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Research article

A unified multi-parameter fractional integral framework for probabilistic inequalities and moment analysis with applications to reliability and financial risk

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Abstract: This paper develops a unified probabilistic framework based on a generalized multi-parameter fractional integral operator. The operator incorporates an auxiliary function, a kernel, a scale parameter, and a fractional order, providing a flexible mechanism for modeling memory-dependent processes. Using this operator, we introduce generalized fractional expectation, variance, and higher-order moments for random variables and establish fundamental inequalities in this setting, including generalized Grüss-, Chebyshev-, Markov-, and Hoeffding-type (concentration) inequalities. The proposed framework unifies many existing fractional models as special cases and extends them through additional parameters and kernel structures. To illustrate applicability, we discuss models arising in reliability analysis and financial risk assessment. These examples demonstrate how generalized fractional operators can be used to quantify uncertainty, volatility, and risk in systems where memory and non-uniform weighting play an important role.

Keywords: fractional expectation; fractional variance; fractional moment; generalized fractional integral operator; integral inequality; random variable
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1. Introduction

Fractional calculus has become an essential analytical toolkit for modeling nonlocal and memory-dependent phenomena across mathematics, physics, engineering, biology, and finance [1]. Classical constructions such as the Riemann-Liouville and Caputo integrals provide rigorous descriptions of hereditary effects, anomalous diffusion, and viscoelasticity [2]. For a modern and comprehensive overview of the theory and applications of fractional calculus, we refer the reader to the recent paper [3]. A large body of work has extended these foundations into many useful variants: For example, k-fractional operators and k-generalizations were developed to introduce additional scaling flexibility [4]. Fractional integrals with respect to another function (ψ or φ-fractional integrals) were introduced to accommodate nonlinear time distortions and change of integration measure [5]. Related recent constructions involving operators with respect to another function further illustrate the importance of such auxiliary-function-based approaches in generalized fractional analysis [6]. Generalized proportional and (ρ, k, ψ)-type proportional operators provide intermediate models between classical and local proportional derivatives [7]. Kernel-based formulations employing non-singular or exponential-type kernels were proposed to model memory with bounded or tempered weighting [8, 9]. These developments both enrich modeling capability and clarify a hierarchical structure in which many well-known fractional models are recovered by specific parameter, kernel, or auxiliary-function choices.

Parallel to operator advances, researchers have recently developed probabilistic versions of fractional integration by defining fractional expectations, variances, and moments and proving operator-specific inequalities [10]. This line of work produced fractional Grüss-, Chebyshev-, Markov-, and Hoeffding-type bounds for random variables under particular fractional operators and parameter cases [11, 12]. These investigations have also been connected with several generalized fractional frameworks, including ψ-Riemann-Liouville fractional integrals [13] and fractional operators involving exponential kernels [14]. In parallel, recent studies on fractional integral inequalities, including Hermite-Hadamard-type inequalities in generalized operator settings, confirm the continuing interest in extending classical inequalities beyond the standard integral framework [15]. However, most existing probabilistic results have been proved inside narrowly specified operator frameworks (often single-parameter or proportional families), so they do not transfer automatically across other fractional structures [16, 17]. Consequently, the literature lacks a unified operator-level probabilistic theory that simultaneously accommodates multi-parameter, φ-distorted, and kernel-flexible fractional integrals; in other words, operator generality and probabilistic inequalities have not been integrated in a single coherent framework. The challenge of developing a unifying perspective that encapsulates these disparate fractional operators has been highlighted in recent review articles [18].

Accurate uncertainty quantification for processes with memory requires bounds that simultaneously capture dispersion, dependence structure, and the nonlocal persistence effects intrinsic to long-range interactions. In contrast to classical Markovian settings, where independence or short-range dependence allows direct application of standard concentration inequalities, memory-driven systems exhibit temporal correlations that fundamentally alter variance growth, deviation behavior, and risk amplification [19, 20]. Consequently, refined probabilistic tools are necessary to control fluctuations and quantify uncertainty in such nonlocal environments.

Theoretical foundations for handling long-range dependence and persistent stochastic correlations have been developed extensively in the literature on weakly dependent sequences and fractional stochastic processes. In particular, the structural analysis of long-memory phenomena and their impact on covariance behavior and variance scaling has been systematically studied in [21]. Complementary developments on deviation inequalities and limit theorems for dependent sequences provide rigorous mechanisms for bounding stochastic fluctuations in the absence of independence assumptions [22]. Moreover, stochastic systems driven by fractional Brownian motion and related non-Markovian noises further demonstrate how memory modifies dispersion properties and risk profiles, requiring generalized integral and variance operators to properly quantify uncertainty [23]. The need for generalized expectation and risk measures in such non-Markovian environments has motivated the development of fractional stochastic volatility models and associated estimation techniques [24, 25].

These developments collectively emphasize that classical expectation and variance operators may be insufficient in memory-sensitive contexts. Instead, generalized operators incorporating kernel-based weighting or fractional integration naturally arise as mathematically consistent tools for modeling persistent dependence. The framework developed in the present work aligns with this perspective by extending classical probabilistic inequalities into a generalized fractional setting, thereby providing dispersion bounds that remain valid under nonlocal and memory-influenced dynamics.

Motivated by these needs, the objective of this paper is to introduce and analyze a highly generalized multi-parameter fractional integral operator ${}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}$ that depends simultaneously on an auxiliary function φ, a kernel g, the scale parameter k, and the fractional order ϑ, and to build a systematic probabilistic inequality and moment theory based on this operator. Concretely, we define generalized fractional expectation, variance, and higher moments via this operator, derive variance identities and Grüss-type, Chebyshev-type, Markov-type, and Hoeffding-type inequalities under minimal boundedness hypotheses, and demonstrate how classical and recent operator-specific results arise as immediate special cases.

The inequalities established below should be understood as operator-level extensions and unifications of known Grüss-, Chebyshev-, Markov-, and Hoeffding-type inequalities. Similar inequalities have been obtained previously in more restrictive fractional settings. The novelty of the present work is not that each inequality is new in its classical form, but rather that these inequalities are derived systematically within a single multi-parameter framework depending simultaneously on the auxiliary function φ, the kernel g, the scale parameter k, and the fractional order ϑ. This unified setting makes it possible to recover several existing fractional probabilistic inequalities as special cases and, at the same time, to incorporate more general memory kernels and nonlinear time distortions.

A central contribution of this paper lies in the broader range of applications enabled by the generalized operator. In reliability theory, we introduce generalized fractional mean lifetime and dispersion measures and derive Grüss-type bounds that quantify uncertainty under heterogeneous aging. In finance, we define fractional expected return and volatility under non-proportional memory and nonlinear market time scaling, providing flexible volatility bounds for long-memory markets. These examples show how kernel shape and φ distortions materially affect probabilistic moments and tail estimates, and thus why operator generality matters in applied modeling.

The key contributions of the paper are fourfold:

- Construction of a comprehensive probabilistic framework including fractional expectation, variance, and moments.

- Establishment of generalized Grüss-type inequalities under multi-parameter settings.
- Recovery of numerous existing results, including proportional fractional inequalities, as special cases.
- Development of broader application perspectives in reliability and finance.

The remainder of the paper is organized as follows. Section 2 introduces preliminary definitions and properties of the generalized operator. Section 3 develops the probabilistic fractional framework, including expectation, variance, moment identities, and related probabilistic inequalities. Section 4 presents numerical verification, parameter sensitivity analysis, and applications to reliability and financial risk modeling. Finally, Section 5 concludes the paper and outlines directions for future research.

2. Preliminaries

The generalized fractional integral operators, definitions, and introductory facts are introduced in this section, which will be used throughout this work.

Let $[a, b]$ be an interval with $0 \leq a < b$, and let $C^n([a, b], \mathbb{R})$ denote the space of n-times continuously differentiable real-valued functions on $[a, b]$. To simplify the notation, we define

$$S_+^1([a, b], \mathbb{R}) = \left\{\varphi \in C^1([a, b], \mathbb{R}) : \varphi(\xi) > 0 \text{ and } \varphi'(\xi) > 0 \text{ for all } \xi \in [a, b]\right\}.$$

For $a \leq s < \xi \leq b$, we set

$$\Psi(\xi, s) = \varphi(\xi) - \varphi(s).$$

Since φ is strictly increasing, $\Psi(\xi, s) > 0$ on this domain. Therefore, for any real exponent $\alpha \in \mathbb{R}$, we define

$$\Psi^{\alpha}(\xi, s) = \left(\varphi(\xi) - \varphi(s)\right)^{\alpha}, \qquad a \leq s < \xi \leq b.$$

In particular, the power $\Psi^{\frac{\vartheta}{k}-1}(\xi, s)$ appearing in the generalized fractional integral is well defined even when $\frac{\vartheta}{k} - 1 < 0$.

Definition 2.1. [26] *For $\theta, k > 0$, the k-gamma function given by*

$$\Gamma_k(\theta) = \int_0^{\infty} \xi^{\theta-1} e^{-\frac{\xi^k}{k}} d\xi,$$

one can easily observe the following standard relations:

$$\Gamma_k(\theta) = k^{\frac{\theta}{k}-1}\Gamma\left(\frac{\theta}{k}\right), \;\; \Gamma_k(\theta + k) = \theta\Gamma_k(\theta), \;\; \Gamma_k(k) = \Gamma(1) = 1.$$

Definition 2.2. [27] *For $1 \leq p < \infty$, we denote by $L^p([0, \infty), \mathbb{R})$ the set of all Lebesgue measurable functions $\mathfrak{z}$ such that*

$$\|\mathfrak{z}\|_p = \left(\int_0^{\infty} |\mathfrak{z}(s)|^p ds\right)^{\frac{1}{p}} < \infty.$$

Definition 2.3. [28] *Let f be integrable on $[a,b]$. The left- and right-sided generalized fractional integral operators are defined by*

$$\mathcal{I}^{g}_{a^+}f(\xi) = \int_a^{\xi} \frac{g(\xi - s)}{\xi - s} f(s)ds, \quad \xi > a,$$

and

$$\mathcal{I}^{g}_{b^-}f(\xi) = \int_{\xi}^{b} \frac{g(s - \xi)}{s - \xi} f(s)ds, \quad \xi < b,$$

where a function $g : [0,\infty) \to [0,\infty)$ satisfies the following conditions:

$$\int_0^1 \frac{g(s)}{s} ds < \infty, \tag{2.1}$$

$$\frac{1}{T_1} \leq \frac{g(\theta)}{g(\zeta)} \leq T_1, \quad \textit{for } \frac{1}{2} \leq \frac{\theta}{\zeta} \leq 2, \tag{2.2}$$

$$\frac{g(\zeta)}{\zeta^2} \leq T_2 \frac{g(\theta)}{\theta^2}, \quad \textit{for } \theta \leq \zeta, \tag{2.3}$$

and

$$\left| \frac{g(\zeta)}{\zeta^2} - \frac{g(\theta)}{\theta^2} \right| \leq T_3 |\zeta - \theta| \frac{g(\zeta)}{\zeta^2}, \quad \textit{for } \frac{1}{2} \leq \frac{\theta}{\zeta} \leq 2, \tag{2.4}$$

where $T_1, T_2, T_3 > 0$ are independent of $\theta, \zeta > 0$.

Example 2.1. If $g(\xi)\xi^{\alpha}$ $(\alpha > 0)$ is increasing and $\frac{g(\xi)}{\xi^{\beta}}$, $(\beta > 0)$ is decreasing, then g satisfies (2.1) to (2.4).

Definition 2.4. [28] *Let $k > 0$, $\varphi \in \mathbb{S}^1_+([a,b],\mathbb{R})$, the function g satisfy (2.1) to (2.4), and $f : [a,b] \to \mathbb{R}$ be a positive and integrable function. The left-sided fractional integral of f with respect to φ of order $\vartheta > 0$ is defined by*

$${}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+} f(\xi) = \frac{1}{k\Gamma_k(\vartheta)} \int_a^{\xi} \frac{g(\Psi(\xi,s))}{\Psi(\xi,s)} \Psi^{\frac{\vartheta}{k}-1}(\xi,s)\varphi'(s)f(s)ds, \quad \xi > a.$$

First, we give the fractional expectation ($\mathbb{FE}$) function of $\mathbb{K}$.

Definition 2.5. *[12] Assume that $\mathbb{K}$ is a random variable with a positive probability density function ($\mathbb{PDF}$) $\mathfrak{f}$ defined on $[a,b]$, and $\varphi, \chi \in \mathbb{S}+^1([a,b],\mathbb{R})$. Then the $\mathbb{FE}$ function ${}^{g}\mathbb{EK}\vartheta(\xi)$ of order $\vartheta \geq 0$ is given by*

$$\begin{aligned} {}^{g}\mathbb{EK}\vartheta(\xi) &= k^{\varphi}\mathcal{I}a^{+\vartheta;g}\left[\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)\mathfrak{f}(\xi)\right] \\ &= \frac{1}{k\Gamma k(\vartheta)} \int_a^{\xi} \frac{g(\Psi(\xi,s))}{\Psi(\xi,s)} \Psi^{\frac{\vartheta}{k}-1}(\xi,s) \frac{g(\varphi(s))}{\varphi(s)} \chi(s)\mathfrak{f}(s)\varphi'(s)ds, \end{aligned} \tag{2.5}$$

where the expectation is given by

$${}^{g}\mathbb{E}(\mathbb{K}) = {}^{g}\mathbb{EK}k(b) = \frac{1}{k} \int a^b \frac{g(\Psi(b,s))}{\Psi(b,s)} \frac{g(\varphi(s))}{\varphi(s)} \chi(s)\mathfrak{f}(s), \varphi'(s)ds. \tag{2.6}$$

Next, we introduce the $\mathbb{FE}$ function of $\mathbb{K} - {}^{g}\mathbb{E}(\mathbb{K})$.

Definition 2.6. *[12] Let $\varphi, \chi \in \mathbb{S}+^{1}([a,b],\mathbb{R})$, and let $\mathfrak{f} : [a,b] \to \mathbb{R}+$ be the $\mathbb{PDF}$ of the random variable $\mathbb{K}$. The generalized $\mathbb{FE}$ function of order $\vartheta \geq 0$ for the centered random variable $\mathbb{K} - {}^{g}\mathbb{E}(\mathbb{K})$ is defined by*

$$\begin{aligned} {}^{g}\mathbb{E}\mathbb{K} - {}^{g}\mathbb{E}(\mathbb{K})\vartheta(\xi) &= k^{\varphi}\mathcal{I}a^{+\vartheta;g}\left[\Big(\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi) - {}^{g}\mathbb{E}(\mathbb{K})\Big)\mathfrak{f}(\xi)\right] \\ &= \frac{1}{k\Gamma_k(\vartheta)}\int_a^{\xi}\frac{g(\Psi(\xi,s))}{\Psi(\xi,s)}\Psi^{\frac{\vartheta}{k}-1}(\xi,s)\Big(\frac{g(\varphi(s))}{\varphi(s)}\chi(s) - {}^{g}\mathbb{E}(\mathbb{K})\Big)\mathfrak{f}(s)\varphi'(s)ds. \end{aligned} \tag{2.7}$$

If $\xi = b$, then we present the following definition.

Definition 2.7. [28] *Let $\varphi, \chi \in \mathbb{S}_+^1([a,b],\mathbb{R})$, and let $\mathfrak{f} : (a,b) \to \mathbb{R}_+$ be the positive $\mathbb{PDF}$ of the random variable $\mathbb{K}$. The generalized $\mathbb{FE}$ function of order $\vartheta \geq 0$ evaluated at $\xi = b$ is defined by*

$${}^{g}\mathbb{E}_{\mathbb{K},\vartheta}(b) = \frac{1}{k\Gamma_k(\vartheta)}\int_a^{b}\frac{g(\Psi(b,s))}{\Psi(b,s)}\Psi^{\frac{\vartheta}{k}-1}(b,s)\frac{g(\varphi(s))}{\varphi(s)}\chi(s)\mathfrak{f}(s)\varphi'(s)\,ds. \tag{2.8}$$

For the generalized $\mathbb{FE}$ function of $\mathbb{K}$, we introduce the following definition.

Definition 2.8. [10] *Let $\mathbb{K}$ be a random variable with a positive $\mathbb{PDF}$ $\mathfrak{f}$ defined on (a,b), and assume that $\varphi, \chi \in \mathbb{S}_+^1([a,b],\mathbb{R})$. Then, the generalized $\mathbb{FE}$ function ${}^{g}\sigma^2_{\mathbb{K},\vartheta}$ of order $\vartheta \geq 0$ is defined by*

$$\begin{aligned} {}^{g}\sigma^2_{\mathbb{K},\vartheta}(\xi) &= {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}\left[\left(\tfrac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi) - {}^{g}\mathbb{E}(\mathbb{K})\right)^2\mathfrak{f}(\xi)\right] \\ &= \tfrac{1}{k\Gamma_k(\vartheta)}\int_a^{\xi}\tfrac{g(\Psi(\xi,s))}{\Psi(\xi,s)}\Psi^{\frac{\vartheta}{k}-1}(\xi,s)\left(\tfrac{g(\varphi(s))}{\varphi(s)}\chi(s) - {}^{g}\mathbb{E}(\mathbb{K})\right)^2\mathfrak{f}(s)\varphi'(s)ds. \end{aligned} \tag{2.9}$$

If $\xi = b$, we obtain the following particular case.

Definition 2.9. [28] *Under the same assumptions as in Definition 2.8, the generalized $\mathbb{FE}$ function of order $\vartheta \geq 0$ at the endpoint b is defined by*

$${}^{g}\sigma^2_{\mathbb{K},\vartheta}(b) = \tfrac{1}{k\Gamma_k(\vartheta)}\int_a^{b}\tfrac{g(\Psi(b,s))}{\Psi(b,s)}\Psi^{\frac{\vartheta}{k}-1}(b,s)\left(\tfrac{g(\varphi(s))}{\varphi(s)}\chi(s) - {}^{g}\mathbb{E}(\mathbb{K})\right)^2\mathfrak{f}(s)\varphi'(s)ds. \tag{2.10}$$

Definition 2.10. [12] *Let $\mathbb{K}$ be a random variable with a positive $\mathbb{PDF}$ $\mathfrak{f}$ defined on (a,b), and assume that $\varphi, \chi \in \mathbb{S}_+^1([a,b],\mathbb{R})$. Then, the generalized fractional moment $(\mathbb{FM})$ function of order (r,ϑ), with $r > 0$ and $\vartheta \geq 0$, is defined by*

$$\begin{aligned} {}^{g}\mathbb{M}_{\mathbb{K}^r,\vartheta}(\xi) &= {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}\left[\left(\tfrac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)\right)^r\mathfrak{f}(\xi)\right] \\ &= \tfrac{1}{k\Gamma_k(\vartheta)}\int_a^{\xi}\tfrac{g(\Psi(\xi,s))}{\Psi(\xi,s)}\Psi^{\frac{\vartheta}{k}-1}(\xi,s)\left(\tfrac{g(\varphi(s))}{\varphi(s)}\chi(s)\right)^r\mathfrak{f}(s)\varphi'(s)ds. \end{aligned} \tag{2.11}$$

Lemma 2.1. *Let $\mathcal{G}$ be an integrable function on $[a,b]$ with constants $M_0 < M_1$ such that $M_0 < \mathcal{G}(\xi) < M_1$ for all $\xi \in [a,b]$, and let ϖ be a positive function on $[a,b]$. Then for all $\xi > 0$ and $k, \vartheta > 0$, the following identity holds:*

$$\begin{aligned} &\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi]\right)(\xi)\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{G}^2]\right)(\xi) - \left[\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{G}]\right)(\xi)\right]^2 \\ =\; &\left[M_1\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi]\right)(\xi) - \left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{G}]\right)(\xi)\right]\left[\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{G}]\right)(\xi) - M_0\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi]\right)(\xi)\right] \\ &-\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi]\right)(\xi)\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[(M_1-\mathcal{G})(\mathcal{G}-M_0)\varpi]\right)(\xi). \end{aligned} \tag{2.12}$$

Proof. For brevity, define the linear operator

$$L_\xi[h] := {}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[h](\xi).$$

Set $A = L_\xi[\varpi]$, $B = L_\xi[\varpi\mathcal{G}]$, and $C = L_\xi[\varpi\mathcal{G}^2]$. By the linearity of L_ξ, we have

$$\begin{aligned} L_\xi[(M_1-\mathcal{G})(\mathcal{G}-M_0)\varpi] &= L_\xi\Big[((M_1+M_0)\mathcal{G}-\mathcal{G}^2-M_0M_1)\varpi\Big] \\ &= (M_1+M_0)B - C - M_0M_1A. \end{aligned}$$

Therefore,

$$\begin{aligned} &[M_1A-B][B-M_0A] - A\,L_\xi[(M_1-\mathcal{G})(\mathcal{G}-M_0)\varpi] \\ &= M_1AB - M_0M_1A^2 - B^2 + M_0AB - A\left((M_1+M_0)B - C - M_0M_1A\right) \\ &= AC - B^2. \end{aligned}$$

Returning to the notation of the generalized fractional integral, this gives

$$\begin{aligned} &\left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi]\right)\left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{G}^2]\right) - \left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{G}]\right)^2 \\ &= \left[M_1\left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi]\right) - \left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{G}]\right)\right]\left[\left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{G}]\right) - M_0\left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi]\right)\right] \\ &\quad - \left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi]\right)\left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[(M_1-\mathcal{G})(\mathcal{G}-M_0)\varpi]\right), \end{aligned}$$

which is exactly (2.12). The proof is complete.

The next theorem presents a Grüss-type inequality formulated via the generalized fractional integral.

Theorem 2.1. *Let $\mathcal{U}$ and $\mathcal{W}$ be two integrable functions on $[a,b]$ with $M_0 < \mathcal{U}(\xi) < M_1$, $N_0 < \mathcal{W}(\xi) < N_1$, and let ϖ be a positive function on $[a,b]$. Then for all $\xi > 0$, $k, \vartheta > 0$, we have*

$$\begin{aligned} &\left|\left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)\left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}\mathcal{W}(\xi)]\right) - \left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}(\xi)]\right)\left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{W}(\xi)]\right)\right| \\ &\leq \frac{1}{4}(M_1-M_0)(N_1-N_0)\left({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)^2. \end{aligned} \tag{2.13}$$

Proof. Define the quantity

$$\mathfrak{N}(\mathfrak{u},\mathfrak{r}) = (\mathcal{U}(\mathfrak{u}) - \mathcal{U}(\mathfrak{r}))(\mathcal{W}(\mathfrak{u}) - \mathcal{W}(\mathfrak{r})),\ \mathfrak{u},\mathfrak{r} \in (a,\xi),\ a < \xi \leq b. \tag{2.14}$$

Multiplying (2.14) by $\frac{\varphi'(s)}{k\Gamma_k(\vartheta)}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})\varpi(\mathfrak{r})$; $\mathfrak{r} \in (a,\xi)$ and integrating the resulting identity with respect to $\mathfrak{r}$ over (a,ξ), we can state that

$$\begin{aligned} &\int_a^\xi \frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})}{k\Gamma_k(\vartheta)}\varphi'(\mathfrak{r})\mathfrak{N}(\mathfrak{u},\mathfrak{r})d\mathfrak{r} \\ &= ({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}\mathcal{W}(\xi)]) - \mathcal{W}(\mathfrak{u})({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}(\xi)]) - \mathcal{U}(\xi)\mathcal{W}(\mathfrak{u})({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{W}(\mathfrak{r})]) \\ &\quad + \mathcal{U}(\xi)\mathcal{W}(\xi)\mathcal{W}(\mathfrak{u})({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\mathfrak{r})]). \end{aligned} \tag{2.15}$$

It follows that

$$\begin{aligned} &\int_a^\xi\int_a^\xi \frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})}{k_k^\Gamma(\vartheta)}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})}{k_k^\Gamma(\vartheta)}\varphi'(\mathfrak{r})\varphi'(\mathfrak{u})\mathfrak{N}(\mathfrak{u},\mathfrak{r})d\mathfrak{r}d\mathfrak{u} \\ &= 2({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)])({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}\mathcal{W}(\xi)]) - 2({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}(\xi)])({}_k^\varphi\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{W}(\xi)]). \end{aligned} \tag{2.16}$$

Using the weighted Cauchy-Schwarz inequality for double integrals, we obtain

$$\begin{aligned}&\left(\int_a^{\xi}\int_a^{\xi}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})}{k_k^{\Gamma}(\vartheta)}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})}{k_k^{\Gamma}(\vartheta)}\varphi'(\mathfrak{r})\varphi'(\mathfrak{u})\mathfrak{R}(\mathfrak{u},\mathfrak{r})d\mathfrak{r}d\mathfrak{u}\right)^2\\ &\leq\int_a^{\xi}\int_a^{\xi}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})}{k_k^{\Gamma}(\vartheta)}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})}{k_k^{\Gamma}(\vartheta)}\varphi'(\mathfrak{r})\varphi'(\mathfrak{u})(\mathcal{U}(\mathfrak{u})-\mathcal{U}(\mathfrak{u}))^2d\mathfrak{r}d\mathfrak{u}\\ &\times\int_a^{\xi}\int_a^{\xi}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})}{k_k^{\Gamma}(\vartheta)}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})}{k_k^{\Gamma}(\vartheta)}\varphi'(\mathfrak{r})\varphi'(\mathfrak{u})(\mathcal{W}(\mathfrak{u})-\mathcal{W}(\mathfrak{u}))^2d\mathfrak{r}d\mathfrak{u}.\end{aligned}\tag{2.17}$$

The right-hand side of (2.17) can be expressed as follows:

$$\begin{aligned}&\int_a^{\xi}\int_a^{\xi}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})}{k_k^{\Gamma}(\vartheta)}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})}{k_k^{\Gamma}(\vartheta)}\varphi'(\mathfrak{r})\varphi'(\mathfrak{u})(\mathcal{U}(\mathfrak{u})-\mathcal{U}(\mathfrak{u}))^2d\mathfrak{r}d\mathfrak{u}\\ &=2({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)])({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{U}^2(\xi)])-2({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{U}(\xi)])^2\end{aligned}\tag{2.18}$$

and

$$\begin{aligned}&\int_a^{\xi}\int_a^{\xi}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})}{k_k^{\Gamma}(\vartheta)}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})}{k_k^{\Gamma}(\vartheta)}\varphi'(\mathfrak{r})\varphi'(\mathfrak{u})(\mathcal{W}(\mathfrak{u})-\mathcal{W}(\mathfrak{u}))^2d\mathfrak{r}d\mathfrak{u}\\ &=2({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)])({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{W}^2(\xi)])-2({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{W}(\xi)])^2.\end{aligned}\tag{2.19}$$

Using (2.16), (2.18), and (2.19), Eq (2.17) can be written as

$$\begin{aligned}&\left(({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)])({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{U}\mathcal{W}(\xi)])-({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{U}(\xi)])({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{W}(\xi)])\right)^2\\ &\leq\left[({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)])({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{U}^2(\xi)])-({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{U}(\xi)])^2\right]\\ &\times\left[({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)])({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{W}^2(\xi)])-({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{W}(\xi)])^2\right].\end{aligned}\tag{2.20}$$

By Lemma 2.1, with $\mathcal{G}=\mathcal{U}$ and $\mathcal{G}=\mathcal{W}$ applied to the right-hand side of (2.20), we obtain

$$\begin{aligned}&\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)]\right)\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}\left[\varpi\mathcal{U}^2(\xi)\right]\right)-\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{U}(\xi)]\right)^2\\ &=\left[M_1\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)]\right)-\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{U}(\xi)]\right)\right]\left[\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{U}(\xi)]\right)-M_0\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)]\right)\right]\\ &-\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)]\right)\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[(M_1-\mathcal{U}(\xi))(\mathcal{U}(\xi)-M_0)\varpi(\xi)]\right),\end{aligned}\tag{2.21}$$

and

$$\begin{aligned}&\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)]\right)\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}\left[\varpi\mathcal{W}^2(\xi)\right]\right)-\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{W}(\xi)]\right)^2\\ &=\left[N_1\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)]\right)-\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{W}(\xi)]\right)\right]\left[\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi\mathcal{W}(\xi)]\right)-N_0\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)]\right)\right]\\ &-\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)]\right)\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[(N_1-\mathcal{W}(\xi))(\mathcal{W}(\xi)-N_0)\varpi(\xi)]\right).\end{aligned}\tag{2.22}$$

On the other hand, since

$$\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\varpi(\xi)]\right)\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[(M_1-\mathcal{U}(\xi))(\mathcal{U}(\xi)-M_0)\varpi(\xi)]\right)\geq 0,$$

and

$$\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[(N_1 - \mathcal{W}(\xi))(\mathcal{W}(\xi) - N_0)\varpi(\xi)]\right) \geq 0,$$

then the following inequalities hold:

$$\begin{aligned}&\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\varpi\mathcal{U}^2(\xi)\right]\right) - \left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}(\xi)]\right)^2\\ \leq{}& \left[M_1\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right) - \left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}(\xi)]\right)\right]\left[\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}(\xi)]\right) - M_0\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)\right],\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}&\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\varpi\mathcal{W}^2(\xi)\right]\right) - \left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{W}(\xi)]\right)^2\\ \leq{}& \left[N_1\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right) - \left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{W}(\xi)]\right)\right]\left[\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{W}(\xi)]\right) - N_0\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)\right].\end{aligned} \tag{2.24}$$

By taking into account (2.20), (2.23), and (2.24), we get the following inequality:

$$\begin{aligned}&\left(({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)])({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}\mathcal{W}(\xi)]) - ({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}(\xi)])({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{W}(\xi)])\right)^2\\ \leq{}& \left[M_1\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right) - \left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}(\xi)]\right)\right]\left[\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}(\xi)]\right) - M_0\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)\right]\\ &\times\left[N_1\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right) - \left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{W}(\xi)]\right)\right]\left[\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{W}(\xi)]\right) - N_0\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)\right].\end{aligned} \tag{2.25}$$

Since for all $A, B \in \mathbb{R}$, we have $4AB \leq (A + B)^2$, it follows that

$$\begin{aligned}&4\left[M_1\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right) - \left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}(\xi)]\right)\right]\left[\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{U}(\xi)]\right) - M_0\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)\right]\\ \leq{}& \left((M_1 - M_0){}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)^2,\end{aligned} \tag{2.26}$$

and

$$\begin{aligned}&4\left[N_1\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right) - \left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{W}(\xi)]\right)\right]\left[\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{W}(\xi)]\right) - N_0\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)\right]\\ \leq{}& \left((N_1 - N_0){}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)^2.\end{aligned} \tag{2.27}$$

Combining (2.25)–(2.27) and simplifying yields (2.13). The proof is complete.

The generalized fractional integral operator ${}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}$ possesses a flexible multi-parameter structure depending on the scale parameter k, the auxiliary function φ, the kernel g, the fractional order ϑ, and the weighting function χ. Through suitable parameter selections, numerous classical fractional operators and associated probabilistic formulations are recovered as particular cases. Table 1 summarizes these reductions in a unified and systematic manner.

Table 1. Special parameter configurations recovering classical operators and probabilistic formulations.

k	$\varphi(\xi)$	$\chi(\xi)$	$g(\xi)$	ϑ	**Recovered result**
1	ξ	ξ	ξ	general	Classical Riemann-Liouville operator
k	ξ	ξ	ξ	general	k-Riemann-Liouville operator
1	general φ	$\varphi(\xi)$	ξ	general	φ-Riemann-Liouville operator
1	ξ	ξ	$\xi^{\rho\vartheta}e^{\frac{\rho-1}{\rho}\xi}$	general	Proportional fractional operator ($\rho \in (0,1]$)
1	general φ	$\varphi(\xi)$	$\xi^{\rho\vartheta}e^{\frac{\rho-1}{\rho}\xi}$	general	φ-proportional fractional operator
$k=\vartheta=1$	ξ	ξ	ξ	1	Classical expectation, variance, and moments
1	ξ	ξ	ξ	general	Definitions 2.2–2.6 in [12]
1	general φ	$\varphi(\xi)$	ξ	general	Definitions 2.5–2.9 in [29]
1	general φ	$\varphi(\xi)$	$\xi^{\rho\vartheta}e^{\frac{\rho-1}{\rho}\xi}$	general	Definitions 4–9 in [30]
general	ξ	arbitrary	1	general	Grüss inequality (Theorem 3 in [7])
1	general	general	general	general	Grüss inequality (Theorem 3.1 in [11])

3. Main result

In this section, we develop the generalized fractional probabilistic framework associated with the operator ${}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}$. Throughout this section, K denotes an ordinary continuous random variable with probability density function $f : [a,b] \to \mathbb{R}_+$. The term "fractional" is used only to describe the generalized operator through which expectation, variance, moments, and probability-type quantities are evaluated. Thus, a fractional random variable is not a new kind of random variable, it rather refers to a classical random variable analyzed under a fractional integral weighting induced by the parameters ϑ, k, φ, and g.

Lemma 3.1. *Let $\mathbb{K}$ be a continuous random variable ($\mathbb{CRV}$) with $\mathbb{PDF}$ $\mathfrak{f} : [a,b] \to \mathbb{R}_+$. Then*

$$ {}^{g}\sigma^2_{\mathbb{K},\vartheta}(b) = {}^{g}\mathbb{M}_{\mathbb{K}^2,\vartheta}(b) - 2\,{}^{g}\mathbb{E}(\mathbb{K})\,{}^{g}\mathbb{E}_{\mathbb{K},\vartheta}(b) + ({}^{g}\mathbb{E}(\mathbb{K}))^2\, {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{f}(b)],\ \forall \vartheta \geq 0. \tag{3.1}$$

Proof. By Definition 2.9, we have

$$\begin{aligned}
{}^{g}\sigma^2_{\mathbb{K},\vartheta}(b) &= \frac{1}{k\Gamma_k(\vartheta)}\int_a^b \frac{g(\Psi(b,s))}{\Psi(b,s)}\Psi^{\frac{\vartheta}{k}-1}(b,s)\left(\frac{g(\varphi(s))}{\varphi(s)}\chi(s) - {}^{g}\mathbb{E}(\mathbb{K})\right)^2 \mathfrak{f}(s)\varphi'(s)ds \\
&= \frac{1}{k\Gamma_k(\vartheta)}\int_a^b \frac{g(\Psi(b,s))}{\Psi(b,s)}\Psi^{\frac{\vartheta}{k}-1}(b,s)\left[\left(\frac{g(\varphi(s))}{\varphi(s)}\chi(s)\right)^2 - 2\frac{g(\varphi(s))}{\varphi(s)}\chi(s)\,{}^{g}\mathbb{E}(\mathbb{K}) + ({}^{g}\mathbb{E}(\mathbb{K}))^2\right]\mathfrak{f}(s)\varphi'(s)ds \\
&= \frac{1}{k\Gamma_k(\vartheta)}\int_a^b \frac{g(\Psi(b,s))}{\Psi(b,s)}\Psi^{\frac{\vartheta}{k}-1}(b,s)\left(\frac{g(\varphi(s))}{\varphi(s)}\chi(s)\right)^2 \mathfrak{f}(s)\varphi'(s)ds \\
&\quad - \frac{2\,{}^{g}\mathbb{E}(\mathbb{K})}{k\Gamma_k(\vartheta)}\int_a^b \frac{g(\Psi(b,s))}{\Psi(b,s)}\Psi^{\frac{\vartheta}{k}-1}(b,s)\frac{g(\varphi(s))}{\varphi(s)}\chi(s)\mathfrak{f}(s)\varphi'(s)ds \\
&\quad + \frac{({}^{g}\mathbb{E}(\mathbb{K}))^2}{k\Gamma_k(\vartheta)}\int_a^b \frac{g(\Psi(b,s))}{\Psi(b,s)}\Psi^{\frac{\vartheta}{k}-1}(b,s)\mathfrak{f}(s)\varphi'(s)ds.
\end{aligned}$$

Therefore,

$$ {}^{g}\sigma^{2}_{\mathbb{K},\vartheta}(b) = {}^{g}\mathbb{M}_{\mathbb{K}^2,\vartheta}(b) - 2{}^{g}\mathbb{E}(\mathbb{K}){}^{g}\mathbb{E}_{\mathbb{K},\vartheta}(b) + ({}^{g}\mathbb{E}(\mathbb{K}))^2\, {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{f}(b)]. $$

Theorem 3.1. *Let $\mathbb{K}$ be a $\mathbb{CRV}$ with $\mathbb{PDF}$ $\mathfrak{f}:[a,b]\to\mathbb{R}_+$. Then the following conditions hold:*

(1) For any $\vartheta\geq 0$ and $\xi\in(a,b]$, one has

$$ \begin{aligned} &{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{f}(\xi)]{}^{g}\sigma^{2}_{\mathbb{K},\vartheta}(\xi) - \left[{}^{g}\mathbb{E}_{\mathbb{K}-{}^{g}\mathbb{E}(\mathbb{K}),\vartheta}(\xi)\right]^2 \\ \leq\ & \|\mathfrak{f}\|_{L^\infty}\left[2\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}1\right){}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}\left[\left(\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)\right)^2\right] - 2\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}\left[\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)\right]\right)^2\right], \end{aligned} \tag{3.2} $$

provided that $\mathfrak{f}\in L^\infty([a,b])$.

(2) The integral inequality

$$ {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{f}(\xi)]{}^{g}\sigma^{2}_{\mathbb{K},\vartheta}(\xi) - \left[{}^{g}\mathbb{E}_{\mathbb{K}-{}^{g}\mathbb{E}(\mathbb{K}),\vartheta}(\xi)\right]^2 \leq \sup_{\mathfrak{u},\mathfrak{r}\in(a,\xi)}\left|\frac{g(\varphi(\mathfrak{u}))}{\varphi(\mathfrak{u})}\chi(\mathfrak{u}) - \frac{g(\varphi(\mathfrak{r}))}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right|^2\left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}\mathfrak{f}(\xi)\right)^2 \tag{3.3} $$

is true for all $\vartheta\geq 0$ and $\xi\in(a,b]$.

Proof. Assume the probability density functions are denoted by $\mathcal{G}$ and $\mathcal{W}$. For $\xi\in(a,b]$ and $\mathfrak{u},\mathfrak{r}\in(a,\xi)$, we have

$$ \begin{aligned} \mathfrak{M}(\mathfrak{u},\mathfrak{r}) &= (\mathcal{G}(\mathfrak{u})-\mathcal{G}(\mathfrak{r}))(\mathcal{W}(\mathfrak{u})-\mathcal{W}(\mathfrak{r})) \\ &= \mathcal{G}(\mathfrak{u})\mathcal{W}(\mathfrak{u}) + \mathcal{G}(\mathfrak{r})\mathcal{W}(\mathfrak{r}) - \mathcal{G}(\mathfrak{u})\mathcal{W}(\mathfrak{r}) - \mathcal{G}(\mathfrak{r})\mathcal{W}(\mathfrak{u}). \end{aligned} \tag{3.4} $$

Inserting a function $\mathfrak{P}:[a,b]\to\mathbb{R}_+$, multiplying both sides of (3.4) by

$$ \frac{1}{k\Gamma_k(\vartheta)}\frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})\varphi'(\mathfrak{u})\mathfrak{P}(\mathfrak{u}),\quad \mathfrak{u}\in(a,\xi), $$

and integrating with respect to $\mathfrak{u}$ from (a,ξ), we have

$$ \begin{aligned} &\frac{1}{k\Gamma_k(\vartheta)}\int_a^\xi \frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})\varphi'(\mathfrak{u})\mathfrak{P}(\mathfrak{u})\mathfrak{M}(\mathfrak{u},\mathfrak{r})d\mathfrak{u} \\ =\ & {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{G}(\xi)\mathcal{W}(\xi)] + {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)]\mathcal{G}(\mathfrak{r})\mathcal{W}(\mathfrak{r}) \\ & - {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{G}(\xi)]\mathcal{W}(\mathfrak{r}) - {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{W}(\xi)]\mathcal{G}(\mathfrak{r}). \end{aligned} \tag{3.5} $$

Repeating the process, taking both sides of (3.5) by

$$ \frac{1}{k\Gamma_k(\vartheta)}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})\varphi'(\mathfrak{r})\mathfrak{P}(\mathfrak{r}),\quad \mathfrak{r}\in(a,\xi), $$

and integrating with respect to $\mathfrak{r}$ from (a,ξ), we have

$$ \begin{aligned} &\frac{1}{(k\Gamma_k(\vartheta))^2}\int_a^\xi\int_a^\xi \frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})\varphi'(\mathfrak{u})\varphi'(\mathfrak{r})\mathfrak{P}(\mathfrak{u})\mathfrak{P}(\mathfrak{r})\mathfrak{M}(\mathfrak{u},\mathfrak{r})d\mathfrak{u}d\mathfrak{r} \\ =\ & {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{G}(\xi)\mathcal{W}(\xi)]{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)] + {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)]{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{G}(\xi)\mathcal{W}(\xi)] \\ & - {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{G}(\xi)]{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{W}(\xi)] - {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{W}(\xi)]{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{G}(\xi)] \\ =\ & 2{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{G}(\xi)\mathcal{W}(\xi)]{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)] - 2{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{G}(\xi)]{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathfrak{P}(\xi)\mathcal{W}(\xi)]. \end{aligned} \tag{3.6} $$

By setting $\mathfrak{P}(\xi) = \mathfrak{f}(\xi)$ and $\mathcal{W}(\xi) = \mathcal{G}(\xi) = \frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi) - {}^{g}\mathbb{E}(\mathbb{K})$ in (3.6), $\xi \in (a, b)$, one has

$$\begin{aligned}
&\frac{1}{(k\Gamma_k(\vartheta))^2}\int_a^\xi\int_a^\xi \frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})\mathfrak{f}(\mathfrak{u})\mathfrak{f}(\mathfrak{r}) \times \left(\frac{g(\varphi(\mathfrak{u}))}{\varphi(\mathfrak{u})}\chi(\mathfrak{u}) - \frac{g(\varphi(\mathfrak{r}))}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right)^2 \varphi'(\mathfrak{u})\varphi'(\mathfrak{r})d\mathfrak{u}d\mathfrak{r} \\
= \;& 2{}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\mathfrak{f}(\xi)\left(\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi) - {}^{g}\mathbb{E}(\mathbb{K})\right)^2\right]{}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}(\xi)] - 2\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\mathfrak{f}(\xi)\left(\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi) - {}^{g}\mathbb{E}(\mathbb{K})\right)\right]\right)^2.
\end{aligned} \tag{3.7}$$

The above quantity admits the estimate

$$\begin{aligned}
&\frac{1}{(k\Gamma_k(\vartheta))^2}\int_a^\xi\int_a^\xi \frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r}) \\
&\times\mathfrak{f}(\mathfrak{u})\mathfrak{f}(\mathfrak{r})\left(\frac{g(\varphi(\mathfrak{u}))}{\varphi(\mathfrak{u})}\chi(\mathfrak{u}) - \frac{g(\varphi(\mathfrak{r}))}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right)^2 \times \varphi'(\mathfrak{u})\varphi'(\mathfrak{r})\,d\mathfrak{u}\,d\mathfrak{r} \\
\leq\;& \|\mathfrak{f}\|_{L^\infty}^2\left[2\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[1]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\left(\frac{g(\varphi)}{\varphi}\chi\right)^2\right]\right)(\xi) - 2\left[\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\frac{g(\varphi)}{\varphi}\chi\right]\right)(\xi)\right]^2\right].
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we get the first inequality of Theorem 3.1.

For any $\mathfrak{u}, \mathfrak{r} \in (a, \xi)$,

$$\begin{aligned}
&\frac{1}{(k\Gamma_k(\vartheta))^2}\int_a^\xi\int_a^\xi \frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r}) \\
&\times\mathfrak{f}(\mathfrak{u})\mathfrak{f}(\mathfrak{r})\left(\frac{g(\varphi(\mathfrak{u}))}{\varphi(\mathfrak{u})}\chi(\mathfrak{u}) - \frac{g(\varphi(\mathfrak{r}))}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right)^2 \times \varphi'(\mathfrak{u})\varphi'(\mathfrak{r})\,d\mathfrak{u}\,d\mathfrak{r} \\
\leq\;& \sup_{\mathfrak{u},\mathfrak{r}\in(a,\xi)}\left|\frac{g(\varphi(\mathfrak{u}))}{\varphi(\mathfrak{u})}\chi(\mathfrak{u}) - \frac{g(\varphi(\mathfrak{r}))}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right|^2 \times \left(\frac{1}{k\Gamma_k(\vartheta)}\int_a^\xi \frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})\mathfrak{f}(\mathfrak{u})\varphi'(\mathfrak{u})\,d\mathfrak{u}\right) \\
&\times\left(\frac{1}{k\Gamma_k(\vartheta)}\int_a^\xi \frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{r})\mathfrak{f}(\mathfrak{r})\varphi'(\mathfrak{r})\,d\mathfrak{r}\right) \\
=\;& \sup_{\mathfrak{u},\mathfrak{r}\in(a,\xi)}\left|\frac{g(\varphi(\mathfrak{u}))}{\varphi(\mathfrak{u})}\chi(\mathfrak{u}) - \frac{g(\varphi(\mathfrak{r}))}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right|^2\left[\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}]\right)(\xi)\right]^2.
\end{aligned} \tag{3.9}$$

From (3.7) and (3.9), we reach the inequality (3.3).

Next, we present a more general version of Theorem 3.1 by introducing two fractional parameters.

Theorem 3.2. *Let $\mathbb{K}$ be a $\mathbb{CRV}$ with $\mathbb{PDF}$ $\mathfrak{f} : [a, b] \to \mathbb{R}_+$. Then the following conditions hold:*

(1) For any $\vartheta, \mu \geq 0$ and $\xi \in (a, b]$, one has

$$\begin{aligned}
&\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}]\right)(\xi){}^{g}\sigma_{\mathbb{K},\mu}^2(\xi) + \left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}[\mathfrak{f}]\right)(\xi){}^{g}\sigma_{\mathbb{K},\vartheta}^2(\xi) - 2\left[{}^{g}\mathbb{E}_{\mathbb{K}-{}^{g}\mathbb{E}(\mathbb{K}),\vartheta}(\xi)\right]\left[{}^{g}\mathbb{E}_{\mathbb{K}-{}^{g}\mathbb{E}(\mathbb{K}),\mu}(\xi)\right] \\
\leq\;& \|\mathfrak{f}\|_{L^\infty}^2\left[\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}[1]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\left(\frac{g(\varphi)}{\varphi}\chi\right)^2\right]\right)(\xi) + \left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[1]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}\left[\left(\frac{g(\varphi)}{\varphi}\chi\right)^2\right]\right)(\xi)\right. \\
&\left.-2\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\frac{g(\varphi)}{\varphi}\chi\right]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}\left[\frac{g(\varphi)}{\varphi}\chi\right]\right)(\xi)\right],
\end{aligned} \tag{3.10}$$

where $\mathfrak{f} \in L^\infty([a, b])$.

(2) The integral inequality

$$\begin{aligned}
&\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}]\right)(\xi)^{g}\sigma^{2}_{\mathbb{K},\mu}(\xi)+\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}[\mathfrak{f}]\right)(\xi)^{g}\sigma^{2}_{\mathbb{K},\vartheta}(\xi)-2\left[{}^{g}\mathbb{E}_{\mathbb{K}-{}^{g}\mathbb{E}(\mathbb{K}),\vartheta}(\xi)\right]\left[{}^{g}\mathbb{E}_{\mathbb{K}-{}^{g}\mathbb{E}(\mathbb{K}),\mu}(\xi)\right]\\
\leq\;& \sup_{\mathfrak{u},\mathfrak{r}\in(a,\xi)}\left|\frac{g\big(\varphi(\mathfrak{u})\big)}{\varphi(\mathfrak{u})}\chi(\mathfrak{u})-\frac{g\big(\varphi(\mathfrak{r})\big)}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right|^{2}\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}[\mathfrak{f}]\right)(\xi)
\end{aligned}\tag{3.11}$$

is true for all $\vartheta,\mu\geq 0$ and $\xi\in(a,b]$.

Proof. Inserting a function $\mathfrak{P}:[a,b]\to\mathbb{R}_+$, taking both sides of (3.5) by

$$\frac{1}{k\Gamma_k(\vartheta)}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\Psi^{\frac{\mu}{k}-1}(\xi,\mathfrak{r})\varphi'(\mathfrak{r})\mathfrak{P}(\mathfrak{r}),\quad \mathfrak{r}\in(a,\xi),$$

and integrating with respect to $\mathfrak{r}$ from (a,ξ), we obtain

$$\begin{aligned}
&\int_a^{\xi}\int_a^{\xi}\frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})}{k\Gamma_k(\vartheta)}\frac{\Psi^{\frac{\mu}{k}-1}(\xi,\mathfrak{r})}{k\Gamma_k(\mu)}\varphi'(\mathfrak{u})\varphi'(\mathfrak{r})\mathfrak{P}(\mathfrak{u})\mathfrak{P}(\mathfrak{r})\mathfrak{M}(\mathfrak{u},\mathfrak{r})\,d\mathfrak{u}\,d\mathfrak{r}\\
=\;&\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{P}\mathcal{G}\mathcal{W}]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}[\mathfrak{P}]\right)(\xi)+\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{P}]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}[\mathfrak{P}\mathcal{G}\mathcal{W}]\right)(\xi)\\
&-\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{P}\mathcal{G}]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}[\mathfrak{P}\mathcal{W}]\right)(\xi)-\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{P}\mathcal{W}]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}[\mathfrak{P}\mathcal{G}]\right)(\xi).
\end{aligned}\tag{3.12}$$

By setting $\mathfrak{P}(\xi)=\mathfrak{f}(\xi)$ and $\mathcal{W}(\xi)=\mathcal{G}(\xi)=\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)-{}^{g}\mathbb{E}(\mathbb{K})$ in (3.12), $\xi\in(a,b)$, yields

$$\begin{aligned}
&\int_a^{\xi}\int_a^{\xi}\frac{g\big(\Psi(\xi,\mathfrak{u})\big)}{\Psi(\xi,\mathfrak{u})}\frac{g\big(\Psi(\xi,\mathfrak{r})\big)}{\Psi(\xi,\mathfrak{r})}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})}{k\Gamma_k(\vartheta)}\frac{\Psi^{\frac{\mu}{k}-1}(\xi,\mathfrak{r})}{k\Gamma_k(\mu)}\mathfrak{f}(\mathfrak{u})\mathfrak{f}(\mathfrak{r})\\
&\times\left(\frac{g(\varphi(\mathfrak{u}))}{\varphi(\mathfrak{u})}\chi(\mathfrak{u})-\frac{g(\varphi(\mathfrak{r}))}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right)^{2}\varphi'(\mathfrak{u})\varphi'(\mathfrak{r})\,d\mathfrak{u}\,d\mathfrak{r}\\
=&\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\mathfrak{f}(\xi)\left(\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)-{}^{g}\mathbb{E}(\mathbb{K})\right)^{2}\right]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}[\mathfrak{f}]\right)(\xi)\\
&+\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}\left[\mathfrak{f}(\xi)\left(\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)-{}^{g}\mathbb{E}(\mathbb{K})\right)^{2}\right]\right)(\xi)\\
&-2\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\mathfrak{f}(\xi)\left(\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)-{}^{g}\mathbb{E}(\mathbb{K})\right)\right]\right)(\xi)\times\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}\left[\mathfrak{f}(\xi)\left(\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)-{}^{g}\mathbb{E}(\mathbb{K})\right)\right]\right)(\xi)\\
=&\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}]\right)(\xi)^{g}\sigma^{2}_{\mathbb{K},\mu}(\xi)+\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}[\mathfrak{f}]\right)(\xi)^{g}\sigma^{2}_{\mathbb{K},\vartheta}(\xi)-2\left[{}^{g}\mathbb{E}_{\mathbb{K}-{}^{g}\mathbb{E}(\mathbb{K}),\vartheta}(\xi)\right]\left[{}^{g}\mathbb{E}_{\mathbb{K}-{}^{g}\mathbb{E}(\mathbb{K}),\mu}(\xi)\right].
\end{aligned}\tag{3.13}$$

Moreover, we get

$$\begin{aligned}
&\int_a^{\xi}\int_a^{\xi}\frac{g\big(\Psi(\xi,\mathfrak{u})\big)}{\Psi(\xi,\mathfrak{u})}\frac{g\big(\Psi(\xi,\mathfrak{r})\big)}{\Psi(\xi,\mathfrak{r})}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})}{k\Gamma_k(\vartheta)}\frac{\Psi^{\frac{\mu}{k}-1}(\xi,\mathfrak{r})}{k\Gamma_k(\mu)}\mathfrak{f}(\mathfrak{u})\mathfrak{f}(\mathfrak{r})\\
&\times\left(\frac{g(\varphi(\mathfrak{u}))}{\varphi(\mathfrak{u})}\chi(\mathfrak{u})-\frac{g(\varphi(\mathfrak{r}))}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right)^{2}\varphi'(\mathfrak{u})\varphi'(\mathfrak{r})\,d\mathfrak{u}\,d\mathfrak{r}\\
\leq&\|\mathfrak{f}\|^{2}_{L^\infty}\Bigg[\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}1\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\left(\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)\right)^{2}\right]\right)(\xi)+\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}1\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}\left[\left(\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)\right)^{2}\right]\right)(\xi)\\
&-2\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)\right]\right)(\xi)\left({}_{k}^{\varphi}\mathcal{I}_{a^+}^{\mu;g}\left[\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)\right]\right)(\xi)\Bigg].
\end{aligned}\tag{3.14}$$

Therefore, (3.10) follows from (3.13) and (3.14).

Now, for $\mathfrak{u}, \mathfrak{r} \in (a, b)$ with (3.12), we have

$$\begin{aligned}
&\int_a^{\xi}\int_a^{\xi} \frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\frac{\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})}{k\Gamma_k(\vartheta)}\frac{\Psi^{\frac{\mu}{k}-1}(\xi,\mathfrak{r})}{k\Gamma_k(\mu)}\mathfrak{f}(\mathfrak{u})\mathfrak{f}(\mathfrak{r})\\
&\times\left(\frac{g(\varphi(\mathfrak{u}))}{\varphi(\mathfrak{u})}\chi(\mathfrak{u})-\frac{g(\varphi(\mathfrak{r}))}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right)^2\varphi'(\mathfrak{u})\varphi'(\mathfrak{r})\,d\mathfrak{u}\,d\mathfrak{r}\\
\leq&\sup_{\mathfrak{u},\mathfrak{r}\in(a,\xi)}\left|\frac{g(\varphi(\mathfrak{u}))}{\varphi(\mathfrak{u})}\chi(\mathfrak{u})-\frac{g(\varphi(\mathfrak{r}))}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right|^2\\
&\times\left[\frac{1}{k\Gamma_k(\vartheta)}\int_a^{\xi}\frac{g(\Psi(\xi,\mathfrak{u}))}{\Psi(\xi,\mathfrak{u})}\Psi^{\frac{\vartheta}{k}-1}(\xi,\mathfrak{u})\varphi'(\mathfrak{u})\mathfrak{f}(\mathfrak{u})\,d\mathfrak{u}\right]\\
&\times\left[\frac{1}{k\Gamma_k(\mu)}\int_a^{\xi}\frac{g(\Psi(\xi,\mathfrak{r}))}{\Psi(\xi,\mathfrak{r})}\Psi^{\frac{\mu}{k}-1}(\xi,\mathfrak{r})\varphi'(\mathfrak{r})\mathfrak{f}(\mathfrak{r})\,d\mathfrak{r}\right]\\
=&\sup_{\mathfrak{u},\mathfrak{r}\in(a,\xi)}\left|\frac{g(\varphi(\mathfrak{u}))}{\varphi(\mathfrak{u})}\chi(\mathfrak{u})-\frac{g(\varphi(\mathfrak{r}))}{\varphi(\mathfrak{r})}\chi(\mathfrak{r})\right|^2\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}]\right)(\xi)\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\mu;g}[\mathfrak{f}]\right)(\xi),
\end{aligned}\tag{3.15}$$

which gives the desired inequality (3.11).

Theorem 3.3. *Let $\mathbb{K}$ be a $\mathbb{CRV}$ with $\mathbb{PDF}$ $\mathfrak{f}:[a,b]\to\mathbb{R}_+$. Hence,*

$$\begin{aligned}
&{}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}(\xi)]{}^g\sigma^2_{\mathbb{K},\vartheta}(\xi)-\left({}^g\mathbb{E}_{\mathbb{K}-{}^g\mathbb{E}(\mathbb{K}),\vartheta}(\xi)\right)^2\\
&\leq\frac{1}{4}\left(\sup_{\xi\in[a,b]}\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)-\inf_{\xi\in[a,b]}\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)\right)^2\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}(\xi)]\right)^2.
\end{aligned}\tag{3.16}$$

Proof. By applying Theorem 2.1, it follows that

$$\left|\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{G}^2(\xi)]\right)-\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi\mathcal{G}(\xi)]\right)^2\right|\leq\frac{(M_1-M_0)^2}{4}\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\varpi(\xi)]\right)^2.\tag{3.17}$$

Substituting $\varpi(\xi)=\mathfrak{f}(\xi)$ and $\mathcal{G}(\xi)=\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)-{}^g\mathbb{E}(\mathbb{K})$ for any $\xi\in[a,b]$,

$$M_1=\sup_{\xi\in[a,b]}\mathcal{G}(\xi)\quad\text{and}\quad M_0=\inf_{\xi\in[a,b]}\mathcal{G}(\xi).$$

Then (3.17) can be rewritten as

$$\begin{aligned}
0\ &\leq\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}]\right)(\xi)\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\mathfrak{f}\left(\frac{g(\varphi)}{\varphi}\chi-{}^g\mathbb{E}(\mathbb{K})\right)^2\right]\right)(\xi)-\left[\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}\left[\mathfrak{f}\left(\frac{g(\varphi)}{\varphi}\chi-{}^g\mathbb{E}(\mathbb{K})\right)\right]\right)(\xi)\right]^2\\
&\leq\frac{1}{4}\left(\sup_{\xi\in[a,b]}\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)-\inf_{\xi\in[a,b]}\frac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)\right)^2\left[\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}]\right)(\xi)\right]^2,
\end{aligned}\tag{3.18}$$

which implies that

$$\begin{aligned}
&\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}]\right)(\xi){}^g\sigma^2_{\mathbb{K},\vartheta}(\xi)-\left({}^g\mathbb{E}_{\mathbb{K}-{}^g\mathbb{E}(\mathbb{K}),\vartheta}(\xi)\right)^2\\
\leq\ &\tfrac{1}{4}\left(\sup_{\xi\in[a,b]}\tfrac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)-\inf_{\xi\in[a,b]}\tfrac{g(\varphi(\xi))}{\varphi(\xi)}\chi(\xi)\right)^2\left[\left({}_k^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[\mathfrak{f}]\right)(\xi)\right]^2.
\end{aligned}\tag{3.19}$$

Hence, (3.16) is obtained.

For clarity, Table 2 lists parameter choices under which Theorems 3.1–3.3 reduce to classical variance identities and to previously established fractional probabilistic inequalities. These reductions hold after identifying the corresponding kernel, auxiliary function, and normalization used in the cited works.

Table 2. Reductions of Theorems 3.1–3.3 under special parameter selections.

k	$\varphi(\xi)$	$\chi(\xi)$	$g(\xi)$	ϑ	**Recovered result**
1	ξ	ξ	ξ	1	Classical variance identity $\sigma^2 = E(K^2) - E^2(K)$
1	ξ	1	ξ	general	Theorem 3.1 in [12]
1	ξ	1	ξ	general	Theorems 3.2 and 3.3 in [12]
1	ξ	ξ	ξ	general	Theorem 3.2 in [29]
1	ξ	ξ	ξ	general	Theorems 3.4 and 3.5 in [29]
1	general φ	$\varphi(\xi)$	ξ	general	φ-fractional variance results in [29]
1	general φ	$\rho^{\vartheta}\varphi(\xi)$	$\xi^{\rho\vartheta}e^{\frac{\rho-1}{\rho}\xi}$	general	Lemma 1 and Theorems 1–3 in [30]

The following results are operator-level probabilistic inequalities (Markov, Chebyshev, Hoeffding types) that follow directly by applying the positive linear operator and the variance identities derived above.

Using the fractional expectation definitions, a Markov-type bound for nonnegative random variables follows by applying the operator to the elementary deterministic inequality $1_{K\geq\lambda} \leq K/\lambda,\ \lambda > 0$. Assume $k > 0$, $\vartheta > 0$, $g(\cdot) \geq 0$ and let K be a nonnegative random variable having positive $\mathbb{PDF}$ $\mathfrak{f}$ on (a, b). Then, we obtain

$$ {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[1_{K\geq\lambda}](\xi) \leq \frac{1}{\lambda}\, {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[K](\xi). $$

Since ${}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[K](\xi) = gE_{K,\vartheta}(\xi)$, this yields the following fractional Markov inequality.

Corollary 3.1. *(Fractional Markov inequality) Let K be a nonnegative random variable with positive $\mathbb{PDF}$ $\mathfrak{f}$ on (a, b). Assume $\vartheta > 0$, $k > 0$, and g is nonnegative. Then for any $\lambda > 0$ and $\xi \in (a, b]$,*

$$ {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}(\mathbf{1}_{\{K\geq\lambda\}})(\xi) \leq \frac{1}{\lambda}\, gE_{K,\vartheta}(\xi). \tag{3.20} $$

Proof. Since $K \geq 0$, we have $\mathbf{1}_{\{K\geq\lambda\}} \leq \frac{K}{\lambda}$. Applying the positive linear operator ${}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}$ to both sides yields

$$ {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}(\mathbf{1}_{\{K\geq\lambda\}}) \leq \frac{1}{\lambda}\, {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}(K). $$

By Definition 2.5, this equals $\frac{1}{\lambda} gE_{K,\vartheta}(\xi)$.

Using the fractional expectation/variance definitions together with the positivity and linearity of the generalized fractional operator ${}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}$, a Chebyshev-type bound follows by applying the operator to the elementary deterministic inequality $1_{|x|\geq\varepsilon} \leq x^2/\varepsilon^2$ with $x = K - gE_{K,\vartheta}(\xi)$. Assume $k > 0$, $\vartheta \geq 0$, $g(\cdot) \geq 0$

and let K be a random variable having finite fractional variance $g\sigma^2_{K,\vartheta}(\xi)$ and positive $\mathbb{PDF}$ $\mathfrak{f}$ on (a,b). Then, we obtain

$$ {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}(\mathbf{1}_{\{|K-gE_{K,\vartheta}(\xi)|\geq\varepsilon\}}) \leq \frac{1}{\varepsilon^2}\, {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}((K-gE_{K,\vartheta}(\xi))^2). $$

By Definition 2.9, the right-hand side equals $g\sigma^2_{K,\vartheta}(\xi)/\varepsilon^2$, which yields the following fractional Chebyshev inequality.

Corollary 3.2. *(Fractional Chebyshev inequality) Let K be a random variable with finite fractional variance. Then for any $\varepsilon > 0$,*

$$ {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}(\mathbf{1}_{\{|K-gE_{K,\vartheta}(\xi)|\geq\varepsilon\}})(\xi) \leq \frac{g\sigma^2_{K,\vartheta}(\xi)}{\varepsilon^2}. \tag{3.21} $$

Proof. Using the elementary inequality $\mathbf{1}_{\{|x|\geq\varepsilon\}} \leq \frac{x^2}{\varepsilon^2}$, with $x = K - gE_{K,\vartheta}(\xi)$, and applying the positive linear operator ${}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}$, we obtain

$$ {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}(\mathbf{1}_{\{|K-gE_{K,\vartheta}(\xi)|\geq\varepsilon\}}) \leq \frac{1}{\varepsilon^2}\, {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}((K-gE_{K,\vartheta}(\xi))^2), $$

which equals $\frac{g\sigma^2_{K,\vartheta}(\xi)}{\varepsilon^2}$.

The Markov- and Chebyshev-type inequalities above are based on polynomial bounds for indicator functions. A closely related exponential approach is obtained through the Chernoff method. To state it conveniently, define the normalized fractional probability measure

$$ \mathbb{P}^{\varphi,k,g}_{\vartheta,\xi}(A) = \frac{{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[\mathbf{1}_A f](\xi)}{{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[f](\xi)}, \qquad {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[f](\xi) > 0. $$

For a measurable function H, the corresponding normalized fractional expectation is

$$ \mathbb{E}^{\varphi,k,g}_{\vartheta,\xi}[H(K)] = \frac{{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[Hf](\xi)}{{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[f](\xi)}. $$

Consequently, for every $t > 0$ and $x \in \mathbb{R}$, the elementary inequality $\mathbf{1}_{\{K\geq x\}} \leq e^{t(K-x)}$ gives the fractional Chernoff-type bound

$$ \mathbb{P}^{\varphi,k,g}_{\vartheta,\xi}(K \geq x) \leq \exp(-tx)\, \mathbb{E}^{\varphi,k,g}_{\vartheta,\xi}\Big[e^{tK}\Big]. $$

Equivalently, if

$$ \Lambda^{\varphi,k,g}_{\vartheta,\xi}(t) = \log \mathbb{E}^{\varphi,k,g}_{\vartheta,\xi}\Big[e^{tK}\Big], $$

then

$$ \mathbb{P}^{\varphi,k,g}_{\vartheta,\xi}(K \geq x) \leq \inf_{t>0} \exp\Big\{-tx + \Lambda^{\varphi,k,g}_{\vartheta,\xi}(t)\Big\}. $$

This estimate shows explicitly how the generalized fractional parameters enter the exponential tail bound through the fractional moment-generating function.

Assume $k > 0$, $\vartheta > 0$, $g(\cdot) \geq 0$, and let K be a random variable with positive $\mathbb{PDF}$ $\mathfrak{f}$ on (a,b) satisfying $m \leq K \leq M$. Applying the positive linear operator ${}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}$ to the Chernoff bound for the centered variable $K - gE_{K,\vartheta}(\xi)$ and optimizing the resulting exponential moment estimate yields the Hoeffding-type inequality below.

Corollary 3.3. *(Fractional Hoeffding-type inequality) Assume $m \leq K \leq M$ almost everywhere. Then for any $t > 0$,*

$$ {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}(\mathbf{1}_{\{K-gE_{K,\vartheta}(\xi)\geq t\}})(\xi) \leq \exp\left(-\frac{2t^2}{(M-m)^2}\right). \tag{3.22}$$

Proof. Using the exponential moment method and the convexity of the exponential function, we apply the fractional operator to $\exp(\lambda K)$ and optimize over λ. The boundedness $m \leq K \leq M$ yields the classical quadratic exponent, preserved under the positive fractional operator.

Remark 3.1. *The exponential factor in the right-hand side of the Hoeffding-type inequality is the classical Hoeffding exponent and therefore does not explicitly contain the parameters ϑ, k, φ, or (g). This occurs because the proof relies on the standard bounded-variable exponential moment estimate. The fractional structure enters through the fractional weighted probability on the left-hand side and through the fractional centering term ${}^{g}E_{K,\vartheta}(\xi)$. Thus, Corollary 3.3 should be interpreted as a fractional operator-level lifting of Hoeffding's inequality, rather than as a sharper parameter-dependent Hoeffding exponent. Obtaining a genuinely parameter-dependent concentration exponent would require additional assumptions on the fractional exponential moment or on a normalized fractional probability measure.*

4. Applications to reliability and finance

This section illustrates the applicability of the proposed generalized fractional probabilistic framework through numerical verification, parameter sensitivity analysis, and two representative applications in reliability theory and financial risk modeling.

4.1. Numerical verification and parameter sensitivity

In this subsection, we present numerical experiments that serve two purposes: (i) to verify the nonnegativity of the quantity $\Omega_\theta(\xi)$ defined in Theorem 3.1, thereby confirming the consistency of the generalized fractional variance identity; and (ii) to illustrate the influence of the parameters k, θ, the auxiliary function φ, and the kernel g on the generalized fractional expectation and variance. All computations are performed for a continuous random variable $K \sim \text{Uniform}(0,1)$ with probability density function $f(s) = 1$ on $[0,1]$. We evaluate the generalized fractional quantities at the right endpoint $\xi = b = 1$.

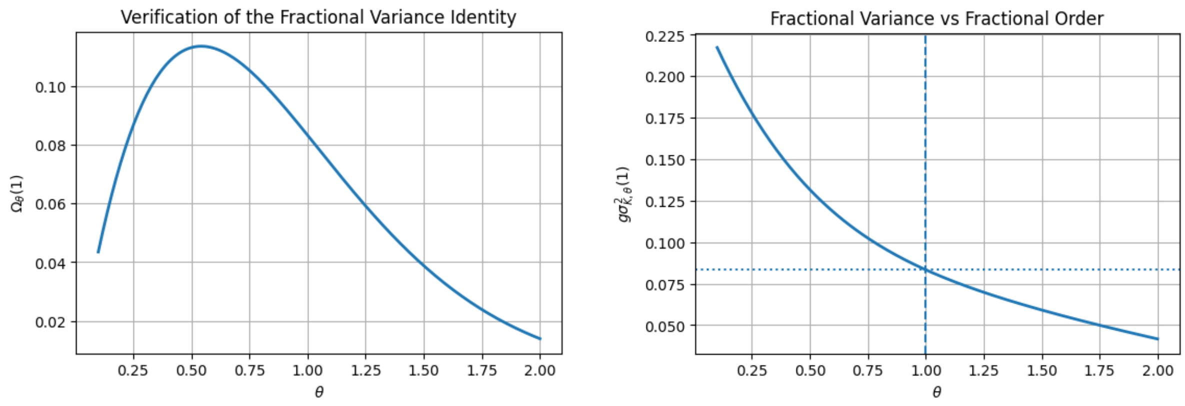
For simplicity, we set $\chi(\xi) = \varphi(\xi)$ throughout the experiments, which is a common choice that recovers many classical cases. Equation (3.7) in the proof of Theorem 3.1 shows that the quantity

$$\Omega_\theta(\xi) = {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[f](\xi)\; g\sigma^2_{K,\theta}(\xi) - \left(gE_{K-gE(K),\theta}(\xi)\right)^2$$

is nonnegative for all admissible parameters. We verify this numerically using the baseline configuration

$$\varphi(\xi) = \xi, \quad \chi(\xi) = \xi, \quad g(\xi) = \xi, \quad k = 1, \quad \theta \in (0.1, 2].$$

Figure 1 shows $\Omega_\theta(1)$ together with the fractional variance $g\sigma^2_{K,\theta}(1)$ as functions of θ.



(a) (b)

Figure 1. Simultaneous verification of the variance identity and the behavior of the fractional variance under the baseline configuration $\varphi(\xi) = \xi$, $\chi(\xi) = \xi$, $g(\xi) = \xi$, $k = 1$, and $K \sim U(0, 1)$. (a) Numerical verification of the nonnegativity of $\Omega_\vartheta(1)$. The curve remains nonnegative for all considered values of ϑ, confirming the variance identity derived in Theorem 3.1. (b) Fractional variance ${}^g\sigma^2_{K,\vartheta}(1)$ as a function of the fractional order ϑ. The classical variance $\mathrm{Var}(K) = 1/12 \approx 0.083$ is recovered at $\vartheta = 1$. The plots show the influence of the fractional order on memory weighting.

Figure 1 displays $\Omega_\theta(1)$, which stays strictly nonnegative and approaches zero as $\theta \to 0^+$, indicating that the inequality is sharp. Figure 2 shows the fractional variance; at $\theta = 1$, we recover the classical variance of the uniform distribution, $\mathrm{Var}(K) = 0.083$. For $\theta > 1$, the variance increases monotonically, reflecting the enhanced memory depth introduced by the fractional operator.

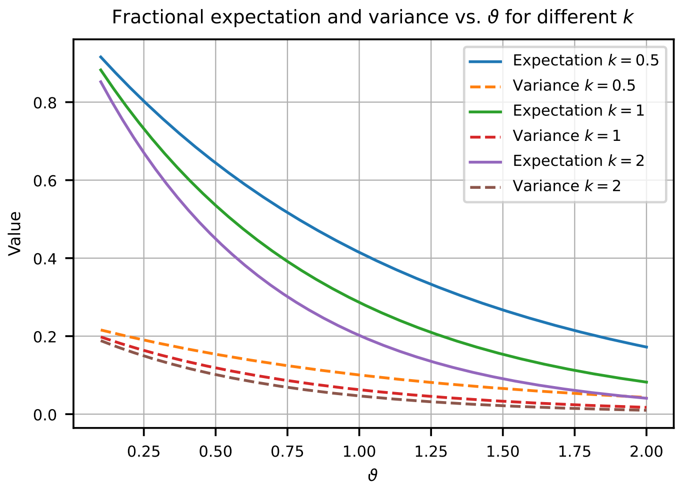


Figure 2. Effect of the scale parameter k on the generalized fractional expectation and variance as functions of the fractional order ϑ. The computations are performed for $\varphi(\xi) = \xi$, $\chi(\xi) = \xi$, $g(\xi) = \xi$, and $K \sim U(0, 1)$. Solid curves represent the fractional expectation, whereas dashed curves represent the fractional variance.

To demonstrate that the proposed framework is not restricted to the symmetric uniform case, we also consider a non-uniform probability density. Let

$$K \sim \mathrm{Beta}(2,5), \qquad f(s) = 30s(1-s)^4, \quad 0 < s < 1.$$

This distribution is asymmetric and concentrated near the lower part of the interval, and therefore provides a more informative test case for the generalized fractional moment framework. Under the baseline choice

$$\varphi(\xi) = \xi, \qquad \chi(\xi) = \xi, \qquad g(\xi) = \xi, \qquad k = 1, \qquad \xi = 1,$$

the generalized fractional expectation and second moment become

$$^{g}E_{K,\vartheta}(1) = \frac{30}{\Gamma(\vartheta)} \int_0^1 s^2(1-s)^{\vartheta+3}\, ds,$$

and

$$^{g}M_{K^2,\vartheta}(1) = \frac{30}{\Gamma(\vartheta)} \int_0^1 s^3(1-s)^{\vartheta+3}\, ds.$$

In particular, when $\vartheta = 1$, the classical mean and variance are recovered:

$$\mathbb{E}(K) = \frac{2}{7}, \qquad \mathrm{Var}(K) = \frac{5}{196}.$$

For $\vartheta \neq 1$, the factor $(1-s)^{\vartheta-1}$ modifies the weighting of the density and changes the corresponding fractional expectation and variance. This example confirms that the proposed operator-based framework remains applicable to asymmetric and non-uniform probability distributions, not only to the uniform distribution used in the baseline verification.

We fix $\varphi(\xi) = \xi, \chi(\xi) = \xi$, $g(\xi) = \xi$ and consider $k \in \{0.5, 1.0, 1.5, 2.0\}$. Figure 2 plots the fractional expectation $gE_{K,\theta}(1)$ and variance $g\sigma^2_{K,\theta}(1)$ as functions of θ.

The plots reveal that k does not act as a simple scaling factor; it modulates the curvature of both curves. Smaller k amplifies the variation with θ, while larger k yields flatter responses. At $\theta = 1$, all curves intersect at the classical values, as expected.

We compare the linear distortion $\varphi(\xi) = \xi$ with the quadratic distortion $\varphi(\xi) = \xi^2$, while keeping $g(\xi) = \xi$, $k = 1$. Figure 3 shows the results.

The quadratic distortion significantly increases both the expectation and the variance, especially for larger θ. This demonstrates that nonlinear time scaling cannot be replicated by merely adjusting k or g; it introduces a distinct memory reshaping effect.

We test three kernel functions: $g_1(\xi) = \xi$, $g_2(\xi) = \xi^{0.7}$, and the proportional kernel $g_3(\xi) = \frac{\xi}{\rho}\exp(\frac{\rho-1}{\rho}\xi)$ with $\rho = 0.8$. We set $\varphi(\xi) = \xi$, $k = 1$. Figure 4 displays the fractional variance for these kernels.

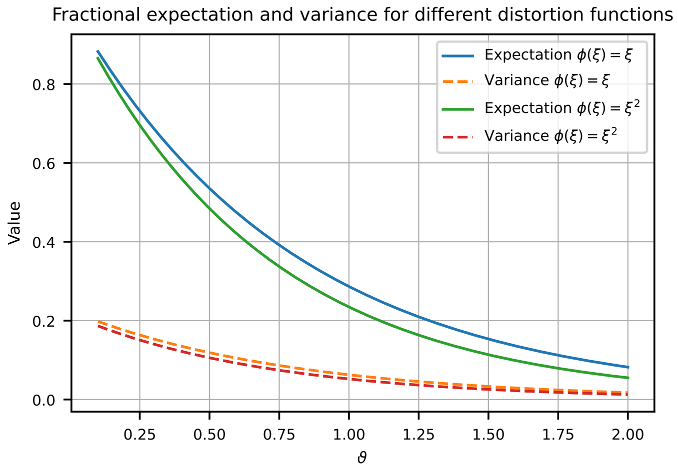


Figure 3. Comparison between the linear distortion $\varphi(\xi) = \xi$ and the quadratic distortion $\varphi(\xi) = \xi^2$ for $g(\xi) = \xi$, $k = 1$, $\chi(\xi) = \varphi(\xi)$, and $K \sim U(0, 1)$. Solid curves correspond to the fractional expectation, whereas dashed curves correspond to the fractional variance.

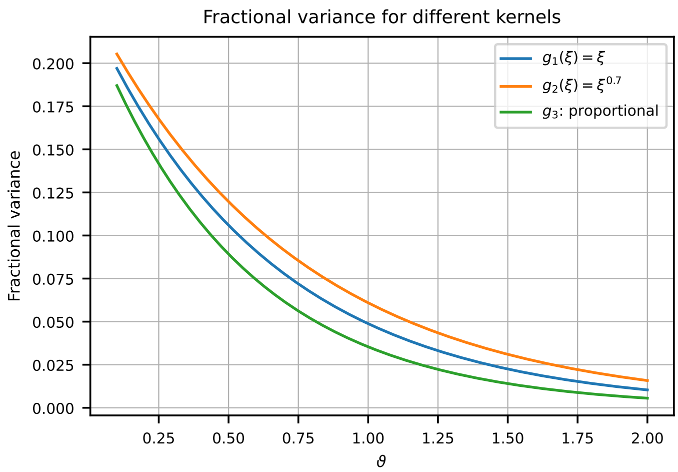


Figure 4. Influence of the kernel function g on the generalized fractional variance for $\varphi(\xi) = \xi$, $\chi(\xi) = \xi$, $k = 1$, and $K \sim U(0, 1)$. The kernels considered are $g_1(\xi) = \xi$, $g_2(\xi) = \xi^{0.7}$, and the proportional-type kernel g_3.

The power-law kernels produce a gradual increase of variance with θ, while the proportional kernel generates a much steeper rise, reflecting its tempered memory characteristics. This illustrates the flexibility gained by choosing different kernel shapes.

From the numerical experiments, we draw the following conclusions:

(i) The classical expectation and variance are recovered when $\theta = 1$, confirming the consistency of the generalized operator.

(ii) Increasing the fractional order θ deepens the memory effect, generally increasing the fractional variance. The growth pattern depends on the kernel and distortion.

(iii) The parameter k interacts nonlinearly with θ and modifies the curvature of the θ-dependence rather than acting as a mere scale factor.
(iv) Nonlinear distortions $\varphi(\xi) = \xi^2$ amplify the sensitivity to θ, particularly in the variance, demonstrating that time-distortion effects are independent of k and g.
(v) The exponential-type proportional kernel exhibits the strongest θ-dependence, offering enhanced flexibility for modeling tempered or bounded memory.

These observations highlight the nontrivial interplay among (θ, k, φ, g) and confirm that the proposed multi-parameter framework provides substantially richer modeling capabilities than classical fractional calculus or single-parameter generalizations.

4.2. Applications

The generalized fractional operator ${}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}$ developed in the previous sections provides a flexible framework for modeling systems with heterogeneous memory, time distortion, and kernel-dependent weighting. The Grüss-type inequalities established earlier yield useful bounds for quantities such as dispersion, volatility, and empirical risk. We illustrate two representative applications.

4.2.1. Reliability modeling

Let T denote the lifetime of a component with $\mathbb{PDF}$ $f(t)$ supported on $[a, b]$.
The classical mean lifetime is

$$E[T] = \int_a^b t f(t)\, dt.$$

To incorporate hereditary effects and distorted time scales, we define the generalized fractional mean lifetime

$$E^{\varphi,k}_{\vartheta}[T] = \frac{{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[t f(t)]}{{}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[f(t)]}.$$

The application of fractional operators in reliability analysis has been increasingly investigated in recent years. For instance, Erden et al. [31] employed local fractional integral inequalities to derive bounds for reliability functions under different aging classes, whereas Aljaaidi et al. [32] demonstrated how generalized proportional fractional operators can effectively capture memory-dependent failure mechanisms in lifetime data. The present framework extends these approaches by allowing for an arbitrary kernel g and distortion function φ, thereby providing a more flexible tool for reliability modeling.

Similarly, the generalized fractional variance becomes

$$\mathrm{Var}^{\varphi,k}_{\vartheta}(T) = E^{\varphi,k}_{\vartheta}[T^2] - \left(E^{\varphi,k}_{\vartheta}[T]\right)^2.$$

If the lifetime is bounded $m \le T \le M$, the Grüss inequality obtained in Section 3 yields

$$\left| {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[f] {}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[T^2 f] - ({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[T f])^2 \right| \le \frac{1}{4}(M - m)^2 \left({}^{\varphi}_{k}\mathcal{I}^{\vartheta;g}_{a^+}[f]\right)^2.$$

This inequality provides a fractional upper bound for the dispersion of the system lifetime under generalized memory effects.

4.2.2. Financial risk modeling

Let X be a bounded financial return random variable with $m \leq X \leq M$ and density $f(x)$. The generalized fractional expected return is defined by

$$E_{\vartheta}^{\varphi,k}[X] = \frac{{}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[xf(x)]}{{}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}[f(x)]}.$$

The Chernoff-type formulation also allows one to introduce the fractional entropic value-at-risk (FEVaR), which serves as a fractional analogue of the classical entropic value-at-risk (EVaR). Let X denote a loss or negative return random variable, and let $\alpha \in (0,1)$ be a confidence level. With respect to the normalized fractional probability measure defined above, we define

$$\mathrm{FEVaR}_{\alpha,\vartheta,\xi}^{\varphi,k,g}(X) = \inf_{t>0} \frac{1}{t}\left[\log \mathbb{E}_{\vartheta,\xi}^{\varphi,k,g}\left(e^{tX}\right) - \log(1-\alpha)\right],$$

provided that the fractional exponential moment exists. This quantity is the FEVaR computed under the fractional weighted probability measure induced by ${}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}$. Hence, the parameters ϑ, k, φ, and g influence the risk estimate through the fractional moment-generating function

$$\Lambda_{\vartheta,\xi}^{\varphi,k,g}(t) = \log \mathbb{E}_{\vartheta,\xi}^{\varphi,k,g}\left(e^{tX}\right).$$

In this sense, the present framework provides a natural fractional analogue of EVaR. When the fractional weighting reduces to the classical probability measure, the usual EVaR is recovered.

Fractional methods have found increasing utility in financial modeling due to their ability to capture long memory and nonlinear time scaling. Beyond the long-memory stochastic volatility models discussed in [24, 25], fractional integral inequalities have been applied to bound financial risks under various convexity and dependence assumptions [33]. The generalized fractional volatility bound derived below complements these existing results by incorporating additional flexibility through the kernel g and the time-distortion function φ.

The corresponding fractional volatility becomes

$$\mathrm{Var}_{\vartheta}^{\varphi,k}(X) = E_{\vartheta}^{\varphi,k}[X^2] - \left(E_{\vartheta}^{\varphi,k}[X]\right)^2.$$

Applying the Grüss-type bound derived earlier gives

$$\mathrm{Var}_{\vartheta}^{\varphi,k}(X) \leq \frac{1}{4}(M-m)^2.$$

Hence, the generalized fractional volatility remains controlled by the essential range of returns while allowing nonlinear time distortions through φ and heterogeneous memory kernels g.

5. Conclusions

In this paper, we introduced and analyzed a generalized multi-parameter fractional integral operator ${}_{k}^{\varphi}\mathcal{I}_{a^+}^{\vartheta;g}$ depending on the auxiliary function φ, the kernel g, the scale parameter k, and the fractional order ϑ. This operator unifies and extends several classical and modern fractional frameworks, including Riemann-Liouville, k-fractional, φ-fractional, and proportional fractional operators, which arise as special cases through suitable parameter choices as summarized in Table 1.

Based on this operator, we developed a probabilistic framework by defining generalized fractional expectation, variance, and higher moments for continuous random variables. Our theoretical contributions included fractional variance identities, generalized Grüss-type inequalities that provide bounds for covariance structures under fractional integration, and multi-parameter inequalities capturing the interaction between different fractional orders. Moreover, classical concentration inequalities were extended to memory-dependent settings through fractional versions of Markov-, Chebyshev-, and Hoeffding-type bounds.

The numerical study presented in Section 4.1 confirms the consistency of the theoretical results. In particular, the classical variance value 0.083 for the uniform distribution is recovered at $\vartheta = 1$, while the quantity $\Omega_\vartheta(\xi)$ remains nonnegative for all tested parameter configurations. The parameter sensitivity analysis further shows that the order ϑ controls the depth of memory effects, whereas the scale parameter k modulates the curvature of the ϑ-dependence. Nonlinear distortions such as $\varphi(\xi) = \xi^2$ introduce distinct memory reshaping effects that cannot be reproduced by adjusting k or g alone, highlighting the independent role of the time-distortion function in the modeling framework.

The practical usefulness of the framework was illustrated through applications in reliability theory and financial risk modeling. In reliability analysis, the generalized fractional mean lifetime and variance provide useful tools for studying systems with heterogeneous aging and memory-dependent failure mechanisms. In financial modeling, the fractional volatility bounds allow the study of long-memory behavior while incorporating nonlinear temporal scaling through the function φ and flexible memory kernels g.

The flexibility of the proposed operator stems from the independent specification of the kernel g, the distortion function φ, the scale parameter k, and the fractional order ϑ. This multi-parameter structure allows the framework to capture complex memory characteristics that are not accessible through classical expectation and variance operators.

Future research may focus on the development of additional fractional concentration inequalities, such as Bernstein-, Chernoff-, and Bennett-type bounds, as well as statistical inference methods for estimating the parameters $(\vartheta, k, \varphi, g)$ from empirical data. Further extensions to multivariate settings and stochastic processes driven by memory-dependent noise also represent promising directions.

Author contributions

A. Moumen: Conceptualization, theoretical analysis, and derivation of the main inequalities; J. Alzabut: Conceptualization, supervision of the research, interpretation of the mathematical results, and development of the application framework; O. Zentar: Contribution to the probabilistic modeling and assistance in the development of the fractional expectation and variance framework; A. Kucukaslan: Contribution to the analytical verification of results and preparation of parts of the mathematical proofs; M. Tayeb : Implementation of the numerical verification and parameter sensitivity analysis; M. Bouye: Contribution to the literature review, manuscript preparation, and technical editing of the paper. All authors contributed to discussion of the results and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The authors declare that they have no conflicts of interest with respect to the research, authorship, and/or publication of this article.

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