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Research article

Inertial viscosity type algorithm for solution of split null fixed point problem with applications

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Abstract: The purpose of this study is to introduce an inertial viscosity type iterative algorithm with a self-adaptive step size to approximate the solution of a split null fixed point problem involving nonexpansive mappings. Under suitable mild assumptions, we establish strong convergence of the iterative sequence generated by the proposed method. As a consequence, we also obtain approximation results for solutions of split feasibility fixed point and split minimization fixed point problems. In addition, we apply our algorithm to solve the signal processing problem. The applicability of the proposed algorithm is demonstrated through a numerical example. Furthermore, a graphical comparison with several existing methods in the literature is presented to illustrate its performance.

Keywords: split null fixed point; split feasibility fixed point; split minimization fixed point; inertial viscosity algorithm
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1. Introduction and preliminaries

The proof of the Banach contraction theorem [1] is derived from the convergence of the Picard iteration scheme, which is one of the most fundamental iteration algorithms. This result provides a solution to the fixed point problem (FPP) of contraction mappings defined on complete metric spaces and has become a fundamental tool to establish the existence as well as the approximation of solutions to several nonlinear problems. In many hypothetical situations, the existence of the FPP is known, however, finding the exact value of a fixed point may not be possible. In this case, finding the approximate solution of the problem is a practical way out that leads to the development of different algorithms [2–4].

Several fixed point iterative techniques have been proposed and employed by many authors for the approximation of solutions to linear and nonlinear equations and system of equations and inclusions. It is preferable to construct an iterative method that outperforms existing schemes for approximating the solution in fewer iterations. In this study, we introduce a method to approximate the solution of a split null fixed point problem (SNFPP).

In what follows, $\mathcal{Z}, \mathcal{Z}_1$, and $\mathcal{Z}_2$ are real Hilbert spaces, and $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are nonempty, closed, and convex subsets of $\mathcal{Z}_1$ and $\mathcal{Z}_2$, respectively. Moreover, $L : \mathcal{Z}_1 \to \mathcal{Z}_2$ is a bounded linear operator and L^* is an adjoint operator of L.

A self mapping S on $\mathcal{Y}_1$ is nonexpansive if

$$\|Sg_1 - Sg_2\| \leq \|g_1 - g_2\|,$$

for all $g_1, g_2 \in \mathcal{Y}_1$. The theory of nonexpansive mappings is motivated by their extensive applications in convex optimization, compressed sensing, variational inequalities, economics, image restoration problems, and some other branches of applied sciences, where numerous equations involve nonexpansive mappings (see, [5, 6]).

Recall that a multivalued mapping $S : \mathcal{Z} \to 2^{\mathcal{Z}}$ is known as monotone, if for any $g_1, g_2 \in \mathcal{Z}$ with $y_1 \in Sg_1$ and $y_2 \in Sg_2$, we have $\langle g_1 - g_2, y_1 - y_2 \rangle \geq 0$. A monotone mapping S is called maximal monotone if there does not exist another monotone mapping whose graph properly contains the graph of S. In what follows, P_1 and P_2 are multivalued maximal monotones mappings on $\mathcal{Z}_1$ and $\mathcal{Z}_2$, respectively.

A mapping $S : \mathcal{Z} \to \mathcal{Z}$ is said to be ν–inverse strongly monotone if $\langle Sg_1 - Sg_2, g_1 - g_2 \rangle \geq \nu \|Sg_1 - Sg_2\|^2$ for each $g_1, g_2 \in \mathcal{Z}$ and $\nu > 0$. If $\nu = 1$, then S is known as inverse strongly monotone (ISM).

For $\chi > 0$, an operator

$$J^S_\chi = (I + \chi S)^{-1}$$

is known as the resolvent of S.

It is known that, for any $g_1 \in \mathcal{Z}$, we have a unique element $P_{\mathcal{Y}_1} g_1 \in \mathcal{Y}_1$ with

$$\|g_1 - P_{\mathcal{Y}_1} g_1\| = \inf\{\|g_1 - g_2\| : g_2 \in \mathcal{Y}_1\}.$$

The above defines an operator $P_{\mathcal{Y}_1}$ from $\mathcal{Z}$ onto the set $\mathcal{Y}_1$, which is known as the metric projection. Moreover, the operator $P_{\mathcal{Y}_1}$ fulfills the properties:

For any $g_1, g_2 \in \mathcal{Z}$,

$$\|P_{\mathcal{Y}_1} g_1 - P_{\mathcal{Y}_1} g_2\|^2 \leq \langle P_{\mathcal{Y}_1} g_1 - P_{\mathcal{Y}_1} g_2, g_1 - g_2 \rangle. \tag{1.1}$$

For each $g_1 \in \mathcal{Z}$,

$$P_{\mathcal{Y}_1} g_1 = g_2 \quad \text{if and only if} \quad \langle g_1 - g_2, g_2 - g_3 \rangle \geq 0, \quad \text{for all} \quad g_3 \in \mathcal{Y}_1. \tag{1.2}$$

Further details of properties of projection operator can be found in [7].

We denote the strong convergence of $\{g_n\}$ to g^* as $g_n \to g^*$, the weak convergence as $g_n \rightharpoonup g^*$, and the set of weak limits of $\{g_n\}$ by $\omega(g_n)$.

The FPP of a self mapping S on $\mathcal{Z}$ is defined as:

$$\text{Find} \quad g^* \in \mathcal{Z} \quad \text{such that} \quad Sg^* = g^*. \tag{1.3}$$

A set $\mathcal{F}(S)$ denotes the set of all fixed points of S.

The split null point problem (SNPP) can be expressed as follows:

Find

$$g^* \in \mathcal{Z}_1 \quad \text{with} \quad 0 \in P_1 g^*,$$

and

$$Lg^* \in \mathcal{Z}_2 \quad \text{with} \quad 0 \in P_2 Lg^*, \tag{1.4}$$

where $P_1 : \mathcal{Z}_1 \to 2^{\mathcal{Z}_1}$ and $P_2 : \mathcal{Z}_2 \to 2^{\mathcal{Z}_2}$.

The split null fixed point problem (SNFPP) can be stated as follows:

Find

$$g^* \in \mathcal{Z}_1 \quad \text{with} \quad 0 \in P_1 g^* \quad \text{and} \quad Sg^* = g^*,$$

and

$$Lg^* \in \mathcal{Z}_2 \quad \text{with} \quad 0 \in P_2 Lg^*, \tag{1.5}$$

where $S : \mathcal{Y}_1 \to \mathcal{Y}_1$ is nonexpansive.

Note that the problems given in (1.3) and (1.4) are particular cases of the SNFPP given by (1.5). We denote the solution set of problem (1.5) by Π.

Remark 1.1. *[8] It is well known that*

- *The mapping S is maximal monotone if and only if J_χ^S is single valued.*
- *$J_\chi^S g^* = g^*$ if and only if $0 \in Sg^*$.*
- *The SNPP (1.4) is equivalent to the following problem:*
  Find $g^* \in \mathcal{Z}_1$ with

$$J_\chi^{P_1} g^* = g^* \quad \textit{such that} \quad Lg^* \in \mathcal{Z}_2 \quad \textit{and} \quad Lg^* = J_\chi^{P_2} Lg^*. \tag{1.6}$$

Lemma 1.1. *[9] If $S : \mathcal{Y}_1 \to \mathcal{Z}_1$ is nonexpansive, then $I - S$ is demiclosed on $\mathcal{Y}_1$, that is, for every $\{g_n\}$ in $\mathcal{Y}_1$ and $g_n \rightharpoonup g'$, then $(I - S)g_n \to g''$ implies $(I - S)g' = g''$.*

Definition 1.1. *The map $S : \mathcal{Z} \to R \cup \{+\infty\}$ is referred to weakly lower semicontinuous at $g^* \in \mathcal{Z}$ if for each $\{g_n\}$ in $\mathcal{Z}$ with $g_n \rightharpoonup g^*$, we get that*

$$\phi(g^*) \leq \liminf_{n\to\infty} S(g_n).$$

Lemma 1.2. *[10] If $g_1, g_2 \in \mathcal{Z}$ and $\theta \in R$, then we have:*

(i) $\|g_1 + g_2\|^2 \leq \|g_1\|^2 + 2\langle g_1 + g_2, g_2\rangle$,
(ii) $\|g_1 + g_2\|^2 = \|g_1\|^2 + 2\langle g_1, g_2\rangle + \|g_2\|^2$,
(iii) $\|g_1 - g_2\|^2 = \|g_1\|^2 - 2\langle g_1, g_2\rangle + \|g_2\|^2$,
(iv) $\|\theta g_1 + (1-\theta)g_2\|^2 = \theta\|g_1\|^2 + (1-\theta)\|g_2\|^2 - \theta(1-\theta)\|g_1 - g_2\|^2$.

Lemma 1.3. *[11] Assuming that $g_1, g_2, g_3 \in \mathcal{Z}$ and $\sigma, \theta, \eta \in [0,1]$ with $\sigma + \theta + \eta = 1$, we have:*

$$\|\sigma g_1 + \theta g_2 + \eta g_3\|^2 = \sigma\|g_1\|^2 + \theta\|g_2\|^2 + \eta\|g_3\|^2 - \sigma\theta\|g_1 - g_2\|^2 - \sigma\eta\|g_1 - g_3\|^2 - \theta\eta\|g_2 - g_3\|^2.$$

Lemma 1.4. *[12] Assume that $\{\bar{g}_n\}, \{\bar{j}_n\} \subset \mathbb{R}^+$, $\sum_{n=0}^{\infty} \bar{j}_n < \infty$, and $\{\bar{h}_n\} \subset \mathbb{R}$, $\{\delta_n\} \subset (0,1)$ with:*

$$\bar{g}_{n+1} \leq (1-\delta_n)\bar{g}_n + \bar{h}_n + \bar{j}_n.$$

Then,

(i) For, $\bar{h}_n \leq \theta\delta_n$ *with* $\theta \geq 0$, *we get that* $\{\bar{g}_n\}$ *is bounded.*

(ii) Conditions $\sum_{n=0}^{\infty} \delta_n = \infty$ *and* $\limsup_{n\to\infty} \frac{\bar{h}_n}{\delta_n} \leq 0$, *give* $\lim_{n\to\infty} \bar{g}_n = 0$.

Lemma 1.5. *[13] Suppose that* $\{\bar{g}_n\} \subset \mathbb{R}^+$, $\{\delta_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} \delta_n = \infty$, $\{\bar{h}_n\} \subset \mathbb{R}$, *and the following holds:*

$$\bar{g}_{n+1} \leq (1-\delta_n)\bar{g}_n + \delta_n \bar{h}_n.$$

If $\limsup_{n\to\infty} \bar{h}_{n_i} \leq 0$ *and for each subsequence* $\{\bar{g}_{n_i}\}$ *we have* $\liminf_{n\to\infty}(\bar{g}_{n_i+1} - \bar{g}_{n_i}) \geq 0$, *then* $\lim_{n\to\infty} \bar{g}_n = 0$.

Since the Picard iteration given by $g_{n+1} = Sg_n$ for nonexpansive mapping S may fail to converge, even when the corresponding fixed point set is nonempty, Mann [14] proposed an iterative method for approximating fixed points of nonexpansive mappings, defined as follows: $g_{n+1} = (1-\kappa_n)x_n + \kappa_n S x_n$, where $\{\kappa_n\} \subset (0,1)$.

In general, the Mann iterative scheme in most of the cases fails to converge strongly. However, in infinite-dimensional spaces, the strong convergence is often more useful than the weak convergence. To get the strong convergence, it was necessary either to modify the Mann iteration scheme or to impose stronger assumptions on the underlying mapping. Considerable research has been dedicated to the development of strongly convergent algorithms by introducing additional conditions on the operator or by constructing modified iterative schemes. To address this problem, Halpern [15] introduced an iterative scheme, which is defined as follows: For $d, g_1 \in \mathcal{Z}$, we have $g_{n+1} = (1-\kappa_n)d + \kappa_n S g_n$, where $\{\kappa_n\} \subset (0,1)$.

An important generalization of Halpern algorithm is the viscosity algorithm introduced by Moudafi [16], and is given in Algorithm 1:

Algorithm 1: Algorithm from [16]

For $g_1 \in \mathcal{Z}$, we have

$$g_{n+1} = (1-\kappa_n)\phi(g_n) + \kappa_n S g_n,$$

where $\{\kappa_n\} \subset (0,1)$, and $\phi : \mathcal{Z} \to \mathcal{Z}$ is a contraction.

Byrne et al. [17] introduced an iterative scheme to approximate the solution of SVIP for maximal monotone mappings S_1 and S_2, which is in Algorithm 2:

Algorithm 2: Algorithm from [17]

Take $\{\kappa_n\} \subset (0,1)$, $\chi > 0$, and $\beta \in (0, \frac{2}{M})$, where $M = \|L^*L\|$.
For $g_1 \in \mathcal{Z}_1$, find g_{n+1} as follows:

$$g_{n+1} = \kappa_n g_n + (1-\kappa_n) J_\chi^{S_1}(g_n + \beta L^*(J_\chi^{S_2} - I)Lg_n),$$

where, $\lim_{n\to\infty}, \kappa_n = 0$ with $\sum_{n=1}^{\infty} \kappa_n = \infty$ and $J_\chi^{S_1}$ and $J_\chi^{S_2}$ are the resolvent operators of S_1 and S_2, respectively.

The problem of approximating common solutions of various nonlinear problems has attracted significant attention of many researchers. For instance, in [18], an iterative scheme for approximating the solution of FPP and SVIP is given by.

Algorithm 3: Algorithm from [18]

Take $\{\kappa_n\} \subset (0,1), \chi > 0$, and $\beta \in (0, \frac{2}{M})$, where $M = \|L^*L\|$.
For $g_1 \in \mathcal{Z}_1$, find g_{n+1} as follows:

$$f_n = J_\chi^{S_1}(g_n + \beta L^*(J_\chi^{S_2} - I)Lg_n),$$

$$g_{n+1} = \kappa_n \theta \phi g_n + (1 - \kappa_n A)S f_n,$$

where $\phi : \mathcal{Z}_1 \to \mathcal{Z}_1$ is contraction, $A : \mathcal{Z}_1 \to \mathcal{Z}_1$ is bounded linear, $S : \mathcal{Z}_1 \to \mathcal{Z}_1$ is a nonexpansive mapping, and S_1 and S_2 are maximal monotone.

The choice of step size plays a crucial role in both the computational efficiency and convergence behavior of an underlying algorithm. In particular, selection of a suitable step size may accelerate the approximation process by reducing the number of required iterations, for details see, [19–21]. Note that the step sizes used in Algorithms 2 and 3 depend on the operator norm, which limits the practical implementation of these methods as the evaluation of the operator norm at each iteration can be computationally expansive.

To address this problem, the following modification of Algorithm 2 was given in [8] in which the step size is updated in each iteration.

Algorithm 4: Algorithm form [8]

Take $\varrho_n \subset (0,4)$ with $\inf \varrho_n(4 - \varrho_n) > 0$.
For $g_1 \in \mathcal{Z}_1$, evaluate g_{n+1} as follows:

$$\gamma_n = \frac{\varrho_n h(g_n)}{\|Gg_n\|^2 + \|Hg_n\|^2}.$$

Then, find

$$g_{n+1} = \kappa_n g_n + (1 - \kappa_n)J_\chi^{S_1}(g_n - \gamma_n L^*(I - J_\chi^{S_2})Lg_n),$$

where $h(g) = \frac{1}{2}\left\|(I - J_{\chi_1}^{S_2})Lg\right\|^2$, $G(g) = A^*(I - J_{\chi_1}^{S_2})Lg$, $H(g) = (I - J_{\chi_1}^{S_2})g$, and $\{\kappa_n\}$ is as given in Algorithm 2.

Similarly, using the assumptions of Algorithm 3, the following scheme is given in [22], which is stated as follows.

Algorithm 5: Algorithm from [22]

Find g_{n+1} as:

$$f_n = J_\chi^{S_1}(g_n + \beta L^*(J_\chi^{S_2} - I)Lg_n),$$

$$g_{n+1} = \kappa_n \phi g_n + (1 - \kappa_n)S f_n.$$

Furthermore, considerable attention has been devoted to the incorporation of the inertial step in algorithms, where each iterative step depends on the two preceding iterates. Several researchers have investigated and developed efficient iterative schemes that combine inertial techniques with adaptive step-size strategies to solve various nonlinear problems, see for example, [23–25].

Motivated by these developments, we propose an efficient inertial and viscosity-type algorithm to solve the SNFPP. We establish strong convergence of the proposed iterative sequence for approximation of solutions to the SNFPP. In addition, to demonstrate the applicability of the proposed method, we solve the split feasibility fixed point and split minimization fixed point problems. These problems arise in various fields, such as network resource allocation and signal processing. For further details, we refer the reader to [26, 27].

2. Convergence analysis

In this section, we prove strong convergence of our proposed iterative sequence for approximation of solutions to the SNFPP. Let us start with the following assumptions.

Assumption 2.1. *Assume $P_1 : \mathcal{Z}_1 \to 2^{\mathcal{Z}_1}$ and $P_2 : \mathcal{Z}_2 \to 2^{\mathcal{Z}_2}$ are maximal monotone, $S : \mathcal{Y}_1 \to \mathcal{Y}_1$ is nonexpansive, and $\phi : \mathcal{Z}_1 \to \mathcal{Z}_1$ is μ-contraction.*
Define mappings as follows:

$$\begin{aligned} h(g) =& \frac{1}{2}\left\|(I - J_{\chi_2}^{P_2})Lg\right\|^2, \\ j(g) =& \frac{1}{2}\left\|(I - J_{\chi_1}^{P_1})g\right\|^2, \\ G(g) =& L^*(I - J_{\chi_2}^{P_2})Lg, \\ H(g) =& (I - J_{\chi_1}^{P_1})g. \end{aligned}$$

Note that, h and j are the convex, differentiable, and weakly lower semicontinuous functions [28] whereas G and H are Lipschitzian continuous [8].

We, now propose an inertial and viscosity-type algorithm as follows.

Algorithm 6: Algorithm for solving SNFPP

Set,

$$\delta_n = \begin{cases} \min\{\delta, \frac{k_n}{\|g_n - g_{n-1}\|}\}, & \text{if} \quad g_n \neq g_{n-1}, \\ \delta, & \text{otherwise.} \end{cases} \tag{2.1}$$

Find

$$a_n = g_n + \delta_n(g_n - g_{n-1}).$$

Evaluate

$$b_n = J_{\chi_1}^{P_1}(I - \gamma_n L^*(I - J_{\chi_2}^{P_2})L)a_n,$$

where

$$\gamma_n = \begin{cases} \frac{\varrho_n h(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} & \text{if } \|G(a_n)\|^2 + \|H(a_n)\|^2 \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Then, define an iterative method as follows:

$$c_n = \theta_n a_n + (1 - \theta_n)b_n,$$

$$d_n = (1 - \eta_n)c_n + \eta_n S c_n,$$
$$e_n = S((1 - \omega_n)d_n + \omega_n S d_n),$$
$$f_n = S((1 - \beta_n)S d_n + \beta_n S e_n),$$
$$g_{n+1} = \kappa_n \phi(g_n) + (1 - \kappa_n)S f_n,$$

where
(i) $\{\kappa_n\} \subset (0, 1)$ with $\sum_{n=0}^{\infty} \kappa_n = \infty$ and $\lim_{n\to\infty} \kappa_n = 0$.
(ii) $\{\theta_n\}, \{\eta_n\}, \{\omega_n\}, \{\beta_n\} \subset (0, 1)$.
(iii) $\delta > 0$ is fixed and $\{k_n\} \subset \mathbb{R}^+$ and $\lim_{n\to\infty} \frac{k_n}{\kappa_n} = 0$.
(iv) $0 < \varrho_n < 4$, with $\inf_{n\geq 0} \varrho_n(4 - \varrho_n) > 0$, and $\chi_i > 0, i = 1, 2$.

Remark 2.1. *Using* (2.1)*, we obtain that*

$$\lim_{n\to\infty} \frac{\delta_n}{\kappa_n} \|g_n - g_{n-1}\| = 0.$$

We now prove the main result of this paper. For the sake of simplicity, the proof of the following theorem is divided into four steps given in the form of lemmas.

Theorem 2.1. *Suppose ϕ, P_1, P_2, and S are as given in Assumption 2.1. The sequence $\{g_n\}$ generated by Algorithm* 6 *converges strongly to fixed point of $P_\Pi o\phi$.*

Lemma 2.1. *The sequence $\{g_n\}$ given by Algorithm* 6 *is bounded.*

Proof. As $P_\Pi o\phi$ is contraction, we have a point $g' \in \mathcal{Z}_1$ with $P_\Pi o\phi(g') = g'$. Also, $g' \in \Pi$ implies $S g^* = g^*$, $J_{\chi_1}^{P_1} g^* = g^*$, and $J_{\chi_2}^{P_2}(Lg^*) = Lg^*$. Now,

$$\begin{aligned} \|a_n - g'\| =& \|g_n + \delta_n(g_n - g_{n-1}) - g'\| \\ \leq& \|g_n - g'\| + \delta_n\|g_n - g_{n-1}\| \\ =& \|g_n - g'\| + \kappa_n \frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\|. \end{aligned} \tag{2.3}$$

From Remark 2.1, $\lim_{n\to\infty} \frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| = 0$. Thus, we have $Q_1 > 0$ with

$$\frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| \leq Q_1.$$

Using (2.3), we get

$$\|a_n - g'\| \leq \|g_n - g'\| + \kappa_n Q_1. \tag{2.4}$$

As $I - J_{\chi_2}^{P_2}$ is firmly nonexpansive, we have

$$\begin{aligned} \langle Ga_n, a_n - g'\rangle =& \langle L^*(I - J_{\chi_2}^{P_2})La_n, a_n - g'\rangle \\ =& \langle (I - J_{\chi_2}^{P_2})La_n, La_n - Lg'\rangle \\ =& \langle (I - J_{\chi_2}^{P_2})La_n - Lg' + Lg', La_n - Lg'\rangle \\ =& \langle (I - J_{\chi_2}^{P_2})La_n - L\underline{p}^* + J_{\chi_2}^{P_2}Lg', La_n - L\underline{p}^*\rangle \end{aligned}$$

$$
\begin{aligned}
&=\langle (I - J_{\chi_2}^{P_2})La_n - (I - J_{\chi_2}^{P_2})Lg', La_n - Lg' \rangle \\
&\geq \|(I - J_{\chi_2}^{P_2})La_n - (I - J_{\chi_2}^{P_2})Lg'\|^2 \\
&= \|(I - J_{\chi_2}^{P_2})La_n\|^2 \\
&= 2h(a_n).
\end{aligned} \tag{2.5}
$$

By Lemma 1.2 and (2.5), we get that

$$
\begin{aligned}
\|b_n - g'\|^2 &= \|J_{\chi_1}^{P_1}(I - \gamma_n L^*(I - J_{\chi_2}^{P_2})L)a_n - g'\|^2 \\
&\leq \|a_n - \gamma_n L^*(I - J_{\chi_2}^{P_2})La_n - g'\|^2 \\
&= \|a_n - g' - \gamma_n G(a_n)\|^2 \\
&= \|a_n - g'\|^2 + \gamma_n^2 \|G(a_n)\|^2 - 2\gamma_n \langle G(a_n), a_n - g' \rangle.
\end{aligned}
$$

Now, by using the value of γ_n, we obtain that

$$
\begin{aligned}
&= \|a_n - g'\|^2 + \frac{\varrho_n^2 h^2(a_n)}{(\|G(a_n)\|^2 + \|H(a_n)\|^2)^2}\|G(a_n)\|^2 - \frac{4\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} \\
&\leq \|a_n - g'\|^2 - \frac{(4 - \varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2}.
\end{aligned} \tag{2.6}
$$

From the assumption on a sequence ϱ_n, we have

$$
\|b_n - g'\| \leq \|a_n - g'\|. \tag{2.7}
$$

Also,

$$
\begin{aligned}
\|c_n - g'\| &= \|\theta_n a_n + (1 - \theta_n)b_n - g'\| \\
&\leq \theta_n \|a_n - g'\| + (1 - \theta_n)\|b_n - g'\| \\
&\leq \theta_n \|a_n - g'\| + (1 - \theta_n)\|a_n - g'\| \\
&= \|a_n - g'\|,
\end{aligned} \tag{2.8}
$$

and

$$
\begin{aligned}
\|d_n - g'\| &= \|(1 - \eta_n)c_n + \eta_n S c_n - g'\| \\
&\leq (1 - \eta_n)\|c_n - g'\| + \eta_n \|S c_n - g'\| \\
&\leq (1 - \eta_n)\|c_n - g'\| + \eta_n \|c_n - g'\| \\
&= \|c_n - g'\| \\
&\leq \|a_n - g'\|.
\end{aligned} \tag{2.9}
$$

Now, we have the following:

$$
\begin{aligned}
\|e_n - g'\| &= \|S((1 - \omega_n)d_n + \omega_n S d_n) - g'\| \\
&\leq \|(1 - \omega_n)d_n + \omega_n S d_n - g'\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\omega_n)\|d_n - g^{'}\| + \omega_n\|S d_n - g^{'}\| \\
&\leq (1-\omega_n)\|d_n - g^{'}\| + \omega_n\|d_n - g^{'}\| \\
&\leq (1-\omega_n)\|d_n - g^{'}\| + \omega_n\|d_n - g^{'}\| \\
&= \|d_n - g^{'}\|. \qquad (2.10)
\end{aligned}
$$

Using (2.10), we get

$$
\begin{aligned}
\|f_n - g^{'}\| =& \|S((1-\beta_n)S d_n + \beta_n S e_n) - g^{'}\| \\
\leq& \|(1-\beta_n)S d_n - \beta_n S e_n - g^{'}\| \\
\leq& (1-\beta_n)\|S d_n - g^{'}\| + \beta_n\|S e_n - g^{'}\| \\
\leq& (1-\beta_n)\|d_n - g^{'}\| + \beta_n\|e_n - g^{'}\| \\
\leq& (1-\beta_n)\|d_n - g^{'}\| + \beta_n\|d_n - g^{'}\| \\
=& \|d_n - g^{'}\|. \qquad (2.11)
\end{aligned}
$$

Finally, we obtain the following:

$$
\begin{aligned}
\|g_{n+1} - g^{'}\| =& \|\kappa_n\phi(g_n) + (1-\kappa_n)S f_n - g^{'}\| \\
=& \|\kappa_n\phi(g_n) - \phi(g^{'}) + \kappa_n(\phi(g^{'}) - g^{'} + (\kappa_n)(S f_n - g^{'}\| \\
\leq& \kappa_n\|\phi(g_n - \phi(g^{'})\| + \kappa_n\|\phi(g^{'}) - g^{'}\| + (1-\kappa_n)S f_n - g^{'}\| \\
\leq& \kappa_n\mu\|g_n - g^{'}\| + \kappa_n\|\phi(g^{'}) - g^{'}\| + (1-\kappa_n)\|f_n - g^{'}\| \\
\leq& \kappa_n\mu\|g_n - g^{'}\| + \kappa_n\|\phi(g^{'}) - g^{'}\| + (1-\kappa_n)\|g_n - g^{'}\| \\
=& (1-\kappa_n(1-\mu))\|g_n - g^{'}\| + \kappa_n(1-\mu)\left\{\frac{\|\phi(g^{'} - g^{'}\|}{1-\mu} + \frac{(1-\mu)Q_1}{1-\mu}\right\} \\
\leq& (1-\kappa_n(1-\mu))\|g_n - g^{'}\| + 2\kappa_n(1-\mu)M^{*}, \qquad (2.12)
\end{aligned}
$$

where $M^{*} = \sup\left\{\frac{\|\phi(g^{'})-g^{'}\|}{1-\mu}, \frac{(1-\mu)Q_1}{1-\mu}\right\}$. Set $\bar{g}_n = \|g_n - g^{'}\|, \bar{h}_n = \kappa_n(1-\mu)M^{*}, \bar{j}_n = 0$, and $\delta_n = \kappa_n(1-\mu)$. Then, using the Lemma 1.4 and the assumption on the parameters, we get that $\{\|g_n - g^{'}\|\}$ is bounded, hence $\{g_n\}$ is bounded. Moreover, $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{e_n\}$, and $\{f_n\}$ are also bounded.

Lemma 2.2. *If $\{g_n\}$ is given by Algorithm* 6 *and the assumptions of Theorem* 2.1 *hold, then we have:*

$$
\begin{aligned}
\left\|g_{n+1} - g^{'}\right\|^2 \leq& \left(1 - \frac{2\kappa_n(1-\mu)}{1-\kappa_n\mu}\right)\|g_n - r^{'}\|^2 + \frac{2\kappa_n(1-\mu)}{1-\kappa_n\mu}\left\{\frac{3Q_2(1-\kappa_n)^2}{2(1-\mu)}\frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| \right. \\
&\left. + \frac{1}{(1-\mu)}\langle\phi(r^{'}) - r^{'}, g_{n+1} - r^{'}\rangle\right\} - \frac{(1-\kappa_n)^2\theta_n(1-\theta_n)}{1-\kappa_n\mu}\|a_n - b_n\|^2 \\
&- \frac{(1-\kappa_n)^2(1-\theta_n)}{(1-\kappa_n\mu)}\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2}.
\end{aligned}
$$

Proof. Note that

$$
\|a_n - g^{'}\|^2 = \|g_n + \delta_n(g_n - g_{n-1}) - g^{'}\|^2
$$

$$
\begin{aligned}
&=\|g_n - g^{'}\|^2 + \delta_n^2\|g_n - g_{n-1}\|^2 + 2\delta_n\langle g_n - g^{'}, g_n - g_{n-1}\rangle \\
&\leq\|g_n - g^{'}\|^2 + \delta_n^2\|g_n - g_{n-1}\|^2 + 2\delta_n\|g_n - g_{n-1}\|\|g_n - g^{'}\| \\
&=\|g_n - g^{'}\|^2 + \delta_n\|g_n - g_{n-1}\|(\delta_n\|g_n - g_{n-1}\| + 2\|g_n - g^{'}\|) \\
&\leq\|g_n - g^{'}\|^2 + 3Q_2\delta_n\|g_n - g_{n-1}\| \\
&=\|g_n - g^{'}\|^2 + 3Q_2\kappa_n\frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\|,
\end{aligned}
\tag{2.13}
$$

where $Q_2 := \sup_{n\in N}\{\|g_n - g^{'}\|, \delta_n\|g_n - g_{n-1}\|\} \geq 0$. Moreover,

$$
\begin{aligned}
\|c_n - g^{'}\|^2 &=\|\theta_n a_n + (1-\theta_n)b_n - g^{'}\|^2 \\
&=\|\theta_n a_n + (1-\theta_n)b_n - \theta_n g^{'} + \theta_n g^{'} - g^{'}\|^2 \\
&=\|\theta_n(a_n - g^{'}) + (1-\theta_n)(b_n - g^{'})\|^2 \\
&=\theta_n\|a_n - g^{'}\|^2 + (1-\theta_n)\|b_n - g^{'}\|^2 - \theta_n(1-\theta_n)\|a_n - b_n\|^2 \\
&\leq\theta_n\|a_n - g^{'}\|^2 + (1-\theta_n)\|a_n - g^{'}\|^2 - \theta_n(1-\theta_n)\|a_n - b_n\|^2 \\
&=\|a_n - g^{'}\|^2 - \theta_n(1-\theta_n)\|a_n - b_n\|^2.
\end{aligned}
\tag{2.14}
$$

Using (2.13) and (2.14), we have

$$
\begin{aligned}
\left\|d_n - g^{'}\right\|^2 &= \left\|(1-\eta_n)\,c_n + \eta_n S c_n - g^{'}\right\|^2 \\
&= \left\|(1-\eta_n)\,(c_n - g^{'}) + \eta_n(S c_n - g^{'})\right\|^2 \\
&= (1-\eta_n)\left\|c_n - g^{'}\right\|^2 + \eta_n\left\|S c_n - g^{'}\right\|^2 - \eta_n\,(1-\eta_n)\,\|c_n - S c_n\|^2 \\
&\leq (1-\eta_n)\left\|c_n - g^{'}\right\|^2 + \eta_n\left\|c_n - g^{'}\right\|^2 - \eta_n\,(1-\eta_n)\,\|c_n - S c_n\| \\
&= \left\|c_n - g^{'}\right\|^2 - \eta_n\,(1-\eta_n)\,\|c_n - S c_n\| \\
&\leq \|a_n - g^{'}\|^2 - \theta_n(1-\theta_n)\|a_n - b_n\|^2 - \eta_n\,(1-\eta_n)\,\|c_n - S c_n\| \\
&\leq \|g_n - g^{'}\|^2 + 3Q_2\kappa_n\frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| - \theta_n(1-\theta_n)\|a_n - b_n\|^2 - \eta_n\,(1-\eta_n)\,\|c_n - S c_n\|\,.
\end{aligned}
\tag{2.15}
$$

Also,

$$
\begin{aligned}
\|e_n - g^{'}\|^2 &=\|S\,((1-\omega_n)d_n + \omega_n S d_n) - g^{'}\|^2 \\
&\leq\|(1-\omega_n)d_n + \omega_n S d_n - g^{'}\|^2 \\
&=\|(1-\omega_n)(d_n - g^{'}) + \omega_n(S d_n - g^{'})\|^2 \\
&=(1-\omega_n)\|d_n - g^{'}\|^2 + \omega_n\|S d_n - g^{'}\|^2 - \omega_n(1-\omega_n)\|d_n - S d_n\|^2 \\
&\leq(1-\omega_n)\|d_n - g^{'}\|^2 + \omega_n\|d_n - g^{'}\|^2 - \omega_n(1-\omega_n)\|d_n - S d_n\|^2 \\
&=\|d_n - g^{'}\|^2 - \omega_n(1-\omega_n)\|d_n - S d_n\|^2.
\end{aligned}
\tag{2.16}
$$

From (2.15) and (2.16), we have

$$
\|f_n - g^{'}\|^2 =\|S\,((1-\beta_n)S d_n + \beta_n S e_n) - g^{'}\|^2
$$

$$
\begin{aligned}
&\leq \|(1-\beta_n)Sd_n + \beta_n Se_n - g^{'}\|^2 \\
&= \|(1-\beta_n)(Sd_n - g^{'}) + \beta_n(Se_n - g^{'})\|^2 \\
&= (1-\beta_n)\|Sd_n - g^{'}\|^2 + \beta_n\|Se_n - g^{'}\|^2 - \beta_n(1-\beta_n)\|Sd_n - Se_n\|^2 \\
&\leq (1-\beta_n)\|d_n - g^{'}\|^2 + \beta_n[\|d_n - g^{'}\|^2 - \omega_n(1-\omega_n)\|d_n - Sd_n\|^2] - \beta_n(1-\beta_n)\|d_n - e_n\|^2 \\
&\leq \|d_n - g^{'}\|^2 - \beta_n\omega_n(1-\omega_n)\|d_n - Sd_n\|^2 - \beta_n(1-\beta_n)\|d_n - e_n\|^2 \\
&\leq \|g_n - g^{'}\|^2 + 3Q_2\kappa_n\frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| - \theta_n(1-\theta_n)\|a_n - b_n\|^2 - \eta_n(1-\eta_n)\|c_n - Sc_n\| \\
&\quad - \beta_n\omega_n(1-\omega_n)\|d_n - Sd_n\|^2 - \beta_n(1-\beta_n)\|d_n - e_n\|^2.
\end{aligned} \tag{2.17}
$$

Thus,

$$
\begin{aligned}
\left\|g_{n+1} - g^{'}\right\|^2 &= \left\|\kappa_n\left(\phi(g_n) - g^{'}\right) + (1-\kappa_n)Sf_n - g^{'}\right\|^2 \\
&= \left\|\kappa_n\left(\phi(g_n) - g^{'}\right) + (1-\kappa_n)\left(Sf_n - g^{'}\right)\right\|^2 \\
&\leq (1-\kappa_n)^2\left\|Sf_n - g^{'}\right\|^2 + 2\kappa_n\left\langle\phi(g_n) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1-\kappa_n)^2\left\|f_n - g^{'}\right\|^2 + 2\kappa_n\left\langle\phi(g_n) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1-\kappa_n)^2\,[\|g_n - g^{'}\|^2 + 3Q_2\kappa_n\frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| - \theta_n(1-\theta_n)\|a_n - b_n\|^2 \\
&\quad - \eta_n(1-\eta_n)\|c_n - Sc_n\| - \beta_n\omega_n(1-\omega_n)\|d_n - Sd_n\|^2 - \beta_n(1-\beta_n)\|d_n - e_n\|^2] \\
&\quad + 2\kappa_n\left\langle\phi(g_n) - g^{'}, g_{n+1} - g^{'}\right\rangle.
\end{aligned} \tag{2.18}
$$

Also,

$$
\begin{aligned}
\|c_n - g^{'}\|^2 &= \|\theta_n a_n + (1-\theta_n)b_n - g^{'}\|^2 \\
&= \|\theta_n a_n + (1-\theta_n)b_n - \theta_n g^{'} + \theta_n g^{'} - g^{'}\|^2 \\
&= \|\theta_n(a_n - g^{'}) + (1-\theta_n)(b_n - g^{'})\|^2 \\
&= \theta_n\|a_n - g^{'}\|^2 + (1-\theta_n)\|b_n - g^{'}\|^2 - \theta_n(1-\theta_n)\|a_n - b_n\|^2 \\
&\leq \theta_n\|a_n - g^{'}\|^2 + (1-\theta_n)[\|a_n - g^{'}\|^2 - \frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2}] - \theta_n(1-\theta_n)\|a_n - b_n\|^2 \\
&\leq \|a_n - g^{'}\|^2 - (1-\theta_n)\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} - \theta_n(1-\theta_n)\|a_n - b_n\|^2.
\end{aligned} \tag{2.19}
$$

Now, by (2.18) and (2.19), we get

$$
\begin{aligned}
\left\|g_{n+1} - g^{'}\right\|^2 &= \left\|\kappa_n\left(\phi(g_n) - g^{'}\right) + (1-\kappa_n)Sf_n - g^{'}\right\|^2 \\
&= \left\|\kappa_n\left(\phi(g_n) - g^{'}\right) + (1-\kappa_n)\left(Sf_n - g^{'}\right)\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\kappa_n)^2 \left\|S f_n - g^{'}\right\|^2 + 2\kappa_n \left\langle \phi\left(g_n\right) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1-\kappa_n)^2 \left\|f_n - g^{'}\right\|^2 + 2\kappa_n \left\langle \phi\left(g_n\right) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1-\kappa_n)^2 \left\|c_n - g^{'}\right\|^2 + 2\kappa_n \left\langle \phi\left(g_n\right) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1-\kappa_n)^2 \left[\|a_n - g^{'}\|^2 - (1-\theta_n)\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} - \theta_n(1-\theta_n)\|a_n - b_n\|^2\right] \\
&+ 2\kappa_n \left\langle \phi\left(g_n\right) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1-\kappa_n)^2 \Bigg[\|g_n - g^{'}\|^2 + 3Q_2\kappa_n \frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| \\
&- (1-\theta_n)\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} - \theta_n(1-\theta_n)\|a_n - b_n\|^2\Bigg] \\
&+ 2\kappa_n \left\langle \phi\left(g_n\right) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1-\kappa_n)^2 \Bigg[\|g_n - g^{'}\|^2 + 3Q_2\kappa_n \frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| \\
&- (1-\theta_n)\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} - \theta_n(1-\theta_n)\|a_n - b_n\|^2\Bigg] \\
&+ 2\kappa_n \left\langle \phi(g_n) + \phi(g^{'}) - \phi(g^{'}) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1-\kappa_n)^2 \Bigg[\|g_n - g^{'}\|^2 + 3Q_2\kappa_n \frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| \\
&- (1-\theta_n)\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} - \theta_n(1-\theta_n)\|a_n - b_n\|^2\Bigg] \\
&+ 2\kappa_n \left\langle \phi(g_n) - \phi(g^{'}), g_{n+1} - g^{'}\right\rangle + 2\kappa_n \left\langle \phi(g^{'}) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1-\kappa_n)^2 \Bigg[\|g_n - g^{'}\|^2 + 3Q_2\kappa_n \frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| \\
&- (1-\theta_n)\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} - \theta_n(1-\theta_n)\|a_n - b_n\|^2\Bigg] \\
&+ 2\kappa_n\mu \left\langle g_n - g^{'}, g_{n+1} - g^{'}\right\rangle + 2\kappa_n \left\langle \phi(g^{'}) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1-\kappa_n)^2 \Bigg[\|g_n - g^{'}\|^2 + 3Q_2\kappa_n \frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| \\
&- (1-\theta_n)\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} - \theta_n(1-\theta_n)\|a_n - b_n\|^2\Bigg] \\
&+ 2\kappa_n\mu[\|g_n - g^{'}\|\|g_{n+1} - g^{'}\|] + 2\kappa_n \left\langle \phi(g^{'}) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1-\kappa_n)^2 \|g_n - g^{'}\|^2 + (1-\kappa_n)^2 \Bigg[3Q_2\kappa_n \frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| \\
&- (1-\theta_n)\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} - \theta_n(1-\theta_n)\|a_n - b_n\|^2\Bigg] \\
&+ \kappa_n\mu[\|g_n - g^{'}\|^2 + \|g_{n+1} - g^{'}\|^2] + 2\kappa_n \left\langle \phi(g^{'}) - g^{'}, g_{n+1} - g^{'}\right\rangle \\
&\leq (1 - 2\kappa_n + \kappa_n^2 + \kappa_n\mu)\|g_n - g^{'}\|^2 + (1-\kappa_n)^2 \Bigg[3Q_2\kappa_n \frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\|
\end{aligned}
$$

$$- (1-\theta_n)\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2+\|H(a_n)\|^2} - \theta_n(1-\theta_n)\|a_n-b_n\|^2\Big]$$
$$+ \kappa_n\mu\|g_{n+1}-g^{'}\|^2 + 2\kappa_n\left\langle \phi(g^{'})-g^{'}, g_{n+1}-g^{'}\right\rangle. \tag{2.20}$$

Finally,

$$\begin{aligned}
\left\|g_{n+1}-g^{'}\right\|^2 \leq & \frac{(1-2\kappa_n+\kappa_n^2+\kappa_n\mu)}{1-\kappa_n\mu}\|g_n-g^{'}\|^2 + \frac{(1-\kappa_n)^2}{1-\kappa_n\mu}\Big[3Q_2\kappa_n\frac{\delta_n}{\kappa_n}\|g_n-g_{n-1}\| \\
& - (1-\theta_n)\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2+\|H(a_n)\|^2} - \theta_n(1-\theta_n)\|a_n-b_n\|^2\Big] \\
& + \frac{2\kappa_n}{1-\kappa_n\mu}\left\langle \phi(g^{'})-g^{'}, g_{n+1}-g^{'}\right\rangle \\
& \leq (1-\frac{2\kappa_n(1-\mu)}{1-\kappa_n\mu})\|g_n-g^{'}\|^2 + \frac{(1-\kappa_n)^2\,(1-\mu)}{1-\kappa_n\mu}\Big[\frac{3Q_2\kappa_n}{1-\mu}\frac{\delta_n}{\kappa_n}\|g_n-g_{n-1}\| \\
& - \frac{(1-\theta_n)}{(1-\mu)}\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2+\|H(a_n)\|^2} - \frac{\theta_n(1-\theta_n)}{(1-\mu)}\|a_n-b_n\|^2\Big] \\
& + \frac{2\kappa_n(1-\mu)}{(1-\kappa_n\mu)(1-\mu)}\left\langle \phi(g^{'})-g^{'}, g_{n+1}-g^{'}\right\rangle \qquad (2.21) \\
= & \Big(1-\frac{2\kappa_n(1-\mu)}{1-\kappa_n\mu}\Big)\|g_n-r^{'}\|^2 + \frac{2\kappa_n(1-\mu)}{1-\kappa_n\mu}\Big\{\frac{3Q_2(1-\kappa_n)^2}{2(1-\mu)}\frac{\delta_n}{\kappa_n}\|g_n-g_{n-1}\| \\
& + \frac{1}{(1-\mu)}\langle\phi(r^{'})-r^{'}, g_{n+1}-r^{'}\rangle\Big\} - \frac{(1-\kappa_n)^2\theta_n(1-\theta_n)}{1-\kappa_n\mu}\|a_n-b_n\|^2 \qquad (2.22) \\
& - \frac{(1-\kappa_n)^2(1-\theta_n)}{(1-\kappa_n\mu)}\frac{(4-\varrho_n)\varrho_n h^2(a_n)}{\|G(a_n)\|^2+\|H(a_n)\|^2}. \qquad (2.23)
\end{aligned}$$

Lemma 2.3. *Under the assumptions of Lemma 2.2, we have the following:*

$$\begin{aligned}
\||g_{n+1}-g^{'}\|^2 \leq & (1-\kappa_n+\kappa_n\mu)\|g_n-g^{'}\|^2 + \kappa_n\|\phi(g^{'})-g^{'}\|^2 + (1-\kappa_n)\Big[3Q_2\delta_n\|g_n-g_{n-1}\|) \\
& - \|b_n-a_n\|^2 + 2Q_4\|L^*(I-J_{\chi_2}^{P_2})La_n\|\Big].
\end{aligned}$$

Proof. If $g^{'}\in\Pi$, then we have

$$\|a_n-\gamma_n L^*(I-J_{\chi_2}^{P_2})La_n-g^{'}\|^2 \leq \|a_n-g^{'}\|^2.$$

As $J_{\chi_1}^{P_1}$ is firmly nonexpansive, using Lemma 1.2, we get that

$$\begin{aligned}
\|b_n-g^{'}\|^2 = & \|J_{\chi_1}^{P_1}(I-\gamma_n L^*(I-J_{\chi_2}^{P_2})La_n)-g^{'}\|^2 \\
\leq & \langle b_n-g^{'}, a_n-\gamma_n L^*(I-J_{\chi_2}^{P_2})La_n-g^{'}\rangle \\
= & \frac{1}{2}(\|b_n-g^{'}\|^2 + \|a_n-\gamma_n L^*(I-J_{\chi_2}^{P_2})La_n-\underline{p}^*\|^2 - \|b_n-a_n \\
& + \gamma_n L^*(I-J_{\chi_2}^{P_2})La_n\|^2)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}(\|b_n - p\|^2 + \|a_n - g'\|^2 - (\|b_n - a_n + \gamma_n L^*(I - J_{\chi_2}^{P_2})La_n\|^2))\\
&= \frac{1}{2}(\|b_n - p\|^2 + \|a_n - \underline{p}^*\|^2 - (\|b_n - a_n\|^2 + \gamma_n^2\|L^*(I - J_{\chi_2}^{P_2})La_n\|^2\\
&- 2\gamma_n\langle a_n - b_n, L^*(I - J_{\chi_2}^{P_2})La_n\rangle))\\
&\leq \frac{1}{2}(\|b_n - \underline{p}^*\|^2 + \|a_n - g'\|^2 - \|b_n - a_n\|^2 - \gamma_n^2\|L^*(I - J_{\chi_2}^{P_2})La_n\|^2\\
&+ 2\gamma_n\|a_n - b_n\|\|L^*(I - J_{\chi_2}^{P_2})La_n\|)\\
&\leq \frac{1}{2}(\|b_n - p\|^2 + \|a_n - g'\|^2 - \|b_n - a_n\|^2 + 2\gamma_n\|a_n - b_n\|\|L^*(I - J_{\chi_2}^{P_2})La_n\|).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|b_n - g'\|^2 &\leq \|a_n - g'\|^2 - \|b_n - a_n\|^2 + 2\gamma_n\|a_n - b_n\|\|L^*(I - J_{\chi_2}^{P_2})La_n\|\\
&\leq \|a_n - g'\|^2 - \|b_n - a_n\|^2 + 2Q_4\|L^*(I - J_{\chi_2}^{P_2})La_n\|,
\end{aligned} \tag{2.24}
$$

where $Q_4 = \sup_{n\in N}\{\gamma_n\|a_n - b_n\|\}$. From (2.24), we obtain that

$$
\begin{aligned}
\left\|g_{n+1} - g'\right\|^2 &= \left\|\kappa_n\left(\phi(g_n) - \phi(g')\right) + \kappa_n\left(\phi(g') - g'\right) + (1-\kappa_n)\left(Sf_n - g'\right)\right\|^2\\
&\leq \kappa_n\|\phi(g_n) - \phi(g')\|^2 + \kappa_n\|\phi(g') - g'\|^2 + (1-\kappa_n)\|Sf_n - g'\|^2\\
&\leq \kappa_n\mu\|g_n - g'\|^2 + \kappa_n\|\phi(g') - g'\|^2 + (1-\kappa_n)\|f_n - g'\|^2\\
&\leq \kappa_n\mu\|g_n - g'\|^2 + \kappa_n\|\phi(g') - g'\|^2 + (1-\kappa_n)\|b_n - g'\|^2\\
&\leq \kappa_n\mu\|g_n - g'\|^2 + \kappa_n\|\phi(g') - g'\|^2 + (1-\kappa_n)\Big[\|a_n - g'\|^2 - \|b_n - a_n\|^2\\
&+ 2Q_4\|L^*(I - J_{\chi_2}^{P_2})La_n\|\Big]\\
&\leq \kappa_n\mu\|g_n - g'\|^2 + \kappa_n\|\phi(g') - g'\|^2 + (1-\kappa_n)\Big[(\|g_n - g'\|^2 + 3Q_2\delta_n\|g_n - g_{n-1}\|)\\
&- \|b_n - a_n\|^2 + 2Q_4\|L^*(I - J_{\chi_2}^{P_2})La_n\|\Big]\\
&= (1 - \kappa_n + \kappa_n\mu)\|g_n - g'\|^2 + \kappa_n\|\phi(g') - g'\|^2 + (1-\kappa_n)\Big[3Q_2\delta_n\|g_n - g_{n-1}\|)\\
&- \|b_n - a_n\|^2 + 2Q_4\|L^*(I - J_{\chi_2}^{P_2})La_n\|\Big].
\end{aligned} \tag{2.25}
$$

The proof of the main result is now given as the proof of the following lemma.

Lemma 2.4. *Under the assumptions of Theorem* 2.1*, the sequence* $\{g_n\}$ *given by Algorithm* 6 *converges to* $g' \in \Pi$*, where* $g' = P_\Pi o\phi(g')$.

Proof. Assume that $r' = P_\Pi o\phi(r')$. Using Lemma 2.2, we get that

$$
\begin{aligned}
\|g_{n+1} - r'\|^2 \leq &\Big(1 - \frac{2\kappa_n(1-\mu)}{1-\kappa_n\mu}\Big)\|g_n - r'\|^2 + \frac{2\kappa_n(1-\mu)}{1-\kappa_n\mu}\Big\{\frac{3Q_2(1-\kappa_n)^2}{2(1-\mu)}\frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\|\\
&+ \frac{1}{(1-\mu)}\langle\phi(r') - r', g_{n+1} - r'\rangle\Big\} - \frac{(1-\kappa_n)^2\theta_n(1-\theta_n)}{1-\kappa_n\mu}\|a_n - b_n\|^2.
\end{aligned} \tag{2.26}
$$

We now show that $\{\|g_n - r'\|\} \to 0$ as $n \to \infty$. Set $\bar{g}_n = \|g_n - r'\|$ and $\bar{h}_n = \left\{\frac{3Q_2(1-\kappa_n)^2}{2(1-\mu)}\frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\| + \frac{1}{(1-\mu)}\langle \phi(r') - r', g_{n+1} - r'\rangle\right\}$ in Lemma 1.5. Now, we prove the following:

$$\limsup_{i\to\infty}\langle \phi(r') - r', g_{n_i+1} - r'\rangle \leq 0,$$

for each subsequence $\{\|g_{n_i} - r'\|\}$ of $\{\|g_n - r'\|\}$, which satisfies

$$\liminf_{i\to\infty}(\|g_{n_i+1} - r'\| - \|g_{n_i} - r'\|) \geq 0. \tag{2.27}$$

Let $\{\|g_{n_i} - r'\|\}$ be a subsequence of the sequence $\{\|g_n - r'\|\}$ with

$$\liminf_{i\to\infty}(\|g_{n_i+1} - r'\| - \|g_{n_i} - r'\|) \geq 0. \tag{2.28}$$

Using (2.26), we get

$$\begin{aligned}\frac{(1-\kappa_{n_i})^2\theta_{n_i}(1-\theta_{n_i})}{(1-\kappa_{n_i}\mu)}\|a_{n_i} - b_{n_i}\|^2 &\leq \left(1 - \frac{2\kappa_{n_i}(1-\mu)}{(1-\kappa_{n_i}\mu)}\right)\|g_{n_i} - r'\|^2\\ &-\|g_{n_i+1} - r'\|^2 + 2\frac{\kappa_{n_i}(1-\mu)}{(1-\kappa_{n_i}\mu)}\left\{\frac{3Q_2(1-\kappa_n)^2}{2(1-\mu)}\frac{\delta_n}{\kappa_n}\|g_n - g_{n-1}\|\right.\\ &\left.+\frac{1}{(1-\mu)}\langle \phi(r') - r', g_{n+1} - r'\rangle\right\}.\end{aligned}$$

From (2.28) and $\lim \kappa_{n_i} = 0$, we have

$$\frac{(1-\kappa_{n_i})^2\theta_{n_i}(1-\theta_{n_i})}{(1-\kappa_{n_i}\mu)}\|a_{n_i} - b_{n_i}\|^2 \to 0,$$

which gives

$$\|a_{n_i} - b_{n_i}\|^2 \to 0 \text{ as } i \to \infty. \tag{2.29}$$

Similarly, we get

$$\frac{(4-\varrho_{n_i})\varrho_{n_i}h^2(a_{n_i})}{\|G(a_{n_i})\|^2 + \|H(a_{n_i})\|^2} \to 0 \text{ as } i \to \infty.$$

Since G and H are Lipschitzian mappings, by the given conditions on ϱ_{n_i} we get that

$$h^2(a_{n_i}) \to 0 \text{ as } i \to \infty,$$

and

$$\lim_{i\to\infty} h(a_{n_i}) = \lim_{i\to\infty}\frac{1}{2}\|(I - J_{\chi_2}^{P_2})La_{n_i}\| = 0. \tag{2.30}$$

Thus,

$$\|(I - J_{\chi_2}^{P_2})La_{n_i}\| \to 0 \quad \text{as} \quad i \to \infty. \tag{2.31}$$

So,

$$\|L^*(I - J_{\chi_2}^{P_2})La_{n_i}\| \leq \|L^*\|\|(I - J_{\chi_2}^{P_2})La_{n_i}\| = \|L\|\|(I - J_{\chi_2}^{P_2})La_{n_i}\| \to 0 \text{ as } i \to \infty. \tag{2.32}$$

Now, from Lemma 2.3, we get that

$$(1-\kappa_{n_i})\|b_{n_i}-a_{n_i}\|^2 \leq (1-\kappa_{n_i}+\kappa_{n_i}\mu)\|g_{n_i}-g'\|^2-\|g_{n_i+1}-g'\|^2+\kappa_{n_i}\|\phi(g')-g'\|^2 \\ +3Q_2(1-\kappa_{n_i})\kappa_{n_i}\frac{\theta_{n_i}}{\kappa_{n_i}}\|g_{n_i}-g_{n_i-1}\|+2Q_4(1-\kappa_{n_i})\|L^*(I-J_{\chi_2}^{P_2})La_{n_i}\|.$$

Using (2.27), (2.32), Remark 2.1, and $\lim_{i\to\infty}\kappa_{n_i}=0$, we have

$$\|b_{n_i}-a_{n_i}\| \to 0 \text{ as } i\to\infty. \tag{2.33}$$

Again, by the Remark 2.1, we have

$$\|a_{n_i}-g_{n_i}\| = \delta_{n_i}\|g_{n_i}-g_{n_i-1}\| \to 0 \text{ as } i\to\infty. \tag{2.34}$$

From (2.33) and (2.34), we obtain that

$$\|b_{n_i}-g_{n_i}\| \to 0 \text{ as } i\to\infty. \tag{2.35}$$

By (2.25), we have

$$(1-\kappa_{n_i})\kappa_{n_i}\mu\|Sf_{n_i}-g_{n_i}\|^2 \leq (1-\kappa_{n_i}+\kappa_{n_i}\mu)\|g_{n_i}-g'\|^2-\|g_{n_i+1}-g'\|^2+\kappa_{n_i}\|\phi(g')-g'\|^2 \\ +3Q_2(1-\kappa_{n_i})\kappa_{n_i}\frac{\theta_{n_i}}{\kappa_{n_i}}\|g_{n_i}-g_{n_i-1}\|+2Q_4(1-\kappa_{n_i})\|L^*(I-J_{\chi_2}^{P_2})La_{n_i}\|.$$

This gives

$$\|Sf_{n_i}-g_{n_i}\| \to 0 \text{ as } i\to\infty. \tag{2.36}$$

Similarly, we have

$$\|f_{n_i}-g_{n_i}\| \to 0 \text{ as } i\to\infty. \tag{2.37}$$

Note that

$$\|c_{n_i}-b_{n_i}\| = \theta_{n_i}\|a_{n_i}-b_{n_i}\| \to 0, \text{ as } i\to\infty. \tag{2.38}$$

Thus,

$$\|c_{n_i}-g_{n_i}\| \leq \|c_{n_i}-b_{n_i}\|+\|b_{n_i}-g_{n_i}\| \to 0, \text{ as } i\to\infty. \tag{2.39}$$

Similarly, we have

$$\|d_{n_i}-g_{n_i}\| \to 0, \quad \|e_{n_i}-g_{n_i}\| \to 0 \quad \text{and} \|d_{n_i}-c_{n_i}\| \to 0, \text{as } i\to\infty. \tag{2.40}$$

Using (2.36) and $\lim_{i\to\infty}\kappa_{n_i}=0$, we obtain that

$$\|g_{n_i+1}-g_{n_i}\| \leq \kappa_{n_i}\|\phi(g_{n_i})-g_{n_i}\|+(1-\kappa_{n_i})\|Sf_{n_i}-g_{n_i}\| \to 0, \quad \text{as } i\to\infty. \tag{2.41}$$

As $\omega(g_n)\subset\Pi$ and $\{g_n\}$ is bounded, $\omega(g_n)\neq\emptyset$. If $r'\in\omega(g_n)$, then there is a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ such that $g_{n_i}\rightharpoonup r'$ as $i\to\infty$.

We now show that $\omega(g_n)\subset\Pi$. For this, let $r'\in\omega(g_n)$. As h is lower semi-continuous, using (2.30), we have

$$0\leq h(r')\leq \lim_{i\to\infty}h(a_{n_i})=\lim_{n\to\infty}h(a_n)=0,$$

which implies that

$$h(r') = \frac{1}{2}\|(I - J_{\chi_2}^{P_2})Lr^{'}\|^2 = 0.$$

From Remark 1.1, we get

$$Lr' \in P_2^{-1}(0) \quad \text{that is} \quad 0 \in P_2(Lr'). \tag{2.42}$$

Now, $b_{n_i} = J_{\chi_1}^{P_1}(a_{n_i} - \gamma_{n_i}L^*(I - J_{\chi_2}^{P_2})La_{n_i})$ implies that $a_{n_i} - \gamma_{n_i}L^*(I - J_{\chi_2}^{P_2})La_{n_i} \in b_{n_i} + \chi_1 P_1(b_{n_i})$, that is,

$$\frac{(a_{n_i} - b_{n_i}) - \gamma_{n_i}L^*(I - J_{\chi_2}^{P_2})La_{n_i}}{\chi_1} \in P_1(b_{n_i}). \tag{2.43}$$

Taking the limit as $i \to \infty$ in (2.43), the using (2.32), (2.33), and the fact that graph of maximal monotone map is weakly strongly closed, we get $0 \in P_1(r')$, and by (2.42), we have $r' \in \Pi$.

Finally, we show that $r' \in \mathcal{F}(S)$. Using (2.35) and (2.40), we get $b_{n_i} \rightharpoonup r'$ and $e_{n_i} \rightharpoonup r'$. From, (2.36) and the demiclosedness principal, we get that $r' \in \mathcal{F}(S)$. So, $\omega(g_n) \subset \Pi$. Moreover, by (2.35) and (2.40), we have $\omega\{g_n\} = \omega\{c_n\}$. As $\{g_{n_i}\}$ is bounded, we have a subsequence $\{g_{k_{i_n}}\}$ of $\{g_{n_i}\}$ with $g_{k_{i_n}} \rightharpoonup r''$ and

$$\begin{aligned}\lim_{i\to\infty}\langle \phi(r^{'}) - r^{'}, g_{k_{i_n}} - r^{'}\rangle &= \limsup_{i\to\infty}\langle \phi(r^{'}) - r^{'}, g_{n_i} - r^{'}\rangle \\ &= \limsup_{i\to\infty}\langle \phi(r^{'}) - r^{'}, b_{n_i} - r^{'}\rangle.\end{aligned}$$

Now, $r^{'} = P_{\Pi} o \phi(r^{'})$ gives

$$\begin{aligned}\limsup_{i\to\infty}\langle \phi(r^{'}) - r^{'}, g_{n_i} - r^{'}\rangle &= \lim_{i\to\infty}\langle \phi(r^{'}) - r^{'}, g_{k_{i_n}} - r^{'}\rangle \\ &= \langle \phi(r^{'}) - r^{'}, r'' - r^{'}\rangle \le 0.\end{aligned} \tag{2.44}$$

From (2.41) and (2.44), we have

$$\begin{aligned}\limsup_{i\to\infty}\langle \phi(r^{'}) - r^{'}, g_{n_i+1} - r^{'}\rangle &= \limsup_{i\to\infty}\langle \phi(r^{'}) - r^{'}, g_{n_i} - r^{'}\rangle \\ &= \langle \phi(r^{'}) - r^{'}, r'' - r^{'}\rangle \le 0.\end{aligned} \tag{2.45}$$

Using Lemma 1.5 and (2.45) with $\lim_{n\to\infty} \frac{\delta_n}{\kappa_n}\|g_n - g_{k-1}\| = 0$ and $\lim_{n\to\infty} \kappa_n = 0$, we obtain that $\lim_{n\to\infty} \|g_n - r^{'}\|^2 = 0$, and hence $\lim_{n\to\infty} \|g_n - r^{'}\| = 0$.

3. Applications

In this section, modifying our proposed iterative method, we approximate the solution of certain nonlinear problems such as the split feasibility fixed point and split minimization fixed point problems.

3.1. Split feasibility fixed point problem

Choose $L, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Y}_1$ and $\mathcal{Y}_2$ as defined in last section. The split feasibility problem is given by

$$\text{Find} \quad g_0 \in \mathcal{Y}_1 \quad \text{such that} \quad Lg_0 \in \mathcal{Y}_2. \tag{3.1}$$

The above problem has applications in various fields such as restoration, inpainting, and diagnosing of an image and radiation therapy [29–36].

The split feasibility fixed point problem (SFFPP) is given as

$$\text{Find a point} \quad g_0 \in \mathcal{Y}_1 \quad \text{such that} \quad Lg_0 \in \mathcal{Y}_2 \quad \text{and} \quad g_0 = Sg_0. \tag{3.2}$$

We shall denote the solution set of split feasibility fixed point problems by Ω_{SFFPP}.

Let us recall the following concepts:

An indicator function $\mathcal{I}_{\mathcal{Y}_1}$ is defined as follows:

$$\mathcal{I}_{\mathcal{Y}_1}(g) = \begin{cases} 0 & \text{if} \quad g \in \mathcal{Y}_1, \\ \infty & \text{if} \quad g \notin \mathcal{Y}_1. \end{cases}$$

A normal cone $\mathcal{N}_{\mathcal{Y}_1} a_0$ at $a_0 \in \mathcal{Y}_1$ is given by

$$\mathcal{N}_{\mathcal{Y}_1} a_0 = \{b \in \mathcal{Z} : \langle b, c - a_0 \rangle \leq 0, \forall c \in \mathcal{Y}_1\}.$$

It is known that $\mathcal{I}_{\mathcal{Y}_1}$ is lower semicontinuous proper and convex. Thus, the subdifferential $\partial \mathcal{I}_{\mathcal{Y}_1}$ of $\mathcal{I}_{\mathcal{Y}_1}$ is maximal monotone. Consider

$$J_{\chi}^{\partial \mathcal{I}_{\mathcal{Y}_1}}(g) = (I + \chi \partial \mathcal{I}_{\mathcal{Y}_1})^{-1} g, \quad \forall g \in \mathcal{Z}.$$

Note that, for any $g \in \mathcal{Y}_1$, we get that

$$\begin{aligned} \partial \mathcal{I}_{\mathcal{Y}_1}(g) =& \{b \in \mathcal{Z} : \mathcal{I}_{\mathcal{Y}_1} g + \langle b, a_0 - g \rangle \leq \mathcal{I}_{\mathcal{Y}_1} a_0 \quad \forall a_0 \in \mathcal{Z}\} \\ =& \{b \in \mathcal{Z} : \langle b, a_0 - g \rangle \leq 0 \quad \forall a_0 \in \mathcal{Y}_1\} \\ =& \mathcal{N}_{\mathcal{Y}_1} g. \end{aligned}$$

Also, for $\chi > 0$, we obtain that

$$\begin{aligned} a_0 = J_{\chi}^{\partial \mathcal{I}_{\mathcal{Y}_1}}(g) &\Leftrightarrow g \in a_0 + \chi \partial \mathcal{I}_{\mathcal{Y}_1} a_0 \\ &\Leftrightarrow g - a_0 \in \chi \partial \mathcal{I}_{\mathcal{Y}_1} a_0 \\ &\Leftrightarrow \langle g - a_0, b - a_0 \rangle \leq 0, \quad \forall b \in \mathcal{Y}_1 \\ &\Leftrightarrow a_0 = P_{\mathcal{Y}_1} g. \end{aligned}$$

We now modify our proposed algorithm to approximate the solution of SFFPP.

Algorithm 7: Algorithm for solving SFFPP

Define

$$\hat{\delta}_n = \begin{cases} \min\{\delta, \frac{k_n}{\|g_n - g_{n-1}\|}\}, & \text{if} \quad g_n \neq g_{n-1}, \\ \delta, & \text{otherwise.} \end{cases}$$

Find

$$a_n = g_n + \delta_n(g_n - g_{n-1}).$$

Evaluate

$$b_n = P_{\mathcal{Y}_1}(I - \gamma_n L^*(I - P_{\mathcal{Y}_2})L)a_n,$$

where

$$\gamma_n = \begin{cases} \frac{\varrho_n h(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} & \text{if } \|G(a_n)\|^2 + \|H(a_n)\|^2 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Generate a sequence $\{g_n\}$ as follows:

$$c_n = \theta_n a_n + (1 - \theta_n)b_n,$$

$$d_n = (1 - \eta_n)c_n + \eta_n S c_n,$$

$$e_n = S((1 - \omega_n)d_n + \omega_n S d_n),$$

$$f_n = S((1 - \beta_n)S d_n + \beta_n S e_n),$$

$$g_{n+1} = \kappa_n \phi(g_n) + (1 - \kappa_n)S f_n,$$

where

$$\begin{aligned} h(g) =& \frac{1}{2}\left\|(I - P_{\mathcal{Y}_2})Lg\right\|^2, \\ j(g) =& \frac{1}{2}\left\|(I - P_{\mathcal{Y}_1})g\right\|^2, \\ G(g) =& L^*(I - P_{\mathcal{Y}_2})Lg, \\ H(g) =& (I - P_{\mathcal{Y}_1})g. \end{aligned}$$

Theorem 3.1. *If S, ϕ are as given in last section and $\Omega_{SFFPP} \neq \emptyset$, then the sequence $\{g_n\}$ obtained from Algorithm 7 converges strongly to $g' \in \Omega_{SFFPP}$.*

Proof. The proof follows using similar arguments given to those in the proof of Theorem 2.1. □

3.2. Split minimization fixed point problem

Let $\chi > 0$ and $\psi : \mathcal{Z} \to R \cup \{\infty\}$ be a proper convex and lower semicontinuous. The proximal operator of ψ is given as:

$$prox_{\chi,\psi}(g) = \arg\min_{c \in \mathcal{Z}} \{\psi c + \frac{1}{2\chi}\|g - c\|^2\} \quad \forall g \quad \in \mathcal{Z}.$$

Note that,

$$prox_{\chi,\psi}(g) = (I + \chi\partial\psi)^{-1}(g) = J_\chi^{\partial\psi}(g), \tag{3.3}$$

where, $\partial\psi$ is the subdifferential of the mapping ψ, and is given by

$$\partial\psi(g) = \{c \in \mathcal{Z} : \psi g - \psi e \leq \langle c, g - e\rangle, \quad \forall \quad e \in \mathcal{Z}, \quad \text{for each} \quad g \in \mathcal{Z}\}.$$

Assume that $\psi_1 : \mathcal{Z}_1 \to R \cup \{\infty\}$ and $\psi_2 : \mathcal{Z}_2 \to R \cup \{\infty\}$ are lower semicontinuous and convex. The split minimization problem is stated as follows:

Find
$$g^{'} \in \mathcal{Z}_1 \quad \text{with} \quad g^{'} \in \arg\min_{g\in\mathcal{Z}_1} \psi_1 g \quad \text{and} \quad Lg^{'} = q^{'} \in \arg\min_{q\in\mathcal{Z}_2} \psi_2 q. \tag{3.4}$$

This problem has been used in regularization, image and signal processing, and inconsistent feasibility problems, (see for example, [37, 38]).

The split minimization problem fixed point problem (SMFPP) can be express as: $g^{'} \in \mathcal{Z}_1$ such that:
$$\text{Find} \quad g^{'} \in \arg\min_{g\in\mathcal{Z}_1} \psi_1 g \quad \text{with} \quad Lg^{'} = q^{'} \in \arg\min_{q\in\mathcal{Z}_2} \psi_2 q \quad \text{and} \quad g^{'} = Sg^{'}. \tag{3.5}$$

We denote the solution set of SMFPP (3.5) by Ω_{SMFPP}.

As $\partial\psi$ is maximal monotone and firmly nonexpansive, we set $\partial\psi_1 = P_1$ and $\partial\psi_2 = P_2$ in Theorem 2.1, modify our proposed algorithm to obtain Algorithm 8, and find an approximate solution of SMFPP.

Algorithm 8: Algorithm for solving SMFPP

Define
$$\hat{\delta}_n = \begin{cases} \min\{\delta, \frac{k_n}{\|g_n - g_{n-1}\|}\}, & \text{if} \quad g_n \neq g_{n-1}, \\ \delta, & \text{otherwise.} \end{cases}$$

Find
$$a_n = g_n + \delta_n(g_n - g_{n-1}).$$

Evaluate
$$b_n = prox_{\chi_1,\psi_1}(I - \gamma_n L^*(I - prox_{\chi_2,\psi_2})L)a_n,$$

where
$$\gamma_n = \begin{cases} \frac{\varrho_n h(a_n)}{\|G(a_n)\|^2 + \|H(a_n)\|^2} & \text{if } \|G(a_n)\|^2 + \|H(a_n)\|^2 \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Produce an iterative sequence $\{g_n\}$ as follows:
$$c_n = \theta_n a_n + (1 - \theta_n)b_n,$$
$$d_n = (1 - \eta_n)c_n + \eta_n S c_n,$$
$$e_n = S((1 - \omega_n)d_n + \omega_n S d_n),$$
$$f_n = S((1 - \beta_n)S d_n + \beta_n S e_n),$$
$$g_{n+1} = \kappa_n \phi(g_n) + (1 - \kappa_n)S f_n,$$

where
$$\begin{aligned} h(g) =& \frac{1}{2}\left\|(I - prox_{\chi_2,\psi_2})Lg\right\|^2, \\ j(g) =& \frac{1}{2}\left\|(I - prox_{\chi_1,\psi_1})g\right\|^2, \\ G(g) =& L^*(I - prox_{\chi_2,\psi_2})Lg, \\ H(g) =& (I - prox_{\chi_1,\psi_1})g. \end{aligned}$$

Theorem 3.2. *Suppose that S, ϕ are as given in the last section, ψ_1 and ψ_1 are as given above, and $\Omega_{SMFPP} \neq \emptyset$. Then, the sequence $\{g_n\}$ obtained from Algorithm 8 converges to $g' \in \Omega_{SFFPP}$.*

Proof. Following arguments similar to those given in the proof of the Theorem 2.1, the result follows. □

4. Numerical computations

We, in this section, compare the convergence rate of our proposed iterative algorithm with the some known comparable approaches. For this purpose, the codes are written in MATLAB R2018a. We compare Algorithm 6 with the Algorithms 2–5.

Example 4.1. *Assume that $\mathcal{Z}_1 = \mathcal{Z}_2 = \mathbb{R}^3$ and $\mathcal{Y}_1 = \{g \in \mathbb{R}^3 : \langle a, g\rangle \geq b\}$. Set $\theta_n = \frac{n}{n+5}, \varrho_n = 3 - \frac{1}{2n-1}, \omega_n = \frac{n^2}{n^2+4}, \kappa_n = \frac{1}{4n+3}, \chi_1 = \chi = \chi_2 = 0.5, \delta = 0.8, \eta_n = \frac{n+3}{2n+7}$, and $\alpha_n = \frac{1}{2n+5}$. Define $\phi(g) = \frac{g}{7}$ and $S(g) = \frac{g}{3}$. Further, take $\gamma = 0.0002$ in the Algorithms 2, 3, and 5, and choose $\sigma = 0.5$ and $P(g) = \frac{g}{4}$ in Algorithm 3. Define L, P_1, P_2 by*

$$P_1 = \begin{pmatrix} 7 & 0 & 0 \\ 5 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad L = \begin{pmatrix} 6 & 3 & 1 \\ 8 & 7 & 5 \\ 3 & 6 & 2 \end{pmatrix}.$$

Note that, the assumptions of Theorem 2.1 hold. In our numerical experiment, we set $b = -1$, $g = (8.5, -3.2, 1.1)$, and use different initial guesses, which are given in the following cases: ***Case-I:*** *$g_0 = (7, -2.3, 3.4)$ and $g_1 = (3.9, 1.6, 3.2)$;* ***Case-II:*** *$g_0 = (5.8, -2.2, 3.6)$, $g_1 = (3.9, 1.7, 2.5)$;* ***Case-III:*** *$g_0 = (3.7, -2.5, 5.4)$ and $g_1 = (4.9, 0.6, -1.5)$; and* ***Case-IV:*** *$g_0 = (22, -11, 32.6)$, $g_1 = (14, 7.7, -20)$. We used the stopping criteria given as: $\|g_{n+1} - g_n\| < 10^{-3}$. We now present the error graphs against the iteration numbers in each case.*

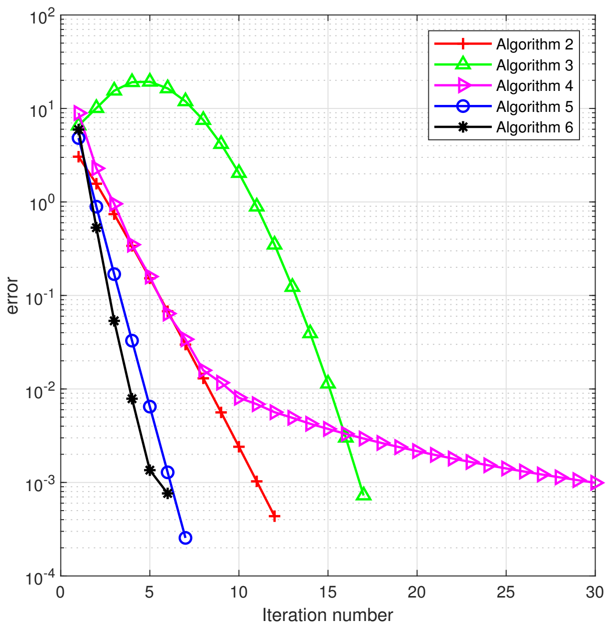

Figure 1. Case-I.

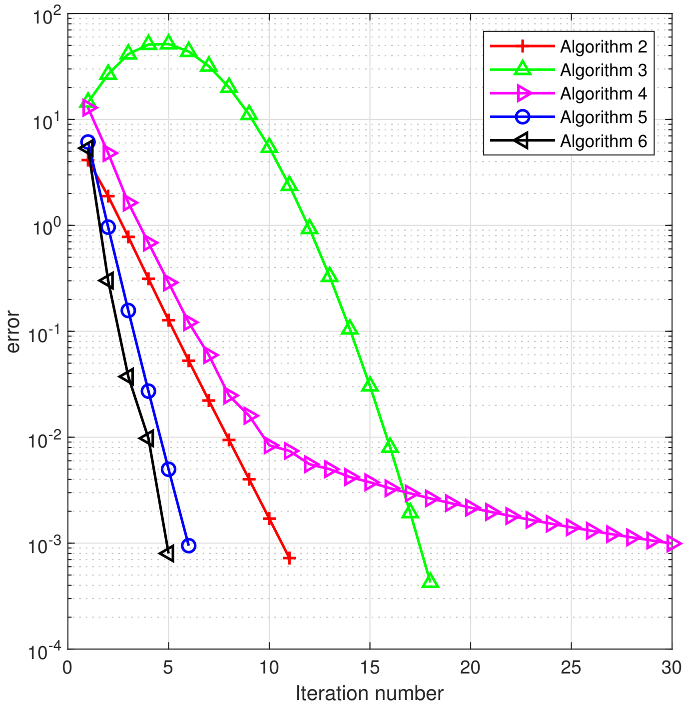

Figure 2. Case-II.

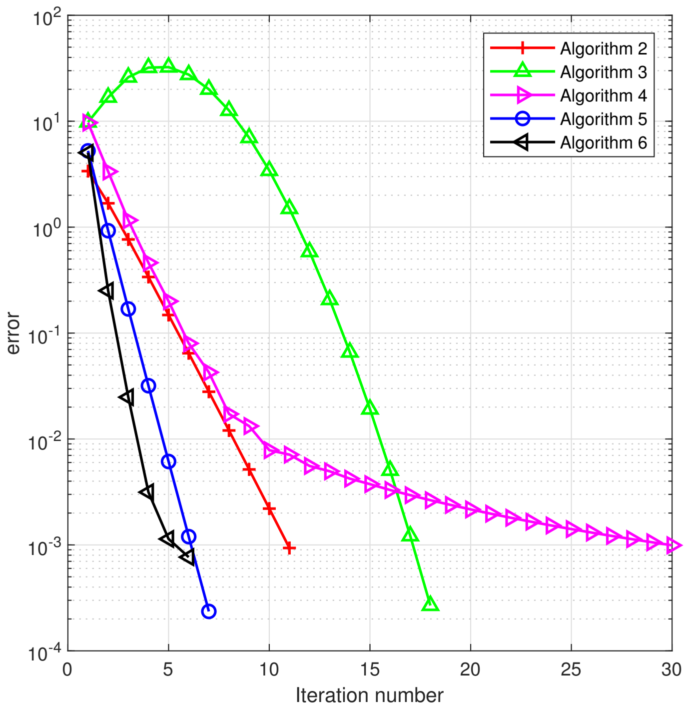

Figure 3. Case-III.

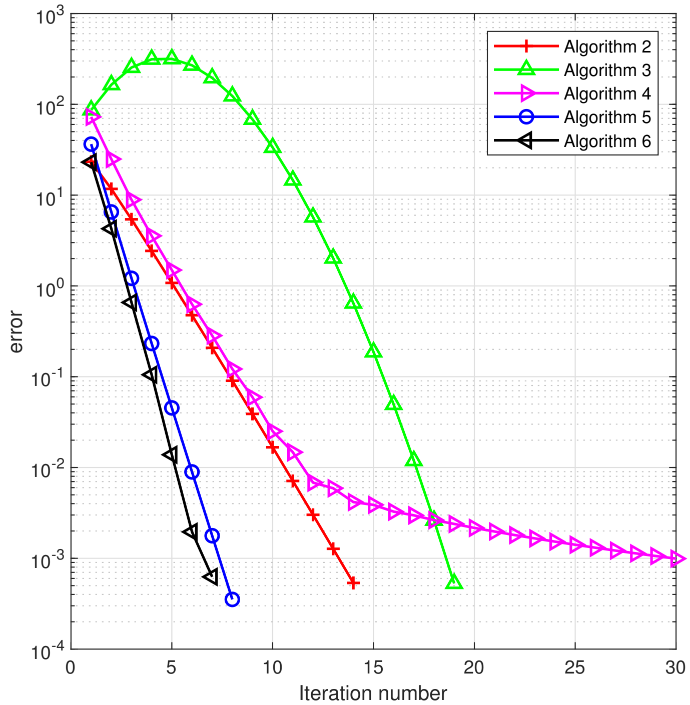

Figure 4. Case-IV.

Remark 4.1. *In the above example, Figures 1–4 illustrate the error analysis with the iteration numbers and conclude that for different choices of initial values, our proposed algorithm shows better performance in the sense of the rate of the convergence relative to other comparable algorithms.*

5. Application to signal processing

Fixed point iterative techniques play an important role in several signal processing tasks, such as, signal reconstruction, denoising, image recovery, and adaptive filtering. In many practical situations, the recovery of a clean signal from corrupted observations can be formulated as a solution of a fixed point equation associated with a nonlinear operator.

In this section, we investigate the applicability and numerical behavior of the proposed inertial viscosity iterative scheme for signal reconstruction problems and compare its performance with the Thakur scheme [4], M iteration [39], and D-iterative processes [40].

Let $\mathbb{R}^N$ denote the Euclidean signal space equipped with the norm $\|\cdot\|_2$. Consider the clean signal defined on the uniform grid $t_i \in [0, 1]$ by

$$s(t_i) = \sin(2\pi f_1 t_i) + \frac{1}{2}\sin(2\pi f_2 t_i), \quad i = 1, 2, \ldots, N, \tag{5.1}$$

where $f_1 = 5$ Hz and $f_2 = 15$ Hz are the frequencies of the harmonic components.

The observed noisy signal is generated by

$$y = s + \sigma\varepsilon, \tag{5.2}$$

where $\varepsilon \sim N(0, I)$ is a Gaussian random vector and $\sigma > 0$ controls the noise intensity.

To reconstruct the signal, we define the nonlinear operator $S : \mathbb{R}^N \to \mathbb{R}^N$ by

$$S(x) = \lambda y + (1 - \lambda)F(x), \quad \lambda \in (0, 1), \tag{5.3}$$

where F denotes a local smoothing operator implemented through a moving-average filter. The operator S combines fidelity of the observed data with local smoothing properties.
The reconstruction problem is therefore reduced to the fixed point equation

$$x^* = S(x^*). \tag{5.4}$$

In addition, we consider the contraction mapping $\phi : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\phi(x) = \rho x, \quad 0 < \rho < 1. \tag{5.5}$$

Starting from the initial approximation $g_0 = y$, take $g_0 = g_1$ and $J_{\chi_1}^{P_1} = J_{\chi_2}^{P_2} = I$ in our proposed inertial viscosity algorithm (Algorithm 6), and generate a sequence $\{g_n\}$ to approximate the solution.

For comparison purposes, we also apply the following iterative processes.

The Thakur iteration [4] is given by

$$\begin{cases} e_n = \beta_n S(g_n) + (1 - \beta_n)g_n, \\ f_n = S(\kappa_n e_n + (1 - \kappa_n)g_n), \\ g_{n+1} = S(f_n). \end{cases} \tag{5.6}$$

The M iteration [39] is defined by

$$\begin{cases} e_n = \kappa_n S(g_n) + (1 - \kappa_n)g_n, \\ f_n = S(e_n), \\ g_{n+1} = S(f_n). \end{cases} \tag{5.7}$$

The D iteration [40] is given by

$$\begin{cases} e_n = S(\beta_n S(g_n) + (1 - \beta_n)g_n), \\ f_n = S(\kappa_n S(e_n) + (1 - \kappa_n)S(g_n)), \\ g_{n+1} = S(f_n). \end{cases} \tag{5.8}$$

All iterative methods are implemented under identical computational conditions and initialized using the noisy observation. The control sequences are chosen as

$$\kappa_n = \frac{2n + 3}{4n + 5}, \quad \beta_n = \frac{3n + 1}{7n + 3}, \tag{5.9}$$

while the parameters associated with the proposed algorithm are selected by

$$\eta_n = \frac{2n + 1}{4n + 3}, \quad \omega_n = \frac{3n + 1}{5n + 2}, \quad \theta_n = \frac{2n + 3}{6n + 5}, \tag{5.10}$$

and

$$k_n = \frac{1}{(n+1)^2}. \tag{5.11}$$

The reconstruction quality is measured through the approximation error

$$E_n = \|g_n - s\|_2, \tag{5.12}$$

and the signal-to-noise ratio

$$\mathrm{SNR}_n = 20\log_{10}\left(\frac{\|s\|_2}{\|g_n - s\|_2}\right). \tag{5.13}$$

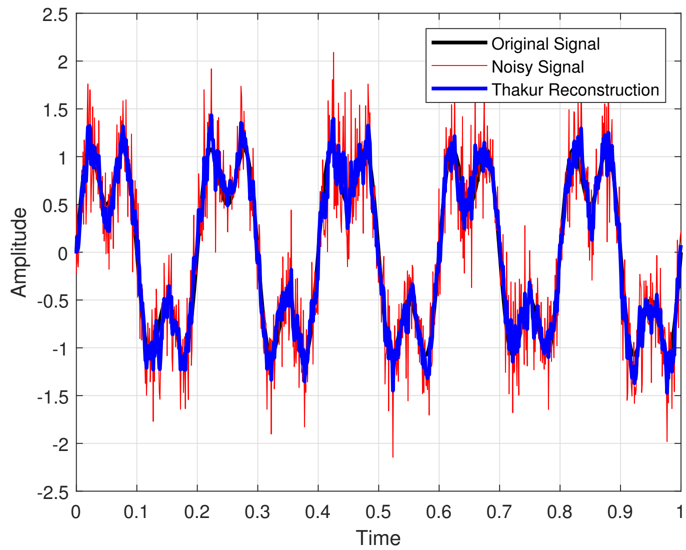

Figure 5. Signal reconstruction using Thakur iteration.

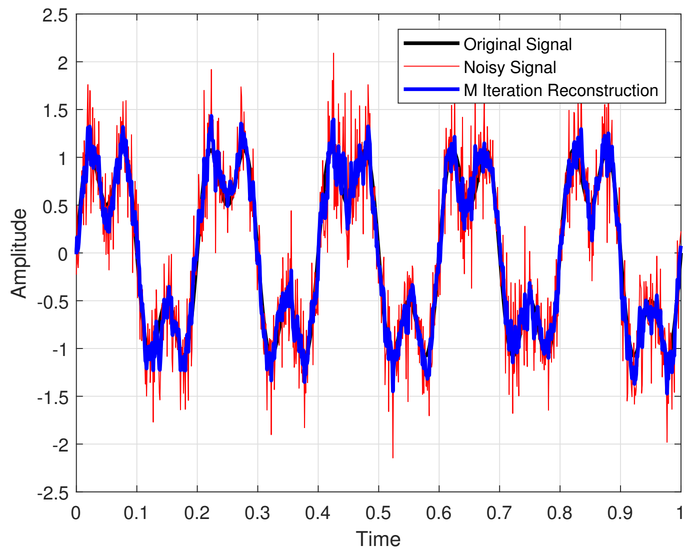

Figure 6. Signal reconstruction using M-iteration.

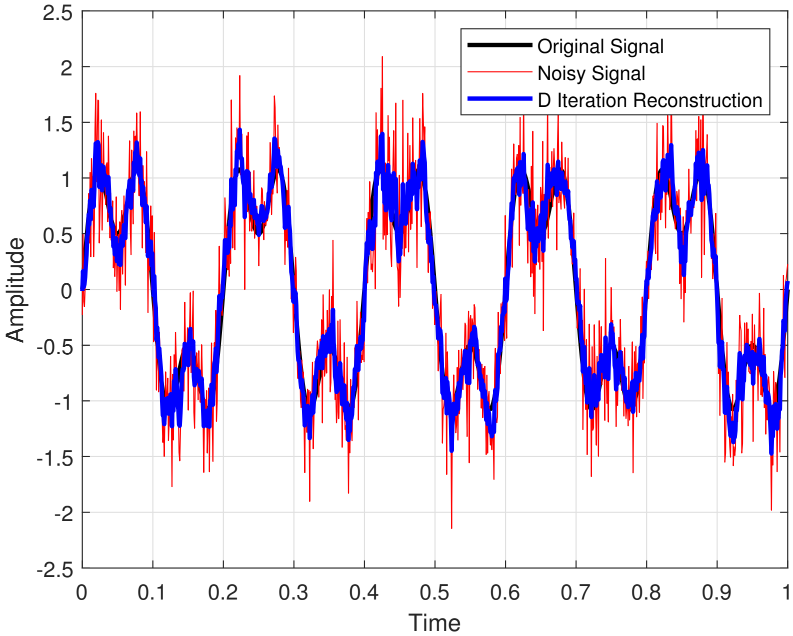

Figure 7. Signal reconstruction using D-iteration.

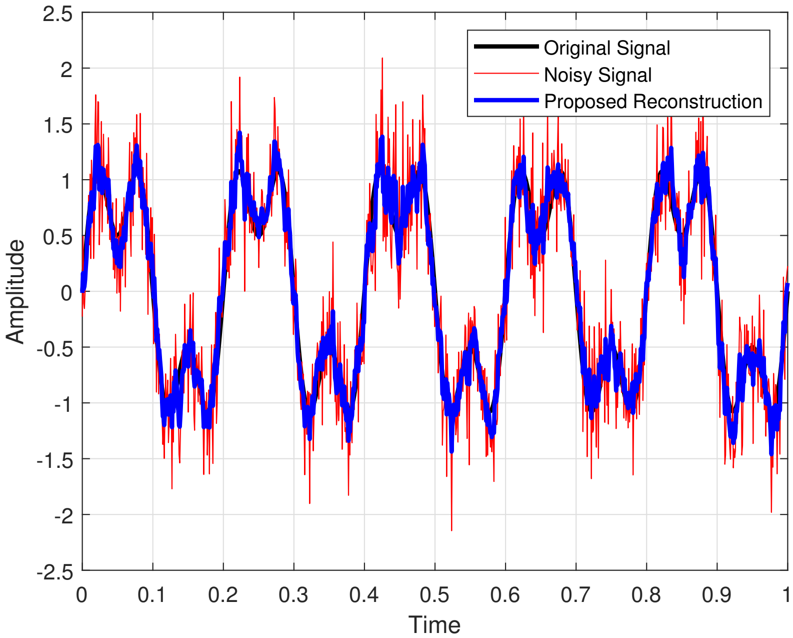

Figure 8. Signal reconstruction using proposed algorithm.

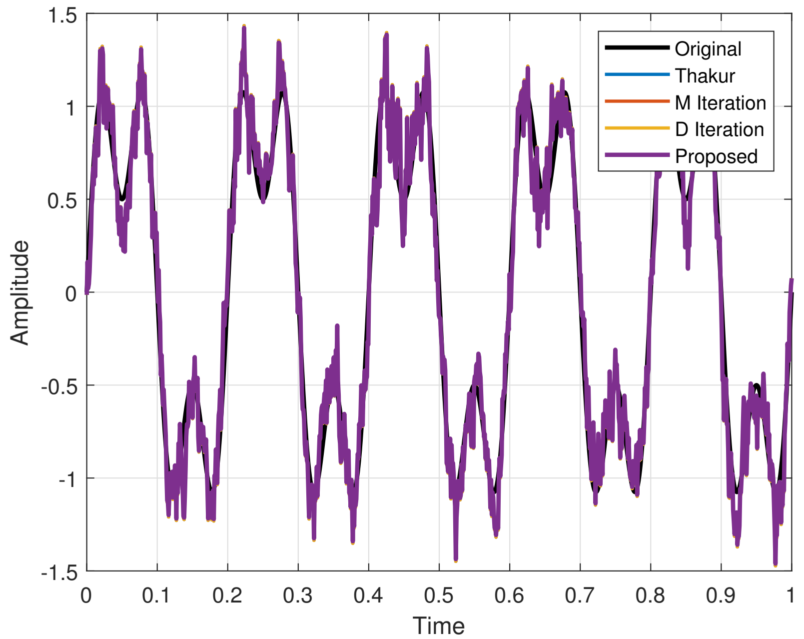

Figure 9. Comparison of reconstruction schemes.

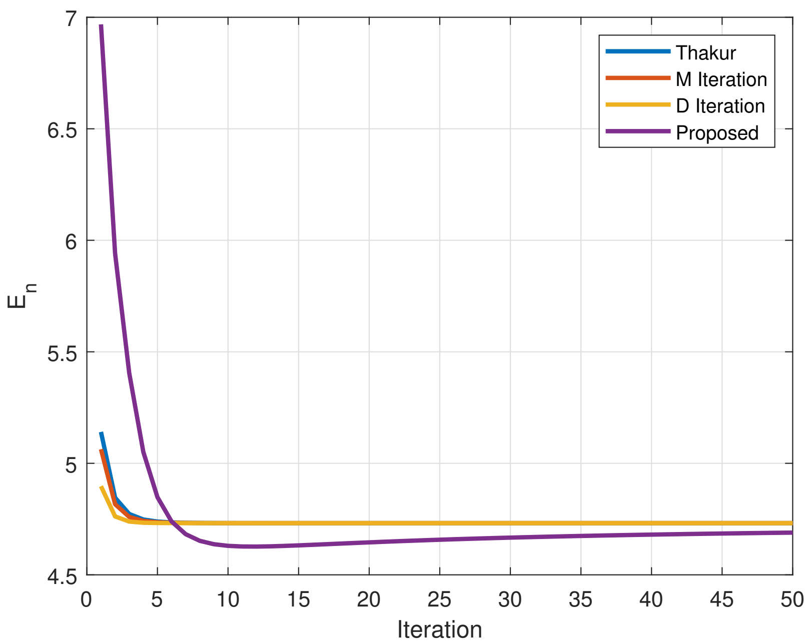

Figure 10. Comparison of approximation errors.

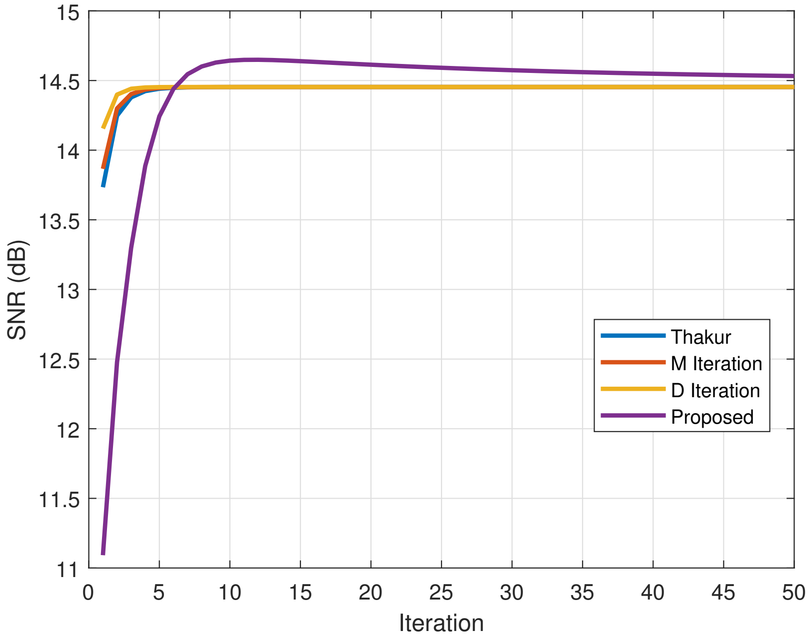

Figure 11. Comparison of signal-to-noise ratios.

Figure 5 illustrates the reconstruction produced by the Thakur iteration together with the original and noisy signals. The reconstructed signal follows the principal oscillatory structure; however, noticeable deviations from the original signal remain visible.

Figure 6 presents the reconstruction obtained through the M iteration. The method exhibits improved smoothing behavior compared to the Thakur process, although some fluctuations produced by the noise are still present in the recovered signal.

The reconstruction corresponding to the D iteration is displayed in Figure 7. The D process provides additional stabilization and yields a more accurate approximation of the clean signal compared to the previous methods.

Figure 8 shows the reconstruction generated by the proposed inertial viscosity algorithm. The recovered signal demonstrates a high level of agreement with the original waveform while effectively suppressing the noisy perturbations. The incorporation of the inertial and viscosity components improves both stability and convergence behavior.

Figure 9 compares the reconstructed signals obtained from all iterative schemes. It is observed that the proposed algorithm produces a reconstruction that is visually closer to the original signal than the competing methods.

The convergence behavior of the approximation errors is presented in Figure 10. Although all iterative methods converge, the proposed algorithm achieves a faster decay rate of the error sequence, indicating accelerated convergence toward the fixed point of the operator S.

Figure 11 illustrates the evolution of the signal-to-noise ratio. The proposed method attains larger SNR values in fewer iterations, demonstrating better reconstruction efficiency and improved numerical stability.

Table 1. Comparison of errors and SNR values.

Iter.	Error (Thakur)	Error(M)	Error(D)	Error(Proposed)	SNR (Thakur)	SNR (M)	SNR (D)	SNR (Proposed)
1	5.1408	5.0632	4.8974	6.9683	13.734	13.866	14.155	11.092
2	4.8458	4.8166	4.7619	5.9408	14.247	14.300	14.399	12.478
3	4.7728	4.7597	4.7388	5.4051	14.379	14.403	14.441	13.298
4	4.7482	4.7420	4.7337	5.0510	14.424	14.435	14.450	13.887
5	4.7388	4.7359	4.7325	4.8481	14.441	14.446	14.453	14.243
6	4.7350	4.7335	4.7321	4.7393	14.448	14.451	14.453	14.440
7	4.7333	4.7326	4.7320	4.6822	14.451	14.452	14.454	14.545
8	4.7326	4.7323	4.7320	4.6525	14.452	14.453	14.454	14.601
9	4.7323	4.7321	4.7320	4.6373	14.453	14.453	14.454	14.629
10	4.7321	4.7320	4.7320	4.6300	14.453	14.453	14.454	14.643
11	4.7320	4.7320	4.7320	4.6271	14.453	14.454	14.454	14.648
12	4.7320	4.7320	4.7320	4.6267	14.454	14.454	14.454	14.649
13	4.7320	4.7320	4.7320	4.6278	14.454	14.454	14.454	14.647
14	4.7320	4.7320	4.7320	4.6297	14.454	14.454	14.454	14.643
15	4.7320	4.7320	4.7320	4.6321	14.454	14.454	14.454	14.639
16	4.7320	4.7320	4.7320	4.6347	14.454	14.454	14.454	14.634
17	4.7320	4.7320	4.7320	4.6375	14.454	14.454	14.454	14.629
18	4.7320	4.7320	4.7320	4.6402	14.454	14.454	14.454	14.624
19	4.7320	4.7320	4.7320	4.6429	14.454	14.454	14.454	14.619
20	4.7320	4.7320	4.7320	4.6456	14.454	14.454	14.454	14.614
21	4.7320	4.7320	4.7320	4.6481	14.454	14.454	14.454	14.609
22	4.7320	4.7320	4.7320	4.6506	14.454	14.454	14.454	14.604
23	4.7320	4.7320	4.7320	4.6530	14.454	14.454	14.454	14.600
24	4.7320	4.7320	4.7320	4.6553	14.454	14.454	14.454	14.596
25	4.7320	4.7320	4.7320	4.6574	14.454	14.454	14.454	14.592
26	4.7320	4.7320	4.7320	4.6595	14.454	14.454	14.454	14.588
27	4.7320	4.7320	4.7320	4.6615	14.454	14.454	14.454	14.584
28	4.7320	4.7320	4.7320	4.6633	14.454	14.454	14.454	14.581
29	4.7320	4.7320	4.7320	4.6651	14.454	14.454	14.454	14.577
30	4.7320	4.7320	4.7320	4.6669	14.454	14.454	14.454	14.574
31	4.7320	4.7320	4.7320	4.6685	14.454	14.454	14.454	14.571
32	4.7320	4.7320	4.7320	4.6700	14.454	14.454	14.454	14.568
33	4.7320	4.7320	4.7320	4.6715	14.454	14.454	14.454	14.565
34	4.7320	4.7320	4.7320	4.6730	14.454	14.454	14.454	14.563
35	4.7320	4.7320	4.7320	4.6743	14.454	14.454	14.454	14.560
36	4.7320	4.7320	4.7320	4.6756	14.454	14.454	14.454	14.558
37	4.7320	4.7320	4.7320	4.6769	14.454	14.454	14.454	14.555
38	4.7320	4.7320	4.7320	4.6781	14.454	14.454	14.454	14.553
39	4.7320	4.7320	4.7320	4.6792	14.454	14.454	14.454	14.551
40	4.7320	4.7320	4.7320	4.6803	14.454	14.454	14.454	14.549
41	4.7320	4.7320	4.7320	4.6814	14.454	14.454	14.454	14.547
42	4.7320	4.7320	4.7320	4.6824	14.454	14.454	14.454	14.545
43	4.7320	4.7320	4.7320	4.6834	14.454	14.454	14.454	14.543
44	4.7320	4.7320	4.7320	4.6843	14.454	14.454	14.454	14.541
45	4.7320	4.7320	4.7320	4.6853	14.454	14.454	14.454	14.540
46	4.7320	4.7320	4.7320	4.6861	14.454	14.454	14.454	14.538
47	4.7320	4.7320	4.7320	4.6870	14.454	14.454	14.454	14.537
48	4.7320	4.7320	4.7320	4.6878	14.454	14.454	14.454	14.535
49	4.7320	4.7320	4.7320	4.6886	14.454	14.454	14.454	14.534
50	4.7320	4.7320	4.7320	4.6893	14.454	14.454	14.454	14.532

The numerical experiments from Table 1 indicate that the inclusion of inertial and viscosity effects significantly enhances the reconstruction process. In particular, the proposed algorithm combines rapid convergence with strong smoothing capability, which makes it effective for practical signal denoising applications.

6. Conclusions and future directions

In this research, we proposed an inertial viscosity iterative algorithm with adaptive step for solving split null fixed point problems involving nonexpansive mappings. The proposed method combines the advantages of inertial techniques and viscosity approximation methods to improve the convergence behavior of the iterative process. We proved the strong convergence of the sequence generated by the algorithm to a solution of the considered problem. As applications of our main results, we also obtained convergence results for split feasibility fixed point problems and split minimization fixed point problems. Moreover, the applicability of the proposed algorithm is demonstrated through a signal processing problem. A numerical example was presented to show the effectiveness and applicability of the proposed algorithm. In addition, graphical comparisons with several existing methods demonstrated that the proposed approach provides better convergence performance and computational efficiency. Overall, the results of this demonstrate that the proposed algorithm is reliable and effective for solving different classes of nonlinear split problems.

Author contributions

All authors have equal contribution for preparation of manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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