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Research article

New results on the Laplacian ABC-eigenvalues of a graph

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Abstract: Let G be a graph of order n with m edges. Let $\tilde{L}(G)$ be the Laplacian atom-bond connectivity (ABC)-matrix of G. The eigenvalues of $\tilde{L}(G)$ are the Laplacian ABC-eigenvalues of G, and its largest eigenvalue is called the Laplacian ABC-spectral radius of G. In this paper, we present some new upper and lower bounds for the Laplacian ABC-spectral radius, and we characterize the extremal graphs attaining these bounds. We also present an upper bound for the second smallest Laplacian ABC-eigenvalue and characterize the extremal graphs. We determine the Laplacian ABC-eigenvalues of the join of two regular graphs and the join of a regular graph with the union of two regular graphs, in terms of the Laplacian eigenvalues of the component graphs.

Keywords: Laplacian ABC-matrix; Laplacian ABC-eigenvalues; spectral radius; extremal graphs; join of graphs
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1. Introduction

In this paper, we consider simple undirected finite graphs. Let $G = (V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Let $n = |V(G)|$ and $m = |E(G)|$ be the order and size of G, respectively. For any vertex $v_i \in V(G)$, the degree d_i, d_{v_i}, or $d(v_i)$ represents the number of vertices adjacent to v_i. A graph G is said to be k-regular if $d_{v_i} = k$, for all $v_i \in V(G)$. For fundamental graph-theoretic terminology and notations not explicitly defined herein, we refer the reader to [1, 2].

The adjacency matrix $A(G)$ of G is an n-square symmetric matrix having the ijth entry 1, if v_i is adjacent to v_j and 0, otherwise. The eigenvalues of $A(G)$ are called the eigenvalues of G, and the largest eigenvalue of $A(G)$ is called the spectral radius of G. Let $D(G)$ be the diagonal matrix of vertex degrees of G. The Laplacian matrix $L(G)$ of G is defined as $L(G) = D(G) - A(G)$. The eigenvalues of the matrix $L(G)$ are called the Laplacian eigenvalues of G, and its largest eigenvalue is called the

Laplacian spectral radius of G. A well-known property of the matrix $L(G)$ is that it is a real symmetric positive semidefinite matrix with the smallest eigenvalue equal to zero. Let

$$\mu_1 \geq \mu_2 \geq \mu_{n-1} \geq \mu_n = 0$$

be the Laplacian eigenvalues of G. The spectral properties of the matrices $A(G)$ and $L(G)$, including bounds on the spectral radius, energy, and other eigenvalues, are well-studied and have become a rich area of study in present-day research. For recent developments on the spectral properties of $A(G)$ and $L(G)$, we refer the reader to [3–5] and the references therein.

In recent decades, the investigation of matrices derived from topological indices has emerged as a prominent area of research, bridging graph theory with mathematical chemistry. One such significant descriptor is the atom-bond connectivity (ABC)-index, introduced by Estrada et al. [6], which is defined as:

$$ABC(G) = \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_id_j}}. \tag{1.1}$$

The ABC-index has gained prominence for its ability to correlate with the thermodynamic stability of alkanes and its precision in modeling molecular heat of formation [7–9]. This success led to the introduction of the ABC-matrix $\tilde{A}(G) = (\tilde{a}_{ij})$, where

$$\tilde{a}_{ij} = \sqrt{(d_i + d_j - 2)/(d_id_j)}$$

if v_i is adjacent to v_j, and equal to 0, otherwise. The eigenvalues of the matrix $\tilde{A}(G)$ are called the ABC-eigenvalues of G, and its largest eigenvalue is called the ABC-spectral radius of G. This matrix was introduced by Estrada [6] as a matrix representation of the probability of visiting a nearest-neighbor edge from one side or the other of a given edge in a graph, which can be related to the polarizing capacity of the bond considered in a molecular context. For recent work on the ABC-spectral radius and ABC-energy, we refer the reader to [10–12], and for the recent work on ABC-spectral radius of trees and unicyclic graphs and for graphs with given maximum degree, we refer the reader to [13–15].

The Laplacian ABC-matrix, recently formulated in [16] as $\tilde{L}(G) = \overline{D}(G) - \tilde{A}(G)$, combines features of both the Laplacian and ABC-matrices. Here, $\overline{D}(G) = \mathrm{diag}(\overline{d}_1, \overline{d}_2, \ldots, \overline{d}_n)$ is the diagonal matrix of ABC-degrees, where

$$\overline{d}_i = \sum_{v_iv_j \in E(G)} \sqrt{(d_i + d_j - 2)/(d_id_j)}.$$

As $\tilde{L}(G)$ is a real symmetric and positive semidefinite matrix, its eigenvalues are called Laplacian ABC-eigenvalues of G and can be ordered as $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq \xi_n = 0$. The largest eigenvalue $\xi_1(G)$ of $\tilde{L}(G)$ is called the Laplacian ABC-spectral radius of G. The eigenvalues of the matrix $\tilde{L}(G)$ are closely related to the ABC-index of a graph G; in fact, the trace of the matrix $\tilde{L}(G)$, which is the same as the sum of its eigenvalues, is twice the ABC-index of G. The study of this matrix is interesting because it brings together two different types of structural information. The ABC-weights describe the local degree distribution around edges, while the Laplacian framework reflects the overall connectivity of the graph. Its eigenvalues may therefore capture structural features that are not fully visible from the ordinary ABC-spectrum or the classical Laplacian spectrum alone. The investigation of Laplacian ABC-eigenvalues is also motivated by the rich theory surrounding the ordinary Laplacian matrix. Classical Laplacian eigenvalues are closely related to important graph parameters such as connectivity, spanning trees, graph partitioning, and network robustness. It is natural to ask whether similar relationships can be established when the ordinary edge weights are replaced by ABC-weights.

In [16], the authors explored some basic properties of the Laplacian ABC-matrix $\tilde{L}(G)$. They derived a lower bound for the Laplacian ABC-spectral radius and the characterization of graphs with two distinct Laplacian ABC-eigenvalues. In [17], the authors considered the problem of characterizing graphs with three distinct Laplacian ABC-eigenvalues, and they solved this problem for bipartite graphs, multipartite graphs, unicyclic graphs, and regular graphs. They showed the non-existence of such graphs with diameter greater than 2. Further, they introduced the concept of trace norm of the matrix $\tilde{L}(G) - \frac{tr(\tilde{L}(G))}{n}I$, called the Laplacian *ABC*-energy of G. Some bounds for the Laplacian ABC-energy were established, and the extremal graphs were characterized. The spectral theory of the Laplacian ABC-matrix is still at an early stage of development. In contrast to the classical Laplacian matrix and the ABC-matrix, whose spectral properties have been extensively investigated, only a limited number of studies have been devoted specifically to the Laplacian ABC-matrix. This leaves several fundamental questions open and provides considerable scope for further research.

One of the interesting problems that has attracted the attention of many researchers in spectral graph theory is the determination of the bounds and characterization of extremal graphs for the largest and smallest eigenvalues of a graph matrix associated with a graph G. This problem has been discussed for almost all the graph matrices introduced so far. We aim to discuss this problem for the Laplacian ABC-matrix, and present some new upper and lower bounds for the Laplacian ABC-spectral radius in terms of various graph parameters associated with the structure of the graph. The extremal graphs attaining these bounds are completely characterized. For the second smallest Laplacian ABC-eigenvalue of a connected graph, we present an upper bound in terms of minimum ABC-degree and characterize the extremal graphs that attain this bound.

Another interesting problem that has been considered for the different graph matrices in the literature is the determination of the eigenvalues of a graph that can be obtained by means of some operation from a class of smaller graphs. We will consider this problem for the determination of the Laplacian ABC-eigenvalues of the join of two regular graphs and the join of a regular graph with the union of two regular graphs. Consequently, we obtain explicit Laplacian ABC-eigenvalue formulae for several families of graphs that can be constructed using the join operation.

Throughout this paper, K_n, $\overline{K}_n$, C_n, and P_n denote, respectively, the complete graph, the empty graph, the cycle, and the path on the n vertices. For a vertex $v_i \in V(G)$, we use $d(v_i)$, d_{v_i}, or d_i to denote its degree, $\overline{d}(v_i)$ or $\overline{d}_i$ to denote its ABC-degree, and $\overline{m}(v_i)$ or $\overline{m}_i$ to denote its 2-ABC-degree. The Laplacian eigenvalues of G are denoted by

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0,$$

while the Laplacian ABC-eigenvalues are denoted by

$$\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq \xi_n = 0.$$

The largest Laplacian ABC-eigenvalue, or equivalently the Laplacian ABC-spectral radius, is denoted by

$$\xi_1(G) = \lambda_1(\tilde{L}(G)),$$

and the signless Laplacian ABC-spectral radius is denoted by $\lambda_1(\tilde{Q}(G))$. Finally, the join operation is denoted by $\vee$.

The rest of the paper is organized as follows. In Section 2, we present some new upper and lower bounds for the Laplacian ABC-spectral radius, and we characterize the extremal graphs for these bounds. We also obtain an upper bound for the second smallest Laplacian ABC-eigenvalue and characterize the extremal graphs. In Section 3, we determine the Laplacian ABC-eigenvalues of the join of two regular graphs and the join of a regular graph with the union of two regular graphs. As a consequence of our results in this section, we determine the Laplacian ABC-eigenvalues of various well-known families of graphs. In Section 4, we present some examples of graphs and highlight the importance and application of our results obtained in Sections 2 and 3. In Section 5, we add a conclusion regarding the contribution of the paper and the future directions.

2. Bounds for the Laplacian ABC-spectral radius

In this section, we obtain some new upper and lower bounds for the Laplacian ABC-spectral radius of a graph. We characterize the extremal graphs attaining these bounds. We also present an upper bound for the second smallest Laplacian ABC-eigenvalue of G and characterize the extremal graphs.

Any column vector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ can be regarded as a function defined on $V(G)$ that relates every v_i to x_i, that is, $X(v_i) = x_i$ for all $i = 1, 2, \ldots, n$. Also, it is easy to see that

$$\begin{aligned} X^T \tilde{\mathrm{L}}(G) X &= \sum_{i=1}^{n} \overline{d}_i x_i^2 - 2 \sum_{v_i v_j \in E(G)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}} x_i x_j \\ &= \sum_{v_i v_j \in E(G)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}} \, (x_i - x_j)^2, \end{aligned}$$

where

$$\overline{d}_i = \overline{d}_{v_i} = \sum_{v_i v_j \in E(G)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}}.$$

A real number ξ is the Laplacian ABC-eigenvalue with its associated eigenvector X if and only if for every $v_i \in V(G)$, we have

$$\xi X(v_i) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}} (X(v_i) - X(v_j)), \tag{2.1}$$

or equivalently

$$\xi X(v_i) - \overline{d}_i X(v_i) = - \sum_{v_i v_j \in E(G)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}} \, X(v_j). \tag{2.2}$$

Equations (2.1) and (2.2) are the (ξ, X)-eigenequations for the Laplacian ABC-matrix.

The signless Laplacian ABC-matrix $\tilde{Q}(G)$ of a graph G was introduced in [18] and is defined as $\tilde{Q}(G) = \overline{D}(G) + \tilde{A}(G)$, where $\tilde{A}(G)$ is the ABC matrix and $\overline{D}(G)$ is the diagonal matrix of ABC-degrees of the graph G. Some interesting results were presented in [18] regarding the spectral properties of the signless Laplacian ABC-matrix of G.

The following result shows that for bipartite graphs, the Laplacian ABC- and the signless Laplacian ABC-matrices share the same spectrum and can be found in [18].

Lemma 2.1. *Let G be a bipartite graph of order $n \geq 2$. Then $\tilde{Q}(G)$ and $\tilde{L}(G)$ are unitarily similar and share the same spectrum. Further, for connected bipartite graphs, the Laplacian ABC-spectral radius is simple.*

The following result is due to Perron and Frobenius [19].

Lemma 2.2. *Let $M = (m_{ij})$ be a complex matrix and N be an irreducible matrix of the same order. Let $|M|$ denote the matrix whose (i, j)-entry is $|m_{ij}|$. If $|M| \leq N$ and M have μ as an eigenvalue, then $|\mu| \leq \lambda_1(N)$. If the equality holds, then $|M| = N$, and there is a diagonal matrix E with diagonal entries of absolute value 1 and a constant c of absolute value 1 such that $M = cENE^{-1}$.*

In the next theorem, we will show that the Laplacian ABC-spectral radius is at most the signless Laplacian ABC-spectral radius, and equality is attained for bipartite graphs.

Theorem 2.3. *Let G be a connected graph of order $n \geq 3$. Then*

$$\lambda_1(\tilde{L}(G)) \leq \lambda_1(\tilde{Q}(G)),$$

with equality if and only if G is a bipartite graph.

Proof. Let $\tilde{L}(G) = (\tilde{l}_{ij})$ be the Laplacian ABC-matrix and $\tilde{Q}(G) = (\tilde{q}_{ij})$ be the signless Laplacian ABC-matrix of G. It is clear from the definitions of $\tilde{L}(G)$ and $\tilde{Q}(G)$ that $|\tilde{l}_{ij}| \leq \tilde{q}_{ij}$, for all i, j. Also, $\tilde{Q}(G)$ is a non-negative irreducible matrix, so it follows from Lemma 2.2 that $\lambda_1(\tilde{L}(G)) \leq \lambda_1(\tilde{Q}(G))$. Assume that equality holds; then equality holds in Lemma 2.2. If equality holds in Lemma 2.2, then $|\tilde{L}(G)| = \tilde{Q}(G)$, which is always true, and there is a diagonal matrix E with diagonal entries of absolute value 1, say, $E = diag(e^{i\theta_u} : u \in V(G))$ such that $\tilde{L}(G) = cE\tilde{Q}(G)E^{-1}$, where c is a number with absolute value 1. Note that $i^2 = -1$ and θ_u is a real number. From equation $\tilde{L}(G) = cE\tilde{Q}(G)E^{-1}$, we get $\tilde{l}_{ij} = ce^{i\theta_i}\tilde{q}_{ij}e^{-i\theta_j} = ce^{i(\theta_i-\theta_j)}\tilde{q}_{ij}$. For $i = j$, this gives that $c\tilde{q}_{ii} = \tilde{l}_{ii}$, implying that $c = 1$ as $\tilde{q}_{ii} = \tilde{l}_{ii}$. Now, for $c = 1$, the equality $\tilde{l}_{ij} = ce^{i(\theta_i-\theta_j)}\tilde{q}_{ij}$ gives that $e^{i(\theta_i-\theta_j)} = 1$ or -1 whenever $v_iv_j \in E(G)$. Since $e^{i(\theta_u-\theta_v)}$, for any edge $uv \in E(G)$ implying that $\theta_u = \theta_v$ or $\theta_u = \pi + \theta_v$, it follows that $e^{i\theta_u} = e^{i\theta_v}$ or $-e^{i\theta_v}$. Now, G is a connected graph, which implies that for any two vertices $u, v \in V(G)$, there exists a path $u = v_1v_2 \ldots v_k = v$ in G. From the above observation, we have $e^{i\theta_{v_2}} = e^{i\theta_{v_1}}$ or $-e^{i\theta_{v_1}}$; $e^{i\theta_{v_3}} = e^{i\theta_{v_2}}$ or $-e^{i\theta_{v_2}} = e^{i\theta_{v_1}}$ or $-e^{i\theta_{v_1}}$. Continuing this way, we arrive at $e^{i\theta_v} = e^{i\theta_{v_k}} = e^{i\theta_{v_1}}$ or $-e^{i\theta_{v_1}} = e^{i\theta_u}$ or $-e^{i\theta_u}$. This gives that $e^{i\theta_v} = e^{i\theta_u}$ or $-e^{i\theta_u}$, for any two vertices in G. So, it follows that $E = e^{i\theta}W$, where W is a diagonal matrix with diagonal entries 1 or -1. Therefore, $\tilde{L}(G) = E\tilde{Q}(G)E^{-1}$ gives that $\tilde{L}(G) = W\tilde{Q}(G)W^{-1}$. Comparing the ijth entries, we get $\tilde{l}_{ij} = s_{ii}\tilde{q}_{ij}s_{jj}$, where s_{ii} is the ith diagonal entry of W. Since $\tilde{q}_{ij} = -\tilde{l}_{ij}$, whenever $v_iv_j \in E(G)$, it follows from $\tilde{l}_{ij} = s_{ii}\tilde{q}_{ij}s_{jj}$ that $s_{ii}s_{jj} = -1$ whenever $v_iv_j \in E(G)$. Let $V_1 = \{v : s_{uu} = 1\}$ and $V_2 = \{v : s_{vv} = -1\}$. It is clear that $V_1 \cap V_2 = \phi$. Let $u, w \in V_1$; we will show that u is not adjacent to w. For if $uw \in E(G)$, then $s_{ii}s_{jj} = -1$, whenever $v_iv_j \in E(G)$ gives that $s_{uu}s_{ww} = -1$, which is not possible as $u, w \in V_1$ implies that $s_{uu} = s_{ww} = 1$ and so $s_{uu}s_{ww} = 1$. Therefore, no two vertices of V_1 are adjacent. Similarly, we can show that no two vertices of V_2 are adjacent. Since G is a connected graph, any edge in G has one end in V_1 and the other in V_2, implying that G is a bipartite graph. Conversely, suppose that G is a bipartite graph, then by Lemma 2.1, the matrices $\tilde{L}(G)$ and $\tilde{Q}(G)$ have the same spectrum, implying that $\lambda_1(\tilde{L}(G)) = \lambda_1(\tilde{Q}(G))$. This completes the proof. □

The following lemma states that the spectral radius of a non-negative matrix lies between the maximum and the minimum row sums of the matrix, and can be found in [20].

Lemma 2.4. [20] *If M is an $n \times n$ non-negative matrix with the spectral radius $\lambda(M)$ and row sums $r_1, r_2, \ldots, r_n$, then*

$$\min_{1\le i\le n} r_i \le \lambda(M) \le \max_{1\le i\le n} r_i.$$

Moreover, if M is irreducible, then both of the equalities hold if and only if the row sums of M are all equal.

The following gives an upper bound for the Laplacian ABC-spectral radius in terms of the maximum ABC-degree of G.

Theorem 2.5. *Let G be a connected graph of order $n \ge 3$ having maximum ABC-degree $\overline{d}_{\max}$ and Laplacian ABC-spectral radius $\xi_1(G)$. Then*

$$\xi_1(G) \le 2\overline{d}_{\max}, \tag{2.3}$$

with equality if and only if G is an ABC-regular bipartite graph.

Proof. Consider the signless Laplacian ABC-matrix $\tilde{Q}(G)$ having $r_i(\tilde{Q}(G))$ as the ith row sum. Then $r_i(\tilde{Q}(G)) = 2\overline{d}_i$, for all $i = 1, 2, \ldots, n$. Therefore, by Lemma 2.4, we get $\lambda_1(\tilde{Q}(G)) \le \max_{1\le i\le n} 2\overline{d}_i = 2\overline{d}_{\max}$. Now, using Theorem 2.3, it follows that $\lambda_1(\tilde{L}(G)) \le 2\overline{d}_{\max}$, that is, $\xi_1(G) \le 2\overline{d}_{\max}$. Equality occurs in (2.3) if and only if equality holds in Lemma 2.4 and equality holds in Theorem 2.3. Since G is a connected graph, it implies that the matrix $\tilde{Q}(G)$ is irreducible; it follows that the equality holds in Lemma 2.4 if and only if all row sums of $\tilde{Q}(G)$ are the same. It is clear from the definition of $\tilde{Q}(G)$ that all its row sums are the same if and only if G is ABC-regular. Also, equality holds in Theorem 2.3 if and only if G is a bipartite graph. It follows that equality holds in (2.3) if and only if G is an ABC-regular bipartite graph. This completes the proof. □

If G is an r-regular bipartite graph, then $d_i = r$, for all i and so

$$\overline{d}_i = \sum_{v_iv_j\in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_id_j}} = \sqrt{\frac{2r-2}{r^2}},$$

for all i, which implies that G is an ABC-regular graph. This shows that for a regular bipartite graph G, equality always holds in (2.3).

For a graph, the average 2-degree of a vertex v_i is denoted by m_i and is defined as

$$m_i = \frac{1}{d_i} \sum_{v_iv_j\in E(G)} d_j.$$

Likewise, we define the average 2-ABC-degree as

$$\overline{m}_i = \frac{1}{\overline{d}_i} \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_id_j}} \overline{d}_j.$$

The following result gives an upper bound for the Laplacian ABC-spectral radius in terms of the average 2-ABC-degrees of G.

Theorem 2.6. *Let G be a connected graph of order $n \geq 3$ having ABC-degrees $\overline{d}_i$ and average 2-ABC-degrees $\overline{m}_i$, $i = 1, 2, \ldots, n$. Then*

$$\xi_1(G) \leq \max_{1 \leq i \leq n} \{\overline{d}_i + \overline{m}_i\}, \tag{2.4}$$

with equality if and only if G is a bipartite graph with $\overline{d}_i + \overline{m}_i = k$, for all i.

Proof. Let $\overline{d}_1 \geq \overline{d}_2 \geq \cdots \geq \overline{d}_n$ be the ABC-degrees of the vertices of G. Since G is a connected graph, it implies that $\overline{d}_i > 0$ for all i. It follows that the matrix $\overline{D}(G)^{-1} = diag((\overline{d}_1)^{-1}, (\overline{d}_2)^{-1}, \cdots, (\overline{d}_n)^{-1})$ is defined. Consider the matrix $\overline{D}^{-1}(G)\tilde{Q}(G)\overline{D}(G)$, where $\tilde{Q}(G) = (\tilde{q}_{ij})$ is the signless Laplacian ABC-matrix of G. Let $r_i(\overline{D}^{-1}(G)\tilde{Q}(G)\overline{D}(G))$ be the ith row sum of $\overline{D}^{-1}(G)\tilde{Q}(G)\overline{D}(G)$. The ijth entry of $\overline{D}^{-1}(G)\tilde{Q}(G)\overline{D}(G)$ is $\overline{d}_i$, if $i = j$ and is

$$(\overline{d}_i)^{-1}\tilde{q}_{ij}\overline{d}_j = \frac{\overline{d}_j}{\overline{d}_i} \sqrt{\frac{d_i + d_j - 2}{d_id_j}}$$

if $i \neq j$. Therefore,

$$r_i(\overline{D}^{-1}(G)\tilde{Q}(G)\overline{D}(G)) = \overline{d}_i + \sum_{v_iv_j \in E(G)} \frac{\overline{d}_j}{\overline{d}_i} \sqrt{\frac{d_i + d_j - 2}{d_id_j}} = \overline{d}_i + \overline{m}_i,$$

for all $i = 1, 2, \ldots, n$. So, by Lemma 2.4, we get

$$\lambda_1(\tilde{Q}(G)) = \lambda_1(\overline{D}^{-1}(G)\tilde{Q}(G)\overline{D}(G)) \leq \max_{1 \leq i \leq n} \{\overline{d}_i + \overline{m}_i\}.$$

Now, using Theorem 2.3, it follows that

$$\lambda_1(\tilde{L}(G)) \leq \max_{1 \leq i \leq n} \{\overline{d}_i + \overline{m}_i\}.$$

Equality occurs in (2.4) if and only if equality holds in Lemma 2.4 and equality holds in Theorem 2.3. Since G is a connected graph, it implies that the matrix $\tilde{Q}(G)$ and so the matrix $\overline{D}^{-1}(G)\tilde{Q}(G)\overline{D}(G)$ are irreducible. It follows that the equality holds in Lemma 2.4 if and only if all the row sums of $\overline{D}^{-1}(G)\tilde{Q}(G)\overline{D}(G)$ are the same. That is, equality holds in Lemma 2.4 if and only if $\overline{d}_i + \overline{m}_i$ is the same for all $i = 1, 2, \ldots, n$. Also, equality holds in Theorem 2.3 if and only if G is a bipartite graph. It follows that equality holds in (2.4) if and only if G is a bipartite graph with $\overline{d}_i + \overline{m}_i = k$, for all $i = 1, 2, \ldots, n$. This completes the proof. □

If G is a r-regular bipartite graph, then $d_i = r$ for all i, and so

$$\overline{d}_i = \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}} = \sqrt{\frac{2r-2}{r^2}}$$

for all i and

$$\overline{m}_i = \sum_{v_iv_j \in E(G)} \frac{\overline{d}_j}{\overline{d}_i} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}} = \overline{d}_i = \sqrt{\frac{2r-2}{r^2}}$$

for all i. This shows that $\overline{d}_i + \overline{m}_i = 2\sqrt{\frac{2r-2}{r^2}}$ for all i, and so equality occurs in (2.4) for regular bipartite graphs.

If G is a (r, s)-semiregular bipartite graph with partite sets V_1 and V_2, then $d_i = r$, for all $v_i \in V_1$ and $d_i = s$, for all $v_i \in V_2$. Therefore, we have

$$\overline{d}_i = \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}} = r\sqrt{\frac{r+s-2}{rs}},$$

for all $v_i \in V_1$ and

$$\overline{d}_i = \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}} = s\sqrt{\frac{r+s-2}{rs}},$$

for all $v_i \in V_2$. Also,

$$\overline{m}_i = \sum_{v_iv_j \in E(G)} \frac{\overline{d}_j}{\overline{d}_i} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}} = \frac{s\sqrt{\frac{r+s-2}{rs}}}{r\sqrt{\frac{r+s-2}{rs}}} r\sqrt{\frac{r+s-2}{rs}} = s\sqrt{\frac{r+s-2}{rs}},$$

for all $v_i \in V_1$ and

$$\overline{m}_i = \sum_{v_iv_j \in E(G)} \frac{\overline{d}_j}{\overline{d}_i} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}} = \frac{r\sqrt{\frac{r+s-2}{rs}}}{s\sqrt{\frac{r+s-2}{rs}}} s\sqrt{\frac{r+s-2}{rs}} = r\sqrt{\frac{r+s-2}{rs}},$$

for all $v_i \in V_2$. Thus, it follows that

$$\overline{d}_i + \overline{m}_i = r\sqrt{\frac{r+s-2}{rs}} + s\sqrt{\frac{r+s-2}{rs}} = (r+s)\sqrt{\frac{r+s-2}{rs}},$$

for all $v_i \in V(G)$. This shows that equality occurs in (2.4) for (r, s)-semiregular bipartite graphs.

Remark 2.7. *The upper bound given by* (2.4) *is better than the upper bound given by* (2.3)*. As*

$$\overline{d}_i + \overline{m}_i = \overline{d}_i + \sum_{v_iv_j \in E(G)} \frac{\overline{d}_j}{\overline{d}_i} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}$$

$$\leq \overline{d}_{\max} + \frac{\overline{d}_{\max}}{\overline{d}_i} \sum_{v_iv_j \in E(G)} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}$$
$$= \overline{d}_{\max} + \frac{\overline{d}_{\max}}{\overline{d}_i}\overline{d}_i = 2\overline{d}_{\max},$$

for all i.

The importance of the upper bound given by (2.3) is that it is easy to use in practice as compared to the upper bound given by (2.4).

Using Lemma 2.1 and proceeding similarly to Theorem 2.5, we obtain the following lower bound for the Laplacian ABC-spectral radius in terms of the minimum ABC-degree of the bipartite graph G.

Theorem 2.8. *Let G be a connected bipartite graph of order $n \geq 3$ having minimum ABC-degree $\overline{d}_{\min}$. Then*

$$\xi_1(G) \geq 2\overline{d}_{\min},$$

with equality if and only if G is an ABC-regular bipartite graph.

Again using Lemma 2.1 and proceeding similarly to Theorem 2.6, we obtain the following lower bound for the Laplacian ABC-spectral radius in terms of the average 2-ABC-degrees of the bipartite graph G.

Theorem 2.9. *Let G be a connected bipartite graph of order $n \geq 3$ having ABC-degrees $\overline{d}_i$ and average 2-ABC-degrees $\overline{m}_i$, $i = 1, 2, \ldots, n$. Then*

$$\xi_1(G) \geq \min_{1 \leq i \leq n} \{\overline{d}_i + \overline{m}_i\},$$

with equality if and only if G graph with $\overline{d}_i + \overline{m}_i = k$, for all $v_i \in V(G)$.

The following result, which is helpful to determine the extremal results for the ABC-index, can be found in [21].

Lemma 2.10. *Let $f(x, y) = \sqrt{\frac{x+y-2}{xy}}$, where $n - 1 \geq x \geq 2, n - 1 \geq y \geq 1$. Then,*

(1) $f(x, 1)$ is an increasing function of x and so $\frac{1}{\sqrt{2}} = f(2, 1) \leq f(x, 1) \leq f(n - 1, 1) = \sqrt{\frac{n-2}{n-1}}$;
(2) $f(x, 2) = \frac{1}{\sqrt{2}}$;
(3) $f(x, y)$ is a decreasing function of x, y for $x \geq y \geq 3$ and so $\frac{\sqrt{2n-4}}{n-1} = f(n - 1, n - 1) \leq f(x, y) \leq f(3, 3) = \frac{2}{3}$.

Let G be a graph of order $n \geq 3$ such that $d_i \geq 3$ for each $v_i \in V(G)$. For the graph G, using part 3 of Lemma 2.10 together with Theorem 2.6, we obtain the following upper bound for the Laplacian ABC-spectral radius in terms of the ABC-degrees.

Corollary 2.11. *Let G be a connected graph of order $n \geq 3$, which satisfies the property that each vertex of G is of a degree greater than or equal to 3. Let $\overline{d}_i$ be the ABC-degree of the vertex $v_i \in V(G)$. Then*

$$\xi_1(G) \leq \max_{1\leq i\leq n}\{\overline{d}_i + \frac{2}{3\overline{d}_i}\sum_{v_iv_j\in E(G)} \overline{d}_j\}, \tag{2.5}$$

with equality if and only if G is a 3-regular bipartite graph.

Proof. The inequality (2.5) directly follows from Theorem 2.6 by utilizing part 3 of Lemma 2.10. Equality occurs in (2.5) if and only if equality occurs in Theorem 2.6 and equality occurs in part 3 of Lemma 2.10. The equality holds in Theorem 2.6 if and only if G is a bipartite graph with $\overline{d}_i + \overline{m}_i$ as the same for all $v_i \in V(G)$, and equality occurs in part 3 of Lemma 2.10 if and only if G is a 3-regular graph. Since for a 3-regular graph, $\overline{d}_i + \overline{m}_i$ is the same for all $v_i \in V(G)$, we conclude that equality occurs in 2.5 if and only if G is a 3-regular bipartite graph. This completes the proof. □

The following result provides another upper bound for the Laplacian ABC-spectral radius in terms of the ABC-degrees of G.

Theorem 2.12. *Let G be a connected graph of order $n \geq 3$ having ABC-degrees $\overline{d}_1 \geq \overline{d}_2 \geq \cdots \geq \overline{d}_n$. Then,*

$$\xi_1(G) \leq \max_{v_iv_j\in E(G)} \left\{\overline{d}_i + \overline{d}_j\right\}, \tag{2.6}$$

with equality if and only if G is a bipartite graph with partite sets V_1 and V_2 such that $\overline{d}_i = k_1,\ \forall\ v_i \in V_1$ and $\overline{d}_j = k_2,\ \forall\ v_j \in V_2$.

Proof. Let G be a connected graph of order $n \geq 3$ and let $\tilde{Q}(G)$ be the signless Laplacian ABC-matrix of G. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the principal eigenvector of $\tilde{Q}(G)$ corresponding to the eigenvalue $\lambda_1(\tilde{Q}(G))$ then by the Perron–Frobenius theorem $x_k > 0$, for all k. We assume that $x_i = \max_{v_k\in V(G)}\{x_k\}$ and $x_j = \max_{v_iv_k\in E(G)}\{x_k\}$. From the i-th equation of $\tilde{Q}(G)X = \lambda_1(\tilde{Q}(G))X$, we get

$$(\lambda_1(\tilde{Q}(G)) - \overline{d}_i)x_i = \sum_{v_iv_k\in E(G)} \sqrt{\frac{d_i + d_k - 2}{d_id_k}}x_k,$$

which implies that

$$(\lambda_1(\tilde{Q}(G)) - \overline{d}_i)x_i \leq \sum_{v_iv_k\in E(G)} \sqrt{\frac{d_i + d_k - 2}{d_id_k}}x_j = \overline{d}_ix_j. \tag{2.7}$$

Also, from the j-th equation of $\tilde{Q}(G)X = \lambda_1(\tilde{Q}(G))X$, we get

$$(\lambda_1(\tilde{Q}(G)) - \overline{d}_j)x_j = \sum_{v_jv_k\in E(G)} \sqrt{\frac{d_j + d_k - 2}{d_jd_k}}x_k,$$

which implies that

$$(\lambda_1(\tilde{Q}(G)) - \overline{d}_j)x_j \leq \sum_{v_jv_k \in E(G)} \sqrt{\frac{d_j + d_k - 2}{d_j d_k}} x_i = \overline{d}_j x_i. \tag{2.8}$$

Multiplying the corresponding sides of the inequalities (2.7) and (2.8) and using the fact that $x_k > 0$ for all k, we arrive at

$$(\lambda_1(\tilde{Q}(G)) - \overline{d}_i)(\lambda_1(\tilde{Q}(G)) - \overline{d}_j) \leq \overline{d}_i \overline{d}_j.$$

From this, it follows that $\lambda_1(\tilde{Q}(G) \leq \overline{d}_i + \overline{d}_j$. Since v_i and v_j are arbitrary adjacent vertices in G, we get

$$\lambda_1(\tilde{Q}(G) \leq \max_{v_iv_j \in E(G)} \{\overline{d}_i + \overline{d}_j\}.$$

Now, using Theorem 2.3, inequality (2.6) follows. Suppose that equality occurs in (2.6). Then, each of the inequalities (2.7) and (2.8) occurs as an equality. The equality in (2.7) gives that $x_k = x_j \ \forall\ k$ with $v_iv_k \in E(G)$, and the equality in (2.8) gives that $x_k = x_i \ \forall\ k$ with $v_jv_k \in E(G)$. Consider the sets $W_1 = \{v_k : x_k = x_i\}$ and $W_2 = \{v_k : x_k = x_j\}$. Clearly, $N(v_i) \subset W_2$ and $N(v_j) \subset W_1$, where $N(v)$ is the set of neighbors of the vertex v. We claim that $V(G) = W_1 \cup W_2$. Let $v_p \in N(v_i)$ and $v_r \in N(v_p)$, then $x_p = x_j$ and $x_r = x_i$. Further, if $s \in N(v_r)$ and $v_r \in N(v_s)$, then $x_s = x_j$ and $x_r = x_i$. Proceeding this way and using the fact that G is a connected graph, we conclude that $x_u = x_i$ or $x_u = x_j$, $\forall\ u \in V(G)$. This proves that $V(G) = W_1 \cup W_2$. If G is a non-bipartite graph, then G contains odd cycles, and so using the above procedure, we arrive at $x_i = x_j$. This gives that $X = (1, 1, \ldots, 1)^T$ is an eigenvector for $\lambda_1(\tilde{Q}(G))$, and so G has all row sums equal. That is, equality occurs in this case if G is ABC-regular. On the other hand, if G is a bipartite graph with partite sets V_1 and V_2, then we have either $x_i = x_j$ or $x_i \neq x_j$. If $x_i = x_j$, then as above, equality occurs if G is ABC-regular. Assume that $x_i \neq x_j$. Since for $v_k \in V_1$, we have $x_k = x_i$ and $x_p = x_j$, $\forall\ v_p \in N(v_k)$, so it follows that

$$(\lambda_1(\tilde{Q}(G)) - \overline{d}_k)x_i = \overline{d}_k x_j.$$

This gives that

$$\overline{d}_k = \frac{\lambda_1(\tilde{Q}(G)))x_i}{x_i + x_j}, \ \forall\ v_k \in V_1.$$

Similarly, for $v_k \in V_2$, we have $x_k = x_j$ and $x_z = x_i$, $\forall\ v_z \in N(v_k)$, so it follows that

$$\overline{d}_k = \frac{\lambda_1(\tilde{Q}(G)))x_j}{x_i + x_j}, \ \forall\ v_k \in V_2.$$

This shows that equality holds in this case if $\overline{d}_k = k_1$, $\forall\ v_k \in V_1$ and $\overline{d}_k = k_2$, $\forall\ v_k \in V_2$. That is, equality occurs in this case if $\overline{d}_i$ is the same for any two vertices belonging to the same partite set. Now, since equality occurs in Theorem 2.3 if and only if G is a bipartite graph, the result follows. □

Let $X = (x_1, x_2, \ldots, x_n)^T$ be the principal eigenvector of $\tilde{Q}(G)$ corresponding to the eigenvalue $\lambda_1(\tilde{Q}(G))$, then by the Perron–Frobenius theorem $x_k > 0$ for all k, we assume that $x_i = \min_{v_k \in V(G)} \{x_k\}$ and $x_j = \min_{v_iv_k \in E(G)} \{x_k\}$. Now, using Theorem 2.3 and proceeding similarly to Theorem 2.12, we obtain the following lower bound for the Laplacian ABC-spectral radius of a bipartite graph.

Theorem 2.13. *Let G be a connected bipartite graph of order $n \geq 3$ having ABC-degrees $\overline{d}_1 \geq \overline{d}_2 \geq \cdots \geq \overline{d}_n$. Then,*

$$\xi_1(G) \geq \min_{v_i v_j \in E(G)} \left\{ \overline{d}_i + \overline{d}_j \right\},$$

with equality if and only if G is a bipartite graph with partite sets V_1 and V_2 such that $\overline{d}_i = k_1,\ \forall\, v_i \in V_1$ and $\overline{d}_j = k_2,\ \forall\, v_j \in V_2$.

The next result gives an upper bound for the second smallest Laplacian ABC-eigenvalue $\xi_{n-1}(G)$ in terms of the minimum ABC-degree of G.

Theorem 2.14. *Let G be a connected graph of order $n \geq 3$ having the minimum ABC-degree $\overline{d}_{\min}$. Then*

$$\xi_{n-1}(G) \leq \frac{n\overline{d}_{\min}}{n-1}, \tag{2.9}$$

with equality holding if and only if $G \cong K_1 \vee H$, where H is a regular graph of order $n-1$ and the minimum ABC-degree of G corresponds to the vertex of K_1 in G.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. By the Rayleigh-Ritz theorem for any non-zero vector $X = (x_1, x_2, \ldots, x_n)^T$, which is perpendicular to $\mathbf{e}$, the all-one vector, we have

$$\xi_{n-1} \leq \frac{X^T \tilde{L}(G) X}{X^T X} = \frac{\sum\limits_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \left(x_u - x_v \right)^2}{\sum\limits_{v \in V(G)} x_v^2}. \tag{2.10}$$

Let v_1 be the vertex having the minimum ABC-degree. Taking $x_1 = -(n-1)$ and $x_i = 1$, for all $i \geq 2$, $X^T\mathbf{e} = -(n-1) + (n-1) = 0$ implies that $X \perp \mathbf{e}$. It follows from (2.10) that

$$\begin{aligned}
\xi_{n-1} &\leq \frac{\sum\limits_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \left(x_u - x_v \right)^2}{\sum\limits_{v \in V(G)} x_v^2} \\
&= \frac{1}{n(n-1)} \Big(\sum_{v_1 v \in E(G)} \sqrt{\frac{d_{v_1} + d_v - 2}{d_{v_1} d_v}} \left(x_{v_1} - x_v \right)^2 + \sum_{v_i v \in E(G), i \neq 1} \sqrt{\frac{d_{v_i} + d_v - 2}{d_{v_i} d_v}} \left(x_{v_i} - x_v \right)^2 \Big) \\
&= \frac{1}{n(n-1)} \left(n^2 \overline{d}_1 + 0 \right) = \frac{n\overline{d}_{\min}}{n-1}.
\end{aligned}$$

For the equality case, let $V(G) = \{v = v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of G. Suppose that equality occurs; then it is clear from Rayleigh-Ritz theorem that $X = (-(n-1), 1, 1, \ldots, 1)^T$ is an eigenvector for the matrix $\tilde{L}(G)$ corresponding to eigenvalue $\xi_{n-1}(G)$ and conversely. From the first eigenequation of $\tilde{L}(G)X = \xi_{n-1}(G)X$, we get

$$(\xi_{n-1} - \overline{d}_1)(n-1) = \sum_{v_1 v_j \in E(G)} \sqrt{\frac{d_{v_1} + d_{v_j} - 2}{d_{v_1} d_{v_j}}} = \overline{d}_1.$$

This gives that $\xi_{n-1} = \frac{n\overline{d}_1}{n-1}$. Again, from the ith eigenequation with $i \geq 2$ of $\tilde{L}(G)X = \xi_{n-1}(G)X$, we get

$$(\xi_{n-1} - \overline{d}_i) = -\sum_{v_i v_j \in E(G)} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_i} d_{v_j}}} x_j. \tag{2.11}$$

If there exists some $v_k, k \geq 2$ that is not adjacent to v_1, then for such a vertex v_k, it follows from Eq (2.11) that $\xi_{n-1} = 0$, giving that G is not connected, which is not possible as G is a connected graph. This shows that, under the given hypothesis, all the vertices v_i, $i \geq 2$, are adjacent to v_1. Now, v_1 is adjacent to v_i, for all $i \geq 2$, so Eq (2.11) gives that

$$\begin{aligned}
(\xi_{n-1} - \overline{d}_i) &= \sqrt{\frac{d_{v_1} + d_{v_i} - 2}{d_{v_1} d_{v_i}}}(n-1) - \sum_{v_i v_j \in E(G), j \neq 1} \sqrt{\frac{d_{v_i} + d_{v_j} - 2}{d_{v_1} d_{v_j}}} \\
&= (n-1)\sqrt{\frac{d_{v_1} + d_{v_i} - 2}{d_{v_1} d_{v_i}}} - \left(\overline{d}_i - \sqrt{\frac{d_{v_1} + d_{v_i} - 2}{d_{v_1} d_{v_i}}}\right) \\
&= n\sqrt{\frac{d_{v_1} + d_{v_i} - 2}{d_{v_1} d_{v_i}}} - \overline{d}_i.
\end{aligned}$$

This gives that

$$\xi_{n-1}(G) = n\sqrt{\frac{d_{v_1} + d_{v_i} - 2}{d_{v_1} d_{v_i}}} \text{ and so } \sqrt{\frac{d_{v_1} + d_{v_i} - 2}{d_{v_1} d_{v_i}}} = \frac{\xi_{n-1}(G)}{n}.$$

Since v_1 is adjacent to all the vertices $v_i, i \geq 2$ and G is connected, it follows that $d_1 = n - 1$. Using this in

$$\sqrt{\frac{d_{v_1} + d_{v_i} - 2}{d_{v_1} d_{v_i}}} = \frac{\xi_{n-1}(G)}{n}$$

gives that $d(v_i)$ is a constant for all $i \geq 2$, say, $d(v_i) = k$, for $i \geq 2$. Also, v_1 is adjacent to all v_i for $i \geq 2$. Now, using $d(v_i) = k$, for $i \geq 2$ and v_1 adjacent to all v_i, it follows that

$$\overline{d}_1 = (n-1)\sqrt{\frac{d_{v_1} + k - 2}{k d_{v_1}}},$$

implying that

$$\frac{\overline{d}_1}{n-1} = \sqrt{\frac{d_{v_1} + k - 2}{k d_{v_1}}}.$$

Thus, it follows that equality holds in (2.9) if and only if $G = K_1 \vee H$, where H is a k-regular graph of order $n-1$ and $\vee$ is the join operation. This completes the proof. □

If G is the complete graph $K_n = K_1 \vee K_{n-1}$ of order n, then for the graph K_n, we have

$$\overline{d}_i = (n-1)\frac{\sqrt{2n-4}}{n-1} = \sqrt{2n-4},$$

for all $i = 1, 2, \ldots, n$. Therefore,

$$\frac{n\overline{d}_{\min}}{n-1} = \frac{n\sqrt{2n-4}}{n-1} = \xi_{n-1}.$$

This shows that for the complete graph, the equality holds in (2.9).

The following upper bound for the second smallest Laplacian ABC-eigenvalue $\xi_{n-1}(G)$ was reported in [16]:

$$\xi_{n-1}(G) \leq \frac{n\sqrt{2n-4}}{n-1}, \tag{2.12}$$

with equality if and only if $G \cong K_n$.

Remark 2.15. *For a graph G of order $n \geq 3$, the upper bound given by Theorem 2.14 is always better than the upper bound given by* (2.12)*. We have*

$$\frac{n\overline{d}_{\min}}{n-1} \leq \frac{n\sqrt{2n-4}}{n-1} \quad \Longrightarrow \quad \overline{d}_{\min} \leq \sqrt{2n-4},$$

which is always true. As

$$\overline{d}_{\min} \leq \frac{2ABC(G)}{n} \leq \sqrt{2n-4},$$

by Lemma 2.4 of [16]. Note that we have used $2ABC(G) = \overline{d}_1 + \overline{d}_2 + \cdots + \overline{d}_n \geq n\overline{d}_{\min}$.

3. Laplacian ABC-eigenvalues of join operation

Let G and H be two disjoint graphs with vertex sets $V(G), V(H)$ and edge sets $E(G), E(H)$. The join of G and H is denoted by $G \vee H$ and is defined as the graph having a vertex set $V(G) \cup V(H)$ and an edge set $E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$. In the following result, we determine the Laplacian ABC-eigenvalues of the join of two regular graphs in terms of the Laplacian eigenvalues of the component graphs.

Theorem 3.1. *Let G_1 be an r_1-regular graph of order n_1 and G_2 be an r_2-regular graph of order n_2. Let $\mu_1(G_i) \geq \mu_2(G_i) \geq \cdots \geq \mu_{n-1}(G_i) \geq 0$ be the Laplacian eigenvalues of G_i, for $i = 1, 2$. The Laplacian ABC-eigenvalues of $G_1 \vee G_2$ consist of the eigenvalue $\alpha_1\mu_i(G_1) + n_2\beta$, for $i = 1, 2, \ldots, n_1 - 1$, the eigenvalue $\alpha_2\mu_i(G_2) + n_1\beta$, for $i = 1, 2, \ldots, n_2 - 1$, the eigenvalue $(n_1 + n_2)\beta$, and the eigenvalue* 0*, where*

$$\alpha_1 = \frac{\sqrt{2r_1 + 2n_2 - 2}}{r_1 + n_2}, \quad \alpha_2 = \frac{\sqrt{2r_2 + 2n_1 - 2}}{r_2 + n_1}, \quad \beta = \sqrt{\frac{r_1 + r_2 + n_1 + n_2 - 2}{(r_1 + n_2)(r_2 + n_1)}}.$$

Proof. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ be the vertex set of G_1 and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$ be the vertex set of G_2. Then, the vertex set of $G_1 \vee G_2$ is $V(G_1 \vee G_2) = \{v_1, v_2, \ldots, v_{n_1}, u_1, u_2, \ldots, u_{n_2}\}$. Let $d_{G_1 \vee G_2}(v)$ be the degree of the vertex $v \in V(G_1 \vee G_2)$. Then $d_{G_1 \vee G_2}(v_i) = r_1 + n_2$, for all $i = 1, 2, \ldots, n_1$ and $d_{G_1 \vee G_2}(u_i) = r_2 + n_1$, for all $i = 1, 2, \ldots, n_2$. Also,

$$\overline{d}_{G_1 \vee G_2}(v_i) = \sqrt{\frac{r_1 + n_2 + r_1 + n_2 - 2}{(r_1 + n_2)(r_1 + n_2)}}\, d_{G_1}(v_i) + n_2\sqrt{\frac{r_1 + n_2 + r_2 + n_1 - 2}{(r_1 + n_2)(r_2 + n_1)}} = \alpha_1 d_{G_1}(v_i) + n_2\beta,$$

for all $i = 1, 2, \ldots, n_1$ and

$$\overline{d}_{G_1 \vee G_2}(u_i) = \sqrt{\frac{r_2 + n_1 + r_2 + n_1 - 2}{(r_2 + n_1)(r_2 + n_1)}} d_{G_2}(u_i) + n_1 \sqrt{\frac{r_1 + n_2 + r_2 + n_1 - 2}{(r_1 + n_2)(r_2 + n_1)}} = \alpha_2 d_{G_2}(u_i) + n_1 \beta,$$

for all $i = 1, 2, \ldots, n_2$. Therefore, using the definition of the Laplacian ABC-matrix, the Laplacian ABC-matrix of $G_1 \vee G_2$ can be put in the form

$$\tilde{L}(G_1 \vee G_2) = \begin{pmatrix} \alpha_1 L(G_1) + n_2 \beta I_{n_1} & -\beta J_{n_1 \times n_2} \\ -\beta J_{n_2 \times n_1} & \alpha_2 L(G_2) + n_1 \beta I_{n_2} \end{pmatrix},$$

where $L(G_i)$ is the Laplacian matrix of G_i, I_{n_i} is the identity matrix of order n_i, and $J_{n_1 \times n_2}$ is the all-one matrix of order $n_1 \times n_2$. Let $\mu_i(G_1)$ be a Laplacian eigenvalue of G_1, for $i = 1, 2, \ldots, n_1 - 1$ and let X be the corresponding eigenvector. Then, $X^T \mathbf{e_{n_1}} = 0$, where $\mathbf{e_{n_1}}$ is the all-one vector of order n_1 and $L(G_1)X = \mu_i(G_1)X$. Consider the vector $Y = \begin{pmatrix} X \\ 0_{n_2} \end{pmatrix}$, where 0_{n_2} is the zero column vector of order n_2. We have

$$\begin{aligned} \tilde{L}(G_1 \vee G_2)Y &= \begin{pmatrix} \alpha_1 L(G_1) + n_2 \beta I_{n_1} & -\beta J_{n_1 \times n_2} \\ -\beta J_{n_2 \times n_1} & \alpha_2 L(G_2) + n_1 \beta I_{n_2} \end{pmatrix} \begin{pmatrix} X \\ 0_{n_2} \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 L(G_1)X + n_2 \beta I_{n_1} X \\ 0_{n_2} \end{pmatrix} = \begin{pmatrix} \alpha_1 \mu_i(G_1)X + n_2 \beta X \\ 0_{n_2} \end{pmatrix} \\ &= (\alpha_1 \mu_i(G_1) + n_2 \beta) \begin{pmatrix} X \\ 0_{n_2} \end{pmatrix} = (\alpha_1 \mu_i(G_1) + n_2 \beta)Y. \end{aligned}$$

This shows that $\alpha_1 \mu_i(G_1) + n_2 \beta$ is an eigenvalue of $\tilde{L}(G_1 \vee G_2)$, for all $i = 1, 2, \ldots, n_1 - 1$. Similarly, if $\mu_i(G_2)$ is a Laplacian eigenvalue of G_2, for $i = 1, 2, \ldots, n_2 - 1$, then we can show that $\alpha_2 \mu_i(G_2) + n_1 \beta$ is an eigenvalue of $\tilde{L}(G_1 \vee G_2)$, for all $i = 1, 2, \ldots, n_2 - 1$. The equitable quotient matrix of $\tilde{L}(G_1 \vee G_2)$ is given by $\begin{pmatrix} n_2 \beta & -n_2 \beta \\ -n_1 \beta & n_1 \beta \end{pmatrix}$. It is easy to check that the remaining two Laplacian ABC-eigenvalues of $G_1 \vee G_2$ are $(n_1 + n_2)\beta$ and 0. This completes the proof. □

Example 3.2. *Consider the graph $G = K_3 \vee P_2$, where K_3 is a complete graph on $n_1 = 3$ vertices and P_2 is a path on $n_2 = 2$ vertices. Since the graph K_3 is 2-regular and the graph P_2 is 1-regular, it follows that $r_1 = 2$ and $r_2 = 1$. Also,*

$$\alpha_1 = \frac{\sqrt{2(2) + 2(2) - 2}}{2 + 2} = \frac{\sqrt{6}}{4}, \ \ \alpha_2 = \frac{\sqrt{2(1) + 2(3) - 2}}{1 + 3} = \frac{\sqrt{6}}{4}, \ \ \beta = \sqrt{\frac{2 + 1 + 5 - 2}{(2 + 2)(1 + 3)}} = \frac{\sqrt{6}}{4},$$

and $\mu_1(K_3) = \mu_2(K_3) = 3, \mu_3(K_3) = 0$; $\mu_1(P_2) = 2, \mu_2(P_2) = 0$. Therefore, by Theorem 3.1, it follows that the Laplacian ABC-eigenvalues of graph $G = K_3 \vee K_2$ consist of the eigenvalue $\frac{5\sqrt{6}}{4}$ repeated four times and the eigenvalue 0 repeated once. On the other hand, the Laplacian ABC-matrix $\tilde{L}(G)$ of G is given by

$$\tilde{L}(G) = \begin{pmatrix} \frac{4\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} \\ -\frac{\sqrt{6}}{4} & \frac{4\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} \\ -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & \frac{4\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} \\ -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & \frac{4\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} \\ -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & \frac{4\sqrt{6}}{4} \end{pmatrix}.$$

Using direct calculations, we see that the eigenvalues of this matrix are $\frac{5\sqrt{6}}{4}$ repeated four times and the eigenvalue 0 repeated once. It is now clear that the Laplacian ABC-eigenvalues provided by Theorem 3.1 for the graph $G = K_3 \vee P_2$ are in agreement with the Laplacian ABC-eigenvalues calculated by direct computation.

The wheel graph of order n is denoted by W_n and is the graph obtained by joining a vertex to each of the vertices of the cycle C_{n-1} of order $n-1$. It is clear that $W_n = K_1 \vee C_{n-1}$. The friendship graph of order N is denoted by F_N and is the graph obtained by joining a vertex to each of the ends of the n copies of the graph K_2. It is clear that $F_N = K_1 \vee nK_2$, where $N = 2n+1$. Let e be an edge in K_n; then the graph obtained by deleting the edge e from K_n is denoted by $K_n - e$.

The following result follows from Theorem 3.1.

Corollary 3.3. *(1) The Laplacian ABC-eigenvalues of $G = K_1 \vee H$, where H is a r-regular graph of order $n-1$, consist of the eigenvalue $\frac{\sqrt{2r}}{r+1}\mu_i(H) + \sqrt{\frac{n+r-2}{(r+1)(n-1)}}$, for $i = 1, 2, \ldots, n-2$, the eigenvalue $n\sqrt{\frac{n+r-2}{(r+1)(n-1)}}$, and the eigenvalue 0, where $\mu_1(H) \geq \mu_2(H) \geq \cdots \geq \mu_{n-2} \geq 0$ are the Laplacian eigenvalues of H.*

(2) The Laplacian ABC-eigenvalues of $W_n = K_1 \vee C_{n-1}$, consist of the eigenvalue $\frac{8}{3}\sin^2\left(\frac{i\pi}{n-1}\right) + \sqrt{\frac{n}{3(n-1)}}$, for $i = 1, 2, \ldots, n-2$, the eigenvalue $n\sqrt{\frac{n}{3(n-1)}}$, and the eigenvalue 0.

(3) The Laplacian ABC-eigenvalues of $F_N = K_1 \vee nK_2$, consist of the eigenvalue $\sqrt{2} + \frac{1}{\sqrt{2}}$ repeated n times, the eigenvalue $\frac{1}{\sqrt{2}}$ repeated $n-1$ times, the eigenvalue $\frac{2n+1}{\sqrt{2}}$, and the eigenvalue 0.

(4) The Laplacian ABC-eigenvalues of $K_n - e = 2K_1 \vee K_{n-2}$, consist of the eigenvalue $(n-2)\sqrt{\frac{2n-5}{(n-1)(n-2)}}$, the eigenvalue $(n-2)\frac{\sqrt{2n-4}}{n-1} + 2\sqrt{\frac{2n-5}{(n-1)(n-2)}}$ repeated $n-3$ times, the eigenvalue $n\sqrt{\frac{2n-5}{(n-1)(n-2)}}$, and the eigenvalue 0.

(5) The Laplacian ABC-eigenvalues of the generalized wheel graph $C_t \vee (n-t)K_1$, consist of the eigenvalue $\frac{4\sqrt{2n-2t+2}}{(n-t+2)}\sin^2\left(\frac{i\pi}{t}\right) + (n-t)\sqrt{\frac{n}{t(n-t+2)}}$, for $i = 1, 2, \ldots, t-1$, the eigenvalue $t\sqrt{\frac{n}{t(n-t+2)}}$ repeated $n-t-1$ times, the eigenvalue $n\sqrt{\frac{n}{t(n-t+2)}}$, and the eigenvalue 0.

(6) The Laplacian ABC-eigenvalues of the graph $C_t \vee K_{n-t}$, consist of the eigenvalue $\frac{4\sqrt{2n-2t+2}}{(n-t+2)}\sin^2\left(\frac{i\pi}{t}\right) + (n-t)\sqrt{\frac{2n-t-1}{(n-1)(n-t+2)}}$, for $i = 1, 2, \ldots, t-1$, the eigenvalue $(n-2)\frac{\sqrt{2n-4}}{n-1} + t\sqrt{\frac{2n-t-1}{(n-1)(n-t+2)}}$ repeated $n-t-1$ times, the eigenvalue $n\sqrt{\frac{2n-t-1}{(n-1)(n-t+2)}}$, and the eigenvalue 0.

Proof. (1) Taking $G_1 = K_1, G_2 = H, n_1 = 1, n_2 = n-1, r_1 = 0$, and $r_2 = r$ in Theorem 3.1, we get

$$\alpha_1 = \frac{\sqrt{2n-4}}{n-1}, \ \ \alpha_2 = \frac{\sqrt{2r}}{r+1}, \ \ \beta = \sqrt{\frac{n+r-2}{(r+1)(n-1)}}.$$

The result now follows from Theorem 3.1.

(2) Taking $H = C_{n-1}$, a cycle of order $n-1$, and using $r = 2$ and $\mu_i(C_{n-1}) = 2 - 2\cos\left(\frac{2k\pi}{n-1}\right) = 4\sin^2\left(\frac{k\pi}{n-1}\right)$ for $i = 0, 1, \ldots, n-2$ in part (1), the result follows.

(3) Taking $H = nK_2$, a union of n copies of K_2, and using $r = 1$ and $\mu_i(nK_2) = 2$ for $i = 1, 2, \ldots, n$ and

$\mu_i(nK_2) = 0$, for $i = n+1, \ldots, 2n$, in part (1), the result follows.
(4) Taking $G_1 = 2K_1, G_2 = K_{n-2}, n_1 = 2, n_2 = n-2, r_1 = 0$, and $r_2 = n-3$ in Theorem 3.1, we get

$$\alpha_1 = \frac{\sqrt{2n-6}}{n-2}, \ \ \alpha_2 = \frac{\sqrt{2n-4}}{n-1}, \ \ \beta = \sqrt{\frac{2n-5}{(n-2)(n-1)}}.$$

The result now follows from Theorem 3.1 by using the fact that $\mu_i(K_{n-2}) = n-2$ for $i = 1, 2, \ldots, n-3$, and $\mu_{n-2}(K_{n-2}) = 0$.
(5) Taking $G_1 = C_t, G_2 = (n-t)K_1, n_1 = t, n_2 = n-t, r_1 = 2$, and $r_2 = 0$ in Theorem 3.1, we get

$$\alpha_1 = \frac{\sqrt{2n-2t+2}}{n-t+2}, \ \ \alpha_2 = \frac{\sqrt{2t-2}}{t}, \ \ \beta = \sqrt{\frac{n}{t(n-t+2)}}.$$

The result now follows from Theorem 3.1 by using the fact the Laplacian eigenvalues of C_t are $4\sin^2(\frac{k\pi}{t})$ for $k = 0, 1, \ldots, t-1$.
(6) Taking $G_1 = C_t, G_2 = K_{n-t}, n_1 = t, n_2 = n-t, r_1 = 2$, and $r_2 = n-t-1$ in Theorem 3.1, we get

$$\alpha_1 = \frac{\sqrt{2n-2t+2}}{n-t+2}, \ \ \alpha_2 = \frac{\sqrt{2n-4}}{n-1}, \ \ \beta = \sqrt{\frac{2n-t-1}{(n-1)(n-t+2)}}.$$

The result now follows from Theorem 3.1 by using the fact the Laplacian eigenvalues of C_t are $4\sin^2(\frac{k\pi}{t})$ for $k = 0, 1, \ldots, t-1$, and the Laplacian eigenvalues of K_{n-t} are $n-t$ repeated $n-t-1$ times 0 repeated once. This completes the proof. □

The following result presents the Laplacian ABC-eigenvalues of the join of a regular graph with the union of two regular graphs, in terms of the Laplacian eigenvalues of the component graphs.

Theorem 3.4. *Let G_1 be an r_1-regular graph of order n_1, G_2 be an r_2-regular graph of order n_2, and G_3 be an r_3-regular graph of order n_3. Let $\mu_1(G_i) \geq \mu_2(G_i) \geq \cdots \geq \mu_{n-1}(G_i) \geq 0$ be the Laplacian eigenvalues of G_i for $i = 1, 2, 3$. The Laplacian ABC-eigenvalues of $G_1 \vee (G_2 \cup G_3)$ consist of the eigenvalue $\alpha_1\mu_i(G_1) + n_2\beta_1 + n_3\beta_2$ for $i = 1, 2, \ldots, n_1 - 1$, the eigenvalue $\alpha_2\mu_i(G_2) + n_1\beta_1$ for $i = 1, 2, \ldots, n_2 - 1$, the eigenvalue $\alpha_3\mu_i(G_3) + n_1\beta_2$ for $i = 1, 2, \ldots, n_3 - 1$, the eigenvalue 0, and the two zeros of the polynomial $p(x) = x^2 - ((n_1+n_2)\beta_1 + (n_1+n_3)\beta_2)x + \beta_1\beta_2(n_1n_2 + n_1n_3 + n_1^2)$, where*

$$\alpha_1 = \frac{\sqrt{2r_1 + 2n_2 + 2n_3 - 2}}{r_1 + n_2 + n_3}, \ \ \alpha_2 = \frac{\sqrt{2r_2 + 2n_1 - 2}}{r_2 + n_1}, \ \ \alpha_3 = \frac{\sqrt{2r_3 + 2n_1 - 2}}{r_3 + n_1},$$

$$\beta_1 = \sqrt{\frac{r_1 + r_2 + n_1 + n_2 + n_3 - 2}{(r_1 + n_2 + n_3)(r_2 + n_1)}}, \ \ \beta_2 = \sqrt{\frac{r_1 + r_3 + n_1 + n_2 + n_3 - 2}{(r_1 + n_2 + n_3)(r_3 + n_1)}}.$$

Proof. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$ be the vertex set of G_1, $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$ be the vertex set of G_2, and $V(G_3) = \{w_1, w_2, \ldots, w_{n_3}\}$ be the vertex set of G_3. Then, the vertex set of $G_1 \vee (G_2 \cup G_3)$ is $V(G_1 \vee (G_2 \cup G_3))G_2) = \{v_1, v_2, \ldots, v_{n_1}, u_1, u_2, \ldots, u_{n_2}, w_1, w_2, \ldots, w_{n_3}\}$. Let $d_{G_1\vee(G_2\cup G_3)}(v)$ be the degree of the vertex $v \in V(G_1 \vee (G_2 \cup G_2))$. Then, $d_{G_1\vee(G_2\cup G_3)}(v_i) = r_1 + n_2 + n_3$, for all $i = 1, 2, \ldots, n_1, d_{G_1\vee(G_2\cup G_3)}(u_i) = r_2 + n_1$, for all $i = 1, 2, \ldots, n_2$, and $d_{G_1\vee(G_2\cup G_3)}(w_i) = r_3 + n_1$, for all $i = 1, 2, \ldots, n_3$. Also,

$$
\begin{aligned}
\overline{d}_{G_1\vee(G_2\cup G_3)}(v_i) = & \sqrt{\frac{2r_1+2n_2+2n_3-2}{(r_1+n_2+n_3)^2}}d_{G_1}(v_i)\\
& + n_2\sqrt{\frac{r_1+r_2+n_1+n_2+n_3-2}{(r_1+n_2+n_3)(r_2+n_1)}}\\
& + n_3\sqrt{\frac{r_1+r_3+n_1+n_2+n_3-2}{(r_1+n_2+n_3)(r_3+n_1)}}\\
& = \alpha_1 d_{G_1}(v_i) + n_2\beta_1 + n_3\beta_2,
\end{aligned}
$$

for all $i = 1, 2, \ldots, n_1$,

$$
\overline{d}_{G_1\vee(G_2\cup G_3)}(u_i) = \sqrt{\frac{2r_2+2n_1-2}{(r_2+n_1)^2}}d_{G_2}(u_i) + n_1\sqrt{\frac{r_1+r_2+n_1+n_2+n_3-2}{(r_1+n_2+n_3)(r_2+n_1)}} = \alpha_2 d_{G_2}(u_i) + n_1\beta_1,
$$

for all $i = 1, 2, \ldots, n_2$, and

$$
\overline{d}_{G_1\vee(G_2\cup G_3)}(w_i) = \sqrt{\frac{2r_3+2n_1-2}{(r_3+n_1)^2}}d_{G_3}(w_i) + n_1\sqrt{\frac{r_1+r_3+n_1+n_2+n_3-2}{(r_1+n_2+n_3)(r_3+n_1)}} = \alpha_3 d_{G_3}(w_i) + n_1\beta_2,
$$

for all $i = 1, 2, \ldots, n_3$. Therefore, using the definition of the Laplacian ABC-matrix, we arrive at the Laplacian ABC-matrix of $G_1 \vee (G_2 \cup G_3)$, given by

$$
\tilde{L}(G_1 \vee (G_2 \cup G_3)) = \begin{pmatrix} \alpha_1 L(G_1) + (n_2\beta_1 + n_3\beta_2)I_{n_1} & -\beta_1 J_{n_1\times n_2} & -\beta_2 J_{n_1\times n_3} \\ -\beta_1 J_{n_2\times n_1} & \alpha_2 L(G_2) + n_1\beta_1 I_{n_2} & 0 \\ -\beta_2 J_{n_3\times n_1} & 0 & \alpha_3 L(G_3) + n_1\beta_2 I_{n_3} \end{pmatrix},
$$

where $L(G_i)$ is the Laplacian matrix of G_i, I_{n_i} is the identity matrix of order n_i, and $J_{n_i\times n_j}$ is the all-one matrix of order $n_i \times n_j$. For $i = 1, 2, \ldots, n_1 - 1$, let $\mu_i(G_1)$ be a Laplacian eigenvalue of G_1 with the corresponding eigenvector X. Then $X^T\mathbf{e_{n_1}} = 0$ and $L(G_1)X = \mu_i(G_1)X$. Consider the vector $Y = \begin{pmatrix} X \\ 0_{n_2} \\ 0_{n_3} \end{pmatrix}$, where 0_{n_i} is the zero column vector of order n_i. We have

$$
\begin{aligned}
\tilde{L}(G_1 \vee (G_2 \cup G_3))Y &= \begin{pmatrix} \alpha_1 L(G_1) + (n_2\beta_1 + n_3\beta_2)I_{n_1} & -\beta_1 J_{n_1\times n_2} & -\beta_2 J_{n_1\times n_3} \\ -\beta_1 J_{n_2\times n_1} & \alpha_2 L(G_2) + n_1\beta_1 I_{n_2} & 0 \\ -\beta_2 J_{n_3\times n_1} & 0 & \alpha_3 L(G_3) + n_1\beta_2 I_{n_3} \end{pmatrix}\begin{pmatrix} X \\ 0_{n_2} \\ 0_{n_3} \end{pmatrix}\\
&= \begin{pmatrix} \alpha_1 L(G_1)X + (n_2\beta_1 + n_3\beta_2)X \\ 0_{n_2} \\ 0_{n_3} \end{pmatrix} = \begin{pmatrix} \alpha_1 \mu_i(G_1)X + (n_2\beta_1 + n_3\beta_2)X \\ 0_{n_2} \\ 0_{n_3} \end{pmatrix}\\
&= (\alpha_1\mu_i(G_1) + n_2\beta_1 + n_3\beta_2)\begin{pmatrix} X \\ 0_{n_2} \\ 0_{n_3} \end{pmatrix} = (\alpha_1\mu_i(G_1) + n_2\beta_1 + n_3\beta_2)Y.
\end{aligned}
$$

This shows that $\alpha_1\mu_i(G_1)+n_2\beta_1+n_3\beta_2$ is an eigenvalue of $\tilde{L}(G_1\vee(G_2\cup G_3))$, for all $i=1,2,\ldots,n_1-1$. For $i=1,2,\ldots,n_2-1$, let $\mu_i(G_2)$ be a Laplacian eigenvalue of G_2 with the corresponding eigenvector Z. Then $Z^T\mathbf{e_{n_2}}=0$ and $L(G_2)Z=\mu_i(G_2)Z$. Consider the vector $Y'=\begin{pmatrix}0_{n_1}\\ Z\\ 0_{n_3}\end{pmatrix}$, where 0_{n_i} is the zero column vector of order n_i. Proceeding as above, it can be seen that

$$\tilde{L}(G_1\vee(G_2\cup G_3))Y'=(\alpha_2\mu_i(G_2)+n_1\beta_1)Y'.$$

This gives that $\alpha_2\mu_i(G_2)+n_1\beta_1$ is an eigenvalue of $\tilde{L}(G_1\vee(G_2\cup G_3))$, for all $i=1,2,\ldots,n_2-1$. Similarly, if $\mu_i(G_3)$ is a Laplacian eigenvalue of G_3, for $i=1,2,\ldots,n_3-1$, then we can show that $\alpha_3\mu_i(G_3)+n_1\beta_2$ is an eigenvalue of $\tilde{L}(G_1\vee(G_2\cup G_3))$, for all $i=1,2,\ldots,n_3-1$. The equitable quotient matrix of $\tilde{L}(G_1\vee G_2\cup G_3))$ is given by $\begin{pmatrix}n_2\beta_1+n_3\beta_2 & -n_2\beta_1 & -n_3\beta_2\\ -n_1\beta_1 & n_1\beta_1 & 0\\ -n_1\beta_2 & 0 & n_1\beta_2\end{pmatrix}$. It is easy to check that the remaining three Laplacian ABC-eigenvalues of $G_1\vee(G_2\cup G_3)$ consist of the eigenvalue 0 and the two zeros of the polynomial $x^2-((n_1+n_2)\beta_1+(n_1+n_3)\beta_2)x+\beta_1\beta_2(n_1n_2+n_1n_3+n_1^2)$. This completes the proof. □

Example 3.5. *Consider the graph $G=K_3\vee(P_2\cup P_2)$, where K_3 is a complete graph on $n_1=3$ vertices and P_2 is a path on $n_2=n_3=2$ vertices. Since the graph K_3 is 2-regular and the graph P_2 is 1-regular, it follows that $r_1=2$ and $r_2=r_3=1$. Also,*

$$\alpha_1=\frac{\sqrt{2(2)+2(2)+2(2)-2}}{2+2+2}=\frac{\sqrt{10}}{6},\quad \alpha_2=\frac{\sqrt{2(1)+2(3)-2}}{1+3}=\frac{\sqrt{6}}{4},$$

$$\alpha_3=\frac{\sqrt{2(1)+2(3)-2}}{1+3}=\frac{\sqrt{6}}{4},\quad \beta_1=\beta_2=\sqrt{\frac{3+7-2}{(2+2+2)(1+3)}}=\sqrt{\frac{8}{24}}=\frac{1}{\sqrt{3}},$$

and $\mu_1(K_3)=\mu_2(K_3)=3,\mu_3(K_3)=0$; $\mu_1(P_2)=2,\mu_2(P_2)=0$. Therefore, by Theorem 3.4, it follows that the Laplacian ABC-eigenvalues of the graph $G=K_3\vee(P_2\cup P_2)$ consist of the eigenvalue $\frac{\sqrt{10}}{2}+\frac{4}{\sqrt{3}}$ repeated two times, the eigenvalue $\frac{\sqrt{6}}{2}+\sqrt{3}$ repeated two times, and the eigenvalue 0 repeated once. The remaining two Laplacian ABC-eigenvalues are the zeros of the polynomial $x^2-\frac{10}{\sqrt{3}}x+7$. By quadratic formula, we see that the zeros of $x^2-\frac{10}{\sqrt{3}}x+7$ are $\sqrt{3}$ and $\frac{7}{\sqrt{3}}$. Thus, by Theorem 3.4, the Laplacian ABC-eigenvalues of the graph G are $\frac{\sqrt{10}}{2}+\frac{4}{\sqrt{3}}$ repeated two times, $\frac{\sqrt{6}}{2}+\sqrt{3}$ repeated two times, $\sqrt{3}$ repeated once, $\frac{7}{\sqrt{3}}$ repeated once, and the eigenvalue 0 repeated once. On the other hand, the Laplacian ABC-matrix $\tilde{L}(G)$ of G is given by

$$\tilde{L}(G)=\begin{pmatrix}
t_1 & -\frac{\sqrt{10}}{6} & -\frac{\sqrt{10}}{6} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}}\\
-\frac{\sqrt{10}}{6} & t_1 & -\frac{\sqrt{10}}{6} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}}\\
-\frac{\sqrt{10}}{6} & -\frac{\sqrt{10}}{6} & t_1 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}}\\
-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & t_2 & -\frac{\sqrt{6}}{4} & 0 & 0\\
-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{\sqrt{6}}{4} & t_2 & 0 & 0\\
-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & t_2 & -\frac{\sqrt{6}}{4}\\
-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 & -\frac{\sqrt{6}}{4} & t_2
\end{pmatrix},$$

where $t_1 = \frac{\sqrt{10}}{3} + \frac{4}{\sqrt{3}}$ *and* $t_2 = \sqrt{3} + \frac{\sqrt{6}}{4}$*. By direct calculations, we see that the eigenvalues of this matrix are* $3.890539 \approx \frac{\sqrt{10}}{2} + \frac{4}{\sqrt{3}}$ *repeated two times,* $2.956795 \approx \frac{\sqrt{6}}{2} + \sqrt{3}$ *repeated two times,* $1.732050 \approx \sqrt{3}$ *repeated once,* $4.041451 \approx \frac{7}{\sqrt{3}}$ *repeated once, and the eigenvalue* 0 *repeated once. It is now clear that the Laplacian ABC-eigenvalues provided by Theorem 3.4 for the graph* $G = K_3 \vee (P_2 \cup P_2)$ *are in agreement with the Laplacian ABC-eigenvalues calculated by direct computation.*

The following result is immediate from Theorem 3.4 and provides the Laplacian ABC-eigenvalues of the graph $K_{n_1} \vee (K_{n_2} \cup K_{n_3})$.

Corollary 3.6. *The Laplacian ABC-eigenvalues of* $K_{n_1} \vee (K_{n_2} \cup K_{n_3})$*, where* $n_1 + n_2 + n_3 = n$*, consist of the eigenvalue* $n_1\frac{\sqrt{2n-4}}{n-1} + n_2\sqrt{\frac{2n-n_3-4}{(n-1)(n-n_3-1)}} + n_3\sqrt{\frac{2n-n_2-4}{(n-1)(n-n_2-1)}}$ *repeated* $n_1 - 1$ *times, the eigenvalue* $n_2\frac{\sqrt{2n-2n_3-4}}{n-n_3-1} + n_1\sqrt{\frac{2n-n_3-4}{(n-1)(n-n_3-1)}}$ *repeated* $n_2 - 1$ *times, the eigenvalue* $n_3\frac{\sqrt{2n-2n_2-4}}{n-n_2-1} + n_1\sqrt{\frac{2n-n_2-4}{(n-1)(n-n_2-1)}}$ *repeated* $n_3 - 1$ *times, the eigenvalue* 0*, and the two zeros of the polynomial* $x^2 - ((n-n_3)\sqrt{\frac{2n-n_3-4}{(n-1)(n-n_3-1)}} + (n-n_2)\sqrt{\frac{2n-n_2-4}{(n-1)(n-n_2-1)}})x + \sqrt{\frac{(2n-n_3-4)(2n-n_2-4)}{(n-1)^2(n-n_3-1)(n-n_2-1)}}nn_1$.

Proof. Taking $G_1 = K_{n_1}, G_2 = K_{n_2}, G_3 = K_{n_3}, r_1 = n_1 - 1, r_2 = n_2 - 1$, and $r_3 = n_3 - 1$ in Theorem 3.4, we get

$$\begin{aligned}
\alpha_1 &= \frac{\sqrt{2(n_1-1) + 2n_2 + 2n_3 - 2}}{n_1 - 1 + n_2 + n_3} = \frac{\sqrt{2n-4}}{n-1},\\
\alpha_2 &= \frac{\sqrt{2(n_2-1) + 2n_1 - 2}}{n_2 - 1 + n_1} = \frac{\sqrt{2n-2n_3-4}}{n-n_3-1},\\
\alpha_3 &= \frac{\sqrt{2(n_3-1) + 2n_1 - 2}}{n_3 - 1 + n_1} = \frac{\sqrt{2n-2n_2-4}}{n-n_2-1},\\
\beta_1 &= \sqrt{\frac{(n_1-1) + (n_2-1) + n_1 + n_2 + n_3 - 2}{(n_1 - 1 + n_2 + n_3)(n_2 - 1 + n_1)}} = \sqrt{\frac{2n-n_3-4}{(n-1)(n-n_3-1)}},\\
\beta_2 &= \sqrt{\frac{(n_1-1) + (n_3-1) + n_1 + n_2 + n_3 - 2}{(n_1 - 1 + n_2 + n_3)(n_3 - 1 + n_1)}} = \sqrt{\frac{2n-n_2-4}{(n-1)(n-n_2-1)}}.
\end{aligned}$$

The Laplacian eigenvalues of K_{n_i} are n_i repeated $n_i - 1$ times and 0 repeated once, for all $i = 1, 2, 3$. Therefore, from Theorem 3.4, it follows that

$$\begin{aligned}
&\alpha_1\mu_i(K_{n_1}) + n_2\beta_1 + n_3\beta_2\\
&\qquad = n_1\frac{\sqrt{2n-4}}{n-1} + n_2\sqrt{\frac{2n-n_3-4}{(n-1)(n-n_3-1)}} + n_3\sqrt{\frac{2n-n_2-4}{(n-1)(n-n_2-1)}}
\end{aligned}$$

is a Laplacian ABC-eigenvalue of $K_{n_1} \vee (K_{n_2} \cup K_{n_3})$ repeated $n_1 - 1$ times. Similarly, by Theorem 3.4,

$$\alpha_2\mu_i(K_{n_2}) + n_1\beta_1 = n_2\frac{\sqrt{2n-2n_3-4}}{n-n_3-1} + n_1\sqrt{\frac{2n-n_3-4}{(n-1)(n-n_3-1)}}$$

is a Laplacian ABC-eigenvalue of $K_{n_1} \vee (K_{n_2} \cup K_{n_3})$ repeated $n_2 - 1$ times, and

$$\alpha_3\mu_i(K_{n_3}) + n_1\beta_2 = n_3\frac{\sqrt{2n-2n_2-4}}{n-n_2-1} + n_1\sqrt{\frac{2n-n_2-4}{(n-1)(n-n_2-1)}}$$

is a Laplacian ABC-eigenvalue of $K_{n_1} \vee (K_{n_2} \cup K_{n_3})$ repeated n_3-1 times. The remaining three Laplacian ABC-eigenvalues of $K_{n_1} \vee (K_{n_2} \cup K_{n_3})$ consist of the eigenvalue 0 and the two zeros of the polynomial $x^2 - ((n-n_3)\sqrt{\frac{2n-n_3-4}{(n-1)(n-n_3-1)}} + (n-n_2)\sqrt{\frac{2n-n_2-4}{(n-1)(n-n_2-1)}})x + \sqrt{\frac{(2n-n_3-4)(2n-n_2-4)}{(n-1)^2(n-n_3-1)(n-n_2-1)}}nn_1$. □

In particular, if $n_1 = n_2 = n_3$, then the Laplacian ABC-eigenvalues of $K_{n_1} \vee (K_{n_1} \cup K_{n_1})$ consist of the eigenvalue $n_1\frac{\sqrt{2n-4}}{n-1} + 2n_1\sqrt{\frac{2n-n_1-4}{(n-1)(n-n_1-1)}}$ repeated $n_1 - 1$ times, the eigenvalue $n_1\frac{\sqrt{2n-2n_1-4}}{n-n_1-1} + n_1\sqrt{\frac{2n-n_1-4}{(n-1)(n-n_1-1)}}$ repeated $2n_1 - 2$ times, the eigenvalue 0, and the two zeros of the polynomial $x^2 - 2(n-n_1)\sqrt{\frac{2n-n_1-4}{(n-1)(n-n_1-1)}}x + \frac{2n-n_1-4}{(n-1)(n-n_1-1)}nn_1$. This shows that the graph $K_{n_1} \vee (K_{n_1} \cup K_{n_1})$ has five distinct Laplacian ABC-eigenvalues.

The next result provides the Laplacian ABC-eigenvalues of the graph $\overline{K}_{n_1} \vee (\overline{K}_{n_2} \cup \overline{K}_{n_3})$.

Corollary 3.7. *The Laplacian ABC-eigenvalues of $\overline{K}_{n_1} \vee (\overline{K}_{n_2} \cup \overline{K}_{n_3})$, where $n_1 + n_2 + n_3 = n$, consist of the eigenvalue $(n-n_1)\sqrt{\frac{n-2}{n_1(n-n_1)}}$ repeated $n_1 - 1$ times, the eigenvalue $n_1\sqrt{\frac{n-2}{n_1(n-n_1)}}$ repeated $n - n_1 - 2$ times, the eigenvalue 0, and the two zeros of the polynomial $x^2 - (n+n_1)\sqrt{\frac{n-2}{n_1(n-n_1)}}x + \frac{n(n-2)}{n-n_1}$.*

Proof. Taking $G_1 = \overline{K}_{n_1}, G_2 = \overline{K}_{n_2}, G_3 = \overline{K}_{n_3}, r_1 = 0, r_2 = 0$, and $r_3 = 0$ in Theorem 3.4, we get

$$\begin{aligned}
\alpha_1 &= \frac{\sqrt{2(0) + 2n_2 + 2n_3 - 2}}{0 + n_2 + n_3} = \frac{\sqrt{2n-2n_1-2}}{n-n_1},\\
\alpha_2 &= \frac{\sqrt{2(0) + 2n_1 - 2}}{0 + n_1} = \frac{\sqrt{2n_1-2}}{n_1},\\
\alpha_3 &= \frac{\sqrt{2(0) + 2n_1 - 2}}{0 + n_1} = \frac{\sqrt{2n_1-2}}{n_1},\\
\beta_1 &= \sqrt{\frac{0+0+n_1+n_2+n_3-2}{(0+n_2+n_3)(0+n_1)}} = \sqrt{\frac{n-2}{(n-n_1)n_1}},\\
\beta_2 &= \sqrt{\frac{0+0+n_1+n_2+n_3-2}{(0+n_2+n_3)(0+n_1)}} = \sqrt{\frac{n-2}{(n-n_1)n_1}}.
\end{aligned}$$

The Laplacian eigenvalues of $\overline{K}_{n_i}$ consist of the eigenvalue 0 repeated n_i times. Therefore, using Theorem 3.4, it follows that

$$\alpha_1\mu_i(\overline{K}_{n_1}) + n_2\beta_1 + n_3\beta_2 = (n-n_1)\sqrt{\frac{n-2}{n_1(n-n_1)}}$$

is a Laplacian ABC-eigenvalue of $\overline{K}_{n_1} \vee (\overline{K}_{n_2} \cup \overline{K}_{n_3})$ repeated $n_1 - 1$ times. Similarly, by Theorem 3.4,

$$\alpha_2\mu_i(\overline{K}_{n_2}) + n_1\beta_1 = n_1\sqrt{\frac{n-2}{n_1(n-n_1)}}$$

is a Laplacian ABC-eigenvalue of $\overline{K}_{n_1} \vee (\overline{K}_{n_2} \cup \overline{K}_{n_3})$ repeated $n_2 - 1$ times, and

$$\alpha_2\mu_i(\overline{K}_{n_3}) + n_1\beta_1 = n_1\sqrt{\frac{n-2}{n_1(n-n_1)}}$$

is a Laplacian ABC-eigenvalue of $\overline{K}_{n_1} \vee (\overline{K}_{n_2} \cup \overline{K}_{n_3})$ repeated $n_3 - 1$ times. The remaining three Laplacian ABC-eigenvalues of $\overline{K}_{n_1} \vee (\overline{K}_{n_2} \cup \overline{K}_{n_3})$ consist of the eigenvalue 0, and the two zeros of the polynomial $x^2 - (n + n_1)\sqrt{\frac{n-2}{n_1(n-n_1)}}x + \frac{n(n-2)}{n-n_1}$. □

It is clear from Corollary 3.7 that the Laplacian ABC-eigenvalues of the graph $\overline{K}_{n_1} \vee (\overline{K}_{n_2} \cup \overline{K}_{n_3})$ are the functions of n and n_1 only.

4. Applications

In this section, we show the application of our results obtained in Sections 2 and 3 for smaller graphs.

Let R be a commutative ring with multiplicative identity $1 \neq 0$. An element x of R is said to be a zero divisor if there exists a non-zero element y such that $xy = 0$. Let $Z(R)$ be the set of zero divisors of R and let $Z(R)^* = Z(R) - \{0\}$ be the set of non-zero zero divisors of R. The zero-divisor graph of R is denoted by $\Gamma(R)$ and defined as the graph with vertex set $Z(R)^*$ such that two vertices $x, y \in Z(R^*)$ are adjacent if $xy = yx = 0$. Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ be the set of integers modulo n. It is well-known that $\mathbb{Z}_n$ is a commutative ring with multiplicative identity $1 \neq 0$ having $Z(\mathbb{Z}_n)^* = \{a \in \mathbb{Z}_n : gcd(a, n) \neq 1\}$. Since every element of $\mathbb{Z}_n$ is either a unit or a zero divisor, it follows that the order of the graph $\Gamma(\mathbb{Z}_n)$ is $n - \phi(n) - 1$, where ϕ is the Euler ϕ-function ($\phi(n)$ is the number of positive integers less than n and relatively prime to n). Regarding the degrees of the vertices of the graph $\Gamma(\mathbb{Z}_n)$, the following result was shown in [22].

Lemma 4.1. *Let $\Gamma(\mathbb{Z}_n)$ be the zero divisor graph of $\mathbb{Z}_n$, the ring of integers modulo n. For any vertex $a \in V(\Gamma(\mathbb{Z}_n))$, we have $d_a = gcd(a, n) - 1$, if $a^2 \not\equiv 0(\mod n)$, and $d_a = gcd(a, n) - 2$, if $a^2 \equiv 0(\mod n)$.*

Example 4.2. *Consider the graph $\Gamma(\mathbb{Z}_{20})$ having vertex set*

$$V(\Gamma(\mathbb{Z}_{20})) = \{2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18, \}$$

and edge set

$$E(\Gamma(\mathbb{Z}_{20})) = \Big\{\{2, 10\}, \{4, 5\}, \{4, 10\}, \{4, 15\}, \{5, 8\}, \{5, 12\}, \{5, 16\}, \{6, 10\}, \{8, 10\}, \{8, 15\},$$
$$\{10, 12\}, \{10, 14\}, \{10, 16\}, \{10, 18\}, \{12, 15\}, \{15, 16\}\Big\}.$$

Let $d(a)$ be the degree, $\overline{d}(a)$ be the ABC-degree, and $\overline{m}(a)$ be the average 2-ABC-degree of the vertex $a \in V(\mathbb{Z}_{20})$. By Lemma 4.1, we have

$$d(2) = 1,\ d(4) = 3,\ d(5) = 4,\ d(6) = 1,\ d(8) = 3,$$

$$d(10) = 8,\ d(12) = 3,\ d(14) = 1,\ d(15) = 4,\ d(16) = 3, d(18) = 1.$$

Therefore,

$$\begin{aligned}
\overline{d}(2) = \overline{d}(6) = \overline{d}(14) = \overline{d}(18) &= \sqrt{\frac{7}{8}} = 0.935414,\\
\overline{d}(4) = \overline{d}(8) = \overline{d}(12) = \overline{d}(16) &= 2\sqrt{\frac{5}{12}} + \sqrt{\frac{9}{24}} = 1.903366,\\
\overline{d}(5) = \overline{d}(15) &= 4\sqrt{\frac{5}{12}} = 2.581988,\\
\overline{d}(10) &= 4\sqrt{\frac{7}{8}} + 4\sqrt{\frac{9}{24}} = 6.191147.
\end{aligned}$$

This gives that $\overline{d}_{\max} = \overline{d}(10) = 6.191147$ *and* $\overline{d}_{\min} = 0.935414$. *Therefore, using the upper bound given by Theorem 2.5, we get*

$$\xi_1(\Gamma(\mathbb{Z}_{20})) \le 2\overline{d}_{\max} = 12.382294.$$

Also,

$$\begin{aligned}
\overline{m}(2) = \overline{m}(6) = \overline{m}(14) = \overline{m}(18) &= \frac{1}{0.935414}\sqrt{\frac{7}{8}}(6.191147) = 6.721956,\\
\overline{m}(4) = \overline{m}(8) = \overline{m}(12) = \overline{m}(16) &= \frac{1}{1.903366}\Big[2\sqrt{\frac{5}{12}}(2.581988) + \sqrt{\frac{9}{24}}(6.191147)\Big] = 3.743168,\\
\overline{m}(5) = \overline{m}(15) &= \frac{1}{2.581988}\Big[4\sqrt{\frac{5}{12}}(1.1.903366)\Big] = 1.903366,\\
\overline{m}(10) &= \frac{1}{6.191147}\Big[4\sqrt{\frac{7}{8}}(0.935414) + 4\sqrt{\frac{9}{24}}(1.903366)\Big] = 1.318378.
\end{aligned}$$

Therefore, by Theorem 2.6, we get

$$\begin{aligned}
\xi_1(\Gamma(\mathbb{Z}_{20})) \le \max\Big\{&\overline{d}(2) + \overline{m}(2), \overline{d}(4) + \overline{m}(4), \overline{d}(5) + \overline{m}(5), \overline{d}(6) + \overline{m}(6), \overline{d}(8) + \overline{m}(8), \overline{d}(10)\\
&+ \overline{m}(10), \overline{d}(12) + \overline{m}(12), \overline{d}(14) + \overline{m}(14), \overline{d}(15) + \overline{m}(15), \overline{d}(16) + \overline{m}(16), \overline{d}(18) + \overline{m}(18)\Big\}\\
&= 7.509525.
\end{aligned}$$

Further,

$$\begin{aligned}
\overline{d}(2) + \overline{d}(10) &= 0.935414 + 6.191147 = 7.126561,\\
\overline{d}(4) + \overline{d}(5) &= 1.903366 + 2.581988 = 4.485354,\\
\overline{d}(4) + \overline{d}(10) &= 1.903366 + 6.191147 = 8.094513,\\
\overline{d}(4) + \overline{d}(15) &= 1.903366 + 2.581988 = 4.485354,\\
\overline{d}(5) + \overline{d}(8) &= 2.581988 + 1.903366 = 4.485354,\\
\overline{d}(5) + \overline{d}(12) &= 2.581988 + 1.903366 = 4.485354,
\end{aligned}$$

$$\begin{aligned}
\overline{d}(5)+\overline{d}(16) &= 2.581988+1.903366=4.485354,\\
\overline{d}(6)+\overline{d}(10) &= 0.935414+6.191147=7.126561,\\
\overline{d}(8)+\overline{d}(10) &= 1.903366+6.191147=8.094513,\\
\overline{d}(8)+\overline{d}(15) &= 1.903366+2.581988=4.485354,\\
\overline{d}(10)+\overline{d}(12) &= 6.191147+1.903366=8.094513,\\
\overline{d}(10)+\overline{d}(14) &= 6.191147+0.9935414=7.126561,\\
\overline{d}(10)+\overline{d}(16) &= 6.191147+1.903366=8.094513,\\
\overline{d}(10)+\overline{d}(18) &= 6.191147+0.935414=7.126561,\\
\overline{d}(12)+\overline{d}(15) &= 1.903366+2.581988=4.485354,\\
\overline{d}(15)+\overline{d}(16) &= 2.581988+1.903366=4.485354.
\end{aligned}$$

Therefore, by Theorem 2.12, it follows that

$$\begin{aligned}
\xi_1(\Gamma(\mathbb{Z}_{20})) \le \max\Big\{&\overline{d}(2)+\overline{d}(10), \overline{d}(4)+\overline{d}(5), \overline{d}(4)+\overline{d}(10), \overline{d}(4)+\overline{d}(15), \overline{d}(5)+\overline{d}(8),\\
&\overline{d}(5)+\overline{d}(12), \overline{d}(5)+\overline{d}(16), \overline{d}(6)+\overline{d}(10), \overline{d}(8)+\overline{d}(10), \overline{d}(8)+\overline{d}(15), \overline{d}(10)+\overline{d}(12),\\
&\overline{d}(10)+\overline{d}(14), \overline{d}(10)+\overline{d}(16), \overline{d}(10)+\overline{d}(18), \overline{d}(12)+\overline{d}(15), \overline{d}(15)+\overline{d}(16)\Big\}\\
=\ & 8.094513.
\end{aligned}$$

Lastly, using the upper bound given by Theorem 2.14, we get

$$\xi_{n-1}(\Gamma(\mathbb{Z}_{20})) \le \frac{20}{19}(0.935414) = 0.984646.$$

The Laplacian ABC-matrix of $\Gamma(\mathbb{Z}_{20})$ *is*

$$\tilde{L}(\Gamma(\mathbb{Z}_{20})) = \begin{pmatrix}
a & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0\\
0 & 2c+b & -c & 0 & 0 & -b & 0 & 0 & -c & 0 & 0\\
0 & -c & 4c & 0 & -c & 0 & -c & 0 & 0 & -c & 0\\
0 & 0 & 0 & a & 0 & -a & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -c & 0 & 2c+b & -b & 0 & 0 & -c & 0 & 0\\
-a & -b & 0 & -a & -b & 4a+4b & -b & -a & 0 & -b & -a\\
0 & 0 & -c & 0 & 0 & -b & 2c+b & 0 & -c & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -a & 0 & a & 0 & 0 & 0\\
0 & -c & 0 & 0 & -c & 0 & -c & 0 & 4c & -c & 0\\
0 & 0 & -c & 0 & 0 & -b & 0 & 0 & -c & 2c+b & 0\\
0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & a
\end{pmatrix},$$

where

$$a=\sqrt{\frac{7}{8}}, b=\sqrt{\frac{9}{24}}, \text{ and } c=\sqrt{\frac{5}{12}}.$$

By using Mathematica, it can be seen that $\xi_1(\Gamma(\mathbb{Z}_{20})) \approx 7.096030$ *and* $\xi_{n-1}(\Gamma(\mathbb{Z}_{20})) \approx 0.582954$.

It is clear from this example that the upper bound provided by Theorem 2.6 is better than the upper bound provided by Theorem 2.12 for the graph $\Gamma(\mathbb{Z}_{20})$. In general, it seems that the upper bound provided by Theorem 2.6 is always better than the upper bound provided by Theorem 2.12.

It is well-known that if $n = pq$, where p and q are distinct primes, then

$$\Gamma(\mathbb{Z}_{pq}) = K_{p-1,q-1} = K_{\phi(p),\phi(q)},$$

and so equality holds in Theorems 2.3, 2.6, 2.9, and 2.12 for the zero divisor graph $\Gamma(\mathbb{Z}_{pq})$, for all $1 < p < q$. Also, taking

$$G_1 = \overline{K}_{\phi(p)}, G_2 = \overline{K}_{\phi(q)}, r_1 = r_2 = 0, n_1 = \phi(p), n_2 = \phi(q), \mu_i(\overline{K}_{\phi(p)}) = 0 = \mu_j(\overline{K}_{\phi(q)}),$$

for all i, j, in Theorem 3.1, it follows that the Laplacian ABC-eigenvalues of $\Gamma(\mathbb{Z}_{pq})$ consist of the eigenvalue $(\phi(p) + \phi(q))\sqrt{\frac{\phi(p)+\phi(q)-2}{\phi(p)\phi(q)}}$, the eigenvalue $\sqrt{\frac{\phi(p)(\phi(p)+\phi(q)-2)}{\phi(q)}}$ repeated $\phi(q) - 1$ times, the eigenvalue $\sqrt{\frac{\phi(q)(\phi(p)+\phi(q)-2)}{\phi(p)}}$ repeated $\phi(p) - 1$ times, and the eigenvalue 0.

5. Conclusions

The Laplacian ABC-matrix, and so the Laplacian ABC-eigenvalues, are closely related to the ABC-index of a graph G. Therefore, the spectral study of the Laplacian ABC-matrix makes sense. This work aims to establish new upper and lower bounds for the Laplacian ABC-spectral radius in terms of various graph parameters and the characterization of the candidate graphs that attain these bounds. Further, we consider the problem of determining the Laplacian ABC-eigenvalues of the graphs, which can be constructed from the join of two regular graphs or the join of a regular graph with the union of two regular graphs. Although this work contributes to the spectral analysis of the Laplacian ABC-matrix of graphs, several aspects remain unexplored, offering promising directions for future research.

Author contributions

Hilal A. Ganie and Amal Alsaluli: Writing-review and editing; conceptualization, methodology, formal analysis, writing-original draft, and writing-review and editing. Both authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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