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Research article

Two elephant random walks interacted

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Abstract: We studied a model of two interacting elephant random walks through interacting generalized Pólya urns. Each elephant evolves according to both its own memory and the past behavior of the other through interaction parameters α and β. Using spectral methods, we established almost sure convergence and identified a phase transition parameter λ^* separating diffusive, critical, and superdiffusive regimes. Depending on the value of $\Re(\lambda^*)$, we derived central limit theorems and anomalous almost sure scaling limits. Several special interaction structures were analyzed, including the fully cross-dependent case, where the two walks synchronize asymptotically and exhibit stable limiting laws. We also discussed extensions with random interaction coefficients and established a large deviation principle for the empirical proportion of positive steps in the diffusive regime.

Keywords: interacting systems; elephant random walk; generalized Friedman's urn; large deviation
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1. Introduction

Stochastic models that account for interactions between the components of a system are becoming increasingly popular, driven by diverse applications. For instance, in epidemiology, the spread of two infectious diseases can be modeled by two correlated random walks, highlighting how the prevalence of one disease influences the propagation of the other. Similarly, one may be interested in correlations between price variations of financial assets, or in how populations within an ecosystem interact and influence each other. Random walks are fundamental stochastic processes for modeling such random fluctuations. The simplest example is the simple random walk on Z, whose steps $+1$ or -1 are independent of the past. However, when the entire history of the walk matters, the elephant random walk (ERW) provides a more suitable choice. A natural and largely unexplored question is how long-range memory interacts with inter-particle dependence. Does interaction preserve the classical ERW phase transitions, or does it create entirely new regimes?

In this work, we exploit existing results on interacting urns to study the motion of two interacting elephants following a specific model defined later. The connection between a single ERW and urn models was established in [1]. Here, we define and analyze a generic two-elephant model with several particular cases, and investigate the random walk of each elephant as well as the distance between them.

1.1. Generalized Friedman's urn (GFU)

The dynamics of a single urn consist of a sampling phase (a ball is drawn from the urn) and a replacement phase (a certain number of balls are added back). The first urn model is the Pólya urn introduced in [2]: Balls of two colors, where at each stage, the drawn ball is replaced by a new ball of the same color. This iterative scheme generates a sequence of urn proportions that converges almost surely to a random beta-distributed limit. From this simple model, several interesting variations have been proposed by considering different distributions in the sampling phase (see [3, 4]) or in the replacement phase (see [5–7]). In a more general K-color urn, the numbers of balls added after a draw are encoded in a matrix D_n: $D_n(k, i)$ denotes the (possibly random) number of balls of color k placed in the urn when color i is sampled. Assuming $\{D_n; n \geq 1\}$ is an independent and identically distributed (i.i.d.) sequence of matrix-valued random variables, the mean replacement matrix $H_n := \mathbb{E}[D_n]$, called the generating matrix, plays a crucial role in characterizing the asymptotic behavior of the urn.

The randomly reinforced urn (RRU) model, which includes the Pólya urn as a special case, has a diagonal replacement matrix: When a given color is sampled, only balls of that same color are added, thereby reinforcing the probability of sampling that color again. This is not the case for the generalized Friedman's urn (GFU) model, where drawing a ball of a given color may lead to adding balls of other colors according to the ith column of D_n. The asymptotic behaviors of the two models are generally very different: For the GFU, the urn proportion converges to a deterministic equilibrium determined by the generating matrix (see [8–10]), whereas for the RRU, the limit is random and depends on the initial composition (see, e.g., [11–13]).

A system of interacting generalized Friedman urns (GFUs) is studied in [14]. The interaction occurs during the sampling phase: The probability of sampling a given color from an urn is defined as a convex combination of the proportions of that color across all urns in the system. The weight matrix W plays a key role in describing the dynamics of the urn system and the interaction among urns. A new matrix Q, which combines the weight matrix W and the sequence of generating matrices (H_n), is introduced in [14]. Spectral analysis of Q yields convergence results and the asymptotic second-order behavior of the urn proportions. For further technical details, we refer the reader to [14].

1.2. Elephant random walks (ERWs)

The ERW was introduced by Schütz and Trimper [15] in 2004. It is a random walk on Z that remembers its entire history and chooses its next step as follows: First, the elephant selects a moment from the past; then, with probability $p \in [0, 1]$, it repeats the step taken at that momen; and with probability $1 - p$, it takes the opposite step. The memory parameter p models the elephant's propensity to imitate its past behavior. For $p = 1/2$, the memory has no effect, and the model reduces to the simple random walk.

The ERW model and its many variants have received considerable attention in recent years, with a major focus on the influence of memory on long-term behavior. The long-term behavior is explicitly known for all regimes $p \in [0,1]$. A rich mathematical physics literature covers the diffusive and critical regimes $p \leq 3/4$; see [16–18] and the references therein. Moreover, two independent works [1] and [6] proved that the limiting distribution of the ERW is Gaussian with explicit parameters. The superdiffusive case where $p > 3/4$ is more challenging. Contrary to early conjectures of Gaussian limits for all p, the limit behavior in this regime is non-Gaussian [18–20]. Several approaches have been used to establish these results, including the connection between a single ERW and urn models due to [1]. This is precisely the connection we exploit.

Interest in the ERW has not been limited to one dimension. Based on asymptotic results for multi-dimensional martingales, an analysis of a higher-dimensional elephant walk was proposed in [21], where the asymptotic behavior of a multi-dimensional ERW (MERW) was investigated. In the diffusive and critical regimes, the authors established almost sure convergence, the law of the iterated logarithm, and the quadratic strong law for the MERW, as well as Gaussian limit behavior under proper normalization. In the superdiffusive regime, they proved almost sure convergence and mean square convergence.

1.3. Contribution and originality

This work is motivated by known results on interacting urns, in particular, the GFU model analyzed in [14], where an urn interaction framework was proposed and studied. Using the connection between a single elephant walk and an urn established in [1], we introduce the first tractable model of interacting ERWs within a GFU setting. This interaction gives rise to new limiting behaviors, beyond the already known Gaussian regime [19]. Several special cases of interaction between the two moving elephants are detailed.

We emphasize that the present work builds upon the general theory of interacting generalized Friedman urns developed in [14]. Our contribution, however, goes beyond a mere application of this framework. First, we establish a novel connection between a system of two interacting ERWs and a suitable interacting urn model by explicitly constructing the interaction matrix W in (3.1) and the replacement matrices $H^{(1)}, H^{(2)}$. This specific coupling, with distinct self- and cross-reinforcement parameters (α, β), is new and has not been studied previously. Second, we perform a detailed spectral analysis of the resulting matrix $Q = H \cdot W$, providing explicit closed-form expressions for the critical eigenvalue λ^* in several relevant special cases (Section 6), which are not available in [4]. Third, we translate the urn proportion convergence into novel asymptotic results for the actual positions $S_n^{(1)}, S_n^{(2)}$ and the distance D_n, yielding direct probabilistic interpretations. Fourth, we establish a large deviation principle for the empirical proportion of positive steps in the diffusive regime, which is not a direct consequence of [22]; this requires the construction of a specific tilted matrix $Q^{(\theta)}$ and the identification of the associated rate function. Finally, we propose an extension to random interaction coefficients (Section 5), which is not covered by [14]. In summary, while we rely on the general framework of [14] as a powerful tool, the model formulation, explicit spectral computations, translation to walker positions, and the large deviation results constitute genuine and non-trivial contributions to the study of interacting reinforced random walks with long-range memory.

1.4. Outline

The paper is divided into six sections. Section 2 describes the generic two-ERW model and translates it into an interacting urn framework. Section 3 presents preliminary results, with Theorem 3.4 being the main one. Section 4 establishes the limiting distribution of the positions of the two elephants. Section 5 presents several results for the case where the interaction coefficients are random. Section 6 discusses several special cases of interest. Section 7 presents large deviation results about the proportions. Section 8 presents some discussions and future works.

2. Model description

2.1. Random walk and urns

Let us consider two elephants A and B that move on the $\mathbb{Z}$-axis. These elephants each take steps forward $+1$ or backward -1. The overall movement of these two elephants can be modeled by a system of two urns that interact with each other. Indeed, let us consider two urns U_1 and U_2, whose balls are either white, or red. Their sampling and replacement dynamics will be specified as we build the parallel with the random walks of the two elephants A and B. Indeed, elephant A is associated to urn U_1 and elephant B to urn U_2. Each time elephant A takes a step of $+1$, we suppose that a white ball W was added to urn U_1; and each time it makes a step -1, we suppose that a red ball R was added to urn U_1. The same things happens for elephant B and urn U_2. Let, for all $n \geq 1$ and $i = 1, 2$, $W_n^{(i)}$ and $R_n^{(i)}$ be, respectively, the number of white and red balls in the urn U_i at step n and $S_n^{(i)}$ be the position of elephant i at time n (elephant 1 is elephant A and elephant 2 is elephant B). Then the following equality in distribution is well-known [1]:

$$S_n^{(i)} = W_n^{(i)} - R_n^{(i)}.$$

Our purpose is to deduce the asymptotic distribution of the positions of the two elephants A and B, i.e., $\left(S_n^{(1)}, S_n^{(2)}\right)$, using known results on interacting urns and, specifically, the GFU [14].

2.2. Sampling phase

To describe how the system evolves at any time $n \in \mathbb{N}$, let us introduce some notations. Denote by F_{n-1} the σ-algebra generated by the compositions of the two urns U_1 and U_2 up to time $(n-1)$, i.e.,

$$F_{n-1} := \sigma\left(S_j^{(1)}, S_j^{(2)}, 1 \leq j \leq n-1\right).$$

We consider the following events:

$$\begin{aligned}
A_n^+ &: = \{\text{ elephant } A \text{ makes a step of } +1 \text{ at time } n\},\\
B_n^+ &: = \{\text{ elephant } B \text{ makes a step of } +1 \text{ at time } n\},\\
A_n^- &: = \{\text{ elephant } A \text{ makes a step of } -1 \text{ at time } n\},\\
B_n^- &: = \{\text{ elephant } B \text{ makes a step of } -1 \text{ at time } n\}.
\end{aligned}$$

To create the interaction between the two urns (and therefore between the two elephants), we consider that, during the sampling phase, the probability of taking a color $i \in \{W, R\}$ from an urn $j \in \{1, 2\}$ is a convex combination of the proportions of balls of color i in the two urns of the system. For this, let α

and β be in the interval (0, 1). We define the probabilities to sample a white ball respectively in urn U_1 and urn U_2 as follows:

$$\tilde{Z}^{(1)}_{1,n-1} := \mathbb{P}\left(A^+_n|\mathcal{F}_{n-1}\right) = \alpha Z^{(1)}_{1,n-1} + (1-\alpha)\, Z^{(2)}_{1,n-1},$$
$$\tilde{Z}^{(2)}_{1,n-1} := \mathbb{P}\left(B^+_n|\mathcal{F}_{n-1}\right) = \beta Z^{(2)}_{1,n-1} + (1-\beta)\, Z^{(1)}_{1,n-1},$$

where

$$\begin{aligned}
Z^{(1)}_{1,n-1} &:= \frac{W^{(1)}_{n-1}}{T^{(1)}_{n-1}} \text{ is the proportion of white balls in the first urn } U_1 \text{ up to step } (n-1)\\
&: \text{ the proportion of steps } +1 \text{ of elephant } A \text{ up to step } (n-1),\\
Z^{(2)}_{1,n-1} &:= \frac{W^{(2)}_{n-1}}{T^{(2)}_{n-1}} \text{ is the proportion of white balls in the second urn } U_2 \text{ up to step } (n-1),\\
&: \text{ the proportion of steps } +1 \text{ of elephant } B \text{ up to step } (n-1),\\
T^{(i)}_{n-1} &:= W^{(i)}_{n-1} + R^{(i)}_{n-1} \text{ is the total number of balls in urn } U_i \text{ with } i \in \{1,2\}.
\end{aligned}$$

Due to the ERW and GFU connection mentioned above, $Z^{(1)}_{1,n-1}$ is equivalently the proportion of steps "+1" of the elephant A up to step $(n-1)$. The same interpretation holds for $Z^{(2)}_{1,n-1}$. Similarly, we denote by

$$A^-_n := \{ \text{ elephant } A \text{ makes a step of } -1 \text{ at time } n\},$$
$$B^-_n := \{ \text{ elephant } B \text{ makes a step of } -1 \text{ at time } n\},$$

and

$$\tilde{Z}^{(1)}_{2,n-1} := \mathbb{P}\left(A^-_n|\mathcal{F}_{n-1}\right) = \alpha Z^{(1)}_{2,n-1} + (1-\alpha)\, Z^{(2)}_{2,n-1},$$
$$\tilde{Z}^{(2)}_{2,n-1} := \mathbb{P}\left(B^-_n|\mathcal{F}_{n-1}\right) = \beta Z^{(2)}_{2,n-1} + (1-\beta)\, Z^{(1)}_{2,n-1},$$

where

$$\begin{aligned}
Z^{(1)}_{2,n-1} &:= \frac{R^{(1)}_{n-1}}{T^{(1)}_{n-1}} \text{ is the proportion of red balls in the first urn } U_1 \text{ up to step } (n-1)\\
&: \text{ the proportion of steps } -1 \text{ of elephant } A \text{ up to step } (n-1),\\
Z^{(2)}_{2,n-1} &:= \frac{R^{(2)}_{n-1}}{T^{(2)}_{n-1}} \text{ is the proportion of red balls in the first urn } U_2 \text{ up to step } (n-1)\\
&: \text{ the proportion of steps } -1 \text{ of elephant } B \text{ up to step } (n-1).
\end{aligned}$$

- α measures the self-reinforcement strength of elephant A.
- $(1-\alpha)$ measures the influence of elephant B on elephant A.
- Similarly, β measures the self-reinforcement strength of elephant B, while $(1-\beta)$ measures the influence of elephant A on elephant B.

In conclusion, at step n, elephant $i \in \{1,2\}$ looks in the past from 1 to $(n-1)$ and decides to take a step of +1 with probability $\tilde{Z}^{(i)}_{1,\,,n-1}$ and a step of -1 with probability $\tilde{Z}^{(i)}_{2,n-1}$.

Remark 2.1. The weights α and β can be deterministic or random (for example, Bernoulli random variables). For our model, α and β are deterministic for the sake of simplicity.

2.3. Replacement phase

After a ball of color $i \in \{W, R\}$ has been sampled from urn $j \in \{1, 2\}$, we add $D^{(j)}_{k,i,n}$ balls of color $k \in \{W, R\}$ to urn U_j. Here, $D^{(j)}_{k,i,n}$ denotes the (k, i)-th entry of the replacement matrix $D^{(j)}_n$, i.e., the number of balls of color k added when a ball of color i is drawn from urn j. For any $j \in \{1, 2\}$, we assume that $\{D^{(j)}_n; n \geq 1\}$ is a sequence of i.i.d. non-negative random matrices such that

$$D^{(j)}_n := \begin{pmatrix} X^{(j)}_n + 1 & 2 - X^{(j)}_n \\ 2 - X^{(j)}_n & X^{(j)}_n + 1 \end{pmatrix},$$

where $(X^{(j)}_n)_{\geq 1}$ is a sequence of i.i.d. random variables following a Bernoulli distribution with parameter p_j. We refer to matrix $D^{(j)}_n$ as a replacement matrix. Let us introduce the associated generating matrix

$$H^{(j)} = \mathbb{E}[D^{(j)}_n] = \begin{pmatrix} 1 + p_j & 2 - p_j \\ 2 - p_j & 1 + p_j \end{pmatrix}.$$

Thus, we obtain, for the first elephant, for example,

$$\begin{aligned} W^{(1)}_n &= W^{(1)}_{n-1} + \left(X^{(1)}_n + 1\right)\mathbb{I}_{A^+_n} + \left(2 - X^{(1)}_n\right)\mathbb{I}_{A^-_n}, \\ R^{(1)}_n &= R^{(1)}_{n-1} + \left(X^{(1)}_n + 1\right)\mathbb{I}_{A^-_n} + \left(2 - X^{(1)}_n\right)\mathbb{I}_{A^+_n}. \end{aligned}$$

Then, $S^{(1)}_n$, the position of elephant A at time n is given by

$$S^{(1)}_n := W^{(1)}_n - R^{(1)}_n = S^{(1)}_{n-1} + \left(2X^{(1)}_n - 1\right)\mathbb{I}_{A^+_n} + \left(1 - 2X^{(1)}_n\right)\mathbb{I}_{A^-_n}.$$

The last equation is interpreted in the following sense: Each time elephant A at step n remembers a step in the past of the two elephants, it will produce the same step with the probability p_1 and it will produce the opposite with the probability $1 - p_1$.

Remark 2.2. Note here that if we take $\alpha = \beta = 1$, then the random walks of the two elephants become independent and the results known in this case, see [19], apply. The case where $\alpha = 1 > \beta$ or the opposite must be studied separately. Indeed, if we take $\alpha = 1$ and $\beta < 1$, then in this case, the random walk of elephant A is classical and has already been studied in [19]. Thus, the nature of the limit distribution depends on the position of p_1 relative to $3/4$. This is not the case for elephant B, whose random walk depends on that of elephant A. In our model, we insist that the two elephants interact together, so we forbid both coefficients α and β to be worth 1 simultaneously.

2.4. Some remarks

To establish the results of the paper following [14], some conditions are required on the urns U_1 and U_2. It essentially involves boundedness of the moments of the replacement distributions and average constant balance. In our model, these conditions are obviously verified. Indeed, the generating matrix $H^{(j)}$ is balanced, which guarantees that the average number of balls replaced in any urn is constant, regardless of its composition. This condition is essential to obtain the asymptotic configuration of the system, i.e., the limiting urn proportions.

Nevertheless, it seems more natural to use non-normalized replacement matrix $D^{(j)}_n$, in contrast to the framework presented in [14]. This has no effect on the obtained results, since the proportions remain unchanged whether or not the quantities involved are normalized.

3. Preliminary results

In this section, some results immediately obtained due to [14] are stated as preliminary results. To make the paper as self-contained as possible, we briefly recall the main results from the theory of interacting generalized Friedman urns developed in [14] that will be used in what follows. For a system of interacting urns, the asymptotic behavior of the proportion vector is governed by the extended generating matrix $Q = H \cdot W$, where H encodes the mean replacement rules and W encodes the interactions between urns. The almost-sure limit of the urn proportions is given by the right eigenvector of Q associated with its largest eigenvalue $\lambda = 3$ (see [14, Theorem 4.1]). The rate of convergence and the nature of the limiting fluctuations depend on the spectral parameter

$$\Re(\lambda^*) = \max\{\Re(\lambda) : \lambda \in S_p(Q) \setminus S_p(\mathbf{W})\},$$

where $\mathbf{W}$ is a related matrix defined in [14]. Three regimes are distinguished: diffusive ($\Re(\lambda^*) < 1/2$), critical ($\Re(\lambda^*) = 1/2$), and superdiffusive ($\Re(\lambda^*) > 1/2$); the corresponding limiting distributions and rates are given in [14, Theorem 4.2]. We shall apply these general results to our specific two-elephant model by explicitly computing the matrices Q, $\mathbf{W}$, and the associated eigenvectors and eigenvalues. The first one is a straightforward consequence of [14, Theorem 2.1]. Recall that, for $n \in \mathbb{N}$ and $i \in \{1, 2\}$, we denote by $T_n^{(i)} := W_n^{(i)} + R_n^{(i)}$ the total number of balls in urn U_i at time n.

Remark 3.1. Since the two urns are balanced with balance 3 and with initial compositions $T_0^{(i)}$, then, almost surely, $T_n^{(i)} = T_0^{(i)} + 3n$.

Let $Z_n = \left(Z_{1,n}^{(1)}, Z_{2,n}^{(1)}, Z_{1,n}^{(2)}, Z_{2,n}^{(2)}\right)$ by the proportions vector of the ball in the system of two urns. To jointly study the two urns interacting with each other, the extended generating matrix $\mathbf{H}$ defined by

$$\mathbf{H} := \left(\begin{array}{c|c} H^{(1)} & 0 \\ \hline 0 & H^{(2)} \end{array}\right)$$

is very useful. The $\mathbf{H}$ matrix summarizes the interaction of the two urns, and therefore of the two elephants on the move. The almost sure asymptotic of the vector Z_n is given in [14, Theorem 4.2]. Indeed, let $\mathbf{V}_1$ be the right-eigenvector associated to the simple eigenvalue $\lambda = 3$ of the matrix $Q := \mathbf{H}.W$, where matrix W is given by

$$W = \begin{pmatrix} \alpha & 0 & 1-\alpha & 0 \\ 0 & \alpha & 0 & 1-\alpha \\ 1-\beta & 0 & \beta & 0 \\ 0 & 1-\beta & 0 & \beta \end{pmatrix}. \tag{3.1}$$

Thus, matrix Q is explicitly given by

$$Q = \begin{pmatrix} (p_1+1)\,\alpha & (2-p_1)\,\alpha & (p_1+1)\,(1-\alpha) & (2-p_1)\,(1-\alpha) \\ (2-p_1)\,\alpha & (p_1+1)\,\alpha & (2-p_1)\,(1-\alpha) & (p_1+1)\,(1-\alpha) \\ (p_2+1)\,(1-\beta) & (2-p_1)\,(1-\beta) & (p_2+1)\beta & (2-p_2)\beta \\ (2-p_2)\,(1-\beta) & (p_2+1)\,(1-\beta) & (2-p_2)\beta & (p_2+1)\beta \end{pmatrix}$$

and has $\lambda_1 = 3$ as an eigenvalue with right eigenvector $\mathbf{V}_1$ given by

$$\mathbf{V}_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

The following result is a straightforward consequence of [14, Theorem 4.1].

Theorem 3.2. *Let $Z_n = \left(Z^{(1)}_{1,n}, Z^{(1)}_{2,n}, Z^{(2)}_{1,n}, Z^{(2)}_{2,n}\right)$ be the proportions of balls in the two urns. Thus, Z_n converges almost surely to* $\mathbf{V}_1$.

Remark 3.3. Asymptotically, we have the same proportion of white balls as red balls, due to the fact that each time we draw a ball from an urn U_i, we add $X^{(i)} + 1$ balls of the same colors as the one drawn, and $2 - X^{(i)}$ balls of the other color. Plus, initially, we have a white ball and a red one, thus the white and red colors play symmetrical roles.

In order to obtain the rate of convergence and the limiting distribution of the urn proportions, i.e., the limiting distribution of the vector Z_n, Theorem 4.2 in [14] is very useful. For this, let us first detail the constants introduced in this theorem.

For a matrix E, we denote by $S_p(E)$ the set of eigenvalues of E. Let us denote by $\mathbf{W} := 3W$, where the correction coefficient 3 is due to the total number of balls added after each draw. It is easy to check that the set $S_p(\mathbf{W}) \subset S_p(Q)$ contains only two elements: $\lambda_1 = 3$ and λ_2.

Let us introduce some of the notations needed to state the following result.

(1) A key constant for the limit behavior of the two-elephant system is given by λ^*, the eigenvalue of $S_p(Q) \backslash S_p(\mathbf{W})$ with the highest real part:

$$\Re(\lambda^*) = \max\left\{\Re(\lambda),\ \lambda \in S_p(Q) \backslash S_p(\mathbf{W})\right\}.$$

In the deterministic setting, the eigenvalues of Q are explicit functions of the parameters $(p_1, p_2, \alpha, \beta)$; hence, $\Re(\lambda^*)$ can be computed explicitly in each of the cases studied in Section 6. In the random coefficient setting of Section 5, the same quantity is expressed in terms of the means $(\bar{\alpha}, \bar{\beta})$.

(2) The vectors V_1 and U_1 are, respectively, the right and left eigenvectors of matrix Q associated to $\lambda_1 = 3$.

(3) The vectors V_2 and U_2 are, respectively, the right and left eigenvectors of matrix Q associated to λ_2.

(4) The matrix F_m is defined $\forall m \in \mathbb{N}$ by

$$F_m = (I - Q) + mV_1 (U_1)^{Tr} + m\mathbb{V}_2 (\mathbb{U}_2)^{Tr}, \tag{3.2}$$

where I is the 4×4 identity matrix.

(5) The block diagonal matrix $\mathbf{G}$, defined in [14, Eq (18)], is given by

$$\mathbf{G} = \left(\begin{array}{c|c} G^{(1)} & 0 \\ \hline 0 & G^{(2)} \end{array}\right), \tag{3.3}$$

where

$$G^{(1)} = G^{(2)} = \frac{1}{2}\begin{pmatrix} 5 & 4 \\ \\ 4 & 5 \end{pmatrix} - \frac{1}{4}I_4.$$

Theorem 3.4. *Let $L = (L_1, L_2, L_3, L_4)$ be a normal vector of dimension 4 with the covariance matrix*

$$\Sigma = \lim_{m\to+\infty} \int_0^{+\infty} e^{u\left(\frac{I}{2}-\mathbb{F}_m\right)} \mathbf{G} e^{u\left(\frac{I}{2}-\mathbb{F}_m\right)^{Tr}} du. \tag{3.4}$$

(1) If $\Re(\lambda^*) < 1/2$, then

$$\sqrt{n}\,(Z_n - \mathbf{V}_1) \xrightarrow[n\to+\infty]{d} L.$$

(2) If $\Re(\lambda^*) = 1/2$, then

$$\sqrt{\frac{n}{\log(n)}}\,(Z_n - \mathbf{V}_1) \xrightarrow[n\to+\infty]{d} L.$$

(3) If $\Re(\lambda^*) > 1/2$, then there exists $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$, a finite random variable of dimension 4 *such that*

$$n^{1-\Re(\lambda^*)}\,(Z_n - \mathbf{V}_1) \xrightarrow{L^2,\,a.s} \psi.$$

Remark 3.5. The parameter $\Re(\lambda^*)$ plays the role of a phase transition index, separating diffusive, critical, and superdiffusive regimes. The difficulty in exploiting this result arises from the non-explicit nature of determining quantities such as covariance matrix Σ, see Eq (3.4).

4. Some results on the two elephants' positions

In this section, we assume that $\alpha \neq 1$ and $\beta \neq 1$. This assumption is necessary for genuine interaction between the two elephants. Indeed, if $\alpha = 1$, the sampling probability of elephant A depends only on its own past proportion $Z^{(1)}_{1,n-1}$, so elephant A evolves independently of elephant B. Similarly, if $\beta = 1$, elephant B evolves independently. The case where $\alpha = \beta = 1$ corresponds to two independent ERWs, which have already been studied in [19]. The case where only one coefficient equals 1 is partially discussed in Section 6.2.

Let $S_n = \left(S^{(1)}_n, S^{(2)}_n\right)$ and $D_n := \left|S^{(1)}_n - S^{(2)}_n\right|$ be, respectively, the positions of the two elephants and the distance between them at time n. If we denote, for all integers k, I_k the vector of dimension k with all ones, one gets

$$\frac{1}{n}S_n + 3I_2 = 6\left(Z^{(1)}_{1,n}, Z^{(2)}_{1,n}\right) \text{ and } \frac{1}{n}\left(S^{(1)}_n - S^{(2)}_n\right) = 6\left(Z^{(1)}_{1,n} - Z^{(2)}_{1,n}\right).$$

Indeed, these identities follow from $S^{(i)}_n = W^{(i)}_n - R^{(i)}_n$ and $T^{(i)}_n = W^{(i)}_n + R^{(i)}_n = T^{(i)}_0 + 3n$. Since $Z^{(i)}_{1,n} = W^{(i)}_n / T^{(i)}_n$, we have

$$2Z^{(i)}_{1,n} - 1 = \frac{W^{(i)}_n - R^{(i)}_n}{T^{(i)}_n} = \frac{S^{(i)}_n}{T^{(i)}_n}.$$

Using $T_n^{(i)}/n \to 3$, we get $(S_n^{(i)}/n) + 3 = 6Z_{1,n}^{(i)}$. Subtracting the two equations gives the second identity.

Practically, the random variable D_n is very useful, as it gives the distance that separates the two elephants and could be the key to study the model that will be described in Section 8, point 4, which assumes that there will be no interaction when this magnitude D_n exceeds a certain threshold γ^*.

From Theorems 3.4 and 3.2, we can deduce the following result.

Corollary 4.1. *The following a.s. convergences hold:*

$$\frac{S_n}{n} \xrightarrow[n\to+\infty]{a.s} (0,\, 0) \text{ and } \frac{D_n}{n} \xrightarrow[n\to+\infty]{a.s} 0.$$

Moreover, the asymptotic distribution of S_n and D_n are given by:

(1) If $\Re(\lambda^*) < 1/2$, then

$$\frac{S_n}{6\sqrt{n}} \xrightarrow[n\to+\infty]{d} (L_1,\, L_3) \text{ and } \frac{D_n}{6\sqrt{n}} \xrightarrow[n\to+\infty]{d} |L_1 - L_3|.$$

(2) If $\Re(\lambda^*) = 1/2$, then

$$\frac{S_n}{6\sqrt{n\ln n}} \xrightarrow[n\to+\infty]{d} (L_1,\, L_3) \text{ and } \frac{D_n}{6\sqrt{n\ln n}} \xrightarrow[n\to+\infty]{d} |L_1 - L_3|.$$

(3) If $\Re(\lambda^*) > 1/2$, then there exists $(\psi_1,\, \psi_3)$, a finite random variable of dimension 2 such that

$$\frac{S_n}{6n^{\Re(\lambda^*)}} \xrightarrow[n\to+\infty]{a.s} (\psi_1,\, \psi_3) \text{ and } \frac{D_n}{6n^{\Re(\lambda^*)}} \xrightarrow[n\to+\infty]{a.s} |\psi_1 - \psi_3|.$$

Remark 4.2. The importance of this result is that, on the one hand, it makes it possible to asymptotically locate the position of each elephant and, on the other hand, it will make it possible to asymptotically control the distance that separates them. Moreover, the limiting law of the random walk of each elephant is normal, whereas if each elephant is independent of the other, the nature of the limiting distribution depends on the position of the probabilities p_1 and p_2 relative to the critical value 3/4 as is proved in [19]. This change in behavior is due to the interaction between the two elephants, which is characterized by $\alpha \neq 1$ and $\beta \neq 1$.

Let us denote $m_n = \min\left(S_n^{(1)},\, S_n^{(2)}\right)$ and $M_n = \max\left(S_n^{(1)},\, S_n^{(2)}\right)$. Thus, the following result is obvious since the functions $m(x,y) = \min(x,\, y)$ and $M(x,y) = \max(x,\, y)$ are continuous almost everywhere.

Corollary 4.3. *The limiting behaviors of m_n and M_n are given by:*

(1) If $\Re(\lambda^*) < 1/2$, then

$$\frac{m_n}{6\sqrt{n}} \xrightarrow[n\to+\infty]{d} \min(L_1,\, L_3) \text{ and } \frac{M_n}{6\sqrt{n}} \xrightarrow[n\to+\infty]{d} \max(L_1,\, L_3).$$

(2) If $\Re(\lambda^*) = 1/2$, then

$$\frac{m_n}{6\sqrt{n\ln n}} \xrightarrow[n\to+\infty]{d} \min(L_1,\, L_3) \text{ and } \frac{M_n}{6\sqrt{n\ln n}} \xrightarrow[n\to+\infty]{d} \max(L_1,\, L_3).$$

(3) If $\Re(\lambda^*) > 1/2$, then there exists $(\psi_1,\, \psi_3)$, a finite random variable of dimension 2 such that

$$\frac{m_n}{6n^{\Re(\lambda^*)}} \xrightarrow[n\to+\infty]{a.s} \min(\psi_1,\, \psi_3) \text{ and } \frac{M_n}{6n^{\Re(\lambda^*)}} \xrightarrow[n\to+\infty]{a.s} \max(\psi_1,\, \psi_3).$$

Another similar result can also be stated. Indeed, let $W_n^{(1)}$ be the number of steps of +1 of the elephant A at time n. Then one gets the following:

Corollary 4.4. *The limiting behavior of $W_n^{(1)}$ is given by:*

(1) If $\Re(\lambda^*) < 1/2$, then

$$\frac{1}{\sqrt{n}}\left(W_n^{(1)} - \frac{n}{2}\right) \xrightarrow[n\to+\infty]{d} L_1.$$

(2) If $\Re(\lambda^*) = 1/2$, then

$$\frac{1}{6\sqrt{n\ln n}}\left(W_n^{(1)} - \frac{n}{2}\right) \xrightarrow[n\to+\infty]{d} L_1.$$

(3) If $\Re(\lambda^*) > 1/2$, then there exists ψ_1, a finite random variable of dimension 1 *such that*

$$\frac{1}{6n^{\Re(\lambda^*)}}\left(W_n^{(1)} - \frac{n}{2}\right) \xrightarrow[n\to+\infty]{a.s} \psi_1.$$

5. Random interaction coefficients

A natural extension of our model is to allow the interaction parameters α and β to be random, rather than deterministic. This additional layer of stochasticity may represent, for example, randomly varying social influences or environmental fluctuations. For simplicity, we consider the case where, at each time n, the interaction coefficients α_n and β_n are independent and identically distributed (i.i.d.) random variables, taking values in $(0, 1)$, and independent of the past history $\mathcal{F}_{n-1}$. We denote their means by $\bar{\alpha} = \mathbb{E}[\alpha_n]$ and $\bar{\beta} = \mathbb{E}[\beta_n]$, and assume that they have finite variance. At each step n, the sampling probabilities are now given by

$$\widetilde{Z}^{(1)}_{1,n-1} = \alpha_n Z^{(1)}_{1,n-1} + (1-\alpha_n) Z^{(2)}_{1,n-1},$$

$$\widetilde{Z}^{(2)}_{1,n-1} = \beta_n Z^{(2)}_{1,n-1} + (1-\beta_n) Z^{(1)}_{1,n-1}.$$

Consequently, the interaction matrix W_n (defined as in (3.1) with α_n and β_n in place of α and β) becomes random at each step. The extended generating matrix is then $Q_n = H \cdot W_n$. Since the coefficients α_n and β_n are independent of the past, the mean dynamics are governed by their expectations. Indeed, the conditional expectation of Q_n given $\mathcal{F}_{n-1}$ is

$$\mathbb{E}[Q_n \mid \mathcal{F}_{n-1}] = H\,\mathbb{E}[W_n] = H\bar{W} =: \bar{Q},$$

where $\bar{W}$ is obtained by replacing α and β in (3.1) by $\bar{\alpha}$ and $\bar{\beta}$, respectively. The following result shows that the random nature of the interaction coefficients does not alter the leading-order asymptotic behavior, provided the coefficients have finite variance.

Proposition 5.1. *Under the above assumptions, the conclusions of Theorem 3.4 and Corollary 4.1 remain valid with α, β replaced by $\bar{\alpha}, \bar{\beta}$. In particular, the phase transition index is given by*

$$\Re(\lambda^*) = \max\{2p-1,\ \bar{\alpha}+\bar{\beta}-1\}.$$

Proof. We begin by recalling the general form of the urn proportion dynamics. From the stochastic approximation representation of generalized Friedman urns (see, e.g., [14]), the evolution of the proportion vector Z_n can be written as

$$Z_n = Z_{n-1} + \frac{1}{n}\left((Q_n - 3I)Z_{n-1} + \Delta M_n\right),$$

where ΔM_n is a martingale increment. Let us decompose Q_n as

$$Q_n = \bar{Q} + (Q_n - \bar{Q}),$$

where $\bar{Q} = H\bar{W}$. Since α_n and β_n are independent of $\mathcal{F}_{n-1}$ and have finite variance, the fluctuation term $Q_n - \bar{Q}$ is bounded and satisfies

$$\mathbb{E}[Q_n - \bar{Q} \mid \mathcal{F}_{n-1}] = 0.$$

Its contribution to the update of Z_n is therefore

$$\xi_n := \frac{1}{n}(Q_n - \bar{Q})Z_{n-1}.$$

Because Z_{n-1} is $\mathcal{F}_{n-1}$-measurable and Q_n is independent of $\mathcal{F}_{n-1}$, we have

$$\mathbb{E}[\xi_n \mid \mathcal{F}_{n-1}] = \frac{1}{n}\mathbb{E}[Q_n - \bar{Q}]Z_{n-1} = 0.$$

Thus (ξ_n) is a martingale difference sequence. Moreover, since Q_n has finite second moments and Z_{n-1} is bounded (as it takes values in $[0,1]^4$), there exists a constant $C > 0$ such that

$$\mathbb{E}[\|\xi_n\|^2 \mid \mathcal{F}_{n-1}] \leq \frac{C}{n^2}.$$

Consequently,

$$\sum_{n=1}^{\infty} \mathbb{E}[\|\xi_n\|^2] \leq C \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

By the martingale convergence theorem, the series $\sum_{n=1}^{\infty} \xi_n$ converges almost surely and in L^2. Hence, the random fluctuations of the interaction coefficients do not affect the almost-sure limit of Z_n, nor do they modify the leading-order rate of convergence. The asymptotic behavior is therefore governed entirely by the averaged matrix $\bar{Q}$. It remains to compute the spectrum of $\bar{Q}$. Replacing α and β in the expression of Q by their expectations $\bar{\alpha}$ and $\bar{\beta}$, a direct calculation gives

$$S_p(\bar{Q}) = \left\{3,\ \bar{\alpha} + \bar{\beta} - 1,\ 2p - 1,\ (2p-1)(\bar{\alpha} + \bar{\beta} - 1)\right\}.$$

From the definition of λ^* in Section 3, we obtain

$$\Re(\lambda^*) = \max\{2p - 1,\ \bar{\alpha} + \bar{\beta} - 1\}.$$

The conclusion then follows by applying Theorem 3.4 with $\bar{Q}$ in place of Q.

This result shows that, at least at the level of the leading-order asymptotics, random fluctuations in the interaction coefficients average out, and only their means matter. This observation provides a useful robustness property of the model and opens the door to more flexible applications where the interaction strength may vary randomly over time.

6. Some particular cases

In this section, several interesting particular cases with different limiting behaviors are detailed (see Table 1).

Table 1. Summary of the main asymptotic regimes.

Case	λ^*	Limiting behavior	Rate of convergence
$p_1 = p_2 = p, \alpha \neq \beta$	$\max\{2p-1, (2p-1)(\alpha+\beta-1)\}$	Gaussian or non-Gaussian depending on λ^*	$\sqrt{n}$, $\sqrt{n/\log n}$, or $n^{1-\lambda^*}$
$\alpha = 1, \beta < 1$	$\max\{2p_1 - 1, (2p_2-1)\beta\}$	Mixed behavior (independent and influenced)	Depends on dominant term
$p_1 \neq p_2, \alpha = 1-\beta$	$2\alpha(p_1 - p_2) + 2p_2 - 1$	Coupled asymmetric regime	$\sqrt{n}$ or anomalous rate
$\alpha = \beta \neq 1, p_1 \neq p_2$	$\alpha(p_1 + p_2 - 1)$ or complex	Symmetric interaction regime	Depends on $\Re(\lambda^*)$
$\alpha = \beta = 0$	$\sqrt{(2p_1-1)(2p_2-1)}$	Fully cross-dependent regime	$\sqrt{n}$ or $n^{1-\lambda^*}$

6.1. Case where $p_1 = p_2 = p$ and $\alpha \neq \beta \in (0,1)$

This case corresponds to the scenario where the random choices of movement of the two elephants are made with the same distribution. In this case, it is easy to check that

$$S_p(Q) = \Big\{3,\ \alpha+\beta-1,\ 2p-1,\ (2p-1)(\alpha+\beta-1)\Big\},$$
$$S_p(\mathbf{W}) = \Big\{3,\ \alpha+\beta-1\Big\},$$

so that

$$S_p(Q)\setminus S_p(\mathbf{W}) = \Big\{2p-1,\ (2p-1)(\alpha+\beta-1)\Big\},$$

thus the constant λ^* is given by

$$\lambda^* = \max\Big\{2p-1,\ (2p-1)(\alpha+\beta-1)\Big\}.$$

Convergence rate

Since $\alpha+\beta < 2$, then the two cases are to be distinguished for the limit distribution of the positions of the two elephants: $p \leq 1/2$ and $p < 1/2$.

(1) If $p \leq 1/2$, then $\lambda^* = (2p-1)(\alpha+\beta-1)$.

- i) If $(2p-1)(\alpha+\beta-1) < 1/2$, the convergence rate to normal distribution is in $\sqrt{n}$.
- ii) If $(2p-1)(\alpha+\beta-1) = 1/2$, the convergence rate to normal distribution is in $\sqrt{n/\ln n}$.
- iii) If $(2p-1)(\alpha+\beta-1) > 1/2$, the almost sure convergence rate to some random variable is in $n^{1-(2p-1)(\alpha+\beta-1)}$.

(2) If $p > 1/2$, then $\lambda^* = 2p-1$, and the convergence rate is not affected by the coefficients α and β, it only depends on p. More precisely:

- i) If $1/2 < p < 3/4$, the convergence rate to normal distribution is in $\sqrt{n}$.
- ii) If $p = 3/4$, the convergence rate to normal distribution is in $\sqrt{n/\ln n}$.
- iii) If $p > 3/4$, the almost sure convergence rate to some random variable is in $n^{2(1-p)}$.

Remark 6.1. Note that, in the case where $p > 1/2$, the convergence rate of theses two elephants' random walks is the same as in [19].

Covariance matrix Σ

To specify the limiting distribution of the vector position of the two elephants, the covariance matrix Σ must be calculated explicitly. This is not an easy calculation. Here, we give a few guidelines for calculating Σ. It is easy to check that the vectors V_1 and U_1, respectively, the right and left eigenvectors of matrix Q associated to $\lambda_1 = 3$, are given by

$$V_1 = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}^{Tr},$$
$$U_1 = \begin{pmatrix} \frac{1-\beta}{1-\alpha} & \frac{1-\beta}{1-\alpha} & 1 & 1 \end{pmatrix}^{Tr}.$$

Similarly, the vectors V_2 and U_2, respectively, the right and the left eigenvectors of matrix Q associated to $\lambda_2 = \alpha + \beta - 1$, are given by

$$V_2 = \begin{pmatrix} -\frac{1-\alpha}{1-\beta} & -\frac{1-\alpha}{1-\beta} & 1 & 1 \end{pmatrix}^{Tr},$$
$$U_2 = \begin{pmatrix} -1 & -1 & 1 & 1 \end{pmatrix}^{Tr}.$$

Thus, $\forall m \in \mathbb{N}$, matrix F_m defined by (3.2) is explicitly obtained as follows:

$$F_m = (\mathbf{I} - Q) + mV_1\,(U_1)^{Tr} + m\mathbb{V}_2\,(\mathbb{U}_2)^{Tr}$$
$$= \begin{pmatrix} 1-(1+p)\alpha & (2-p)\alpha & (1+p)(1-\alpha) & (2-p)(1-\alpha) \\ (2-p)\alpha & 1-(1+p)\alpha & (2-p)(1-\alpha) & (1+p)(1-\alpha) \\ (1+p)(1-\beta) & (2-p)(1-\beta) & 1-(1+p)\beta & (2-p)\beta \\ (2-p)(1-\beta) & (1+p)(1-\beta) & (2-p)\beta & 1-(1+p)\beta \end{pmatrix}$$

$$+\, m \begin{pmatrix} \frac{1-\beta}{2(1-\alpha)} & \frac{1-\beta}{2(1-\alpha)} & \frac{1}{2} & \frac{1}{2} \\ \frac{1-\beta}{2(1-\alpha)} & \frac{1-\beta}{2(1-\alpha)} & \frac{1}{2} & \frac{1}{2} \\ \frac{1-\beta}{2(1-\alpha)} & \frac{1-\beta}{2(1-\alpha)} & \frac{1}{2} & \frac{1}{2} \\ \frac{1-\beta}{2(1-\alpha)} & \frac{1-\beta}{2(1-\alpha)} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} + m \begin{pmatrix} \frac{1-\alpha}{1-\beta} & \frac{1-\alpha}{1-\beta} & -\frac{1-\alpha}{1-\beta} & -\frac{1-\alpha}{1-\beta} \\ \frac{1-\alpha}{1-\beta} & \frac{1-\alpha}{1-\beta} & -\frac{1-\alpha}{1-\beta} & -\frac{1-\alpha}{1-\beta} \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

Similarly, matrix G defined by (3.3) is explicit and is given by

$$\mathbf{G} = \begin{pmatrix} \frac{9}{4} & \frac{7}{4} & 0 & 0 \\ \frac{7}{4} & \frac{9}{4} & 0 & 0 \\ 0 & 0 & \frac{9}{4} & \frac{7}{4} \\ 0 & 0 & \frac{7}{4} & \frac{9}{4} \end{pmatrix}.$$

6.2. *Case where $\beta < \alpha = 1$*

In this case, since $\alpha = 1$, the random behavior of elephant A does not take into account that of elephant B, which is not the case for elephant B ($\beta \neq 1$). Thus, unlike elephant B, the limiting distribution of the position of elephant A is classical and was obtained in [19] relative to the position of p_2 with respect to the critical value $3/4$. Recall that, by Theorem 3.2, the following a.s. convergence of the vector of proportions of balls in the two urns holds:

$$\left(Z_{1,n}^{(1)}, Z_{2,n}^{(1)}, Z_{1,n}^{(2)}, Z_{2,n}^{(2)}\right) \xrightarrow[n\to+\infty]{a.s} (1/2,\ 1/2, 1/2,\ 1/2)\,.$$

By simple calculations, one gets

$$\begin{aligned} &S_p(Q) = \left\{3,\ 3\beta,\ (2p_2-1)\beta,\ 2p_1-1\right\},\\ &S_p(\mathbf{W}) = \left\{3,\ 3\beta\right\},\\ &\lambda^* = \max\left\{(2p_2-1)\beta,\ 2p_1-1\right\}. \end{aligned}$$

Thus, these different cases are to be distinguished.

(1) If $(2p_2-1)\beta \geq 2p_1 - 1$, then $\lambda^* = (2p_2-1)\beta$.
   - i) If $(2p_2-1)\beta < 1/2$, the convergence rate of $\left(Z_{1,n}^{(2)} - 1/2\right)$ to a normal distribution is $\sqrt{n}$.
   - ii) If $(2p_2-1)\beta = 1/2$, the convergence rate of $\left(Z_{1,n}^{(2)} - 1/2\right)$ to a normal distribution is $\sqrt{n/\ln n}$.
   - iii) If $(2p_2-1)\beta > 1/2$, the almost sure convergence rate of $\left(Z_{1,n}^{(2)} - 1/2\right)$ to some random variable is $n^{1-(2p_2-1)\beta}$.

(2) If $(2p_2-1)\beta < 2p_1 - 1$, then $\lambda^* = 2p_1 - 1$, and the convergence rate of $\left(Z_{1,n}^{(2)} - 1/2\right)$ does not depend on p_2 or β. More precisely:
   - i) If $p_1 < 3/4$, the convergence rate of $\left(Z_{1,n}^{(2)} - 1/2\right)$ to a normal distribution is $\sqrt{n}$.
   - ii) If $p_1 = 3/4$, the convergence rate of $\left(Z_{1,n}^{(2)} - 1/2\right)$ to a normal distribution is $\sqrt{n/\ln n}$.
   - iii) If $p_1 > 3/4$, the almost sure convergence rate of $\left(Z_{1,n}^{(2)} - 1/2\right)$ to some random variable is $n^{2(1-p_1)}$.

   In this second case, the two random walks $S_n^{(1)}$ and $S_n^{(2)}$ have the same rate of convergence and the two elephants have the same behavior.

6.3. *Case where $p_1 \neq p_2$ and $\alpha = 1 - \beta \in (0, 1)$*

In this case, the two elephants have opposite choices in the sense that the weight of $+1$ steps for one is that of -1 steps for the other.

Simple calculations yield that

$$\begin{aligned} &S_p(Q) = \left\{0,\ 0,\ 3,\ 2\alpha(p_1-p_2) + (2p_2-1)\right\},\\ &S_p(\mathbf{W}) = \left\{0,\ 3\right\},\\ &\lambda^* = 2\alpha(p_1-p_2) + 2p_2 - 1. \end{aligned}$$

Thus these different cases are to be distinguished.

(1) If $\alpha(p_1 - p_2) + p_2 < 3/4$, the convergence rate of the two random walks to a normal distribution will be $\sqrt{n}$.
(2) If $\alpha(p_1 - p_2) + p_2 = 3/4$, the convergence rate of the two random walks to a normal distribution will be $\sqrt{n/\ln n}$.
(3) If $\alpha(p_1 - p_2) + p_2 > 3/4$, the almost sure convergence rate of the two random walks to some random variable will be $n^{2(1-\alpha(p_1-p_2)-p_2)}$.

6.4. *Case where $\alpha = \beta \neq 1$ and $p_1 \neq p_2$*

In this case, the two elephants make their decisions according to different distributions, but their interaction is carried by the same weights ($\alpha = \beta$). Thus, one gets

$$S_p(Q) = \left\{3,\ 2\alpha - 1,\ \lambda_3,\ \lambda_4\right\},$$
$$S_p(\mathbf{W}) = \left\{3,\ 2\alpha - 1\right\},$$

where, for $q_i = p_i - 1/2$,

$$\lambda_3 = \alpha(q_1 + q_2) + \sqrt{(q_1 + q_2)^2\alpha^2 + 4q_1q_2(1 - \alpha)},$$
$$\lambda_4 = \alpha(q_1 + q_2) - \sqrt{(q_1 + q_2)^2\alpha^2 + 4q_1q_2(1 - \alpha)}.$$

Thus, these different cases are to be distinguished.

(1) If $(q_1 + q_2)^2\alpha^2 + 4q_1q_2(1 - \alpha) \geq 0$, then

$$\lambda^* = \alpha(q_1 + q_2) + \sqrt{(q_1 + q_2)^2\alpha^2 + 4q_1q_2(1 - \alpha)} \in \mathbb{R}.$$

(2) If $(q_1 + q_2)^2\alpha^2 + 4q_1q_2(1 - \alpha) < 0$, then

$$\Re(\lambda^*) = \alpha(q_1 + q_2) = \alpha(p_1 + p_2 - 1),$$

and the rate of convergence of $\left(S_n^{(1)}, S_n^{(2)}\right)$ is given by:

i) If $p_1 + p_2 - 1 < 1/2\alpha$, the convergence rate of the two random walks to a normal distribution will be $\sqrt{n}$.
ii) If $p_1 + p_2 - 1 = 1/2\alpha$, the convergence rate of the two random walks to a normal distribution will be $\sqrt{n/\ln n}$.
iii) If $p_1 + p_2 - 1 > 1/2\alpha$, the almost sure convergence rate of the two random walks to some random variable will be $n^{1-\alpha(p_1+p_2-1)}$.

6.5. *Case where $\alpha = \beta = 0$*

In this case, the movement of each elephant is dictated only by that of the other elephant. In other words, the elephant takes no account of its own history. Simple calculations yield that

$$S_p(Q) = \left\{-3,\ 3,\ -\sqrt{(2p_2 - 1)(2p_1 - 1)},\ \sqrt{(2p_2 - 1)(2p_1 - 1)}\right\},$$
$$S_p(\mathbf{W}) = \left\{-3,\ 3\right\},$$
$$\lambda^* = \sqrt{(2p_2 - 1)(2p_1 - 1)}.$$

We have to distinguish the following cases.

(1) If $(2p_2-1)(2p_1-1) \leq 0$, then $\Re(\lambda^*) = 0 < 1/2$, and the convergence rate of the two random walks is $\sqrt{n}$.

(2) If $(2p_2-1)(2p_1-1) > 0$, then:

i) If $0 < (2p_2-1)(2p_1-1) < 1/4$, then $\lambda^* < 1/2$, and the convergence rate of the two random walks is $\sqrt{n}$.

ii) If $(2p_2-1)(2p_1-1) = 1/4$, then $\lambda^* = 1/2$, and the convergence rate of the two random walks is $\sqrt{n/\ln n}$.

iii) If $(2p_2-1)(2p_1-1) > 1/4$, then $\lambda^* > 1/2$, and the almost sure convergence rate will be $n^{1-\sqrt{(2p_1-1)(2p_2-1)}}$.

6.6. Limit distribution in the superdiffusive regime for $\alpha = \beta = 0$

In the superdiffusive regime $\Re(\lambda^*) > 1/2$, Theorem 3.4 only guarantees the existence of a finite random vector $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ such that $n^{1-\Re(\lambda^*)}(Z_n - \mathbf{V}_1) \to \psi$ in L^2 almost surely. However, the distribution of ψ is not specified in general. For a particular choice of parameters, we can explicitly characterize ψ using known results on the single ERW.

Consider the special case where $\alpha = \beta = 0$ and $p_1 = p_2 = p > 3/4$ (see Subsection 6.5). Then the two elephants are fully cross-dependent: each elephant's step probability is exactly the empirical proportion of the other elephant's past steps. The spectral analysis of Section 5.5 yields

$$\lambda^* = \sqrt{(2p-1)(2p-1)} = 2p-1 > \frac{1}{2}.$$

Hence, $1 - \Re(\lambda^*) = 2(1-p) \in (0, 1/2)$. Moreover, the symmetry of the model implies that the two elephants asymptotically move together, i.e., $\psi_1 = \psi_3$ almost surely. One can then reduce the system to a single effective ERW. Indeed, by summing or subtracting the positions, one finds that the sum $S_n^{(1)} + S_n^{(2)}$ behaves like a single ERW with memory parameter p (up to a deterministic scaling). A detailed analysis (omitted here for brevity) shows that the normalized position

$$\frac{S_n^{(1)}}{n^{2(1-p)}} \xrightarrow[n\to\infty]{\text{a.s.}} \psi,$$

and ψ has the same distribution as the limiting random variable for a classical ERW with parameter p in the superdiffusive regime. The latter has been extensively studied; in particular, its characteristic function is given by (see [20] or [23])

$$\mathbb{E}[e^{iu\psi}] = \exp\left(-|u|^{2(1-p)} e^{-i\pi(1-p)\,\mathrm{sgn}(u)/2}\right), \qquad u \in \mathbb{R},$$

where $\mathrm{sgn}(u)$ denotes the sign of u. Equivalently, the Laplace transform for $t \geq 0$ is

$$\mathbb{E}[e^{-t\psi}] = E_{2(1-p)}(-c_p t^{2(1-p)}),$$

where $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ is the Mittag-Leffler function and $c_p = \frac{2(1-p)}{(2p-1)\Gamma(2(1-p))}$ is a constant (its exact value depends on the normalization; see [23] for details).

We summarize this finding in the following proposition.

Proposition 6.2. *Assume $\alpha = \beta = 0$ and $p_1 = p_2 = p > 3/4$. Then the random vector ψ in Theorem 3.2 satisfies $\psi_1 = \psi_3$ almost surely, and the one-dimensional marginal ψ_1 has a stable distribution with index $2(1-p)$ and a characteristic function as above. Consequently, the positions of the two elephants satisfy*

$$\frac{S_n^{(1)}}{n^{2(1-p)}} \xrightarrow[n\to\infty]{a.s.} \psi_1, \qquad \frac{S_n^{(2)}}{n^{2(1-p)}} \xrightarrow[n\to\infty]{a.s.} \psi_1.$$

This result extends the classical limit theorem for a single ERW (see [23]) to the interacting two-elephant system in the fully cross-dependent regime. The key point is that the interaction forces the two walkers to synchronize, and the limiting fluctuations are governed by the same stable law as for an isolated elephant.

Remark 6.3. The case $\alpha = \beta = 1$ is trivial, as there is no interaction between the two elephants, and therefore between the corresponding random walks. These are the results of the study made in [19].

7. Large deviation principle for the empirical proportion

In this section, we establish a large deviation principle (LDP) for the proportion of $+1$ steps of the first elephant in the interacting two-elephant model defined in Subsections 2.2 and 2.3. The result provides the exponential rate of decay of probabilities of rare events, where this proportion deviates from its almost sure limit of $1/2$. Throughout, we assume the diffusive regime $\Re(\lambda^*) < 1/2$; the critical and superdiffusive regimes require different scalings and are briefly discussed at the end.

7.1. Full LDP for the proportion vector

Recall that $Z_n = (Z_{1,n}^{(1)}, Z_{2,n}^{(1)}, Z_{1,n}^{(2)}, Z_{2,n}^{(2)}) \in [0,1]^4$ is the vector of urn proportions. We define the tilted matrix $Q^{(\theta)}$ as follows. Let $W^{(\theta)}$ be the matrix obtained from W in (1) by multiplying the entries corresponding to the probability of drawing a white ball from urn U_1 by e^{θ_1} and the probability of drawing a red ball from urn U_1 by e^{θ_2}, while leaving the draws from urn U_2 untilted. More precisely, the tilted sampling matrix $W^{(\theta)}$ is given by

$$W^{(\theta)} = \begin{pmatrix} \alpha e^{\theta_1} & 0 & (1-\alpha)e^{\theta_1} & 0 \\ 0 & \alpha e^{\theta_2} & 0 & (1-\alpha)e^{\theta_2} \\ (1-\beta) & 0 & \beta & 0 \\ 0 & (1-\beta) & 0 & \beta \end{pmatrix}.$$

Then, we set $Q^{(\theta)} = H \cdot W^{(\theta)}$, where H is the block-diagonal replacement matrix defined in Section 3.

Theorem 7.1. *(Joint LDP for Z_n) In the diffusive regime $\Re(\lambda^*) < 1/2$, the sequence $\{Z_n\}_{n\geq 1}$ satisfies a large deviation principle on $[0,1]^4$ with speed n and good rate function*

$$\tilde{I}(z) = \sup_{\theta\in\mathbb{R}^4} \{\langle \theta, z\rangle - \Lambda(\theta)\},$$

where $\Lambda(\theta) = \log \lambda_{\max}(Q^{(\theta)})$.

Proof. We apply the Gärtner-Ellis theorem [24, Theorem 2.3.6]. To this end, we study the scaled cumulant generating function of Z_n:

$$\Lambda_n(\theta) := \frac{1}{n} \log \mathbb{E}\left[e^{\langle \theta, nZ_n\rangle}\right].$$

By the spectral theory for generalized Friedman urns (see [22] and [14, Theorem 5.1]), the following limit exists for all $\theta \in \mathbb{R}^4$:

$$\lim_{n\to\infty} \Lambda_n(\theta) = \log \lambda_{\max}(Q^{(\theta)}) =: \Lambda(\theta),$$

where $Q^{(\theta)}$ is the tilted matrix defined above. The convergence is uniform on compact subsets of $\mathbb{R}^4$. Moreover, since all entries of $Q^{(\theta)}$ are strictly positive for θ in a neighborhood of 0, the Perron-Frobenius eigenvalue $\lambda_{\max}(Q^{(\theta)})$ is strictly positive, real, and analytic in θ. Consequently, $\Lambda(\theta)$ is finite for all $\theta \in \mathbb{R}^4$, real analytic, and strictly convex. In particular, Λ is essentially smooth: Its gradient maps $\mathbb{R}^4$ onto the interior of the effective domain of its Legendre-Fenchel transform, and $\|\nabla\Lambda(\theta)\| \to \infty$ as $\|\theta\| \to \infty$. The compactness of the state space $[0,1]^4$ and the boundedness of the urn increments ensure that the conditions of the Gärtner-Ellis theorem are satisfied. Therefore, the sequence (Z_n) satisfies an LDP with speed n and a good rate function given by the Legendre-Fenchel transform of Λ:

$$\tilde{I}(z) = \sup_{\theta\in\mathbb{R}^4}\{\langle\theta, z\rangle - \Lambda(\theta)\}.$$

This completes the proof.

7.2. *LDP for the proportion of* $+1$ *steps of elephant A*

Let $X_n = Z^{(1)}_{1,n}$ be the proportion of $+1$ steps of elephant A. By the contraction principle [24, Theorem 4.2.1] applied to the projection $\pi_1(z) = z_1$, we obtain the following result.

Theorem 7.2. *(LDP for X_n) In the diffusive regime $\Re(\lambda^*) < 1/2$, the sequence $\{X_n\}_{n\geq 1}$ satisfies a large deviation principle with speed n and good rate function*

$$I(x) = \inf\{\tilde{I}(z) : z_1 = x\}.$$

Moreover, near the minimum $x = 1/2$, we have

$$I(x) = \frac{(2x-1)^2}{2\sigma^2} + o((2x-1)^2),$$

where σ^2 is the asymptotic variance from Theorem 3.4.

7.3. *LDP for the distance between the two elephants*

We now turn to the distance $D_n = |S^{(1)}_n - S^{(2)}_n|$. Recall that

$$\frac{D_n}{6n} = \left|Z^{(1)}_{1,n} - Z^{(2)}_{1,n}\right| = |z_1 - z_3|.$$

The contraction principle applied to the continuous map $\phi(z) = |z_1 - z_3|$ yields the following rigorous result.

Corollary 7.3. *(LDP for the distance) In the diffusive regime $\Re(\lambda^*) < 1/2$, the sequence $\{D_n/(6n)\}_{n\geq 1}$ satisfies a large deviation principle with speed n and good rate function*

$$J(d) = \inf\left\{\tilde{I}(z) : z \in [0,1]^4,\ |z_1 - z_3| = d\right\}.$$

Equivalently,

$$J(d) = \inf_{x,y\in[0,1],|x-y|=d} \inf\{\tilde{I}(z) : z_1 = x,\ z_3 = y\}.$$

In particular, for any $c > 0$*,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{D_n}{6n} \geq c\right) = -\inf_{d\geq c} J(d).$$

Proof. By Theorem 7.1, the sequence (Z_n) satisfies an LDP on $[0,1]^4$ with speed n and rate function $\tilde{I}$. Consider the continuous function $\phi : [0,1]^4 \to [0,1]$ defined by

$$\phi(z) = |z_1 - z_3|.$$

The contraction principle (see [24, Theorem 4.2.1]) states that if (Z_n) satisfies an LDP with rate function $\tilde{I}$, then for any continuous function ϕ, the sequence $(\phi(Z_n))$ satisfies an LDP with rate function

$$J(y) = \inf\{\tilde{I}(z) : \phi(z) = y\}.$$

Applying this with $\phi(z) = |z_1 - z_3|$ gives

$$J(d) = \inf\{\tilde{I}(z) : z \in [0,1]^4,\ |z_1 - z_3| = d\}.$$

Since

$$\phi(Z_n) = |Z_{1,n}^{(1)} - Z_{1,n}^{(2)}| = \frac{|S_n^{(1)} - S_n^{(2)}|}{6n} = \frac{D_n}{6n},$$

we obtain the desired LDP for the normalized distance. Equivalently, we may write

$$J(d) = \inf_{x,y\in[0,1],|x-y|=d} \inf\{\tilde{I}(z) : z_1 = x,\ z_3 = y\}.$$

This formulation makes explicit that the cost of the distance being d is the minimum over all possible pairs of proportions (x, y) that differ by d. Finally, for any $c > 0$, the LDP upper bound for the closed set $[c, \infty)$ gives

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{D_n}{6n} \geq c\right) \leq -\inf_{d\geq c} J(d),$$

and the lower bound for open sets gives the corresponding liminf inequality. The result follows.

Remark 7.4. In the special case $\alpha = \beta = 0$, the two proportions $Z_{1,n}^{(1)}$ and $Z_{1,n}^{(2)}$ are perfectly correlated (since each elephant's step probability is exactly the other elephant's empirical proportion). In this case, the rate function simplifies to $J(d) = I(d)$ for $d \geq 0$, where I is the rate function for a single ERW. This follows from the fact that $Z_{1,n}^{(1)} = Z_{1,n}^{(2)}$ almost surely when $\alpha = \beta = 0$, which is a direct consequence of the sampling probabilities in Subsection 2.2.

7.4. Remarks on other regimes

In the critical regime $\Re(\lambda^*) = 1/2$, the fluctuations of X_n are of order $\sqrt{\log n/n}$, and the scaled cumulant generating function $\Lambda_n(\theta)$ converges to the same $\Lambda(\theta)$ but with a different normalization: One must replace n by $n/\log n$ as the speed. The Gärtner-Ellis theorem still applies, yielding an LDP with speed $n/\log n$ and the same rate function I (up to a constant factor). In the superdiffusive regime $\Re(\lambda^*) > 1/2$, the typical fluctuations are of order $n^{\Re(\lambda^*)}$ and the large deviation speed is $n^{2(1-\Re(\lambda^*))}$, which is slower than n; the rate function then degenerates and the probabilities decay only polynomially. The analysis of these regimes is more delicate and is left for future work.

8. Discussion and further work

In this paper, we introduced a model of two interacting ERWs, in which each walker evolves according to a convex combination of its own past and that of the other. By exploiting the connection with interacting generalized Friedman's urns, we derived a detailed description of the asymptotic behavior of the system.

Our results highlight the crucial role of the interaction parameters (α, β) in shaping the long-term dynamics. In particular, we showed that the interaction can significantly alter the nature of the limiting behavior, leading to regimes that differ from those observed in the classical (independent) ERW. Notably, the interaction may restore Gaussian fluctuations in parameter regimes where independent ERWs exhibit non-Gaussian behavior.

A central quantity governing the asymptotic behavior is the spectral parameter $\Re(\lambda^*)$, which acts as a phase transition index separating diffusive, critical, and superdiffusive regimes. This provides a unified framework for understanding the interplay between memory effects and interaction.

Despite these advances, several questions remain open and suggest directions for further research:

- **Partial or delayed memory.** Another interesting direction is to consider variants of the model where memory is restricted (e.g., finite memory window) or evolves over time. This could lead to intermediate regimes between Markovian and fully non-Markovian dynamics.
- **Distance-dependent interaction.** One may introduce a threshold γ^* such that the interaction between the two elephants vanishes when their distance exceeds γ^*. This would produce a hybrid model combining interacting and independent phases, whose analysis remains largely open.

Overall, the present work provides a first step toward a systematic understanding of interacting reinforced random walks with long memory. We believe that the combination of reinforcement mechanisms and interaction effects offers a rich field for future investigations.

Author contributions

Mohamed Abdelkader: Conceptualization, Methodology, Software, Formal analysis, Writing–original draft; Rafik Aguech: Conceptualization, Software, Formal analysis, Writing–original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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