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Research article

On a chemorepulsion elliptic system with nonlinear production and weak data

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Abstract: Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$, with $N \geq 3$ and $\chi > 0$. Let $M(x)$ be a symmetric positive definite matrix with coefficients $m_{ij} \in L^\infty(\Omega)$. We study the problem:

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + u = \chi \mathrm{div}(uM(x)\,\nabla\psi) + f(x), \\ -\mathrm{div}(M(x)\nabla\psi) + \psi = u^\theta + g(x). \end{cases}$$

Under the assumptions

1. $\theta < \frac{N+2}{N-2}$;
2. $f \geq 0$, $f \in L^m(\Omega)$, for $m > \frac{2N}{N+2}$;
3. $g \geq 0$, $g \in L^q(\Omega)$, for $q > \frac{2N}{N+2}$,

we prove the existence of at least one weak solution in $W^{1,\frac{N}{N-1}}(\Omega) \times W^{1,2}(\Omega)$. Moreover, under the assumptions

4. $\theta < \frac{2}{N-2}$;
5. $g \in L^q(\Omega)$ for some $q > \frac{N}{2}$,

we have

$$u \in W^{1,2}(\Omega), \quad \psi \in W^{1,2}(\Omega) \cap L^\infty(\Omega).$$

Keywords: chemotaxis; chemorepulsion; elliptic systems; bounded solutions; weak data
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1. Introduction

Chemotaxis has been extensively studied from a mathematical point of view since the pioneering works of Keller and Segel [16, 17] published in 1970 and 1971, respectively. The authors proposed a system of two parabolic equations describing the movement of a biological species towards the higher concentration of a chemical substance (positive taxis) or away from it (chemorepulsion). In [8], the authors study a parabolic chemorepulsion system

$$\begin{cases} u_t - \Delta u = \chi \mathrm{div}(u\nabla\psi), \\ \psi_t - D\Delta\psi + \psi = u, \end{cases}$$

and prove the existence of solutions and their convergence to the constant steady state for $N = 2$ and $N = 3$ in $L^2(\Omega)$−norm. Nevertheless, for $N = 4$, the convergence to the constant steady state is given in a weaker sense, see also [9] for the 3-dimensional case. In [7, 11, 18–20, 27], other chemorepulsion systems have also been studied from a mathematical point of view, along with [12–15, 22] from the viewpoint of optimization and control. In [15], the global existence of solutions to the fully parabolic chemorepulsion system

$$\begin{cases} u_t - \Delta u = \chi \mathrm{div}(u\nabla\psi), \\ \tau\psi_t - D\Delta\psi + \psi = u^\theta, \end{cases}$$

for $\theta \in (1, 2)$ and $N \leq 3$ is established, together with the numerical analysis of the system.

The original Keller-Segel system proposed in [16, 17] models the movement of bacteria following a chemical gradient in biological tissues (such as immune system cells moving towards an infection site). Cell motility is influenced by the fiber alignment and other characteristics of the extracellular matrix. This leads to different migration velocities of the biological species even under the same chemical gradient. Such a movement is described by an anisotropic matrix, see for instance [10, 24, 26] and the references therein for more details. Other authors have considered $M = M(u)$, combining the porous media equation with chemotaxis systems and leading to additional mathematical difficulties. We denote by $M(x) \in \mathcal{M}_{N\times N}$ the anisotropic symmetric matrix of coefficients $\{m_{ij}\}_{i,j=1}^N$, where

$$M(x)\,\xi \cdot \xi \geq \alpha|\xi|^2, \quad |M(x)| \leq \beta\,, \tag{1.1}$$

for some positive constants $\alpha \leq \beta$ and any ξ in $\mathbb{R}^N$. The coefficients $\{m_{ij}\}_{i,j=1}^N$ satisfy

$$m_{ij} \in L^\infty(\Omega),\ \text{for } i, j = 1, \ldots, N. \tag{1.2}$$

For simplicity, and without loss of generality, we assume that $\alpha = 1$.

We denote by u the biological species and by ψ the chemical signal satisfying the system

$$\begin{cases} u_t - \mathrm{div}(M(x)\nabla u) = \chi \mathrm{div}(uM(x)\nabla\psi), \quad x \in \Omega, \\ \psi_t - \mathrm{div}(M(x)\nabla\psi) = u^\theta, \quad x \in \Omega, \\ \dfrac{\partial u}{\partial n_M} = \dfrac{\partial \psi}{\partial n_M} = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.3}$$

under the assumption

$$\theta \in \left(0, \frac{N+2}{N-2}\right), \tag{1.4}$$

where $\dfrac{\partial u}{\partial n_M} = M(x)\nabla u \cdot \vec{n}$, and $\vec{n}$ is the outward unit normal to the boundary of the domain.

Notice that the x-dependent coefficients of the matrix M are bounded but do not necessarily possess the C^1 regularity required to apply the Calderón–Zygmund theory of elliptic equations. That lack of regularity creates an additional difficulty in the problem because the solution does not belong to $W^{2,p}(\Omega)$. For that reason, the boundedness of the solutions is obtained by using Stampacchia′s method.

We introduce an implicit time discretization $\{t_j\}_{j=1}^{n}$ in the problem and approximate u_t and ψ_t as follows:

$$u_t \sim \frac{\tilde{u}(t_j) - \tilde{u}(t_{j-1})}{t_j - t_{j-1}}, \quad \psi_t \sim \frac{\tilde{\psi}(t_j) - \tilde{\psi}(t_{j-1})}{t_j - t_{j-1}}.$$

We rescale the problem and denote by f and g the terms $\tilde{u}(t_{j-1})$ and $\tilde{\psi}(t_{j-1})$, respectively. For simplicity in the notation we omit the tilde notation and the dependence on t_j to obtain the system

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + u = \chi \mathrm{div}(uM(x)\nabla\psi) + f(x), & x \in \Omega \\ -\mathrm{div}(M(x)\nabla\psi) + \psi = u^{\theta} + g(x), & x \in \Omega \\ \dfrac{\partial u}{\partial n_M} = \dfrac{\partial \psi}{\partial n_M} = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.5}$$

(1.5) for $g \equiv 0$ when the chemotaxis term has the form $-\chi\mathrm{div}(uM(x)\nabla\psi)$ was initially studied in [2], where the elliptic system was proposed following previous parabolic systems with constant diffusion and using Dirichlet boundary conditions. In [2], the model involves two main issues: the weak regularity of the source term f and the regularity of the anisotropic matrix, whose coefficients are assumed to be in $L^\infty(\Omega)$. Following [2], in [5], the authors analyze the system

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + u = \chi \mathrm{div}(uM(x)\nabla\psi) + f(x), & x \in \Omega \\ -\mathrm{div}(A(x)\nabla\psi) + \psi = B(u) + E\nabla\psi, & x \in \Omega. \end{cases}$$

with Dirichlet boundary conditions under the assumptions

- $f(x) \geq 0$ and $f\ln(1+f) \in L^1(\Omega)$,
- B is a bounded, positive and continuous function,
- $E \in \left[L^N(\Omega)\right]^N$.

The authors prove the existence of weak solutions (u, ψ) in $W^{1,\frac{N}{N-1}}(\Omega) \times W^{1,2}(\Omega)$ for $N \geq 3$ for $M(x) = Id$. Moreover, for bounded data E and f, the solution u is bounded for $N = 3$. The case $B(u) = u^\theta$ (for $\theta < 2/N$), and $A(x) = M(x)$ under the same assumptions in f and E has been studied in [3] obtaining the existence of weak solutions in $W^{1,\frac{N}{N-1}}(\Omega) \times W^{1,2}(\Omega)$. The flux limitation elliptic system

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + u = -\chi \mathrm{div}(u|\nabla\psi|^{p-2}\nabla\psi) + f(x), & x \in \Omega \\ -\mathrm{div}(M(x)\nabla\psi) + \psi = u^{\theta}, & x \in \Omega \end{cases}$$

for $p < \frac{N\theta}{N\theta-1}$ for $\theta \in (1/N, 1]$ has been considered in [4] for $N \geq 3$ and $f \in L^m(\Omega)$ $(m > 2/N)$. For that problem, the authors use $C^1(\overline{\Omega})$ coefficients of M and obtain that $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and

$\psi \in W^{2,q}(\Omega) \cap W_0^{1,2}(\Omega)$ for any $q < +\infty$, see also [28]. The question of uniqueness remains open in the mentioned elliptic systems studied in [2–5].

In this article, the existence of a solution $(u, \psi) \in W^{1,\frac{N}{N-1}}(\Omega) \times W^{1,2}(\Omega)$ of system (1.5) is studied under the following assumptions on f and g:

$$0 \le f(x), \quad f(x) \in L^m(\Omega), \text{ for } m > \frac{2N}{N+2}, \tag{1.6}$$

and

$$0 \le g(x) \in L^q(\Omega), \quad q > \frac{2N}{N+2}. \tag{1.7}$$

Moreover, if θ satisfies

$$\theta \in \left(0, \frac{2}{N-2}\right), \tag{1.8}$$

and g fulfills

$$g(x) \in L^q(\Omega), \text{ for } q > \frac{N}{2}, \tag{1.9}$$

then ψ is bounded and $u \in W^{1,2}(\Omega)$.

We notice that assumption (1.8) is satisfied in [9] in view of $1 < \frac{2}{N-2}$ for $N = 3$, and is the critical case for $N = 4$ in [8]. Notice also that in [15], the constraint $\theta < 2$ and $N = 3$ is the same as (1.8). So, it is expected that the fully parabolic chemorepulsion system (1.3) admits global-in-time existence of bounded weak solutions if θ satisfies (1.8). To the best of the author′s knowledge, the question of existence of solutions for $\theta > \frac{2}{N-2}$ remains open for the chemorepulsion parabolic problem and $\theta > \frac{N+2}{N-2}$ for the chemorepulsion elliptic system.

Throughout the article, the standard notion of a weak solution is used.

Definition 1.1. *A weak solution (u, ψ) to (1.5) is a pair*

$$(u, \psi) \in W^{1,\frac{N}{N-1}}(\Omega) \times W^{1,2}(\Omega)$$

satisfying $u\nabla\psi \in [L^1(\Omega)]^N$ and

$$\int_\Omega M(x)\nabla u\nabla v + \int_\Omega uv = -\chi \int_\Omega M(x)u\nabla\psi\nabla v + \int_\Omega f(x)v,$$

$$\int_\Omega M(x)\nabla\psi\nabla\phi + \int_\Omega \psi\phi = \int_\Omega u^\theta\phi + \int_\Omega g(x)\phi$$

for any $v, \phi \in C^1(\Omega) \cap W^{1,\frac{N}{N-2}}(\Omega) \cap L^\infty(\Omega)$.

The main result of the article is stated in the following theorem.

Theorem 1.1. *Let f and g be given functions satisfying (1.6) and (1.7), respectively. Under the assumptions (1.1)–(1.4), there exists at least one weak solution, in the sense of Definition 1.1. Moreover, if in addition θ and g satisfy (1.8) and (1.9), respectively, we have that*

$$u \in W^{1,2}(\Omega), \qquad \psi \in W^{1,2}(\Omega) \cap L^\infty(\Omega).$$

The article is organized as follows. In Section 2, we approximate the system by using a truncation function. We then obtain the existence of solutions (u_n, ψ_n) of the approximate problem. In Section 3, we deduce some a priori estimates first under assumptions (1.4) and (1.7), and then under the stronger assumptions (1.8) and (1.9). Finally, in Section 4, using the estimates obtained in the previous section, we prove the existence of solutions by passing to the limit in the weak formulation of the approximate system. To do so, we use classical results of measure theory: Fatou′s lemma, Egorov′s theorem and the generalized Lebesgue′s dominated convergence theorem that can be found in standard textbooks on measure theory.

We use standard notation and denote by $|\Omega|$ the Lebesgue measure of Ω given by $|\Omega| = \int_\Omega 1$ and

$$\mu(k < u < h) = \int_{k<u<h} 1, \text{ for } k < h.$$

For any $k > 0$, the truncation function $T_k : \mathbb{R} \to [-k, k]$ defined by $T_k(s) = \max\{-k, \min\{s, k\}\}$ is used throughout the article. We also use the following functions:

- T_k^*, for $k > 0$ is defined by

$$T_k^*(s) = \begin{cases} 0, \text{ if } 0 < s; \\ s, \text{ if } -k < s \leq 0; \\ -k, \text{ if } s \leq -k. \end{cases}$$

- The positive part function defined by $(s)_+ = \max\{0, s\}$.

We assume, throughout the manuscript, (1.1) and (1.2) for the matrix M and (1.6) for the data f. Assumption (1.4) for θ is used in Lemma 3.5 to apply Hölder′s inequality and to obtain the boundedness in $W^{1,2}(\Omega)$ of $\{\psi_n\}_{n\in\mathbb{N}}$. The consequences of the lemma, presented in the subsequent corollaries, is the boundedness of $\{T_n(u_n)\}_{n\in\mathbb{N}}$ in $L^{\theta+1}(\Omega)$. Assumptions (1.8) and (1.9) are needed to apply the Stampacchia method and to provide the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ in $W^{1,2}(\Omega)$ and $L^\infty(\Omega)$, respectively (Lemmas 3.11 and 3.12). These estimates are used in Section 4 to pass to the limit in the weak formulation of the approximated problem to prove the main result.

2. Approximate problem

For simplicity of notation, we define the linear elliptic operator $\mathcal{L}$ as follows:

$$\mathcal{L}(v) := -\mathrm{div}(M(x)\nabla v), \tag{2.1}$$

where M satisfies assumptions (1.1) and (1.2).

The main difficulties of the problem are as follows: The nonlinear terms $\chi\mathrm{div}(u\nabla\psi)$ and u^θ and the weakness of the given data f and g. To avoid such difficulties, we approximate the nonlinearities by truncated terms: u^θ by $|T_n(u)|^\theta$, and the chemotaxis term by

$$\chi\mathrm{div}\Big(\frac{u_n}{1 + \frac{1}{n}|u_n|} M(x)\nabla\psi_n\Big).$$

The weak data f and g, are approximated by their truncations

$$f_n(x) = \min\{f(x), n\}, \quad g_n(x) = \min\{g(x), n\} \tag{2.2}$$

respectively. We consider the auxiliary problem

$$\begin{cases} \mathcal{L}(u_n) + u_n = \chi \mathrm{div}\Big(\dfrac{u_n}{1+\frac{1}{n}|u_n|} M(x)\nabla\psi_n\Big) + f_n(x), & x \in \Omega \\ \mathcal{L}(\psi_n) + \psi_n = |T_n(u_n)|^\theta + g_n(x), & x \in \Omega \\ \dfrac{\partial u_n}{\partial n_M} = \dfrac{\partial \psi_n}{\partial n_M} = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

We now define the notion of a weak solution to (2.3).

Definition 2.1. *Let $n \in \mathbb{N}$, a weak solution of (2.3) is a pair of functions $(u_n, \psi_n) \in [W^{1,2}(\Omega)]^2$, satisfying*

$$\int_\Omega M(x)\nabla u_n \nabla v + \int_\Omega u_n v = -\int_\Omega \chi\left(\frac{u_n}{1+\frac{1}{n}|u_n|} M(x)\nabla\psi_n\nabla v\right) + \int_\Omega f_n v,$$

$$\int_\Omega M(x)\nabla\psi_n\nabla\phi + \int_\Omega \psi_n\phi = \int_\Omega |T_n(u_n)|^\theta\phi + \int_\Omega g_n\phi$$

for any $v, \phi \in W^{1,2}(\Omega)$.

System (2.3) was studied in [2] where the chemotaxis term appears with the opposite sign, i.e., when u_n satisfies the equation

$$\mathcal{L}(u_n) + u_n = -\chi \mathrm{div}\left(\frac{u_n}{1+\frac{1}{n}|u_n|} M(x)\nabla\psi_n\right) + f_n(x)$$

for $\theta < \frac{2}{N}$ when the boundary condition is of Dirichlet type. Even though the proof is similar, we present the details of the existence of a solution of (2.3) in the following theorem.

Theorem 2.1. *Under assumptions (1.4) and (1.7), there exists at least one solution $(u_n, \psi_n) \in [W^{1,2}(\Omega)]^2$ to (2.3).*

Proof. We proceed in several steps.

Step 1. We take $w \in W^{1,2}(\Omega)$ and solve the problem

$$\begin{cases} \tilde{\psi} \in W^{1,2}(\Omega), \quad \mathcal{L}(\tilde{\psi}) + \tilde{\psi} = |T_n(w)|^\theta + g_n, & x \in \Omega, \\ \dfrac{\partial\tilde{\psi}}{\partial n_M} = 0, & x \in \partial\Omega. \end{cases} \tag{2.4}$$

Since

$$|T_n(w)|^\theta \le n^\theta \quad \text{and} \quad g_n \le n,$$

thanks to the Lax-Milgram theorem and the maximum principle, there exists a unique solution to (2.4) $\tilde{\psi} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Moreover,

$$\|\tilde{\psi}\|_{W^{1,2}(\Omega)} \le [n^\theta + n]|\Omega|^{\frac{1}{2}}, \quad \|\tilde{\psi}\|_{L^\infty(\Omega)} \le n^\theta + n.$$

Step 2. We consider the problem

$$\begin{cases} \tilde{u} \in W^{1,2}(\Omega), \quad \mathcal{L}(\tilde{u}) + \tilde{u} = \operatorname{div}\chi\left(\frac{w}{1+\frac{1}{n}|w|}M(x)\nabla\tilde{\psi}\right) + f_n(x) \\ \frac{\partial \tilde{u}}{\partial n_M} = 0, \quad x \in \partial\Omega, \end{cases} \tag{2.5}$$

where $\tilde{\psi}$ is the unique solution to (2.4). Then, thanks to the estimates obtained in the previous step and the Lax-Milgram theorem, there exists a unique solution $\tilde{u} \in W^{1,2}(\Omega)$ to (2.5). Using $\tilde{u}$ as a test function in (2.5), we have

$$\|\tilde{u}\|^2_{W^{1,2}(\Omega)} \le n\chi\|\tilde{\psi}\|_{W^{1,2}(\Omega)}\|\tilde{u}\|_{W^{1,2}(\Omega)} + n\|\Omega|^{\frac{1}{2}}\|\tilde{u}\|_{L^2(\Omega)}.$$

Therefore, we deduce

$$\|\tilde{u}\|_{W^{1,2}(\Omega)} \le n\chi[n^\theta + n]|\Omega|^{\frac{1}{2}} + n\|\Omega|^{\frac{1}{2}}.$$

Step 3. We define the operator $S_n : L^2(\Omega) \to L^2(\Omega)$ such that $S_n(w) = \tilde{u}$, where $\tilde{u}$ is the solution to (2.5). We denote by B_n the set

$$B_n = \{v \in W^{1,2}(\Omega), \quad \|v\|_{W^{1,2}(\Omega)} \le n\chi[n^\theta + n]|\Omega|^{\frac{1}{2}} + n|\Omega|^{\frac{1}{2}}\}.$$

To prove the continuity of S_n, we take a sequence $\{w_j\}_{j\in\mathbb{N}}$ which converges strongly to w in $L^2(\Omega)$. We denote by $\{\tilde{\psi}_j\}_{j\in\mathbb{N}}$ the solution to the problem

$$\begin{cases} \tilde{\psi}_j \in W^{1,2}(\Omega), \quad \mathcal{L}(\tilde{\psi}_j) + \tilde{\psi}_j = |T_n(w_j)|^\theta + g_n, \quad x \in \Omega, \\ \frac{\partial \tilde{\psi}_j}{\partial n_M} = 0, \quad x \in \partial\Omega. \end{cases}$$

Then, by continuous dependence of the data, $\{\tilde{\psi}_j\}_{j\in\mathbb{N}}$ converges strongly in $W^{1,2}(\Omega)$ to $\tilde{\psi}$ solution to the equation

$$\begin{cases} \tilde{\psi} \in W^{1,2}(\Omega), \quad \mathcal{L}(\tilde{\psi}) + \tilde{\psi} = |T_n(w)|^\theta + g_n, \quad x \in \Omega, \\ \frac{\partial \tilde{\psi}}{\partial n_M} = 0, \quad x \in \partial\Omega. \end{cases}$$

Now, we consider $\tilde{u}_j$ the solution of

$$\begin{cases} \tilde{u}_j \in W^{1,2}(\Omega), \quad \mathcal{L}(\tilde{u}_j) + \tilde{u}_j = \operatorname{div}\chi\left(\frac{w_j}{1+\frac{1}{n}|w_j|}M(x)\nabla\tilde{\psi}_j\right) + f_n(x) \\ \frac{\partial \tilde{u}_j}{\partial n_M} = 0, \quad x \in \partial\Omega \end{cases}$$

and $\tilde{u}$ the solution of

$$\begin{cases} \tilde{u} \in W^{1,2}(\Omega), \quad \mathcal{L}(\tilde{u}) + \tilde{u} = \operatorname{div}\chi\left(\frac{w}{1+\frac{1}{n}|w|}M(x)\nabla\tilde{\psi}\right) + f_n(x) \\ \frac{\partial \tilde{u}}{\partial n_M} = 0, \quad x \in \partial\Omega. \end{cases} \tag{2.6}$$

Then, we have that

$$\mathcal{L}(\tilde{u} - \tilde{u}_j) + \tilde{u} - \tilde{u}_j = \mathrm{div}\chi\left(\frac{w}{1 + \frac{1}{n}|w|} M(x)\nabla\tilde{\psi} - \frac{w_j}{1 + \frac{1}{n}|w_j|} M(x)\nabla\tilde{\psi}_j\right).$$

Since the function $\frac{s}{1+|s|}$ is Lipschitz, we have that $\left\{\frac{w_j}{1 + \frac{1}{n}|w_j|}\right\}_{j\in\mathbb{N}}$ converges strongly in $L^2(\Omega)$ to $\frac{w}{1 + \frac{1}{n}|w|}$. Moreover, since the sequence is bounded in $L^\infty(\Omega)$, we have that it converges strongly in $L^p(\Omega)$ for any $p < +\infty$. Therefore, $\left\{\frac{w_j}{1 + \frac{1}{n}|w_j|} M(x)\nabla\tilde{\psi}_j\right\}$ converges strongly to $\left\{\frac{w}{1 + \frac{1}{n}|w|} M(x)\nabla\tilde{\psi}\right\}_{j\in\mathbb{N}}$ in $[L^q(\Omega)]^N$ for $q \in [1, 2)$. Moreover, since $\left\{\frac{w_j}{1 + \frac{1}{n}|w_j|} M(x)\nabla\tilde{\psi}_j\right\}$ is bounded in $[L^2(\Omega)]^N$, there exists a sub-subsequence such that it converges to $\left\{\frac{w}{1 + \frac{1}{n}|w|} M(x)\nabla\tilde{\psi}\right\}$ weakly in $[L^2(\Omega)]^N$ by uniqueness of limit in $[L^p(\Omega)]^N$ and the embedding $[L^2(\Omega)]^N \hookrightarrow [L^p(\Omega)]^N$ for $p \in [1, 2)$. If there exists a subsequence which does not converge to $\left\{\frac{w}{1 + \frac{1}{n}|w|} M(x)\nabla\tilde{\psi}\right\}$ weakly in $[L^2(\Omega)]^N$, then there exists $\xi \in [L^2(\Omega)]^N$ such that

$$\lim_{n\to+\infty} \int_\Omega \left(\frac{w_j}{1 + \frac{1}{n}|w_j|} M(x)\nabla\tilde{\psi}_j\right)\xi = c \neq 0$$

which contradicts the existence of a subsequence which weakly converges in $[L^2(\Omega)]^N$ to $\left\{\frac{w}{1 + \frac{1}{n}|w|} M(x)\nabla\tilde{\psi}\right\}$ and proves that $\left\{\frac{w_j}{1 + \frac{1}{n}|w_j|} M(x)\nabla\tilde{\psi}_j\right\}$ converges to $\left\{\frac{w}{1 + \frac{1}{n}|w|} M(x)\nabla\tilde{\psi}\right\}$ weakly in $[L^2(\Omega)]^N$. We use $\xi \in W^{1,2}(\Omega)$ as a test function in (2.6) to obtain that

$$\int_\Omega M(x)\nabla(\tilde{u} - \tilde{u}_j)\nabla\xi + \int_\omega (\tilde{u} - \tilde{u}_j)\xi = \chi \int_\Omega M(x)\left(\frac{w_j}{1 + \frac{1}{n}|w_j|} M(x)\nabla\tilde{\psi}_j - \frac{w}{1 + \frac{1}{n}|w|} M(x)\nabla\tilde{\psi}\right)\nabla\xi.$$

Passing to the limit as $j \to +\infty$, we deduce

$$\lim_{j\to+\infty} \int_\Omega M(x)\nabla(\tilde{u} - \tilde{u}_j)\nabla\xi + \int_\omega (\tilde{u} - \tilde{u}_j)\xi = 0.$$

Hence, we have the weak convergence of $\tilde{u} - \tilde{u}_j$ to 0 in $W^{1,2}(\Omega)$. By the compact embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$, we conclude that $\tilde{u}_j$ converges strongly to $\tilde{u}$ in $L^2(\Omega)$.
Since S_n is continuous, $S_n(L^2(\Omega)) \subset B_n$, and B_n is a relatively compact set in $L^2(\Omega)$, we have, by Schauder′s fixed point theorem, that there exists at least one solution to (2.3).

□

3. Estimates of the approximate problem

Lemma 3.1. *Under assumption (1.7), the solutions to (2.3) satisfy*

$$u_n \geq 0, \quad \psi_n \geq 0. \tag{3.1}$$

Proof. We consider $T_k^*(u_n)$. Since T_k^* is a Lipschitz function and $u_n \in W^{1,2}(\Omega)$, we deduce that $T_k^*(u_n)$ is an admissible test function. We then use $T_k^*(u_n)$ as a test function in (2.3) to obtain, in view of $f \geq 0$, that

$$\begin{aligned}\int_\Omega |\nabla[T_k^*(u_n)]|^2 + \int_\Omega u_n T_k^*(u_n) &\leq \beta\chi \int_{-k\leq u_n<0} |u_n||\nabla\psi_n||\nabla[T_k^*(u_n)]| \\ &\leq k\beta\chi \|\psi_n\|_{W^{1,2}(\Omega)} \|\nabla T_k^*(u_n)\|_{[L^2(\Omega)]^N}.\end{aligned}$$

Then,

$$\int_\Omega u_n T_k^*(u_n) \leq \frac{k^2\beta^2\chi^2}{2}\|\psi_n\|^2_{W^{1,2}(\Omega)}.$$

We divide by k and take the limit as $k \to 0$ to obtain, in view of

$$\lim_{k\to 0} \frac{u_n T_k^*(u_n)}{k} = \begin{cases} 0 \text{ if } u_n \geq 0 \\ |u_n|, \text{ otherwise} \end{cases}$$

that

$$\int_{u_n<0} |u_n| \leq 0$$

which implies that $u_n \geq 0$ a.e. $x \in \Omega$. Since $g_n(x) \geq 0$ and

$$[T_n(u_n)]^\theta \geq 0$$

the weak maximum principle implies that $\psi_n \geq 0$. □

Remark 3.2. *Integrating the first equation and using the boundary condition, we have*

$$\int_\Omega u_n \leq \int_\Omega f.$$

Since $u_n \geq 0$, we obtain a uniform bound of $\{u_n\}_{n\in\mathbb{N}}$ in $L^1(\Omega)$, i.e.

$$\|u_n\|_{L^1(\Omega)} \leq c.$$

Lemma 3.3. *Let (u_n, ψ_n) be a solution to (2.3). Then, under assumption (1.7) and for any positive θ, we have*

$$\int_\Omega [T_n(u_n)]^\theta u_n + \chi \int_\Omega \frac{u_n}{1 + \frac{1}{n}u_n}|\nabla\psi_n|^2 \leq \int_\Omega f\psi_n. \tag{3.2}$$

Proof. We use ψ_n as a test function in the first equation of (2.3) and u_n as a test function in the second equation, to obtain the identities:

$$\int_\Omega M(x)\nabla u_n \nabla\psi_n + \int_\Omega u_n\psi_n = -\chi \int_\Omega \frac{u_n}{1 + \frac{1}{n}u_n} M(x)\nabla\psi_n\nabla\psi_n + \int_\Omega f_n\psi_n$$

and

$$\int_\Omega M(x)\nabla\psi_n\nabla u_n + \int_\Omega \psi_n u_n = \int_\Omega [T_n(u_n)]^\theta u_n + \int_\Omega g_n u_n.$$

Subtracting the two identities, we deduce

$$\int_\Omega [T_n(u_n)]^\theta\, u_n + \int_\Omega g_n u_n + \chi \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n} M(x)\nabla\psi_n\nabla\psi_n = \int_\Omega f_n\psi_n.$$

Since

$$0 \le g_n u_n, \quad \int_\Omega f_n\psi_n \le \int_\Omega f\psi_n$$

and

$$\chi \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^2 \le \chi \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n} M(x)\nabla\psi_n\nabla\psi_n$$

we deduce the desired result. □

Corollary 3.4. *Under assumption (1.7) and for any positive θ, we have*

$$\int_\Omega [T_n(u_n)]^{\theta+1} \le \int_\Omega f\psi_n.$$

Proof. The result follows immediately from the previous lemma and the inequality

$$0 \le T_n(u_n) \le u_n.$$

□

Lemma 3.5. *Letting $n \in \mathbb{N}$, then under assumptions (1.4) and (1.7), we have*

$$\|\psi_n\|_{W^{1,2}(\Omega)} \le c, \ \textit{where c is independent of n.}$$

Proof. We use ψ_n as a test function in the second equation of (2.3) to obtain, after straightforward computations,

$$\int_\Omega |\nabla\psi_n|^2 + \int_\Omega \psi_n^2 \le \int_\Omega [T_n(u_n)]^\theta\,\psi_n + \int_\Omega g_n\psi_n.$$

We apply Hölder′s inequality to each term on the right-hand side of the previous identity using exponents $\frac{\theta+1}{\theta}$ and $\theta+1$ to obtain:

$$\int_\Omega [T_n(u_n)]^\theta\,\psi_n \le \left[\int_\Omega [T_n(u_n)]^{\theta+1}\right]^{\frac{\theta}{\theta+1}} \left[\int_\Omega \psi_n^{\theta+1}\right]^{\frac{1}{\theta+1}}. \tag{3.3}$$

Thanks to Corollary 3.4, we have that

$$\left[\int_\Omega [T_n(u_n)]^{\theta+1}\right]^{\frac{\theta}{\theta+1}} \le \left[\int_\Omega f\psi_n\right]^{\frac{\theta}{\theta+1}}$$

and applying Hölder′s inequality to the terms on the right-hand side of the previous inequality with exponents $\frac{2N}{N+2}$ and $\frac{2N}{N-2}$, we obtain

$$\begin{aligned}\left[\int_\Omega f\psi_n\right]^{\frac{\theta}{\theta+1}} &\leq \left[\left[\int_\Omega f^{\frac{2N}{N+2}}\right]^{\frac{N+2}{2N}}\left[\int_\Omega \psi_n^{\frac{2N}{N-2}}\right]^{\frac{N-2}{2N}}\right]^{\frac{\theta}{\theta+1}}\\ &= \left[\int_\Omega f^{\frac{2N}{N+2}}\right]^{\frac{N+2}{2N}\frac{\theta}{(1+\theta)}}\left[\int_\Omega \psi_n^{\frac{2N}{N-2}}\right]^{\frac{N-2}{2N}\frac{\theta}{(1+\theta)}}\\ &= c(f,\theta)\left[\int_\Omega \psi_n^{\frac{2N}{N-2}}\right]^{\frac{N-2}{2N}\frac{\theta}{(1+\theta)}}.\end{aligned}$$

Since $\theta+1<\frac{N+2}{N-2}+1=\frac{2N}{N-2}$, it follows that

$$\left[\int_\Omega \psi_n^{\theta+1}\right]^{\frac{1}{\theta+1}} \leq c(\Omega)\left[\int_\Omega \psi_n^{\frac{2N}{N-2}}\right]^{\frac{N-2}{2N}}$$

Hence, (3.3) becomes

$$\int_\Omega [T_n(u_n)]^\theta\,\psi_n \leq c(\Omega)\left[\int_\Omega \psi_n^{\frac{2N}{N-2}}\right]^{\frac{N-2}{2N}\left(1+\frac{\theta}{\theta+1}\right)}.$$

In view of

$$\int_\Omega g_n\psi_n \leq \int_\Omega g\psi_n \leq \left[\int_\Omega g^{\frac{2N}{N+2}}\right]^{\frac{N+2}{2N}}\left[\int_\Omega \psi_n^{\frac{2N}{N-2}}\right]^{\frac{N-2}{2N}}$$

we have

$$\int_\Omega |\nabla\psi_n|^2+\int_\Omega \psi_n^2 \leq c(\Omega)\left[\int_\Omega \psi_n^{\frac{2N}{N-2}}\right]^{\frac{N-2}{2N}\left(1+\frac{\theta}{\theta+1}\right)}+c\left[\int_\Omega \psi_n^{\frac{2N}{N-2}}\right]^{\frac{N-2}{2N}}.$$

Since $1+\frac{\theta}{1+\theta}<2$, together with the Sobolev embedding $W^{1,2}(\Omega)\hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$, we deduce

$$\|\psi_n\|_{W^{1,2}(\Omega)} \leq c,\ \text{where } c \text{ is independent of } n.$$

□

Corollary 3.6. *Thanks to Hölder′s inequality and the Sobolev embedding* $W^{1,2}(\Omega)\hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$*, we have, under the assumptions of Lemma 3.5,*

$$\int_\Omega f\psi_n \leq \left[\int_\Omega f^{\frac{2N}{N+2}}\right]^{\frac{N+2}{2N}}\left[\int_\Omega \psi_n^{\frac{2N}{N-2}}\right]^{\frac{N-2}{2N}} \leq c\left[\int_\Omega f^{\frac{2N}{N+2}}\right]^{\frac{N+2}{2N}}\|\psi_n\|_{W^{1,2}(\Omega)}.$$

The previous lemma together with the boundedness of f *in* $L^{\frac{2N}{N+2}}(\Omega)$ *implies:*

$$\int_\Omega f\psi_n \leq c.$$

Corollary 3.7. *By Corollaries 3.6 and 3.4, we have, under the assumptions of Lemma 3.5,*

$$\int_\Omega [T_n(u_n)]^{\theta+1} \leq \int_\Omega f\psi_n \leq c.$$

COROLLARY 3.8. *Under the assumptions of Lemma 3.5, Corollary 3.4 and (3.2) imply the uniform integrability*

$$\int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^2 \le c.$$

LEMMA 3.9. *Under assumptions (1.4) and (1.7), we have that*

$$\int_\Omega \frac{|\nabla u_n|^2}{1+u_n} \le c.$$

Proof. We consider the function $\ln(1+u_n)$ and note that $\ln(1+s)$ is a Lipschitz function for $s>0$. Thanks to Theorem 2.1 and Lemma 3.1, we have that $\ln(1+u_n)$ is an admissible test function. We use $\ln(1+u_n)$ as a test function in the first equation of (2.3) to obtain, after straightforward computations,

$$\int_\Omega \frac{|\nabla u_n|^2}{1+u_n} + \int_\Omega u_n \ln(1+u_n) \le \chi \int_\Omega \frac{u_n}{(1+\frac{1}{n}u_n)(1+u_n)} M(x)\nabla\psi_n\nabla u_n + \int_\Omega f\ln(1+u_n). \tag{3.4}$$

Since

$$\begin{aligned}\int_\Omega \frac{u_n M(x)\nabla\psi_n\nabla u_n}{(1+\frac{1}{n}u_n)(1+u_n)} &\le \frac{\beta^2\chi^2}{2}\int_\Omega \frac{u_n}{(1+\frac{1}{n}u_n)}|\nabla\psi_n|^2 + \frac{1}{2}\int_\Omega \frac{u_n}{(1+\frac{1}{n}u_n)(1+u_n)^2}|\nabla u_n|^2 \\ &\le \frac{\chi^2\beta^2}{2}\int_\Omega \frac{u_n}{(1+\frac{1}{n}u_n)}|\nabla\psi_n|^2 + \frac{1}{2}\int_\Omega \frac{1}{1+u_n}|\nabla u_n|^2.\end{aligned}$$

Substituting the previous estimate into (3.4) yields

$$\frac{1}{2}\int_\Omega \frac{|\nabla u_n|^2}{1+u_n} + \int_\Omega u_n\ln(1+u_n) \le \frac{\chi^2\beta^2}{2}c + \int_\Omega f\ln(1+u_n)$$

where the constant c is provided by Corollary 3.8. Since

$$\int_\Omega f\ln(1+u_n) \le \|f\|_{L^{\frac{2N}{N+2}}(\Omega)}\left[\int_\Omega [\ln(1+u_n)]^{\frac{2N}{N-2}}\right]^{\frac{N-2}{2N}} \le 6!\|f\|_{L^{\frac{2N}{N+2}}(\Omega)}\left[\int_\Omega u_n\right]^{\frac{N-2}{2N}} \le 6!\|f\|_{L^{\frac{2N}{N+2}}(\Omega)}\|f\|_{L^1(\Omega)}^{\frac{N-2}{2N}},$$

we deduce

$$\frac{1}{2}\int_\Omega \frac{|\nabla u_n|^2}{1+u_n} \le \frac{\chi^2\beta^2}{2}c + 6!\|f\|_{L^{\frac{2N}{N+2}}(\Omega)}\|f\|_{L^1(\Omega)}^{\frac{N-2}{2N}},$$

(for c given in Corollary 3.8) which concludes the proof. □

LEMMA 3.10. *Under assumptions (1.4) and (1.7), we have*

$$\int_\Omega |\nabla u_n|^{\frac{N}{N-1}} + \int_\Omega u_n^{\frac{N}{N-2}} \le c.$$

Proof. We first notice that

$$\frac{|\nabla u_n|^2}{1+u_n} = 4|\nabla(1+u_n)^{\frac{1}{2}}|^2$$

which implies, in view of Lemma 3.9 and Remark 3.2, that

$$\{(1+u_n)^{\frac{1}{2}}\}_{n\in\mathbb{N}} \text{ is uniformly bounded in } W^{1,2}(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$$

and therefore

$$\{u_n\}_{n\in\mathbb{N}} \text{ is uniformly bounded in } L^{\frac{N}{N-2}}(\Omega). \tag{3.5}$$

Thanks to Young's inequality for exponents $\frac{2(N-1)}{N}$ and $\frac{2(N-1)}{N-2}$, we deduce

$$|\nabla u_n|^{\frac{N}{N-1}} = \frac{|\nabla u_n|^{\frac{N}{N-1}}}{(1+u_n)^{\frac{N}{2(N-1)}}}(1+u_n)^{\frac{N}{2(N-1)}} \leq \frac{|\nabla u_n|^2}{(1+u_n)} + c(1+u_n)^{\frac{N}{N-2}}.$$

Integrating the above inequality and using (3.5), we obtain the desired result. □

Lemma 3.11. *Under assumptions (1.8) and (1.9), the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$, i.e.*

$$\{\psi_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega).$$

Proof. Let $k>0$, and let G_k be the function

$$G_k(s) = \left(\frac{s}{s+1} - \frac{k}{k+1}\right)_+.$$

Notice that G_k is a Lipschitz function for $s \geq 0$, and in view of $\psi_n \in W^{1,2}(\Omega)$ and $\psi_n \geq 0$, we have that $G_k(\psi_n)$ is an admissible test function. We use $G_k(\psi_n)$ as a test function in the second equation to obtain, after some computations and using that $G_k(\psi_n) \leq 1$, we have that

$$\int_{\psi_n>k} \frac{|\nabla\psi_n|^2}{(1+\psi_n)^2} + \int_{\psi_n>k} \psi_n G_k(\psi_n) \leq \int_{\psi_n>k} [T_n(u_n)]^\theta + \int_{\psi_n>k} g_n.$$

We add the term

$$\int_{\psi_n>k} (\ln(1+\psi_n) - \ln(1+k))_+^2$$

to both sides of the inequality, to deduce

$$\int_{\psi_n>k} \frac{|\nabla\psi_n|^2}{(1+\psi_n)^2} + \int_{\psi_n>k} (\ln(1+\psi_n) - \ln(1+k))_+^2$$

$$\leq \int_{\psi_n>k} [T_n(u_n)]^\theta + \int_{\psi_n>k} g_n + \int_{\psi_n>k} (\ln(1+\psi_n) - \ln(1+k))_+^2.$$

Setting $w_k = (\ln(1+\psi_n) - \ln(1+k))_+$, we have

$$\int_{\psi_n>k} |\nabla w_k|^2 + \int_{\psi_n>k} w_k^2 \leq \int_{\psi_n>k} [T_n(u_n)]^\theta + \int_{\psi_n>k} g + \int_{\psi_n>k} w_k^2.$$

Notice that

$$\int_{\psi_n>k} [T_n(u_n)]^\theta \leq \|T_n(u_n)\|^\theta_{L^{\theta+1}(\Omega)} [\mu(\psi_n>k)]^{\frac{1}{1+\theta}} \leq c[\mu(\psi_n>k)]^{\frac{1}{1+\theta}}$$

and thanks to assumption (1.4), we have

$$\frac{1}{\theta+1} > \frac{N-2}{N}.$$

We also notice that

$$\int_{\psi_n>k} g_n \leq \|g\|_{L^q(\Omega)}[\mu(\psi_n>k)]^{\frac{1}{q'}},\ \text{for } q>\frac{N}{2}.$$

By assumption (1.9), $q>\frac{N}{2}$, and therefore

$$\frac{1}{q'} = 1-\frac{1}{q} > 1-\frac{2}{N} = \frac{N-2}{N}. \tag{3.6}$$

Finally, for sufficiently large k, it follows that

$$\begin{aligned}\int_{\psi_n>k}(\ln(1+\psi_n)-\ln(1+k))_+^2 &\leq \int_{\psi_n>k}[1+\psi_n]^{\frac{1}{q}}\\ &\leq [\mu(\psi_n>k)]^{\frac{1}{q'}}\|1+\psi_n\|_{L^1(\Omega)}^{\frac{1}{q}}\\ &\leq c[\mu(\psi_n>k)]^{\frac{1}{q'}},\end{aligned}$$

where q and q' satisfy (3.6). Then, thanks to the previous inequalities, we obtain

$$\int_{\psi_n>k}|\nabla w_k|^2+\int_{\psi_n>k} w_k^2 \leq c[\mu(\psi_n>k)]^s,\ \text{for } s=\min\left\{\frac{1}{q'},\frac{1}{1+\theta}\right\}$$

provided k is large enough. Let $h>k$, and notice

$$\begin{aligned}[\ln(1+h)-\ln(1+k)]^2[\mu(\psi_n>h)]^{\frac{N-2}{N}} &\leq \|w_k\|^2_{L^{\frac{2N}{N-2}}(\psi_n>h)}\\ &\leq \|w_k\|^2_{L^{\frac{2N}{N-2}}(\psi_n>k)}\\ &\leq \int_{\psi_n>k}|\nabla w_k|^2+\int_{\psi_n>k} w_k^2\\ &\leq c[\mu(\psi_n>k)]^s,\end{aligned}$$

i.e.

$$[\ln(1+h)-\ln(1+k)]^2[\mu(\psi_n>h)]^{\frac{N-2}{N}} \leq c[\mu(\psi_n>k)]^s$$

where

$$s=\min\left\{\frac{1}{q'},\frac{1}{1+\theta}\right\} > \frac{N-2}{N}.$$

We denote by ϕ_n the function:

$$\phi_n(k) = [\mu(\psi_n>k)]^{\frac{N-2}{N}}$$

which satisfies

$$[\ln(1+h)-\ln(1+k)]^2\phi_n(h) \leq c[\phi_n(k)]^{\frac{N}{N-2}s}.$$

We now apply Lemma 4.1(i) of Stampacchia [25], to obtain that there exists h large enough independent of n such that $\phi_n(h)=0$, which yields the uniform boundedness of the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ in $L^\infty(\Omega)$. □

Lemma 3.12. *Under the assumptions (1.8) and (1.9), we have*

$$\{u_n\}_{n\in\mathbb{N}} \subset W^{1,2}(\Omega).$$

Proof. We use u_n as a test function in the first equation to obtain, after some computations,

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^2 \leq -\chi \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n} M(x)\nabla\psi_n \nabla u_n + \int_\Omega f_n u_n. \tag{3.7}$$

Now, we define the function $H_n(s)$ such that

$$H_n'(s) = \chi \frac{s}{1+\frac{1}{n}s}; \quad H_n(0) = 0$$

and note that H_n is a Lipschitz function for $s > 0$. In view of Lemma 3.1 and Theorem 2.1, $H_n(u_n)$ belongs to the admissible set of test functions. We use it in the second equation of (2.3) to obtain

$$\chi \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n} M(x)\nabla\psi_n \nabla u_n + \int_\Omega \psi_n H_n(u_n) = \int_\Omega [T_n(u_n)]^\theta H_n(u_n) + \int_\Omega g_n H_n(u_n).$$

Substituting this identity into (3.7), we have:

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^2 + \int_\Omega [T_n(u_n)]^\theta H_n(u_n) + \int_\Omega g_n H_n(u_n) \leq \int_\Omega \psi_n H_n(u_n) + \int_\Omega f_n u_n.$$

The positivity of g_n and u_n implies

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^2 \leq \int_\Omega \psi_n u_n^2 + \int_\Omega f_n u_n. \tag{3.8}$$

The terms on the right-hand side are estimated as in [4]. First, we consider

$$\begin{aligned}
\int_\Omega f_n u_n &\leq \|f\|_{L^{\frac{2N}{N+2}}(\Omega)} \|u_n\|_{L^{\frac{2N}{N-2}}(\Omega)} \\
&\leq \frac{1}{c_\Omega} \|f\|_{L^{\frac{2N}{N+2}}(\Omega)} \|u_n\|_{W^{1,2}(\Omega)} \\
&\leq \frac{1}{c_\Omega^2} \|f\|_{L^{\frac{2N}{N+2}}(\Omega)} + \frac{1}{4}\|u_n\|^2_{W^{1,2}(\Omega)}
\end{aligned}$$

where c_Ω is the constant of the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$.

We split the term $\int_\Omega \psi_n u_n^2$ into two parts: Letting $k > 0$,

$$\begin{aligned}
\int_\Omega \psi_n u_n^2 &\leq \int_{u_n<k} \psi_n u_n^2 + \int_{k\leq u_n} \psi_n u_n^2 \\
&\leq k^2 \|\psi_n\|_{L^1(\Omega)} + \|\psi_n\|_{L^\infty(\Omega)} \|u_n\|^2_{L^{\frac{2N}{N-2}}(\Omega)} [\mu(u_n > k)]^{\frac{2}{N}}.
\end{aligned}$$

Since the sequence $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^1(\Omega)$, Chebyshev′s inequality gives

$$\lim_{k\to+\infty} \mu(u_n > k) \leq \lim_{k\to+\infty} \frac{c}{k} = 0.$$

Therefore, there exists $k_0 > 0$ large enough such that

$$[\mu(u_n > k)]^{\frac{2}{N}} < \frac{c_\Omega^2}{4\|\psi_n\|_{L^\infty(\Omega)}}.$$

Substituting the above estimate into (3.8) yields

$$\int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^2 \leq \frac{1}{2}\|u_n\|^2_{W^{1,2}(\Omega)} + \frac{1}{c_\Omega^2}\|f\|_{L^{\frac{2N}{N+2}}(\Omega)} \tag{3.9}$$

which implies the result.

□

4. Existence of solutions

The boundedness of $\{(u_n, \psi_n)\}_{n\in\mathbb{N}}$ in $W^{1,\frac{N}{N-1}}(\Omega) \times W^{1,2}(\Omega)$ implies the existence of a subsequence, still denoted by $\{(u_n, \psi_n)\}_{n\in\mathbb{N}}$, and a pair

$$(u, \psi) \in W^{1,\frac{N}{N-1}}(\Omega) \times W^{1,2}(\Omega)$$

such that:

1. $\{(u_n, \psi_n)\}_{n\in\mathbb{N}}$ converges weakly to (u, ψ) in $W^{1,\frac{N}{N-1}}(\Omega) \times W^{1,2}(\Omega)$;
2. $\{(u_n, \psi_n)\}_{n\in\mathbb{N}}$ converges strongly to (u, ψ) in $L^p(\Omega) \times L^q(\Omega)$ for $p < \frac{N}{N-2}$ and $q < \frac{2N}{N-2}$;
3. $\{(u_n, \psi_n)\}_{n\in\mathbb{N}}$ converges a.e. to (u, ψ).

For simplicity, we still denote this subsequence by $\{(u_n, \psi_n)\}_{n\in\mathbb{N}}$.

LEMMA 4.1. *Under assumptions (1.4) and (1.7), we have that there exists a subsequence of $\{\psi_n\}_{n\in\mathbb{N}}$, denoted with the same numbering, such that it converges strongly in $W^{1,2}(\Omega)$.*

Proof. Let $k, j \in \mathbb{N}$ such that $k < j$, and consider the sequence $\{\psi_n\}_{n\in\mathbb{N}}$, which converges strongly to ψ in $L^q(\Omega)$ for $q < \frac{2N}{N-2}$. Then, we use $\psi_j - \psi_k$ as a test function in the equation satisfied by $\psi_j - \psi_k$ to obtain:

$$\begin{aligned}
\int_\Omega |\nabla\psi_j - \nabla\psi_k|^2 + (\psi_j - \psi_k)^2 &\leq \int_\Omega \left[[T_j(u_j)]^\theta - [T_k(u_k)]^\theta \right| |\psi_j - \psi_k| + \int_{k<g} g|\psi_j - \psi_k| \\
&\leq \left(\|T_j(u_j)\|^\theta_{L^{\theta+1}(\Omega)} + \|T_k(u_k)\|^\theta_{L^{\theta+1}(\Omega)} \right) \|\psi_j - \psi_k\|_{L^{\theta+1}(\Omega)} \\
&\quad + \left[\int_{k<g} g^{\frac{2N}{N+2}} \right]^{\frac{N+2}{2N}} \|\psi_j - \psi_k\|_{L^{\frac{2N}{N-2}}(\Omega)} \\
&\leq c\|\psi_j - \psi_k\|_{L^{\theta+1}(\Omega)} + \frac{1}{2c_\Omega^2} \left[\int_{k<g} g^{\frac{2N}{N+2}} \right]^{\frac{N+2}{N}} \\
&\quad + \frac{c_\Omega^2}{2} \|\psi_j - \psi_k\|^2_{L^{\frac{2N}{N-2}}(\Omega)}.
\end{aligned}$$

Since $g \in L^q(\Omega)$ for some $q > \frac{2N}{N+2}$, it follows that

$$\left[\int_{k<g} g^{\frac{2N}{N+2}}\right]^{\frac{N+2}{N}} \leq \|g\|^2_{L^q(\Omega)}[\mu(k < g)]^{\frac{q(N+2)-2N}{Nq}}.$$

We then deduce, thanks to the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$, that

$$\frac{1}{2}\|\psi_j - \psi_k\|^2_{W^{1,2}(\Omega)} \leq c\|\psi_j - \psi_k\|_{L^{1+\theta}(\Omega)} + c[\mu(k < g)]^{\frac{q(N+2)-2N}{Nq}}. \tag{4.1}$$

Since

$$\theta + 1 < \frac{N+2}{N-2} + 1 = \frac{2N}{N-2},$$

the compact Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{\theta+1}(\Omega)$ together with the boundedness of $\{\psi_n\}_{n\in\mathbb{N}}$ in $W^{1,2}(\Omega)$ implies (up to a subsequence) that for any $\varepsilon > 0$, there exists n_0 such that

$$\|\psi_j - \psi_k\|_{L^{\theta+1}(\Omega)} < \varepsilon, \text{ provided } n_0 < k.$$

Since $g \in L^1(\Omega)$ and ψ_k converges strongly in $L^{1+\theta}(\Omega)$ to ψ, taking limits as $k \to +\infty$ in (4.1), we obtain the desired result. □

Corollary 4.2. *Under assumptions (1.4) and (1.7), there exists a subsequence of $\{\psi_n\}_{n\in\mathbb{N}}$, still denoted by $\{\psi_n\}_{n\in\mathbb{N}}$, such that*

$$\{\nabla\psi_n\}_{n\in\mathbb{N}} \text{ converges to } \nabla\psi \text{ a.e. } x \in \Omega.$$

The result follows from the strong convergence of $\{\psi_n\}_{n\in\mathbb{N}}$ to ψ in $W^{1,2}(\Omega)$.

Lemma 4.3. *Under assumptions (1.4) and (1.7), we have that*

$$\left\{\frac{u_n}{1+\frac{1}{n}u_n}\nabla\psi_n\right\} \to u\nabla\psi \quad \textit{strongly in } [L^1(\Omega)]^N.$$

Proof. We consider the sequence

$$\left\{\left|\frac{u_n}{1+\frac{1}{n}u_n}\nabla\psi_n\right|\right\}_{n\in\mathbb{N}}.$$

Thanks to Young′s inequality for the exponents $\frac{2(N-1)}{N}$ and $\frac{2(N-1)}{N-2}$, we have

$$\int_\Omega \left|\frac{u_n}{1+\frac{1}{n}u_n}\nabla\psi_n\right|^{\frac{N}{N-1}} \leq c\int_\Omega \left|\frac{u_n}{1+\frac{1}{n}u_n}\right|^{\frac{N}{N-2}} + c\int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}|\nabla\psi_n|^2,$$

Lemma 3.10 and Corollary 3.8 imply the uniform integrability

$$\int_\Omega \left|\frac{u_n}{1+\frac{1}{n}u_n}\nabla\psi_n\right|^{\frac{N}{N-1}} \leq c. \tag{4.2}$$

Moreover, from the convergence of $\{u_n\}_{n\in\mathbb{N}}$ to u in $L^{\frac{N}{N-1}}(\Omega)$, we deduce the existence of a subsequence, which for simplicity is denoted by the same index, such that

$$\{u_n\}_{n\in\mathbb{N}} \to u, \text{ a.e.} x \in \Omega, \text{ and therefore } \left\{\frac{u_n}{1+\frac{1}{n}u_n}\right\}_{n\in\mathbb{N}} \to u, \text{ a.e. } x \in \Omega.$$

Thanks to Lemma 4.1, we have that $\{\nabla\psi_n\}_{n\in\mathbb{N}}$ converges strongly to $\nabla\psi$ in $[L^2(\Omega)]^N$. Therefore, there exists a subsequence (still denoted with the same numbering) which converges to $\nabla\psi$ a.e. $x \in \Omega$. Then, it follows that

$$\lim_{n\to+\infty}\left|\frac{u_n}{1+\frac{1}{n}u_n}\nabla\psi_n\right|^{\frac{N}{N-1}} \longrightarrow |u\nabla\psi|^{\frac{N}{N-1}}, \quad \text{a.e. } x \in \Omega.$$

The previous convergence guarantees the convergence in μ-measure of the sequence:

$$\lim_{n\to+\infty}\left|\frac{u_n}{1+\frac{1}{n}u_n}\nabla\psi_n\right|^{\frac{N}{N-1}} \xrightarrow{\mu} |u\nabla\psi|^{\frac{N}{N-1}}. \tag{4.3}$$

(4.2) and (4.3) imply, applying Vitali′s theorem, the desired result. □

LEMMA 4.4. *Under the assumptions (1.6) and (1.7), we have that*

$$\lim_{n\to+\infty}[T_n(u_n)]^\theta \to u^\theta \quad \textit{strongly in } L^1(\Omega).$$

Proof. Since $W^{1,\frac{N}{N-1}}(\Omega) \hookrightarrow L^p(\Omega)$ is a compact embedding for $p < \frac{N}{N-2}$, we have that there exists a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, such that

$$\{u_n\}_{n\in\mathbb{N}} \to u, \text{ strongly in } L^p(\Omega), \text{ for } p < \frac{N}{N-2}.$$

Hence, there exists a subsequence (denoted with the same index) such that

$$\{u_n\}_{n\in\mathbb{N}} \to u, \text{ pointwise a.e. } x \in \Omega$$

and, therefore, for any $\gamma \in (0, \theta+1]$ we have

$$\{[T_n(u_n)]^\gamma\}_{n\in\mathbb{N}} \to u^\gamma, \text{ pointwise a.e. } x \in \Omega,$$

which guarantees the convergence in μ-measure of the sequence:

$$\left\{[T_n(u_n)]^\theta\right\}_{n\in\mathbb{N}} \xrightarrow{\mu} u^\theta.$$

Since $\left\{[T_n(u_n)]^\theta\right\}_{n\in\mathbb{N}}$ is bounded in $L^{\frac{\theta+1}{\theta}}(\Omega) \subset L^1(\Omega)$ and thanks to Vitali′s theorem, we have that $[T_n(u_n)]^\theta$ converges strongly to u^θ in $L^1(\Omega)$, which completes the proof. □

End of the proof of Theorem 1.1

Under the assumption (1.7), thanks to the previous results, we have that:

$$
\begin{aligned}
&u_n \rightharpoonup u \quad \text{weakly in } W^{1,\frac{N}{N-1}}(\Omega).\\
&\psi_n \to \psi \quad \text{strongly in } W^{1,2}(\Omega).\\
&u_n \to u \quad \text{strongly in } L^q(\Omega),\ \text{for } q < \tfrac{N}{N-2}.\\
&\left\{\tfrac{u_n}{1+\frac{1}{n}u_n}\nabla\psi_n\right\} \to u\nabla\psi \text{ strongly in } L^1(\Omega).\\
&[T_n(u_n)]^\theta \to u^\theta \quad \text{strongly in } L^1(\Omega).
\end{aligned}
$$

Following the order in which the subsequences were extracted, i.e. using the subsequence obtained in Lemma 4.1 in Corollary 4.2 and that in Lemma 4.3, etc. we have a subsequence that verifies all the previous convergences. Then, letting $n \to +\infty$, we obtain:

$$
\begin{aligned}
&\int_\Omega M(x)\nabla u_n\nabla v \longrightarrow \int_\Omega M(x)\nabla u\nabla v,\ \text{for any } v \in C^1(\Omega)\cap W^{1,\frac{N}{N-2}}(\Omega)\cap L^\infty(\Omega),\\
&\int_\Omega u_n v \longrightarrow \int_\Omega uv,\ \text{for any } v \in C^1(\Omega)\cap W^{1,\frac{N}{N-2}}(\Omega)\cap L^\infty(\Omega),\\
&\int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}\nabla\psi_n\nabla v \longrightarrow \int_\Omega u\nabla\psi\nabla v,\ \text{for any } v \in C^1(\Omega)\cap W^{1,\frac{N}{N-2}}(\Omega)\cap L^\infty(\Omega),\\
&\int_\Omega f_n v \longrightarrow \int_\Omega fv,\ \text{for any } v \in C^1(\Omega)\cap W^{1,\frac{N}{N-2}}(\Omega)\cap L^\infty(\Omega),\\
&\int_\Omega M(x)\nabla\psi_n\nabla\phi \longrightarrow \int_\Omega M(x)\nabla\psi\nabla\phi,\ \text{for any } \phi \in C^1(\Omega)\cap W^{1,\frac{N}{N-2}}(\Omega)\cap L^\infty(\Omega),\\
&\int_\Omega \psi_n\phi \longrightarrow \int_\Omega \psi\phi,\ \text{for any } \phi \in C^1(\Omega)\cap W^{1,\frac{N}{N-2}}(\Omega)\cap L^\infty(\Omega),\\
&\int_\Omega [T_n(u_n)]^\theta\phi \longrightarrow \int_\Omega u^\theta\phi,\ \text{for any } \phi \in C^1(\Omega)\cap W^{1,\frac{N}{N-2}}(\Omega)\cap L^\infty(\Omega),\\
&\int_\Omega g_n\phi \longrightarrow \int_\Omega g\phi,\ \text{for any } \phi \in C^1(\Omega)\cap W^{1,\frac{N}{N-2}}(\Omega)\cap L^\infty(\Omega).
\end{aligned}
$$

Passing to the limit in the weak formulation of (2.3), given in Definition 2.1, we conclude that (u,ψ) is a weak solution of the problem.

Moreover, if assumption (1.9) is satisfied, we have that:

1. $\{\psi_n\}_{n\in\mathbb{N}}$ is uniformly bounded.
2. $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(\Omega)$.

Therefore, $\psi \in L^\infty(\Omega)$ and $u \in W^{1,2}(\Omega)$, which completes the proof.

5. Conclusions

In this article, an elliptic chemorepulsion system of two PDEs is considered in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ (for $N \geq 3$):

$$
\begin{cases}
-\mathrm{div}(M(x)\nabla u) + u = \chi\mathrm{div}(uM(x)\,\nabla\psi) + f(x),\\
-\mathrm{div}(M(x)\nabla\psi) + \psi = u^\theta + g(x).
\end{cases}
$$

The main result shows the existence of solutions in $W^{1,\frac{N}{N-1}}(\Omega) \times W^{1,2}(\Omega)$ provided

1. $\theta < \frac{N+2}{N-2}$;
2. $f \geq 0$, $f \in L^m(\Omega)$, for $m > \frac{2N}{N+2}$;
3. $g \geq 0$, $g \in L^q(\Omega)$, for $q > \frac{2N}{N+2}$.

The nonlinear term u^θ may exhibit superlinear growth while the standard chemotaxis system with $g \equiv 0$,

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + u = -\chi \mathrm{div}(uM(x)\,\nabla\psi) + f(x), \\ -\mathrm{div}(M(x)\nabla\psi) + \psi = u^\theta, \end{cases}$$

requires $\theta < \frac{2}{N}$ (see [2]).

Since the coefficients of M are merely bounded and the data, f and g, exhibit weak regularity, it is not possible to apply standard elliptic regularity. To overcome these difficulties, we employ a duality method combined with an appropriate truncation of f and g.

If additional regularity is assumed for g,

$$g \in L^q(\Omega) \text{ for some } q > \frac{N}{2},$$

and for $\theta < \frac{2}{N-2}$, the solution satisfies

$$u \in W^{1,2}(\Omega), \quad \psi \in W^{1,2}(\Omega) \cap L^\infty(\Omega).$$

The existence of solutions for the fully parabolic chemorepulsion system remains open for $\theta > \frac{2}{N-2}$ and $N \geq 3$, the results obtained in this article for the elliptic system suggest the global in time existence of weak solutions for $\theta \in [2/(N-2), (N+2)/(N-2))$ and $N \geq 3$. The elliptic chemorepulsion system for $N = 1, 2$ and for low regular data, f and g, is not studied in this article.

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Use of Generative-AI tools declaration

The author declare he have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no competing interests.

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