Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 25814–25838.
DOI: 10.3934/math.20261034
Received: 18 June 2026
Revised: 04 August 2026
Accepted: 13 August 2026
Published: 20 August 2026

Research article

On the multiplicative Zagreb indices of zero-divisor graphs of finite commutative reduced rings

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Abstract: Topological indices serve as key numerical invariants for characterizing graph structures and play an important role in chemical graph theory. In this work, we investigated degree-based topological indices of zero-divisor graphs arising from finite commutative reduced rings. Closed-form expressions were established for the Narumi–Katayama index and the first and second multiplicative Zagreb indices for this class of rings. The results obtained here extend and refine earlier studies on degree-based indices of zero-divisor graphs, and we additionally identify and correct certain inaccuracies present in previous analyses by other researchers. Furthermore, we provide Python-based computations to validate the theoretical findings, accompanied by comparative numerical and graphical illustrations.

Keywords: zero-divisor graph; degree-based topological index, multiplicative Zagreb index; finite commutative reduced ring; Narumi–Katayama index
Mathematics Subject Classification: 05C07, 05C25, 13A99, 13M99

1. Introduction

Algebraic graph theory provides an effective framework for investigating algebraic structures through graph-theoretic methods. Among the various graph constructions associated with commutative rings, the zero-divisor graph introduced by Anderson and Livingston [1] has attracted considerable attention due to its ability to encode algebraic properties in a combinatorial setting. Degree-based topological indices, originally developed in chemical graph theory as molecular descriptors, have subsequently found broad applications in chemistry, materials science, network analysis, and combinatorics [2,3]. Their effectiveness in characterizing graph structures has motivated extensive research on a variety of graph classes.

Among these descriptors, the Narumi–Katayama index [4] and the first and second multiplicative Zagreb indices [5–7] are fundamental multiplicative degree-based invariants that capture the global degree distribution of graphs. These indices, together with other modern descriptors such as the Zagreb, Sombor, atom-bond connectivity, and forgotten indices, have been widely studied because of their ability to distinguish structural characteristics of graphs. More recently, they have also been investigated for zero-divisor graphs of finite commutative rings with regard to $L(2,1)$-labeling [8], Steiner numbers [9,10], and Wiener index and entropy [11,12], revealing strong connections between algebraic structure and graph-theoretic properties.

The selection of the Narumi–Katayama, first multiplicative Zagreb, and second multiplicative Zagreb indices is driven by their mathematical compatibility with the direct product structure of rings. While alternative modern descriptors—such as the Sombor, forgotten, or atom-bond connectivity indices—offer valuable structural insights for standard chemical graphs, their non-linear radical or complex sum-product compositions (e.g., $\sqrt{d(u)^2 + d(v)^2}$) resist clean analytical factorization over coordinate subset lattices. Conversely, the multiplicative nature of NK, Π_1, and Π_2 mirrors the multiplicative nature of field cardinalities in

$$R = \prod F_i,$$

enabling exact, closed-form derivations that highlight the interplay between commutative algebra and graph topology.

Although several topological indices of zero-divisor graphs have been reported, the existing literature is largely restricted to specific ring classes or particular graph invariants. In particular, Gaded and Narayana [13] studied certain topological indices of zero-divisor graphs arising from direct products of three finite fields, while Selvakumar and Gangaeswari [14] investigated multiplicative Zagreb indices for finite reduced rings. However, some published derivations contain computational inaccuracies, resulting in incorrect closed-form expressions. Furthermore, a unified framework for deriving multiplicative degree-based indices of zero-divisor graphs over finite commutative reduced rings has not yet been established.

Recent advances in computational mathematics have strengthened the integration of rigorous mathematical modelling with computational verification, graph-theoretical descriptors, quantitative structure-property relationship and quantitative structure-activity relationship analysis, graph entropy, and algorithmic evaluation, thereby enhancing the efficiency and reproducibility of structural characterization. These developments underscore the growing importance of exact analytical formulations as computationally efficient tools for evaluating graph invariants in mathematical chemistry and computational graph theory [15].

Motivated by these developments, the present work derives exact closed-form expressions for the Narumi–Katayama index and the first and second multiplicative Zagreb indices of zero-divisor graphs associated with finite commutative reduced rings. The proposed analytical formulas rectify inconsistencies in earlier studies, provide rigorous mathematical justification for the derived results, and reinforce the theoretical findings through Python-based computational verification and systematic numerical comparisons. The principal differences between the existing literature and the present work are summarized in Table 1.

Table 1. Comparison of the present work with existing literature.

Reference	Results obtained	Limitation	Contribution of the present work
Gaded and Narayana [13]	Narumi–Katayama, first multiplicative Zagreb, and second multiplicative Zagreb indices of zero-divisor graphs of direct products of three finite fields.	The derivations of Results 5, 6, and 7 contain incorrect treatment of degree contributions within equi-neighbor classes, leading to incorrect closed-form expressions.	The incorrect derivations are corrected, rigorous closed-form expressions are established, and the results are extended from direct products of three finite fields to arbitrary finite commutative reduced rings.
Selvakumar and Gangaeswari [14]	First and second multiplicative Zagreb indices of zero-divisor graphs of finite reduced rings	The derivations involve incorrect exponent manipulations, resulting in erroneous formulas in Theorems 1 and 2, consequently affecting Examples 2 and 3 in Section 3.	Correct derivations are provided together with rigorous proofs and computational verification.
Present work	Narumi–Katayama, first multiplicative Zagreb and second multiplicative Zagreb indices	–	Develops unified closed-form formulas for arbitrary finite reduced rings, corrects previously published inaccuracies, and validates all results through theoretical proofs and Python computations.

The Narumi–Katayama index, introduced by Narumi and Katayama in 1984 [4], is a classical and remarkably simple degree-based topological index, defined as

$$NK(G) = \prod_{u \in V(G)} d(u),$$

where $d(u)$ denotes the degree of the vertex u. Despite its simplicity, the index has proved useful in capturing global degree distributions in a compact multiplicative form. Todeschini et al. [2, 3] later introduced a broader framework for constructing multiplicative analogues of additive graph invariants. When this approach is applied specifically to the Zagreb indices, one obtains the first and second multiplicative Zagreb indices, defined respectively by

$$\Pi_1(G) = \prod_{u \in V(G)} (d(u))^2$$

and

$$\Pi_2(G) = \prod_{uv \in E(G)} d(u)d(v).$$

The multiplicative Zagreb indices $\Pi_1(G)$ and $\Pi_2(G)$ have since been extensively studied and applied in various domains (see [5–7]), owing to their sensitivity to degree patterns and their ability to distinguish a wide range of graph structures.

The closed-form expressions obtained in this paper reveal how the algebraic decomposition of a finite commutative reduced ring determines the corresponding degree-based graph invariants. In particular, the indices are governed by the distribution of equi-neighbor classes and the orders of the constituent finite fields, thereby establishing a direct connection between the algebraic structure of the ring and the global degree distribution of its zero-divisor graph. Furthermore, the analytical formulas eliminate the need for explicit graph construction, providing an efficient computational approach for evaluating these invariants.

The remainder of this paper is organized as follows: Section 2 presents the necessary preliminaries and structural properties of zero-divisor graphs of finite commutative reduced rings. Section 3 derives closed-form expressions for the Narumi–Katayama index, the first multiplicative Zagreb index, and the second multiplicative Zagreb index, together with illustrative examples. Section 4 provides computational verification of the obtained formulas through Python implementations and compares the performance of the closed-form approach with graph-construction-based computations. Section 5 discusses the theoretical implications of the obtained results, including their structural analysis, monotonicity properties, and extremal configurations. Finally, Section 6 concludes the paper and outlines directions for future research.

2. Structural analysis of zero-divisor graph of reduced ring

An effective approach to analyze the structure of $\Gamma(R)$ is through an equivalence relation on $V(\Gamma(R))$ defined by open neighborhoods. For a vertex $v \in V(\Gamma(R))$, let

$$N(v) = \{u \in V(\Gamma(R)) \mid uv \in E(\Gamma(R))\}$$

denote the open neighborhood of v. Two vertices $u, v \in V(\Gamma(R))$ are said to be equivalent, written $u \sim v$, if

$$N(u) \setminus \{v\} = N(v) \setminus \{u\}.$$

This relation partitions $V(\Gamma(R))$ into equivalence classes, denoted by $E_{u_1}, E_{u_2}, \ldots, E_{u_h}$, called *equi-neighbor classes*. These classes provide a natural structural decomposition of the zero-divisor graph and play a fundamental role in the analysis of its degree distribution and adjacency structure.

Let R be a finite commutative reduced ring with unity. By the classical structure theorem for finite reduced rings,

$$R \cong F_{q_1} \times F_{q_2} \times \ldots \times F_{q_h},$$

where each $q_i = p_i^{a_i}$ for a prime p_i and an integer $a_i \geq 1$. Since every finite field has only the trivial zero-divisor, namely 0, the nonzero zero-divisors of R are precisely those elements having at least one zero coordinate in this direct product representation. Consequently, the vertices of $\Gamma(R)$ admit an explicit coordinate-wise description.

The equi-neighbor partition provides a convenient framework for studying the structure of $\Gamma(R)$. In particular, it enables a systematic analysis of vertex degrees and adjacency relationships, which forms the basis for the closed-form expressions of the topological indices derived in the subsequent section. The following lemmas describe the structure and properties of these equi-neighbor classes.

Lemma 2.1. *[9] Each equi-neighbor class contains exactly one element from E, where* $E = \{a = (a_1, a_2, \ldots, a_h) \in Z^*(R) : a_i = 0 \text{ or } a_i = 1\}$.

Lemma 2.2. *All vertices within the same equi-neighbor class have the same degree.*

Proof. Let $x, y \in E_a$. Since E_a is an equi-neighbor class, $N(x) \setminus \{y\} = N(y) \setminus \{x\}$. Therefore, x and y are adjacent to the same number of vertices, which implies $d(x) = d(y)$. □

Lemma 2.3. *Let E_a be an equi-neighbor class, where a has zeros in exactly the coordinates $i_1, i_2, \ldots, i_k$ and let $S = \{i_1, i_2, \ldots, i_k\}$. Then the number of vertices in E_a is*

$$|E_a| = \prod_{j \notin S} (|F_j| - 1).$$

Proof. The equivalence relation on the reduced ring R ensures that every element of the equi-neighbor class E_a has the same zero-coordinate support as a. Accordingly, for each coordinate $j \notin S$, the corresponding entry can be any nonzero element of F_j, and thus contributes $|F_j| - 1$ choices. □

Lemma 2.4. *[16] Let $x_i = a_i - 1$ for $i = 1, \ldots, n$. Then the sum of all nonempty products of $x_1, \ldots, x_n$ can be expressed as*

$$\sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} \prod_{i \in I} x_i = \prod_{i=1}^{n} a_i - 1.$$

Lemma 2.5. *Let $R = F_1 \times F_2 \times \ldots \times F_h$. Then for any $u \in E_a$, where $a = (a_1, a_2, \ldots, a_h)$ has exactly n zero coordinates in positions $i_1, i_2, \ldots, i_n$, the degree of u is $d(u) = |F_{i_1}||F_{i_2}| \ldots |F_{i_n}| - 1$.*

Proof. **Case 1:** Suppose $u \in L_1$.

If $u = (u_1, u_2, \ldots, u_h)$ has a zero at i-th coordinate, then u is adjacent to all vertices belonging to the equi-neighbor class E_{e_i}. Therefore, $d(u) = |E_{e_i}|$ and by Lemma 2.3, we have $|E_{e_i}| = |F_i| - 1$.

Case 2: Suppose $u \in L_k$, for $2 \leq k \leq h - 2$.

Let u have zeros exactly in the coordinates $i_1, i_2, \ldots, i_k$ and let $S = \{i_1, i_2, \ldots, i_k\}$. Then, u is adjacent to all vertices in the equi-neighbor classes $E_{e_{i_1}}, E_{e_{i_2}}, \ldots, E_{e_{i_k}}$ in L_{h-1}. Any sum of two distinct e_{i_j} and e_{i_l} belongs to L_{h-2} and is adjacent to u, with the number of vertices given by

$$|E_{e_{i_j}+e_{i_l}}| = (|F_{i_j}| - 1)(|F_{i_l}| - 1).$$

Therefore, there are $\binom{k}{2}$ equi-neighbor classes in L_{h-2} adjacent to u. More generally, any sum of n distinct e_{i_j} belongs to L_{h-n}, and there are $\binom{k}{n}$ equi-neighbor classes in L_{h-n} adjacent to u, each class containing the n product of $|F_{i_j}| - 1$ vertices corresponding to the indices in the sum e_{i_j}. Hence,

$$\begin{aligned} d(u) &= \sum_{i_j \in S} (|F_{i_j}| - 1) + \sum_{i_j \neq i_l \in S} (|F_{i_j}| - 1)(|F_{i_l}| - 1) + \ldots \\ &+ \sum_{I \subset S} \prod_{i_j \in I} (|F_{i_j}| - 1) + \ldots + \prod_{i_j \in S} (|F_{i_j}| - 1) \\ &= \sum_{\emptyset \neq I \subseteq S} \prod_{i_j \in I} (|F_{i_j}| - 1) \end{aligned}$$

By Lemma 2.4, $d(u) = \prod_{j=1}^{k} |F_{i_j}| - 1$.

Case 3: Suppose $u \in L_{h-1}$.

Let $u \in E_{e_i}$ and let $S = \{1, 2, \ldots, i-1, i+1, \ldots, h\}$. Then u is adjacent to vertices in the $h-1$ equi-neighbor classes $E_{e_1}, E_{e_2}, \ldots, E_{e_{i-1}}, E_{e_{i+1}}, \ldots E_{e_h}$ in L_{h-1}. By the same argument in Case 2, $\binom{h-1}{n}$ classes in L_{h-n} are adjacent to u. Therefore,

$$d(u) = \sum_{\emptyset \neq I \subseteq S} \prod_{j \in I} (|F_j| - 1) = \prod_{j=1, j \neq i}^{h} |F_j| - 1.$$

This completes the proof. □

3. Closed-form expressions for the Narumi–Katayama and multiplicative Zagreb indices

In this section, we derive explicit closed formulas for the topological indices under consideration and present natural generalizations of several key results from [13]. These refinements broaden the applicability of earlier formulations and provide a unified framework for computing degree-based invariants of zero-divisor graphs arising from reduced rings. By systematically analyzing the structure of equi-neighbor classes and their degree patterns, we obtain expressions that remain valid across a wide family of direct product representations of finite fields. Furthermore, Remarks 3.5 and 3.7 highlight and correct specific inaccuracies in the statements and computational procedures reported in [13, 14], thereby clarifying the theoretical landscape and ensuring the consistency of subsequent applications.

Theorem 3.1. *The Narumi–Katayama index of the zero-divisor graph of the reduced ring $R = F_1 \times F_2 \times \ldots \times F_h$ is given by:*

$$NK(\Gamma(R)) = \prod_{\emptyset \neq I \subset S} \left(\prod_{j \in I} |F_j| - 1 \right)^{\left[\prod_{j \notin I} \left(|F_j| - 1 \right) \right]},$$

where $S = \{1, 2, \ldots, h\}$.

Proof. **Case 1.** Suppose $u \in L_1$.

If u has a zero at the ith coordinate, then $d(u) = |F_i| - 1$. By Lemma 2.3, there are $\prod_{j \neq i} (|F_j| - 1)$ vertices having the same degree. Therefore,

$$\prod_{v \in E_u} d(v) = (|F_i| - 1)^{\prod_{j \neq i} \left(|F_j| - 1 \right)}.$$

Hence,

$$\begin{aligned}
\prod_{u \in L_1} d(u) &= \prod_{u \in E_u, E_u \subset L_1} d(u) \\
&= (|F_1| - 1)^{\prod_{j \neq 1} \left(|F_j| - 1 \right)} (|F_2| - 1)^{\prod_{j \neq 2} \left(|F_j| - 1 \right)} \ldots (|F_h| - 1)^{\prod_{j \neq h} \left(|F_j| - 1 \right)} \qquad (3.1) \\
&= \prod_{|I| = 1, I \subset S} \left(\prod_{j \in I} |F_j| - 1 \right)^{\prod_{j \notin I} \left(|F_j| - 1 \right)}.
\end{aligned}$$

Case 2. Suppose $u \in L_k$, where $2 \leq k \leq h-1$.

Let $I = \{i_1, i_2, \ldots i_k\}$ and suppose that u has zeros in the coordinates corresponding to the indices in I. By Lemma 2.2 and 2.5, $d(u) = \prod_{j\in I} |F_j| - 1$ for all $u \in E_u$. Since $|E_u| = \prod_{j\notin I} (|F_j| - 1)$,

$$\prod_{v\in E_u} d(v) = \Big(\prod_{j\in I} |F_j| - 1\Big)^{\prod_{j\notin I}\left(|F_j|-1\right)}.$$

Since there are $\binom{h}{k}$ equi-neighbor classes in L_k, the set I can be chosen in $\binom{h}{k}$ ways, corresponding to all subsets of S of size k. Therefore,

$$\prod_{u\in L_k} d(u) = \prod_{u\in E_u, E_u\subset L_k} d(u) = \prod_{|I|=k, I\subset S} \Big(\prod_{j\in I} |F_j| - 1\Big)^{\prod_{j\notin I}\left(|F_j|-1\right)}. \tag{3.2}$$

Observe that the family of equi-neighbor classes $\{E_u : u \in V(\Gamma(R))\}$ forms a partition of the vertex set $V(\Gamma(R))$. Furthermore, by Lemmas 2.2, 2.3, and 2.5, all vertices belonging to the same equi-neighbor class have the same degree, while distinct subsets $I \subset S$ determine distinct equi-neighbor classes. Consequently, each vertex degree contributes exactly once to the product defining the Narumi–Katayama index, and no duplication occurs in the subset indexing. Since this decomposition depends only on the equi-neighbor class structure of the finite reduced ring $R \cong F_1 \times F_2 \times \ldots \times F_h$, the resulting multiplicative expression is valid for every finite reduced ring.

Since

$$V(\Gamma(R)) = \bigcup_{k=1}^{h-1} L_k,$$

where the union is disjoint, the Narumi–Katayama index factors as the product of the contributions from each level. Hence, combining (3.1) and (3.2), we obtain

$$\begin{aligned}\prod_{u\in V(\Gamma(R))} d(u) &= \prod_{u\in L_1} d(u).\prod_{u\in L_2} d(u)\ldots\prod_{u\in L_h} d(u)\\ &= \prod_{\emptyset\neq I\subset S}\Big(\prod_{j\in I} |F_j| - 1\Big)^{\prod_{j\notin I}\left(|F_j|-1\right)}.\end{aligned}$$

This completes the proof. □

Theorem 3.2. *The first multiplicative Zagreb index of the zero-divisor graph of the reduced ring $R = F_1 \times F_2 \times \ldots \times F_h$ is given by:*

$$\Pi_1(\Gamma(R)) = \prod_{\emptyset\neq I\subset S}\Big(\prod_{j\in I} |F_j| - 1\Big)^{2\left[\prod_{j\notin I}\left(|F_j|-1\right)\right]},$$

where $S = \{1, 2, \ldots, h\}$.

Proof. By definition,

$$\Pi_1(\Gamma(R)) = \prod_{u\in V(\Gamma(R))} (d(u))^2 = \left(\prod_{u\in V(\Gamma(R))} d(u)\right)^2.$$

The result follows immediately from Theorem 3.1 together with Lemmas 2.2–2.5, since each degree contribution is counted exactly once under the equi-neighbor class decomposition. □

Theorem 3.3. *For the reduced ring $R = F_1 \times F_2 \times \ldots \times F_h$, the second multiplicative Zagreb index of the zero-divisor graph $\Gamma(R)$ is given as follows:*

$$\Pi_2(\Gamma(R)) = \left[\prod_{\emptyset\neq I\subset S}\left(\prod_{\emptyset\neq J\subseteq I}\left[(\prod_{i\in I}|F_i| - 1)(\prod_{j\in J^c}|F_j| - 1)\right]^{\prod_{s\in J\cup I^c}\left(|F_s|-1\right)}\right)\right]^{\frac{1}{2}},$$

where $S = \{1, 2, \ldots, h\}$ and $J^c = S \setminus J$.

Proof. **Case 1.** Let $u \in L_1$.

Suppose u has a zero at the ith coordinate, $u \in E_{e_1+e_2+\ldots e_{i-1}+e_{i+1}+\ldots e_h}$. By Lemma 2.5, $d(u) = |F_i| - 1$. Clearly, u is adjacent to all the vertices in $v \in E_{e_i}$ and by Lemma 2.5, $d(v) = \prod_{j\neq i} |F_j| - 1$. Number of edges between $u \in E_{e_1+e_2+\ldots e_{i-1}+e_{i+1}+\ldots e_h}$ and $v \in E_{e_i}$ is

$$\begin{aligned}|E_{e_1+e_2+\ldots e_{i-1}+e_{i+1}+\ldots e_h}||E_{e_i}| &= (|F_1| - 1)(|F_2| - 1)\ldots(|F_{i-1}| - 1)(|F_{i+1}| - 1)\ldots(|F_h| - 1)(|F_i| - 1)\\ &= \prod_{j\in S}(|F_j| - 1).\end{aligned}$$

Therefore, for $u \in E_{e_1+e_2+\ldots e_{i-1}+e_{i+1}+\ldots e_h}$,

$$\begin{aligned}\prod_{uv\in E(\Gamma(R))} d(u)d(v) &= \prod_{uv\in E(\Gamma(R))} (|F_i| - 1)(\prod_{j\neq i}|F_j| - 1)\\ &= \left[(|F_i| - 1)(\prod_{j\neq i}|F_j| - 1)\right]^{\prod_{s\in S}(|F_s|-1)}.\end{aligned}$$

For any $u \in L_1$,

$$\begin{aligned}\prod_{uv\in E(\Gamma(R)),u\in L_1} d(u)d(v) &= \prod_{i\in S}\left[(|F_i| - 1)(\prod_{j\neq i}|F_j| - 1)\right]^{\prod_{s\in S}(|F_s|-1)}\\ &= \prod_{|I|=1,I\subset S}\left[(\prod_{i\in I}|F_i| - 1)(\prod_{l\in I^c}|F_l| - 1)\right]^{\prod_{s\in S}(|F_s|-1)}\\ &= \prod_{|I|=1,I\subset S}\left(\prod_{J\subseteq I}\left[(\prod_{i\in I}|F_i| - 1)(\prod_{m\in j^c}|F_m| - 1)\right]^{\prod_{s\in S}(|F_s|-1)}\right).\end{aligned}$$

Case 2. Let $u \in L_k$, where $2 \le k \le h-1$. Assume that u has zeros exactly at the coordinates $i_1, i_2, \ldots, i_k$. With $I = \{i_1, i_2, \ldots, i_k\} \subset S$, the corresponding vertex u is adjacent to the classes listed below.

- There is a unique equi-neighbor class $E_{e_{i_1}+e_{i_2}+\ldots+e_{i_k}} \subset L_{h-k}$ adjacent to u. By Lemma 2.5,

$$d(u) = \prod_{i_j\in I}|F_{i_j}| - 1 \quad \text{and} \quad d(v) = \prod_{i_j\notin I}|F_{i_j}| - 1,$$

and the number of edges between the corresponding equi-neighbor classes is $\prod_{s\in S}(|F_s|-1)$. Hence,

$$\Big[(\prod_{i_l\in I}|F_{i_l}|-1)(\prod_{i_j\notin I}|F_{i_j}|-1)\Big]^{\prod_{s\in S}(|F_s|-1)}.$$

- There are $\binom{k}{k-1}$ equi-neighbor classes in $L_{h-(k-1)}$ adjacent to u. For $v\in E_{e_{i_1}+e_{i_2}+\ldots+e_{i_{k-1}}}$ with $J=\{i_1,i_2,\ldots,i_{k-1}\}\subseteq I$, we have $d(v)=\prod_{i_j\in J^c}|F_{i_j}|-1$, and the number of edges between the equi-neighbor classes of u and v is $\prod_{i_j\in I^c\cup J}(|F_{i_j}|-1)$. Hence,

$$\Big[(\prod_{i_l\in I}|F_{i_l}|-1)(\prod_{i_j\in J^c}|F_{i_j}|-1)\Big]^{\prod_{i_s\in I^c\cup J}\left(|F_{i_s}|-1\right)}.$$

  The second multiplicative index associated with the edges linking $v\in L_{h-(k-1)}$ and the equi-neighbor class of u is given by:

$$\prod_{|J|=k-1,J\subseteq I}\Big[(\prod_{i_l\in I}|F_{i_l}|-1)(\prod_{i_j\in J^c}|F_{i_j}|-1)\Big]^{\prod_{i_s\in I^c\cup J}\left(|F_{i_s}|-1\right)}.$$

- There exist $\binom{k}{k-2}$ adjacent classes in $L_{h-(k-2)}$. For $v\in E_{e_{i_1}+e_{i_2}+\ldots+e_{i_{k-2}}}$ with $J=\{i_1,i_2,\ldots,i_{k-2}\}\subseteq I$, we obtain

$$\Big[(\prod_{i_l\in I}|F_{i_l}|-1)(\prod_{i_j\in J^c}|F_{i_j}|-1)\Big]^{\prod_{i_s\in I^c\cup J}\left(|F_{i_s}|-1\right)}.$$

  The second multiplicative Zagreb index associated with the edges linking $v\in L_{h-(k-2)}$ and the equi-neighbor class of u is given by:

$$\prod_{|J|=k-2,J\subseteq I}\Big[(\prod_{i_l\in I}|F_{i_l}|-1)(\prod_{i_j\in J^c}|F_{i_j}|-1)\Big]^{\prod_{i_s\in I^c\cup J}\left(|F_{i_s}|-1\right)}.$$

- More generally, $\binom{k}{k-n}$ classes in $L_{h-(k-n)}$ are adjacent to u. We now evaluate the second multiplicative Zagreb index contribution arising from the edges joining any vertex $v\in L_{h-(k-n)}$ with the equi-neighbor class containing u

$$\prod_{|J|=k-n,J\subseteq I}\Big[(\prod_{i_l\in I}|F_{i_l}|-1)(\prod_{i_j\in J^c}|F_{i_j}|-1)\Big]^{\prod_{i_s\in I^c\cup J}\left(|F_{i_s}|-1\right)}.$$

  Consequently, the second multiplicative Zagreb index associated with the collection of edges connecting the equi-neighbor class containing u to all such vertices v can be expressed as:

$$\prod_{\emptyset\neq J\subseteq I}\Big[(\prod_{i_l\in I}|F_{i_l}|-1)(\prod_{i_j\in J^c}|F_{i_j}|-1)\Big]^{\prod_{i_s\in I^c\cup J}\left(|F_{i_s}|-1\right)}.$$

For $2 \leq k \leq h-1$, the second multiplicative Zagreb index of edges between L_k and any vertex v is given by:

$$\prod_{|I|=k, I\subset S} \Big(\prod_{\emptyset \neq J \subseteq I} \Big[(\prod_{i_l \in I} |F_{i_l}| - 1)(\prod_{i_j \in J^c} |F_{i_j}| - 1) \Big]^{\prod_{i_s \in I^c \cup J} \left(|F_{i_s}|-1\right)} \Big).$$

Case 3. Since each edge $uv \in E(\Gamma(R))$ is counted once from each of its endpoints, every edge contribution appears exactly twice in the above product. Hence, taking the square root removes this double counting, and combining the contributions from Cases 1 and 2 yields the desired closed-form expression

$$\Pi_2(\Gamma(R)) = \Bigg[\prod_{\emptyset \neq I \subset S} \Bigg(\prod_{\emptyset \neq J \subseteq I} \Big[(\prod_{i \in I} |F_i| - 1)(\prod_{j \in J^c} |F_j| - 1) \Big]^{\prod_{s \in J \cup I^c} \left(|F_s|-1\right)} \Bigg) \Bigg]^{\frac{1}{2}}.$$

This completes the proof. □

The following example demonstrates the application of Theorem 3.3 to the computation of the second multiplicative Zagreb index.

Example 3.4. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, where $S = \{1, 2, 3, 4\}$. For every subset I with $\emptyset \neq I \subsetneq S$, the contribution of the associated equi-neighbor class is computed according to Theorem 3.3. Let

$$\begin{aligned}
L_1 &= \{E_{(0,1,1,1)}, E_{(1,0,1,1)}, E_{(1,1,0,1)}, E_{(1,1,1,0)}\},\\
L_2 &= \{E_{(0,0,1,1)}, E_{(0,1,0,1)}, E_{(0,1,1,0)}, E_{(1,0,0,1)}, E_{(1,0,1,0)}, E_{(1,1,0,0)}\},\\
L_3 &= \{E_{e_1}, E_{e_2}, E_{e_3}, E_{e_4}\},
\end{aligned}$$

where $e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)$.

The equi-neighbor classes of the vertices are given as follows:

$$\begin{aligned}
E_{(0,1,1,1)} &= \{(0, 1, 1, 1), (0, 1, 1, 2)\}, \quad E_{1,0,1,1)} = \{(1, 0, 1, 1), (1, 0, 1, 2)\},\\
E_{(1,1,0,1)} &= \{(1, 1, 0, 1), (1, 1, 0, 2)\}, \quad E_{(1,1,1,0)} = \{(1, 1, 1, 0)\},\\
E_{(0,0,1,1)} &= \{(0, 0, 1, 1), (0, 0, 1, 2)\}, \quad E_{(0,1,0,1)} = \{(0, 1, 0, 1), (0, 1, 0, 2)\},\\
E_{(0,1,1,0)} &= \{(0, 1, 1, 0)\}, \quad E_{(1,0,0,1)} = \{(1, 0, 0, 1), (1, 0, 0, 2)\},\\
E_{(1,0,1,0)} &= \{(1, 0, 1, 0)\}, \quad E_{(1,1,0,0)} = \{(1, 1, 0, 0)\},\\
E_{e_1} &= e_1, \quad E_{e_2} = e_2, \quad E_{e_3} = e_3, \quad E_{e_4} = \{e_4, (0, 0, 0, 2)\}.
\end{aligned}$$

The adjacency relationships among the equi-neighbor classes are as follows. Every vertex in L_1 is adjacent to the vertices of exactly one equi-neighbor class in L_3, whereas every vertex in L_2 is adjacent to the vertices of exactly two equi-neighbor classes in L_3.

Consider I to be any subset of S except $\emptyset$ and S. Let J be the subset of I, except $\emptyset$.

Case 1: Let $u \in L_1$.

- Consider $I = \{1\}$ and $J = \{1\}$. Suppose $u \in E_{(0,1,1,1)}$. Then u is adjacent to every vertex in the equi-neighbor class E_{e_1}. Moreover, each vertex of E_{e_1} is adjacent to the equi-neighbor classes

$$E_{(0,0,1,1)}, \ E_{(0,1,0,1)}, \ E_{(0,1,1,0)}, \ E_{(0,1,1,1)}, \ E_{e_2}, \ E_{e_3}, \ E_{e_4}.$$

Hence,

$$d(u) = |F_1| - 1 = 1, \quad d(v) = |F_2||F_3||F_4| - 1 = 11,$$

for every $u \in E_{(0,1,1,1)}$ and $v \in E_{e_1}$. Since

$$|E_{(0,1,1,1)}| = (|F_2| - 1)(|F_3| - 1)(|F_4| - 1) \quad \text{and} \quad |E_{e_1}| = |F_1| - 1,$$

the total number of edges joining these two equi-neighbor classes is

$$|E_{(0,1,1,1)}|\,|E_{e_1}| = (|F_2| - 1)(|F_3| - 1)(|F_4| - 1)(|F_1| - 1) = 2 \times 1$$

and

$$\begin{aligned}\prod_{uv \in E(\Gamma(R)), u \in E_{(0,1,1,1)}} d(u)d(v) &= \prod_{uv \in E(\Gamma(R)), u \in E_{(0,1,1,1)}} 1 \times 11 \\ &= (1 \times 11)^2 \\ &= \Big[(|F_1| - 1)(|F_2||F_3||F_4| - 1)\Big]^{(|F_2|-1)(|F_3|-1)(|F_4|-1)(|F_1|-1)} \\ &= \Big[(|F_1| - 1)\Big(\prod_{i \notin I} |F_i| - 1\Big)\Big]^{\prod_{j \in S}(|F_j|-1)}.\end{aligned}$$

- Consider $I = \{2\}$ and $J = \{2\}$. Suppose $u \in E_{(1,0,1,1)}$. Then u is adjacent to every vertex in the equi-neighbor class E_{e_2}. Moreover, each vertex of E_{e_2} is adjacent to the equi-neighbor classes

$$E_{(0,0,1,1)},\ E_{(1,0,0,1)},\ E_{(1,0,1,0)},\ E_{(1,0,1,1)},\ E_{e_1},\ E_{e_3},\ E_{e_4}.$$

Hence,

$$d(u) = |F_2| - 1 = 1, \quad d(v) = |F_1||F_3||F_4| - 1 = 11,$$

for every $u \in E_{(1,0,1,1)}$ and $v \in E_{e_2}$. Since

$$|E_{(1,0,1,1)}| = (|F_1| - 1)(|F_3| - 1)(|F_4| - 1) \quad \text{and} \quad |E_{e_2}| = |F_2| - 1,$$

the total number of edges joining these two equi-neighbor classes is

$$|E_{(1,0,1,1)}|\,|E_{e_2}| = (|F_1| - 1)(|F_3| - 1)(|F_4| - 1)(|F_2| - 1) = 2 \times 1$$

and

$$\begin{aligned}\prod_{uv \in E(\Gamma(R)), u \in E_{(1,0,1,1)}} d(u)d(v) &= \prod_{uv \in E(\Gamma(R)), u \in E_{(0,1,1,1)}} 1 \times 11 \\ &= (1 \times 11)^2 \\ &= \Big[(|F_2| - 1)(|F_1||F_3||F_4| - 1)\Big]^{(|F_1|-1)(|F_3|-1)(|F_4|-1)(|F_2|-1)} \\ &= \Big[(|F_2| - 1)\Big(\prod_{i \notin I} |F_i| - 1\Big)\Big]^{\prod_{j \in S}(|F_j|-1)}.\end{aligned}$$

- Consider $I = \{3\}$ and $J = \{3\}$. Suppose $u \in E_{(1,1,0,1)}$. Then u is adjacent to every vertex in the equi-neighbor class E_{e_3}. Moreover, each vertex of E_{e_3} is adjacent to the equi-neighbor classes

$$E_{(1,0,0,1)},\ E_{(0,1,0,1)},\ E_{(1,1,0,0)},\ E_{(1,1,0,1)},\ E_{e_1},\ E_{e_2},\ E_{e_4}.$$

Hence,

$$d(u) = |F_3| - 1 = 1, \quad d(v) = |F_1||F_2||F_4| - 1 = 11,$$

for every $u \in E_{(1,1,0,1)}$ and $v \in E_{e_3}$. Since

$$|E_{(1,1,0,1)}| = (|F_1| - 1)(|F_2| - 1)(|F_4| - 1) \quad \text{and} \quad |E_{e_3}| = |F_3| - 1,$$

the total number of edges joining these two equi-neighbor classes is

$$|E_{(1,1,0,1)}|\,|E_{e_3}| = (|F_1| - 1)(|F_2| - 1)(|F_4| - 1)(|F_3| - 1) = 2 \times 1$$

and

$$\begin{aligned}\prod_{uv\in E(\Gamma(R)),u\in E_{(1,1,0,1)}} d(u)d(v) &= \prod_{uv\in E(\Gamma(R)),u\in E_{(1,1,0,1)}} 1 \times 11 \\ &= (1 \times 11)^2 \\ &= \Big[(|F_3| - 1)(|F_1||F_2||F_4| - 1)\Big]^{(|F_1|-1)(|F_2|-1)(|F_4|-1)(|F_1|-1)} \\ &= \Big[(|F_3| - 1)\Big(\prod_{i\notin I} |F_i| - 1\Big)\Big]^{\prod_{j\in S}(|F_j|-1)}.\end{aligned}$$

- Consider $I = \{4\}$ and $J = \{4\}$. Suppose $u \in E_{(1,1,1,0)}$. Then u is adjacent to every vertex in the equi-neighbor class E_{e_4}. Moreover, each vertex of E_{e_4} is adjacent to the equi-neighbor classes

$$E_{(1,0,1,0)},\ E_{(1,1,0,0)},\ E_{(0,1,1,0)},\ E_{(1,1,1,0)},\ E_{e_1},\ E_{e_2},\ E_{e_3}.$$

Hence,

$$d(u) = |F_4| - 1 = 2, \quad d(v) = |F_1||F_2||F_3| - 1 = 7,$$

for every $u \in E_{(1,1,1,0)}$ and $v \in E_{e_4}$. Since

$$|E_{(1,1,1,0)}| = (|F_1| - 1)|F_2| - 1)(|F_3| - 1) \quad \text{and} \quad |E_{e_4}| = |F_4| - 1,$$

the total number of edges joining these two equi-neighbor classes is

$$|E_{(1,1,1,0)}|\,|E_{e_4}| = (|F_1| - 1)(|F_2| - 1)(|F_3| - 1)(|F_4| - 1) = 1 \times 2$$

and

$$\begin{aligned}\prod_{uv\in E(\Gamma(R)),u\in E_{(1,1,1,0)}} d(u)d(v) &= \prod_{uv\in E(\Gamma(R)),u\in E_{(0,1,1,1)}} 2 \times 7 \\ &= (2 \times 7)^2 \\ &= \Big[(|F_4| - 1)(|F_1||F_2||F_3| - 1)\Big]^{((|F_1|-1)|F_2|-1)(|F_3|-1)(|F_4|-1)} \\ &= \Big[(|F_4| - 1)\Big(\prod_{i\notin I} |F_i| - 1\Big)\Big]^{\prod_{j\in S}(|F_j|-1)}.\end{aligned}$$

Therefore,

$$\prod_{uv\in E(\Gamma(R)),u\in L_1} d(u)d(v) = 11^2\cdot 11^2\cdot 11^2\cdot 14^2$$
$$= \prod_{|I|=1,I\subsetneq S,i\in I}\Big[(|F_i|-1)\Big(\prod_{l\notin I}|F_l|-1\Big)\Big]^{\prod_{j\in S}(|F_j|-1)}.$$

Case 2: Let $u\in L_2$.

- Assume $I=\{1,2\}$ and let $u\in E_{(0,0,1,1)}$. Then u is adjacent to one equi-neighbor class in L_2 and to the two equi-neighbor classes E_{e_1} and E_{e_2} in L_3. Consequently,

$$d(u) = |F_1||F_2|-1 = 3.$$

(1) Consider $J=\{1\}$.
Since every vertex of $E_{(0,0,1,1)}$ is adjacent to every vertex of E_{e_1}, the number of edges joining these two equi-neighbor classes is

$$|E_{(0,0,1,1)}|\,|E_{e_1}| = (|F_3|-1)(|F_4|-1)(|F_1|-1) = 2$$

and

$$\prod_{uv\in E(\Gamma(R)),u\in E_{(0,0,1,1)},v\in E_{e_1}} d(u)d(v) = \prod_{uv\in E(\Gamma(R)),u\in E_{(0,0,1,1)},v\in E_{e_1}} 3\times 11$$
$$= (3\times 11)^2$$
$$= \Big[(|F_1||F_2|-1)(|F_2||F_3||F_4|-1)\Big]^{(|F_3|-1)(|F_4|-1)(|F_1|-1)}$$
$$= \Big[\Big(\prod_{i\in I}|F_i|-1\Big)\Big(\prod_{j\in J^c}|F_i|-1\Big)\Big]^{\prod_{l\in I^c\cup J}\big(|F_l|-1\big)}.$$

(2) Consider $J=\{2\}$.
The number of edges from u to E_{e_2} is

$$2 = (|F_3|-1)(|F_4|-1)(|F_2|-1)$$

and

$$\prod_{uv\in E(\Gamma(R)),u\in E_{(0,0,1,1)},v\in E_{e_2}} d(u)d(v) = \prod_{uv\in E(\Gamma(R)),u\in E_{(0,0,1,1)},v\in E_{e_2}} 3\times 11$$
$$= (3\times 11)^2$$
$$= \Big[(|F_1||F_2|-1)(|F_1||F_3||F_4|-1)\Big]^{(|F_3|-1)(|F_4|-1)(|F_2|-1)}$$
$$= \Big[\Big(\prod_{i\in I}|F_i|-1\Big)\Big(\prod_{j\in J^c}|F_i|-1\Big)\Big]^{\prod_{l\in I^c\cup J}\big(|F_l|-1\big)}.$$

(3) Consider $J = \{1, 2\}$.
The number of edges from u to $E_{(1,1,0,0)}$ is

$$2 = (|F_3| - 1)(|F_4| - 1)(|F_1| - 1)(|F_2| - 1)$$

and

$$\begin{aligned}\prod_{uv\in E(\Gamma(R)),u\in E_{(0,0,1,1)},v\in E_{(1,1,0,0)}} d(u)d(v) &= \prod_{uv\in E(\Gamma(R)),u\in E_{(0,0,1,1)},v\in E_{(1,1,0,0)}} 3\times 3\\ &= \Big[(|F_1||F_2| - 1)(|F_3||F_4| - 1)\Big]^{(|F_3|-1)(|F_4|-1)(|F_1|-1)(|F_2|-1)}\\ &= \Big[(\prod_{i\in I}|F_i| - 1)(\prod_{j\in J^c}|F_i| - 1)\Big]^{\prod_{l\in I^c\cup J}\left(|F_l|-1\right)}.\end{aligned}$$

- The same argument applies to $u \in E_{(0,1,0,1)}$, where $I = \{1, 3\}$ and J ranges over all nonempty subsets of I. The remaining cases with $|I| = 2$ are treated analogously.
- Finally, let $u \in E_{(0,0,0,1)}$ and take $I = \{1, 2, 3\}$. For each nonempty subset $J \subseteq I$, the corresponding adjacent vertex v belongs to the equi-neighbor class determined by J. In particular, when $J = \{1\}, \{2\}, \{3\}$, we have $v \in E_{e_1}, E_{e_2}, E_{e_3}$, respectively. For $J = \{1, 2\}$, $v \in E_{(1,1,0,0)}$; for $J = \{1, 3\}$, $v \in E_{(1,0,1,0)}$; for $J = \{2, 3\}$, $v \in E_{(0,1,1,0)}$; and for $J = \{1, 2, 3\}$, $v \in E_{(1,1,1,0)}$. The corresponding edge contributions are computed in the same manner as in the previous cases.

Combining the contributions from all possible choices of I and J, every edge contributes twice to the product. Consequently, taking the square root yields

$$\begin{aligned}\Pi_2(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)) =&7449443176322864126066754725897829348335\\ &2832393327186169433593 7500.\end{aligned}$$

Remark 3.5 below points out that several steps in the arguments of [14] concerning the first multiplicative Zagreb and second multiplicative Zagreb indices admit a clearer and more transparent presentation. In particular, certain intermediate justifications and degree computations can be streamlined to enhance readability and avoid potential ambiguity. The remark provides a refined interpretation of these steps, ensuring that the underlying reasoning is explicit and consistent with the structural properties of zero-divisor graphs.

Remark 3.5. In the proof of Theorem 2 in [14], a logical error appears in the second line of the expressions for $\Pi_1(\Gamma(R))$ and $\Pi_2(\Gamma(R))$, where the power of a sum is inadvertently treated as though it distributes over the summands. This misapplication leads to incorrect intermediate expressions and affects the subsequent computations. The corrected formulations are provided in Theorems 3.2 and 3.3, where the appropriate multiplicative structure is carefully preserved. As a consequence, the conclusions in [14, Examples 2 and 3] are also influenced, and the adjustments presented here restore the accuracy and consistency of those results.

In the sequel, we revisit the examples presented in [14], correct the computational oversight identified earlier, and supply detailed and fully verified calculations. This not only rectifies the specific inaccuracies but also provides a clearer illustration of the methodology underlying the computation of the relevant multiplicative Zagreb indices.

Example 3.6. Consider $R = F_1 \times F_2$ as a particular case of Theorems 3.1–3.3. In this setting, the index set is $S = \{1, 2\}$, and we obtain the following results

$$\begin{aligned} NK(\Gamma(R)) &= (|F_1| - 1)^{(|F_2|-1)}(|F_2| - 1)^{(|F_1|-1)}, \\ \Pi_1(\Gamma(R)) &= (|F_1| - 1)^{2(|F_2|-1)}(|F_2| - 1)^{2(|F_1|-1)}, \\ \Pi_2(\Gamma(R)) &= \left[(|F_1| - 1)(|F_2| - 1)\right]^{(|F_1|-1)(|F_2|-1)}. \end{aligned}$$

Consider $F_1 = \{0, 1\}$ and $F_2 = \{0, 1, a, b\}$, we have $V(\Gamma(F_1 \times F_2)) = \{(1, 0), (0, 1), (0, a), (0, b)\}$, $d((1, 0)) = 3$ and $d((0, 1) = d((0, a) = d((0, b) = 1$. Therefore,

$$\begin{aligned} \Pi_1(\Gamma(R)) &= \prod_{u \in V(\Gamma(R))} (d(u))^2 = 3^2 \cdot 1^2 \cdot 1^2 \cdot 1^2 = 9, \\ \Pi_2(\Gamma(R)) &= \prod_{uv \in E(\Gamma(R))} d(u)d(v) \\ &= (3 \cdot 1) \cdot (3 \cdot 1) \cdot (3 \cdot 1) = 27. \end{aligned}$$

According to our results,

$$\begin{aligned} \Pi_1(\Gamma(R)) &= (|F_1| - 1)^{2(|F_2|-1)}(|F_2| - 1)^{2(|F_1|-1)} \\ &= (2 - 1)^{2(4-1)} \cdot (4 - 1)^{2(2-1)} \\ &= 1^6 \cdot 3^2 = 9, \\ \Pi_2(\Gamma(R)) &= \left[(|F_1| - 1)(|F_2| - 1)\right]^{(|F_1|-1)(|F_2|-1)} \\ &= \left[(2 - 1)(4 - 1)\right]^{(2-1)(4-1)} \\ &= 3^3 = 27. \end{aligned}$$

According to the result mentioned in [14],

$$\begin{aligned} \Pi_1(\Gamma(R)) &= (|F_1| - 1)^{2(|F_2|-1)} + (|F_2| - 1)^{2(|F_1|-1)} \\ &= (2 - 1)^{2(4-1)} + (4 - 1)^{2(2-1)} \\ &= 1^6 + 3^2 = 10, \\ \Pi_2(\Gamma(R)) &= [(|F_1| - 1)]^{(|F_2|-1)}(|F_2| - 1)^{(|F_1|-1)} \\ &= (2 - 1)^{(4-1)}(4 - 1)^{(2-1)} \\ &= 1^3 \cdot 3^1 = 3. \end{aligned}$$

A direct computation confirms that our result is correct, indicating that the value given in [14] is erroneous.

Remark 3.7 below highlights inconsistencies in the arguments presented in [13, Results 5–7] concerning the computation of the Narumi–Katayama index and the first and second multiplicative Zagreb index for zero-divisor graphs. The discrepancy arises from an incorrect treatment of degree contributions within certain equi-neighbor classes, which leads to inaccuracies in the final expressions. The corrected formulations are established in Theorems 3.1–3.3 of the present paper.

In addition, the computations provided in [14, Example 3] for the first and second multiplicative Zagreb indices contain similar errors in the handling of degree products. These issues are examined and corrected in the subsequent remark, where we supply revised calculations and clarify the correct structural interpretation needed for obtaining the proper index values.

Remark 3.7. Consider $R = F_1 \times F_2 \times F_3$ as a specific instance of Theorems 3.1–3.3, for which the associated set is $S = \{1, 2, 3\}$ and

$$\begin{aligned} NK(\Gamma(R)) =& (|F_1| - 1)^{(|F_2|-1)(|F_3|-1)} \cdot (|F_2| - 1)^{(|F_1|-1)(|F_3|-1)} \\ &\cdot (|F_3| - 1)^{(|F_1|-1)(|F_2|-1)}(|F_1||F_2| - 1)^{(|F_3|-1)} \\ &\cdot (|F_1||F_3| - 1)^{(|F_2|-1)} \cdot (|F_2||F_3| - 1)^{(|F_1|-1)}, \\ \Pi_1(\Gamma(R)) =& (|F_1| - 1)^{2(|F_2|-1)(|F_3|-1)} \cdot (|F_2| - 1)^{2(|F_1|-1)(|F_3|-1)} \\ &\cdot (|F_3| - 1)^{2(|F_1|-1)(|F_2|-1)} \cdot (|F_1||F_2| - 1)^{2(|F_3|-1)} \\ &\cdot (|F_1||F_3| - 1)^{2(|F_2|-1)} \cdot (|F_2||F_3| - 1)^{2(|F_1|-1)}, \\ \Pi_2(\Gamma(R)) =& [(|F_1| - 1)(|F_2||F_3| - 1)]^{(|F_1|-1)(|F_2|-1)(|F_3|-1)} \\ &\cdot [(|F_2| - 1)(|F_1||F_3| - 1)]^{(|F_1|-1)(|F_2|-1)(|F_3|-1)} \\ &\cdot [(|F_3| - 1)(|F_1||F_2| - 1)]^{(|F_1|-1)(|F_2|-1)(|F_3|-1)} \\ &\cdot [(|F_1||F_2| - 1)(|F_1||F_3| - 1)]^{(|F_3|-1)(|F_2|-1)} \\ &\cdot [(|F_1||F_2| - 1)(|F_2||F_3| - 1)]^{(|F_3|-1)(|F_1|-1)} \\ &\cdot [(|F_1||F_3| - 1)(|F_2||F_3| - 1)]^{(|F_2|-1)(|F_1|-1)}. \end{aligned}$$

4. Numerical analysis

This section presents a numerical study of the Narumi–Katayama, first multiplicative Zagreb, and second multiplicative Zagreb indices for zero-divisor graphs of finite commutative reduced rings. The numerical results illustrate the behaviour of these degree-based invariants and provide computational verification of the closed-form formulas established in Theorems 3.1–3.3.

The computational verification was carried out using Python in the Anaconda3 distribution with the Spyder integrated development environment. The implementation employs the standard Python libraries `itertools` and `math`. The input consists of the orders of the finite fields in the decomposition

$$R \cong F_1 \times F_2 \times \cdots \times F_h.$$

Two independent Python implementations were developed. The first constructs the zero-divisor graph directly from the ring structure and computes the corresponding degree-based quantities and topological indices. The second evaluates the Narumi–Katayama, first multiplicative Zagreb, and second multiplicative Zagreb indices using the closed-form formulas established in Theorems 3.1–3.3. The agreement between the graph-based computations and the formula-based computations confirms the correctness of the theoretical results. The complete Python implementations are provided in Appendix A.

4.1. Comparative study using multiplicative Zagreb topological indices

This subsection presents graphical and numerical comparisons of the Narumi–Katayama, first multiplicative Zagreb, and second multiplicative Zagreb indices for representative finite commutative reduced rings. The values obtained from the graph-construction approach are compared with those computed using the closed-form formulas, providing an independent computational verification of the theoretical expressions. Figure 1 illustrates the behaviour of these indices, while Table 2 summarizes the corresponding numerical results.

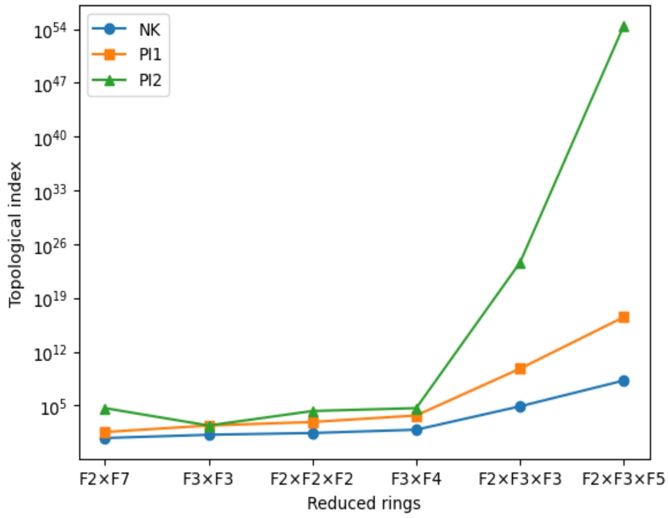

Figure 1. Comparison of the Narumi–Katayama and the first and second multiplicative Zagreb indices.

Table 2. Numerical comparison of the Narumi–Katayama index and the first and second multiplicative Zagreb indices.

$\lvert F_1\rvert = p, \lvert F_2\rvert = q, \lvert F_3\rvert = r$	$NK(\Gamma(R))$	$\Pi_1(\Gamma(R))$	$\Pi_2(\Gamma(R))$
$p = 2, q = 7$	6	36	46656
$p = 3, q = 3$	16	256	256
$p = 2, q = 2, r = 2$	27	729	19683
$p = 3, q = 4$	72	5184	46656
$p = 3, q = 3, r = 3$	80000	64×10^8	4096×10^{20}
$p = 2, q = 3, r = 5$	181440000	329204736×10^8	266856417903983 254812942573840 62976×10^{20}

4.2. Computational performance comparison

To further validate the proposed closed-form formulas, we compare their computational performance with that of the graph-construction-based Python implementation. The latter explicitly constructs the zero-divisor graph from the ring elements, computes the vertex degrees, and evaluates the corresponding topological indices, whereas the closed-form implementation directly evaluates the formulas established in Theorems 3.1–3.3.

Table 3 reports the mean execution time and standard deviation over 100 independent runs for both approaches, together with the corresponding speed-up factors and verification results for representative finite commutative reduced rings. In every case, the values obtained from the graph-construction algorithm agree exactly with those computed using the closed-form formulas, thereby confirming the correctness of the theoretical results. The timing comparison further illustrates the relative computational efficiency of the two approaches for the representative examples considered.

Table 3. Comparison of execution times for graph-construction and closed-form computations (100 runs).

Reduced ring R	Graph construction (mean ± SD)	Closed-form (mean ± SD)	Speed-up	Verification
$\mathbb{F}_2 \times \mathbb{F}_7$	$(6.132 \pm 0.721) \times 10^{-5}$	$(2.667 \pm 0.615) \times 10^{-5}$	2.30	Match
$\mathbb{F}_3 \times \mathbb{F}_3$	$(3.170 \pm 1.023) \times 10^{-5}$	$(2.657 \pm 1.987) \times 10^{-5}$	1.19	Match
$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$	$(5.359 \pm 1.184) \times 10^{-5}$	$(1.2098 \pm 0.2457) \times 10^{-4}$	0.44	Match
$\mathbb{F}_3 \times \mathbb{F}_4$	$(3.351 \pm 1.465) \times 10^{-5}$	$(4.396 \pm 0.607) \times 10^{-5}$	0.76	Match
$\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3$	$(3.5541 \pm 0.7059) \times 10^{-4}$	$(1.2052 \pm 0.1255) \times 10^{-4}$	2.95	Match
$\mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_5$	$(4.3746 \pm 0.7624) \times 10^{-4}$	$(1.2291 \pm 0.4082) \times 10^{-4}$	3.56	Match

The results demonstrate that the graph-construction-based implementation confirms the correctness of the derived closed-form formulas, while the closed-form approach is computationally more efficient, particularly for rings of larger order.

5. Theoretical implications and structural analysis

Our closed-form expression for the second multiplicative Zagreb index $\Pi_2(\Gamma(R))$ of the direct product ring

$$R = \prod_{i=1}^{h} F_i$$

allows us to analyze how algebraic properties of finite fields and ring structures dictate topological graph invariants.

5.1. *Behavior under field component growth ($h \to \infty$)*

The expression of the index depends on the product terms associated with the subset of $S = \{1, 2, \ldots, h\}$. Since the power set 2^S contains 2^h subsets, its size increases exponentially as the number of field components h grows. This exponential increase simultaneously enlarges the degree values $\prod_{i\in I} |F_i| - 1$ and the corresponding equi-neighbor classes. As a result, provided that each field satisfies $|F_i| > 2$, the index $\Pi_2(\Gamma(R))$ grows at a superexponential rate with respect to h.

5.2. *Monotonicity properties*

The invariant $\Pi_2(\Gamma(R))$ satisfies strict monotonicity behavior relative to the field sizes:

- **Monotonicity with respect to field orders:** Let $k \in S$. If the finite field F_k is replaced by a larger field F'_k satisfying $|F'_k| > |F_k|$, then the corresponding vertex degrees and the cardinalities of the associated equi-neighbor classes increase. Since the second multiplicative Zagreb index is expressed as a product of power-weighted vertex degrees, it follows that $\Pi_2(\Gamma(R))$ is strictly increasing with respect to the order of each constituent field $|F_i|$.

- **Monotonicity with respect to the number of field components:** Introducing an additional direct factor F_{h+1} enlarges the indexing set from S to $S \cup \{h+1\}$, thereby generating additional subset interactions and increasing the number of zero-divisor configurations. Consequently, the value of $\Pi_2(\Gamma(R))$ increases with the number of field components.

5.3. Extremal configurations

The closed-form expression for $\Pi_2(\Gamma(R))$ also enables the identification of rings that yield extremal values of the index.

- **Minimum index configuration:** Among all nontrivial finite commutative reduced rings, the smallest value of $\Pi_2(\Gamma(R))$ is attained when $R \cong F_2 \times F_2$, where $h = 2$ and $|F_1| = |F_2| = 2$. In this case, every relevant vertex has degree one, and hence $\Pi_2(\Gamma(R)) = 1$.
- **Maximum index configuration:** Suppose that the number of field components h is fixed and the field orders satisfy
$$\max_{1\le i\le h} |F_i| = q.$$
The largest value of $\Pi_2(\Gamma(R))$ is obtained when each component field is isomorphic to the finite field F_q. Under this configuration, both the vertex degrees and the associated equi-neighbor classes attain their largest possible values, resulting in the maximum distribution of zero-divisors and, consequently, the maximum value of the second multiplicative Zagreb index.

6. Conclusions

In this paper, we derived closed-form expressions for the Narumi–Katayama index and the first and second multiplicative Zagreb indices of zero-divisor graphs associated with arbitrary finite commutative reduced rings. By exploiting the structural decomposition of the zero-divisor graph into equi-neighbor classes, we established a unified framework for computing these multiplicative degree-based invariants and extended several previously known results.

The derived formulas correct inaccuracies in earlier literature and provide an efficient alternative to explicit graph construction. Their correctness was further confirmed through independent Python-based graph construction and computational comparisons with the closed-form computations.

The obtained results also reveal how the algebraic decomposition of finite commutative reduced rings influences the corresponding graph invariants, providing a basis for studying their behaviour under varying ring structures. Future work may focus on deriving closed-form expressions for other degree-based topological indices and extending the proposed framework to broader classes of commutative rings, including non-reduced rings, finite local rings, finite chain rings, infinite commutative rings and other algebraic structures.

Author contributions

R. Sankari Alias Deepa: conceptualization, investigation, data curation, writing; R. Gurusamy: conceptualization, investigation, methodology, writing; S. Arockiaraj: investigation, validation, writing; Yilun Shang: investigation, data curation, writing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

AI-assisted tools were used only for language polishing. All mathematical results and proofs were developed and verified by the authors.

Conflict of interest

Prof. Yilun Shang is an editorial board member for AIMS Mathematics and was not involved in the editorial review and/or the decision to publish this article. All authors declare no conflicts of interest in this paper.

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Appendix

Python implementation

This appendix presents two Python implementations developed in this work. The first independently constructs the zero-divisor graph of a finite commutative reduced ring and computes the corresponding degree-based quantities directly from the graph. The second evaluates the Narumi–Katayama, first multiplicative Zagreb, and second multiplicative Zagreb indices using the closed-form formulas established in Theorems 3.1–3.3. A comparison between the graph-based computations and the theoretical formula-based computations is presented in Section 4, confirming the correctness of the derived results.

A. Python code for constructing the zero-divisor graph

The following Python implementation constructs the zero-divisor graph of a finite commutative reduced ring directly from its ring elements and computes the associated degree-based quantities and topological indices.

```
import itertools
from math import isqrt
def is_prime_power(n):
    if n < 2:
        return False
    for p in range(2, n + 1):
        # Check whether p is prime
        prime = True
        for d in range(2, int(p**0.5) + 1):
```

```
            if p % d == 0:
                prime = False
                break
        if not prime:
            continue
        power = p
        while power < n:
            power *= p
        if power == n:
            return True
    return False
print("Enter the number of product of finite field")
n=int(input("Enter an positive integer:"))
A=[]
for i in range(1,n+1):
    A.append(i)
v=[ ]
p=[]
for i in range(0,n):
    a= int(input("Enter the cardinality of field"))
    if not is_prime_power(a):
        raise ValueError(f"{a} is not a valid
        finite field order (prime power).")
    v.append(a)
elements = list(itertools.product(*[range(q) for q in v]))
def zero_divisors(elements):
    Z=[]
    zero = tuple(0 for _ in elements[0])
    for x in elements:
        if x!=zero and 0 in x:
            Z.append(x)
    return Z
V=zero_divisors(elements)
edges=[]
for i in range(len(V)):
    for j in range(i+1,len(V)):
        u=V[i]
        v=V[j]
        ok=True
        for a,b in zip(u,v):
            if a*b!=0:
                ok=False
                break
```

```
        if ok:
            edges.append((u,v))
degree={v:0 for v in V}
for u,v in edges:
    degree[u]+=1
    degree[v]+=1
NK=1
for d in degree.values():
    NK*=d
print("Narumi katayama index of a graph",NK)
Pi1=1
for d in degree.values():
    Pi1*=d**2
print("First Multiplicative zagreb index of a graph",Pi1)
Pi2=1
for u,v in edges:
    Pi2*=degree[u]*degree[v]
print("Second Multiplicative Zagreb index of a graph",Pi2)
```

B. Python code for computing the topological indices using Theorems 3.1–3.3

The following Python implementation evaluates the Narumi–Katayama, first multiplicative Zagreb, and second multiplicative Zagreb indices using the closed-form expressions established in Theorems 3.1–3.3. The results are compared with those obtained from the graph-construction approach to verify the correctness of the theoretical formulas.

```
import itertools
from math import isqrt
def is_prime_power(n):
    if n < 2:
        return False
    for p in range(2, n + 1):
        # Check whether p is prime
        prime = True
        for d in range(2, int(p**0.5) + 1):
            if p % d == 0:
                prime = False
                break
        if not prime:
            continue
        power = p
        while power < n:
            power *= p
        if power == n:
            return True
```

```
    return False
print("Enter the number of product of finite field")
n=int(input("Enter an positive integer:"))
A=[]
for i in range(1,n+1):
    A.append(i)
v=[ ]
p=[]
for i in range(0,n):
    a= int(input("Enter the cardinality of field"))
    if a <= 1:
    raise ValueError("Field order must be greater than 1.")
    if not is_prime_power(a):
    raise ValueError("Field order must be a prime power.")
    v.append(a)
# Generate all subsets
power_set = [list(s) for r in range(1,len(A) + 1)
for s in itertools.combinations(A, r)]
print(power_set)
a,b,c,d1,d2,d3,d4,s,s2=0,1,1,1,1,1,1,1,1
for i in range(len(power_set)-1):
    power_set1 = [list(s) for r in range(1,len(power_set[i]) + 1)
    for s in itertools.combinations(power_set[i], r)]
    diff = list(set(A) - set(power_set[i]))
    for l in range(len(power_set1)):
        diff1=list(set(power_set[i])-set(power_set1[l]))
        c=1
        for n in power_set[i]:
            c=c*v[n-1]
        f=c-1
        b=1
        diff2=diff+diff1
        for m in diff2:
            b=b*v[m-1]
        e1=b-1
        d=f*e1
        s1=diff+list(set(power_set1[l]))
        d1=1
        for m in s1:
            d1=d1*(v[m-1]-1)
        d4=pow(d,d1)
        d2=d2*d4
    for j in power_set[i]:
```

```
        if len(power_set[i])==1:
            a=(v[j-1] - 1)
        else:
            c=1
            for l in power_set[i]:
                c=c*v[l-1]
            a=(c-1)
            break
    b=1
    for k in diff:
            b=b*(v[k-1]-1)
    d5=pow(a,b)
    d6=pow(a,2*b)
    s=s*d5
    s2=s2*d6
d3 = isqrt(d2)
if d3 * d3 != d2:
    raise ValueError("The computed value is not a perfect square.")
print("Narumi--katayama index of a graph",s)
print("First Multiplicative Zagreb index of a graph",s2)
print("Second Multiplicative Zagreb index of a graph",d3)
```