} - X^{(m,k)}_{1:M}\right| < 10^{-8}$ **then**

      continue (skip degenerate case)

    end

    Step 4: Estimate shape parameter $\tilde{\delta}^{(b)}$ via (3.4) using first r order stats: $Y^{(m,k)}_{1:M} \leq \cdots \leq Y^{(m,k)}_{r:M}$.

    if $0 < \tilde{\delta}^{(b)} < \infty$ **then**

      Step 5: Set $\tau = 1 - \alpha$. Compute quantiles:

$$q_{\tau/2} = G^{-1}(\tfrac{\tau}{2}, \nu, \tilde{\delta}^{(b)}) \text{ and } q_{1-\tau/2} = G^{-1}(1 - \tfrac{\tau}{2}, \nu, \tilde{\delta}^{(b)}).$$

      Step 6: Construct interval (L_b, U_b) by inverting $Q^*_{\nu,M,n}$, namely:

$$L_b = Y^{(m,k)}_{1:M} - q_{1-\tau/2} \cdot (Y^{(m,k)}_{\nu:M} - Y^{(m,k)}_{1:M}), \quad U_b = Y^{(m,k)}_{1:M} - q_{\tau/2} \cdot (Y^{(m,k)}_{\nu:M} - Y^{(m,k)}_{1:M}).$$

      Append (L_b, U_b) to C_{rep}.

    end

  end

  Step 7: Set $B_{rep} = |C_{rep}|$.

  if $B_{rep} > 0$ **then**

$$\mathrm{CP}_{rep} = \frac{1}{B_{rep}} \sum_{(L,U)\in C_{rep}} \mathbf{I}(L \leq X^{(m,k)}_{1:n} \leq U), \quad \mathrm{AIW}_{rep} = \frac{1}{B_{rep}} \sum_{(L,U)\in C_{rep}} (U - L).$$

  end

end

Step 8: Aggregate final results

$\widehat{\mathrm{CP}} = \frac{1}{\tilde{R}} \sum_{rep=1}^{\tilde{R}} \mathrm{CP}_{rep}$ $\quad\widehat{\mathrm{AIW}} = \frac{1}{\tilde{R}} \sum_{rep=1}^{\tilde{R}} \mathrm{AIW}_{rep}$

return $\widehat{\mathrm{CP}}$, $\widehat{\mathrm{AIW}}$

3.2. Computational complexity

Algorithm 1 is computationally efficient and scales well with the bootstrap sample size. Generating a single bootstrap sample of size M requires $O(M)$ operations, whereas sorting the resulting sample, which is needed to compute the lower extreme m-GOSes, requires $O(M \log M)$ operations using standard sorting algorithms. Consequently, one bootstrap replication has an overall computational complexity of

$$O(M \log M).$$

If B bootstrap replications are performed, the total computational complexity becomes

$$O(BM \log M).$$

Because the proposed asymptotic framework requires only that $M = o(n)$, the bootstrap sample size remains substantially smaller than the original sample size for large n. Therefore, the computational burden grows moderately with M, making the proposed procedure computationally feasible, even for large datasets.

4. Results and discussion

The simulation results provide important insights into the behavior of asymptotic bootstrap CIs under different configurations.

4.1. Effect of tuning parameters

The tuning parameters r and ν, along with the accuracy of the estimator for the shape parameter δ, play an important role in influencing the effectiveness of the proposed asymptotic bootstrap CIs for the left endpoint. Increasing the number of observed lower m-GOSes, r, improves both the CP and the AIW. This is due to the fact that larger values of r improve the estimation of the shape parameter δ. On the other hand, increasing ν consistently reduces the AIW. This reflects the use of wider spacings $Y^{(m,k)}_{\nu:M} - Y^{(m,k)}_{1:M}$, which produce tighter intervals. However, large values of ν tend to reduce CP, particularly in small samples, due to increased variability in higher- order spacings. These highlight a fundamental tradeoff between accuracy and efficiency: Smaller values of ν favor coverage, whereas larger values improve precision.

For the exponential distribution, the asymptotic CIs perform well when appropriate parameters are selected, as it is expected. In particular, for $r \geq 25$ and small values of ν (such as $\nu = 2$ or 4), the CP approaches the nominal 95% level. This behavior can be attributed to the compatibility between the exponential distribution and the structure of the pivotal quantity, which relies on spacing properties that are well characterized under exponential models. Increasing the sample size improves the CP and reduces the AIW, indicating increased efficiency.

4.2. Performance under the gamma distribution

In contrast, the asymptotic confidence intervals consistently exhibit modest undercoverage for the Gamma(1.35, 1) distribution across the simulation settings considered. Although the coverage probabilities remain below the nominal level, this behavior should not be interpreted as a failure of the proposed asymptotic theory. The Gamma$(1.35, 1)$ distribution satisfies the assumptions required by Theorems 2.1–2.3 and therefore lies within the theoretical framework developed in Section 2.

The observed undercoverage is instead attributable to finite-sample effects. Although the asymptotic approximation remains valid, the convergence to the limiting distribution is comparatively slower than for the exponential benchmark, so higher-order remainder terms have a more noticeable influence on the finite-sample distribution of the bootstrap statistic. Consequently, the empirical coverage probabilities are slightly below the nominal level for the sample sizes considered. As the sample size increases, these higher-order effects are expected to diminish, in accordance with the asymptotic theory.

These findings indicate that the finite-sample performance of the proposed procedure depends not only on the validity of the asymptotic assumptions but also on the rate at which the limiting approximation is attained. Nevertheless, the simulation results confirm that the theoretical conclusions remain applicable to the Gamma$(1.35, 1)$ model.

4.3. Bootstrap performance

The bootstrap CIs reveal a highly satisfactory performance compared to their asymptotic counterparts, consistently achieving CPs closer to the nominal level across almost all simulated scenarios. This improvement stems from the data-driven nature of the bootstrap, which directly approximates the finite sample distribution of the pivotal quantity without relying on rigid asymptotic assumptions. Consequently, bootstrap methods effectively capture relatively small sample variability and mitigate estimation bias.

4.4. Effect of sample size and bootstrap size

Increasing the sample size n leads to improved performance for both asymptotic and bootstrap methods. CPs become more stable and closer to the nominal level, and interval widths decrease due to increased information in the data. Similarly, increasing the bootstrap sample size M enhances the stability of the pivotal quantity and improves the overall accuracy of the intervals.

4.5. Practical implications

Based on the simulation results, several practical recommendations can be made:

- The bootstrap method is highly suited for general applications, particularly when the exact functional form of the underlying distribution is unknown, yet specific qualitative or structural shape properties are available.
- For exponential data, asymptotic methods can be used as computationally efficient alternatives when $r \geq 25$, and ν is small.
- Moderate values of ν (such as $\nu = 4$ or 8) provide a good balance between coverage and interval width.
- Although larger values of r enhance the stability when estimating the tail or shape parameter of the asymptotic limit, r must still be restricted to a lower extreme to preserve asymptotic validity.

Overall, the simulation study demonstrates that although asymptotic CIs can perform adequately under ideal conditions, their performance deteriorates under model misspecification. In contrast, bootstrap intervals offer a robust and reliable alternative for left endpoint estimation, particularly when relying on a single sample with fewer restrictions on the underlying distribution.

4.6. Choice of the tuning parameter ν

Based on the simulation study, values such as $\nu = 4$ or $\nu = 8$ provide a satisfactory compromise between interval stability and computational efficiency and may therefore be recommended for routine applications. When the underlying distribution is unknown, it is advisable to carry out a preliminary assessment of the lower-tail behavior, for example by means of a quantile-quantile (Q-Q) plot or an appropriate goodness-of-fit procedure. If the data exhibit noticeable departures from an exponential-type model, practitioners should interpret the resulting confidence intervals with additional caution, as finite-sample performance may be influenced by the rate of convergence to the asymptotic regime. Nevertheless, provided that the assumptions of the proposed methodology are satisfied, the asymptotic validity of the procedure remains unaffected.

4.7. Future research

The present work focuses on establishing the asymptotic theory of bootstrap procedures for extreme m-generalized order statistics under unknown normalizing constants and on developing the corresponding bootstrap confidence intervals. The numerical study is intended to illustrate the finite-sample performance of the proposed methodology rather than to provide an exhaustive comparison with alternative inferential procedures. A systematic comparison with direct parametric approaches, such as methods based on L-moments or other parametric endpoint estimation techniques, would require a substantially different simulation framework and performance criteria. Such a comprehensive comparative study is beyond the scope of the present paper and constitutes an interesting direction for future research.

Author contributions

M. E. Sobh: Conceptualization, Methodology, Formal analysis, Writing original draft, review, and editing; H. M. Barakat: Conceptualization, Methodology, Supervision, Writing, review and editing; Magdy E. El-Adll: Methodology, Formal analysis, Software, Formal analysis, Writing, review, and editing; Amany E. Aly: Software, Validation, Writing, Software, Formal analysis, review and editing; Asamh Saleh M. Al Luhayb: Funding acquisition, Investigation, Writing, review and editing, Project administration. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2026).

Conflict of interest

The authors declare no conflict of interest.

References

1. K. B. Athreya, J. Fukuchi, Bootstrapping extremes of i.i.d. random variables, *Proceedings of the Conference on Extreme Value Theory and Applications*, 1994, 23–30.
2. H. M. Barakat, Limit theory of generalized order statistics, *J. Stat. Plan. Infer.*, **137** (2007), 1–11. https://doi.org/10.1016/j.jspi.2005.10.003
3. H. M. Barakat, M. E. El-Adll, Asymptotic theory of extreme dual generalized order statistics, *Stat. Probabil. Lett.*, **79** (2009), 1252–1259. https://doi.org/10.1016/j.spl.2009.01.015
4. H. M. Barakat, E. M. Nigm, M. E. El-Adll, Bootstrap for extreme generalized order statistics, *Arab. J. Sci. Eng.*, **36** (2011), 1083–1090. https://doi.org/10.1007/s13369-011-0105-1
5. H. M. Barakat, M. E. El-Adll, On the limit distribution of lower extreme generalized order statistics, *Proc. Math. Sci.*, **122** (2012), 297–311. https://doi.org/10.1007/s12044-012-0064-9
6. H. M. Barakat, M. E. El-Adll, M. E. Sobh, Bootstrapping m-generalized order statistics with variable rank, *AIMS Mathematics*, **7** (2022), 13704–13732. https://doi.org/10.3934/math.2022755
7. M. Bieniek, T. Rychlik, Conditions for finiteness and bounds on moments of generalized order statistics, *Stat. Papers*, **65** (2024), 2289–2312. https://doi.org/10.1007/s00362-023-01482-5
8. E. Cramer, Contributions to generalized order statistics, Ph.D. Thesis, University of Oldenburg, 2003.
9. J. Daniel, K. Ayinde, A. F. Lukman, O. Albalawi, J. Allohibi, A. A. Alharbi, Optimised block bootstrap: an efficient variant of the circular block bootstrap method with application to South African economic time series data, *AIMS Mathematics*, **9** (2024), 30781–30815. https://doi.org/10.3934/math.20241487
10. A. C. Davison, D. V. Hinkley, *Bootstrap methods and their application*, Cambridge: Cambridge University Press, 1997. https://doi.org/10.1017/CBO9780511802843
11. B. Efron, Bootstrap methods: another look at the jackknife, In: *Breakthroughs in statistics*, New York: Springer, 1992, 569–593. https://doi.org/10.1007/978-1-4612-4380-9_41
12. B. Efron, R. J. Tibshirani, *An introduction to the bootstrap*, New York: Chapman & Hall, 1993. https://doi.org/10.1201/9780429246593
13. M. E. El-Adll, H. M. Barakat, A. E. Aly, Asymptotic prediction for future observations of a random sample of unknown continuous distribution, *Complexity*, **2022** (2022), 4073799. https://doi.org/10.1155/2022/4073799
14. J. I. Fukuchi, Bootstrapping extremes of random variables, Ph.D. Thesis, Iowa State University, 1994. https://doi.org/10.31274/rtd-180813-10322
15. U. Kamps, A concept of generalized order statistics, *J. Stat. Plan. Infer.*, **48** (1995), 1–23. https://doi.org/10.1016/0378-3758(94)00147-N

16. U. Kamps, Generalized order statistics, In: *Wiley statsref: statistics reference online*, Hoboken: John Wiley & Sons, Inc., 2016, 1–12. https://doi.org/10.1002/9781118445112.stat00832.pub2

17. U. Kamps, Generalized order statistics, In: *International encyclopedia of statistical science*, Heidelberg: Springer, 2025, 1042–1047. https://doi.org/10.1007/978-3-662-69359-9_251

18. U. Kamps, E. Cramer, On distributions of generalized order statistics, *Statistics*, **35** (2001), 269–280. https://doi.org/10.1080/02331880108802736

19. J. G. MacKinnon, Bootstrap hypothesis testing, In: *Handbook of computational econometrics*, New York: John Wiley & Sons, 2009, 183–213. https://doi.org/10.1002/9780470748916.ch6

20. D. Nasri-Roudsari, Extreme value theory of generalized order statistics, *J. Stat. Plan. Infer.*, **55** (1996), 281–297. https://doi.org/10.1016/S0378-3758(95)00200-6

21. D. Nasri-Roudsari, Limit distributions of generalized order statistics under power normalization, *Commun. Stat.-Theor. M.*, **28** (1999), 1379–1389. https://doi.org/10.1080/03610929908832362

22. E. M. Nigm, Bootstrapping extremes of random variables under power normalization, *TEST*, **15** (2006), 257–269. https://doi.org/10.1007/BF02595427

23. T. Schendel, R. Thongwichian, Flood frequency analysis: confidence interval estimation by test inversion bootstrapping, *Adv. Water Resour.*, **83** (2015), 1–9. https://doi.org/10.1016/j.advwatres.2015.05.004

24. A. B. Schmiedt, Domains of attraction of asymptotic distributions of extreme generalized order statistics, *Commun. Stat.-Theor. M.*, **45** (2016), 2089–2104. https://doi.org/10.1080/03610926.2013.870206

25. M. E. Sobh, H. M. Barakat, Bootstrapping order statistics with variable rank, *REVSTAT-Stat. J.*, **22** (2024), 545–570. https://doi.org/10.57805/revstat.v22i4.543

26. M. E. Sobh, H. M. Barakat, M. E. El-Adll, A. E. Aly, Asymptotic behavior of bootstrapped extreme order statistics under unknown power normalizing constants, *Stat. Papers*, **66** (2025), 44. https://doi.org/10.1007/s00362-025-01665-2

27. S. Tedim, V. Afreixo, M. Felgueiras, R. P. Leitão, S. J. Pinheiro, C. J. Silva, Evaluating COVID-19 in Portugal: bootstrap confidence interval, *AIMS Mathematics*, **9** (2023), 2756–2765. https://doi.org/10.3934/math.2024136

28. I. Weissman, Confidence intervals for the threshold parameter, *Commun. Stat.-Theor. M.*, **10** (1981), 549–557. https://doi.org/10.1080/03610928108828057

AIMS Press