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Research article

Riemann–Liouville and Caputo local fractional operators in Yang's calculus with applications to linear fractional differential equations

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Abstract: This paper developed Riemann–Liouville and Caputo-type operators within Yang's local fractional calculus on the formal fractional-number set $\mathbb{R}^r$, where $0 < r \leq 1$. Here r is the local/fractal parameter associated with the underlying fractal geometry (for example, $r = \log 2/\log 3$ for the ternary Cantor set), and it is kept distinct from the independent operator order α. Starting from Yang's local fractional integral, derivative, and Laplace transform, we introduced left- and right-sided two-parameter operators of order (α, r). We established power rules, transform formulas, semigroup and composition properties, and the relation between the Riemann–Liouville and Caputo versions. A generalized local Mittag–Leffler function associated with the local gamma function was introduced, leading to distinct and mathematically consistent solution kernels for Riemann–Liouville and Caputo initial-value problems. Several linear examples were presented together with Cantor-supported graphical illustrations for different values of r and α, including the smooth classical limit $r = 1$. When $\alpha = 1$, the proposed construction reduces to Yang's basic local fractional operator.

Keywords: local fractional calculus; Riemann–Liouville local fractional derivative; Caputo local fractional derivative; Yang–Laplace transform; local Mittag–Leffler function; fractal set
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1. Introduction

Fractional calculus extends integer-order differentiation and integration, and is widely used in models with hereditary effects, anomalous transport, and long-time dependence [1–5]. The Riemann–Liouville (RL) and Caputo derivatives are two classical constructions. They share the same nonlocal power-law kernel but differ in the placement of the integer derivative and, consequently, in their natural initial data.

A different line of research, usually called Yang's local fractional calculus, was developed for nondifferentiable functions represented on fractal sets [6–8]. Its elementary derivative D^r is a local operator of order r, and its integral is taken with respect to the local fractional element $(d\vartheta)^r$. The associated Mittag–Leffler functions and Yang–Laplace transform are described in detail in [7]. More general nonlocal fractional derivatives and kernels, which should not be confused with the local derivative D^r, are surveyed in [9].

In Yang's notation, $\mathbb{R}^r$ is a formal fractional-number set used to encode quantities associated with a fractal geometry; it is not the Cartesian power of $\mathbb{R}$. The parameter $r \in (0, 1]$ is interpreted as the local/fractal order (or, in common examples, a fractal dimension). The limit $r = 1$ returns the ordinary real setting. In the present work, we introduce a second parameter, $\alpha > 0$, which is the order of the RL/Caputo-type history operator. This separation of roles is essential: r describes the local fractal geometry, whereas α controls the nonlocal memory kernel.

The elementary local derivative itself is pointwise. The memory effect studied here is introduced by the new local fractional integrals

$$\frac{1}{\Gamma^r(\alpha)\Gamma(1+r)}\int_a^{\vartheta}(\vartheta^r - s^r)^{\alpha-1}x(s)(ds)^r,$$

which aggregate the past values $x(s)$ for $a \leq s \leq \vartheta$. Thus, for $0 < \alpha < 1$, the present formulation combines a fractal local measure with an RL/Caputo-type history kernel. This interpretation is useful in applications, but the analysis below is carried out inside Yang's formal local fractional framework.

The present construction differs from the basic Yang calculus in two ways. First, the local order r and the RL/Caputo order α are independent. Second, instead of using only D^r and the order-r local integral, we build a family $({}_aI^{\alpha,r}, {}_aD^{\alpha,r}, {}_a^CD^{\alpha,r})$ and derive their transform and solution theory. Existing local-fractional studies mainly provide the elementary local operators, integral transforms, and equation-solving tools [6, 7, 10, 11]; our purpose is to formulate the RL and Caputo constructions as a consistent two-parameter extension and to make their different initial-value kernels explicit.

Recent work shows that Cantor-supported local fractional models remain actively used. Wang and Shi constructed nondifferentiable exact solutions of a local fractional modified Benjamin–Bona–Mahony equation on Cantor sets [12], Li and Wang treated a nonlinear local fractional heat-conduction equation on Cantor sets using Yang's nondifferentiable transformation [13], Wang studied a local fractional low-pass transmission-line model and explicitly displayed the dynamics on the ternary Cantor set with fractal dimension $\log 2/\log 3$ [14], and Du et al. presented exact solutions of a local fractional Korteweg-de Vries (KdV) equation with dual-power-law nonlinearity on Cantor sets [15]. These papers support the use of a Cantor mass coordinate in the graphical realization of local fractional solutions and motivate the Cantor-supported plots used in Section 6.

Numerical analysis for the classical RL/Caputo setting is developing rapidly. Recent examples include time–space two-grid methods for nonlinear nonlocal transport [16], robust alternating direction implicit (ADI) schemes on graded meshes for multiterm subdiffusion [17], fourth-order schemes for time-fractional Burgers-type equations [18], and discrete-maximum-principle-preserving finite-volume schemes for subdiffusion [19]. These studies concern classical nonlocal fractional derivatives rather than Yang's local calculus, but they motivate the numerical and structure-preserving questions posed for the present operators.

The main contributions are as follows:

- A two-parameter notation separates the local/fractal parameter r from the RL/Caputo order α;
- the local gamma function is retained in its original integral form, and its computational representation is derived from the Yang–Laplace transform;
- left- and right-sided RL and Caputo local fractional operators are introduced in a consistent two-parameter framework;
- transform, power, semigroup, inverse, and RL–Caputo relations are established under stated transformability and regularity assumptions;
- the local Mittag–Leffler transform is proved in detail, leading to distinct RL and Caputo initial-value formulas; and
- three representative linear problems are illustrated on Cantor-supported geometries and compared with the smooth classical limit $r = 1$.

The paper is organized as follows. Section 2 recalls the classical RL and Caputo operators. Section 3 summarizes Yang's local fractional tools and recalls the local gamma function used later. Section 4 develops the RL local operators and their transform theory. Section 5 treats the Caputo local operators. Applications and their Cantor-supported graphical illustrations are presented in Section 6, and Section 7 concludes the paper.

2. Classical fractional operators in $\mathbb{R}$

We briefly recall the classical definitions; see [2–4]. Let $x : [a, b] \to \mathbb{R}$ and let $\alpha > 0$. Define

$$n_\alpha = \begin{cases} \lceil \alpha \rceil, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}. \end{cases}$$

For noninteger α, the left and right RL fractional integrals are

$$_aI^\alpha x(\vartheta) = \frac{1}{\Gamma(\alpha)} \int_a^\vartheta (\vartheta - s)^{\alpha-1} x(s)\, ds, \tag{2.1}$$

$$I_b^\alpha x(\vartheta) = \frac{1}{\Gamma(\alpha)} \int_\vartheta^b (s - \vartheta)^{\alpha-1} x(s)\, ds. \tag{2.2}$$

The RL derivatives are

$$_aD^\alpha x = \frac{d^{n_\alpha}}{d\vartheta^{n_\alpha}} {}_aI^{n_\alpha-\alpha} x, \tag{2.3}$$

$$D_b^\alpha x = (-1)^{n_\alpha} \frac{d^{n_\alpha}}{d\vartheta^{n_\alpha}} I_b^{n_\alpha-\alpha} x, \tag{2.4}$$

and the Caputo derivatives are

$$ {}_a^C D^\alpha x = {}_a I^{n_\alpha - \alpha} \frac{d^{n_\alpha} x}{d\vartheta^{n_\alpha}}, \tag{2.5}$$

$$ {}^C D_b^\alpha x = I_b^{n_\alpha - \alpha} (-1)^{n_\alpha} \frac{d^{n_\alpha} x}{d\vartheta^{n_\alpha}}. \tag{2.6}$$

For $0 < \alpha < 1$, the distinction is already visible: the RL formulation naturally uses a fractional-integral trace, whereas the Caputo formulation uses $x(a)$ as the initial datum.

3. Yang local fractional preliminaries

3.1. Notation, limiting cases, and standing assumptions

The two orders that appear in the sequel play different roles and will never be identified implicitly. The parameter $r \in (0, 1]$ belongs to Yang's local fractional structure, whereas $\alpha > 0$ denotes the order of the newly introduced RL/Caputo-type operator. To avoid ambiguity, throughout the paper, we use

$$ n_\alpha = \begin{cases} \lceil \alpha \rceil, & \alpha \notin \mathbb{N}, \\ \alpha, & \alpha \in \mathbb{N}, \end{cases} \qquad D^{0,r} = \mathrm{Id}, \qquad D^{m,r} = \underbrace{D^r \circ \cdots \circ D^r}_{m \text{ times}}. $$

All transform identities are understood on a class of locally fractionally integrable functions for which the indicated Yang–Laplace transforms exist and for which the local convolution theorem is applicable. Whenever traces at a occur, we assume that the corresponding local derivatives or fractional integrals admit finite right limits at a. These assumptions are the local-fractal analogues of the hypotheses routinely imposed in the classical RL and Caputo transform theory.

It is useful to record the two principal consistency checks already at this stage. If $r = 1$, then the local measure $(ds)^r/\Gamma(1 + r)$ reduces to the ordinary measure ds, $p^r = p$, and the local coordinate $\xi_a(\vartheta) = \vartheta - a$. Consequently, every operator introduced below reduces to its classical RL or Caputo counterpart of order α. On the other hand, if $\alpha = 1$, the newly introduced memory operator collapses to the first local Yang operator. Thus the construction interpolates between the classical nonlocal fractional calculus and the basic local fractional calculus without forcing the two orders to coincide. For later reference, the main limiting cases are summarized in Table 1.

Table 1. Interpretation of the two parameters and principal limiting cases.

Choice	Resulting framework	Interpretation
$0 < r < 1$, $0 < \alpha < 1$	local-fractal RL/Caputo model	local geometry is encoded by r and memory by α
$r = 1$, $0 < \alpha < 1$	classical RL/Caputo calculus	ordinary real line with nonlocal memory of order α
$0 < r < 1$, $\alpha = 1$	Yang local fractional calculus	local derivative/integral on the fractional-number setting
$r = 1$, $\alpha = 1$	ordinary first-order calculus	standard derivative and integral

3.2. The role of the parameter r

Following Yang's formalism [6, 7], let $0 < r \leq 1$. The notation $\mathbb{R}^r$ represents the fractional-number setting associated with a local fractal order r. A commonly used illustration is the ternary Cantor set, whose Hausdorff dimension is $r_C = \log 2/\log 3$. In engineering language, r therefore controls the geometric/local scaling, not the RL/Caputo memory order introduced later.

3.3. Cantor mass coordinate and the meaning of the local power

The role of the Cantor function [7] is important for interpreting the local variable used below. Let $C \subset [0, 1]$ be the ternary Cantor set and let F_C denote the Cantor (Lebesgue–Cantor) function. Equivalently, $F_C(\vartheta)$ is the cumulative Cantor mass of the interval $[0, \vartheta]$. If a point of C has a ternary expansion containing only the digits 0 and 2,

$$\vartheta = \sum_{\ell=1}^{\infty} \frac{2B_\ell}{3^\ell}, \qquad B_\ell \in \{0, 1\}, \tag{3.1}$$

then the corresponding Cantor mass coordinate is

$$F_C(\vartheta) = \sum_{\ell=1}^{\infty} \frac{B_\ell}{2^\ell}. \tag{3.2}$$

Thus F_C converts the geometric location on the disconnected Cantor support into an accumulated fractal coordinate in $[0, 1]$. It is continuous and nondecreasing and is constant on every complementary interval removed in the construction of C. Consequently, a response expressed as a function of $F_C(\vartheta)$ naturally develops the plateaus and rapid changes associated with the Cantor geometry rather than the smooth appearance obtained from an ordinary uniform Euclidean mesh.

This interpretation is stated explicitly in Yang, Baleanu, and Srivastava [7]. In Chapter 1, Section 1.1.2, their fractal relaxation solution is written in the form $\Phi(\mu) = \exp[-\omega F_C(\mu)]$ and the authors state the scaling relation $F_C(\mu) \sim \mu^r$; the same section also presents the Mittag–Leffler solution $E_r(-\omega\mu^r)$ on the Cantor set and illustrates it in Figures 1.2–1.3. Readers interested in the original graphical and modeling convention are referred directly to that section of [7].

The symbol ϑ^r used in Yang's formal fractional-number notation should therefore not be interpreted, for a Cantor-supported graph, merely as the ordinary real power evaluated at every point of an interval. In the specific Cantor realization, we use the mass coordinate

$$\Xi_{C,a}(\vartheta) := F_C(\vartheta) - F_C(a) \tag{3.3}$$

as the graphical representative of the formal increment $\vartheta^r - a^r$. Yang's relation is a scaling correspondence rather than the literal identity $F_C(\vartheta) = \vartheta^r$; hence the analytical definitions below remain in the standard local-fractional notation, while the Cantor mass is used when the formulas are visualized on a concrete fractal support.

This convention is also consistent with recent local-fractional studies. Cantor-set Mittag–Leffler functions and nondifferentiable solutions are used in [12, 13, 15], while the electrical transmission-line model in [14] explicitly displays dynamics on the Cantor set for the classical ternary dimension $r =$

log 2/ log 3. Therefore the Cantor-supported figures in the present paper follow both Yang's original construction and a continuing convention in the recent literature.

For convenience, define the local coordinate increment

$$\xi_a(\vartheta) := \vartheta^r - a^r, \qquad \vartheta \geq a. \tag{3.4}$$

Definition 1 (Yang local fractional integral [6,7]). *If x is locally fractionally integrable on* $[a,b]$, *then*

$${}_aI_b^{(r)}x = \frac{1}{\Gamma(1+r)} \int_a^b x(\vartheta)(d\vartheta)^r. \tag{3.5}$$

In the Riemann-sum representation used in Yang's calculus,

$${}_aI_b^{(r)}x = \frac{1}{\Gamma(1+r)} \lim_{\max \Delta\vartheta_j \to 0} \sum_j x(\vartheta_j)(\Delta\vartheta_j)^r.$$

Definition 2 (Yang local fractional derivative [6,7]). *The local fractional derivative of order r at* ϑ_0 *is*

$$D^r x(\vartheta_0) = \lim_{\vartheta \to \vartheta_0} \frac{\Delta^r(x(\vartheta) - x(\vartheta_0))}{(\vartheta - \vartheta_0)^r}, \quad \Delta^r(\Delta x) \simeq \Gamma(1+r)\Delta x, \tag{3.6}$$

whenever the local fractional limit exists.

The basic identities used below include

$$D^r c = 0, \qquad D^r(\vartheta - a)^{rk} = \frac{\Gamma(1+rk)}{\Gamma(1+r(k-1))}(\vartheta - a)^{r(k-1)}, \tag{3.7}$$

and the local Mittag–Leffler function

$$E_r(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+rk)}, \qquad D^r E_r(\lambda(\vartheta - a)^r) = \lambda E_r(\lambda(\vartheta - a)^r), \tag{3.8}$$

see [7].

3.4. *Local fractional Laplace transform*

The Yang–Laplace transform based at a is written as

$$\mathcal{L}_a^r\{x(\vartheta)\}(p) = \frac{1}{\Gamma(1+r)} \int_a^{\infty} E_r(-p^r \xi_a(\tau))\, x(\tau)(d\tau)^r \tag{3.9}$$

for functions for which the integral converges. This is the shifted form of the transform in [7,20]. We use the standard local convolution theorem associated with (3.9).

Lemma 1 (Power transform). *For* $\mu > 0$ *and admissible p,*

$$\mathcal{L}_a^r\{\xi_a(\vartheta)^{\mu-1}\}(p) = \frac{\Gamma(1+r(\mu-1))}{(p^r)^{\mu}}. \tag{3.10}$$

Proof. The standard formula in Yang's transform calculus is

$$\mathcal{L}_0^r\{\vartheta^{rz}\}(p) = \frac{\Gamma(1+rz)}{(p^r)^{z+1}}$$

for the admissible values of z [7, 10]. Put $z = \mu - 1$. Since $(\vartheta^r)^{\mu-1} = \vartheta^{r(\mu-1)}$, we obtain

$$\mathcal{L}_0^r\{(\vartheta^r)^{\mu-1}\}(p) = \frac{\Gamma(1+r(\mu-1))}{(p^r)^{\mu}}.$$

The time-shift property of the local transform replaces ϑ^r by $\vartheta^r - a^r = \xi_a(\vartheta)$, yielding (3.10). □

The coefficient $\Gamma(1 + r(\mu - 1))$ appearing in the power-transform formula will be identified below with the local gamma function $\Gamma^r(\mu)$. Thus, using the notation of Definition 3, equation (3.10) may equivalently be written as

$$\mathcal{L}_a^r\{\xi_a^{\mu-1}\}(p) = \frac{\Gamma^r(\mu)}{(p^r)^{\mu}}. \tag{3.11}$$

Definition 3. *The local gamma function is defined for* $r \in [0, 1]$ *as*

$$\Gamma^r(r_2) = \frac{1}{\Gamma(1+r)} \int_0^{\infty} (\tau^r)^{r_2-1} E_r(-\tau^r)(d\tau)^r.$$

Proposition 1. *For every admissible* $\mu > 0$*,*

$$\Gamma^r(\mu) = \Gamma(1 + r(\mu - 1)), \qquad \mu > 0. \tag{3.12}$$

Consequently,

$$\Gamma^r(1) = 1, \qquad \Gamma^r(m+1) = \Gamma(1+rm), \quad m \in \mathbb{N}_0,$$

and, in particular, $\Gamma^r(m + 1) = m!$ *only when* $r = 1$.

Proof. Starting from Definition 3, observe that the defining integral is exactly the Yang–Laplace transform at the transform parameter $p = 1$ of the local power $(\tau^r)^{\mu-1}$. Indeed,

$$\Gamma^r(\mu) = \frac{1}{\Gamma(1+r)} \int_0^{\infty} E_r(-\tau^r)(\tau^r)^{\mu-1}(d\tau)^r = \mathcal{L}_0^r\{(\tau^r)^{\mu-1}\}(1).$$

By Lemma 1 with $a = 0$,

$$\mathcal{L}_0^r\{(\tau^r)^{\mu-1}\}(p) = \frac{\Gamma(1+r(\mu-1))}{(p^r)^{\mu}}.$$

Setting $p = 1$ gives the claimed formula. The normalization $\Gamma^r(1) = 1$ follows immediately from $\Gamma(1) = 1$. For an integer $m \geq 0$, substituting $\mu = m + 1$ yields $\Gamma^r(m + 1) = \Gamma(1 + rm)$. Finally, if $r = 1$, then $\Gamma(1 + m) = m!$, which gives the classical identity. □

Remark 1. *The distinction in Proposition 1 propagates through all power rules. For example, the denominator attached to a local monomial of degree m is* $\Gamma^r(m + 1) = \Gamma(1 + rm)$*, not* $m!$ *unless* $r = 1$*. Retaining this factor is essential in the local Mittag–Leffler series, in the transform of powers, and in the numerical implementation in Section 6.*

3.5. Local convolution and the transform mechanism

For a kernel k depending on the local coordinate, define the left local convolution by

$$(k *_{r,a} x)(\vartheta) := \frac{1}{\Gamma(1+r)} \int_a^{\vartheta} k(\vartheta^r - s^r)x(s)(ds)^r. \tag{3.13}$$

Within the standard hypotheses of Yang's transform calculus, the local convolution theorem gives

$$\mathcal{L}_a^r\{k *_{r,a} x\}(p) = K(p)X_a(p), \tag{3.14}$$

where K and X_a are the corresponding Yang–Laplace transforms. This simple product rule is the operational reason for the normalization used in Definition 4: If

$$k_\alpha(\xi) = \frac{\xi^{\alpha-1}}{\Gamma^r(\alpha)},$$

then Lemma 1 gives

$$\mathcal{L}^r\{k_\alpha\}(p) = \frac{1}{(p^r)^\alpha}.$$

Hence the new RL integral is precisely the convolution operator whose transform multiplier is $(p^r)^{-\alpha}$. All semigroup and inversion formulas below are consequences of this multiplier together with the trace terms generated by repeated local differentiation.

4. Riemann–Liouville local fractional operators

We now introduce the independent operator order α. Unless otherwise stated, $\alpha > 0$ and $0 < r \le 1$.

Definition 4. *The left and right RL local fractional integrals are*

$${}_aI^{\alpha,r}x(\vartheta) = \frac{1}{\Gamma^r(\alpha)\Gamma(1+r)} \int_a^{\vartheta} (\vartheta^r - s^r)^{\alpha-1}x(s)(ds)^r, \tag{4.1}$$

$$I_b^{\alpha,r}x(\vartheta) = \frac{1}{\Gamma^r(\alpha)\Gamma(1+r)} \int_{\vartheta}^{b} (s^r - \vartheta^r)^{\alpha-1}x(s)(ds)^r. \tag{4.2}$$

We also set ${}_aI^{0,r} = I_b^{0,r} = \mathrm{Id}$.

When $\alpha = 1$, (4.1) is Yang's local fractional integral over $[a, \vartheta]$. When $r = 1$, $\Gamma^r(\alpha) = \Gamma(\alpha)$ and (4.1), (4.2) reduce to the classical RL integrals (2.1) and (2.2).

For $n \in \mathbb{N}$, denote the n-fold Yang local derivative by

$$D^{n,r} := \underbrace{D^r \circ \cdots \circ D^r}_{n \text{ times}}, \qquad D^{0,r} = \mathrm{Id}.$$

Definition 5. *Let $n_\alpha = \lceil \alpha \rceil$ for noninteger α and $n_\alpha = \alpha$ for integer α. The left and right RL local fractional derivatives are*

$${}_aD^{\alpha,r}x = D^{n_\alpha,r}\,{}_aI^{n_\alpha-\alpha,r}x, \tag{4.3}$$

$$D_b^{\alpha,r}x = (-1)^{n_\alpha}D^{n_\alpha,r}\,I_b^{n_\alpha-\alpha,r}x. \tag{4.4}$$

For integer $\alpha = n$, these are interpreted as $D^{n,r}$ and $(-1)^nD^{n,r}$, respectively.

Proposition 2. *Let $\alpha, \beta > 0$. Then*

$$
{}_aI^{\alpha,r}\xi_a(\vartheta)^{\beta-1} = \frac{\Gamma^r(\beta)}{\Gamma^r(\alpha+\beta)}\xi_a(\vartheta)^{\alpha+\beta-1}, \tag{4.5}
$$

provided the involved local fractional integrals exist. The analogous right-sided identity is obtained by replacing $\xi_a(\vartheta)$ with $b^r - \vartheta^r$.

Proof. By (3.11) and the convolution theorem, the transform of the left-hand side is

$$
\frac{1}{(p^r)^\alpha}\frac{\Gamma^r(\beta)}{(p^r)^\beta} = \frac{\Gamma^r(\beta)}{(p^r)^{\alpha+\beta}}.
$$

Applying (3.11) with $\mu = \alpha + \beta$ gives (4.5). The right-sided formula follows by the corresponding reflection of the kernel. □

Theorem 1. *Let $X_a(p) = \mathcal{L}_a^r\{x\}(p)$. If x and ${}_aI^{\alpha,r}x$ are transformable, then*

$$
\mathcal{L}_a^r\{{}_aI^{\alpha,r}x\}(p) = \frac{X_a(p)}{(p^r)^\alpha}. \tag{4.6}
$$

Proof. The kernel of (4.1) in the local coordinate is

$$
k_\alpha(\xi) = \frac{\xi^{\alpha-1}}{\Gamma^r(\alpha)}.
$$

By Lemma 1, $\mathcal{L}^r\{k_\alpha\} = (p^r)^{-\alpha}$. Equation (4.1) is the local convolution of k_α and x; hence the local convolution theorem yields (4.6). □

Corollary 1 (Semigroup property)**.** *On a class where the local Laplace transform is injective,*

$$
{}_aI^{\alpha,r}\,{}_aI^{\beta,r}x = {}_aI^{\alpha+\beta,r}x, \qquad \alpha, \beta > 0. \tag{4.7}
$$

The same relation holds for the right-sided integrals.

Proof. Using Theorem 1, we have

$$
\mathcal{L}_a^r\{{}_aI^{\alpha,r}\,{}_aI^{\beta,r}x\} = (p^r)^{-\alpha}(p^r)^{-\beta}X_a = (p^r)^{-(\alpha+\beta)}X_a.
$$

The last expression is the transform of ${}_aI^{\alpha+\beta,r}x$. Injectivity gives (4.7). □

Corollary 2 (Left inverse)**.** *Under the regularity needed in Definition 5,*

$$
{}_aD^{\alpha,r}\,{}_aI^{\alpha,r}x = x. \tag{4.8}
$$

Proof. Let $n = n_\alpha$. By the semigroup property, we get

$$
{}_aD^{\alpha,r}\,{}_aI^{\alpha,r}x = D^{n,r}\,{}_aI^{n-\alpha,r}\,{}_aI^{\alpha,r}x = D^{n,r}\,{}_aI^{n,r}x = x,
$$

where the last identity is the repeated local fundamental theorem within Yang's calculus. □

Lemma 2 (Yang–Laplace transform of repeated local derivatives)**.** *Assume that $D^{k,r} f$ is transformable for $k = 0, \ldots, n$ and the traces at a exist. Then*

$$\mathcal{L}_a^r\{D^{n,r} f\}(p) = (p^r)^n F_a(p) - \sum_{k=0}^{n-1} (p^r)^{n-1-k} D^{k,r} f(a), \tag{4.9}$$

where $F_a = \mathcal{L}_a^r\{f\}$. This is the standard derivative rule for the Yang–Laplace transform [7, 11].

Theorem 2 (RL local fractional transform)**.** *Let $n = n_\alpha$. If the required transforms and traces exist, then*

$$\mathcal{L}_a^r\{{}_aD^{\alpha,r} x\}(p) = (p^r)^\alpha X_a(p) - \sum_{k=0}^{n-1} (p^r)^{n-1-k} D^{k,r} ({}_aI^{n-\alpha,r} x)(a). \tag{4.10}$$

In particular, for $0 < \alpha < 1$,

$$\mathcal{L}_a^r\{{}_aD^{\alpha,r} x\}(p) = (p^r)^\alpha X_a(p) - ({}_aI^{1-\alpha,r} x)(a). \tag{4.11}$$

Proof. Apply Lemma 2 to $f = {}_aI^{n-\alpha,r} x$. By Theorem 1, we have

$$F_a(p) = (p^r)^{-(n-\alpha)} X_a(p).$$

The leading term in (4.9) is therefore $(p^r)^n F_a = (p^r)^\alpha X_a$, which gives (4.10). □

4.1. Additional calculus rules and detailed inversion formulas

The transform multiplier obtained above makes it possible to recover, in a transparent form, several identities that are useful in applications. We state them explicitly because they also clarify the domain restrictions and the boundary terms that distinguish a true inverse from a one-sided inverse.

Proposition 3. *Let $\alpha > 0$, $n = n_\alpha$, and assume $\beta > n$ so that the lower-end traces generated in the RL transform vanish. Then*

$${}_aD^{\alpha,r} \xi_a(\vartheta)^{\beta-1} = \frac{\Gamma^r(\beta)}{\Gamma^r(\beta-\alpha)} \xi_a(\vartheta)^{\beta-\alpha-1}. \tag{4.12}$$

Whenever the right-sided power is admissible, the analogous formula is

$$D_b^{\alpha,r} (b^r - \vartheta^r)^{\beta-1} = \frac{\Gamma^r(\beta)}{\Gamma^r(\beta-\alpha)} (b^r - \vartheta^r)^{\beta-\alpha-1}.$$

Proof. Set $f(\vartheta) = \xi_a(\vartheta)^{\beta-1}$. Lemma 1 gives

$$F_a(p) = \frac{\Gamma^r(\beta)}{(p^r)^\beta}.$$

By the hypothesis $\beta > n$, the traces

$$D^{k,r}({}_aI^{n-\alpha,r} f)(a), \qquad k = 0, \ldots, n-1,$$

vanish. Theorem 2 therefore reduces to

$$\mathcal{L}_a^r\{{}_aD^{\alpha,r} f\}(p) = (p^r)^\alpha F_a(p) = \frac{\Gamma^r(\beta)}{(p^r)^{\beta-\alpha}}.$$

On the other hand, applying Lemma 1 with the exponent parameter $\beta - \alpha$ yields

$$\mathcal{L}_a^r\left\{\frac{\Gamma^r(\beta)}{\Gamma^r(\beta-\alpha)}\xi_a^{\beta-\alpha-1}\right\}(p) = \frac{\Gamma^r(\beta)}{(p^r)^{\beta-\alpha}}.$$

Injectivity of the transform proves (4.12). The right-sided statement follows by the same argument after reflecting the local coordinate from the left endpoint to the right endpoint. □

Proposition 4. *Let $m \in \mathbb{N}$ and $\alpha > m$. If the required transforms exist, then*

$$D^{m,r}{}_aI^{\alpha,r}x = {}_aI^{\alpha-m,r}x. \tag{4.13}$$

In particular, for $m = 1$,

$$D^r{}_aI^{\alpha,r}x = {}_aI^{\alpha-1,r}x.$$

Proof. Let $F = {}_aI^{\alpha,r}x$. By Theorem 1, we have

$$\mathcal{L}_a^r\{F\}(p) = (p^r)^{-\alpha}X_a(p).$$

Because $\alpha > m$, the local integral F vanishes at the lower endpoint with enough order to remove the m trace terms in Lemma 2. Hence

$$\mathcal{L}_a^r\{D^{m,r}F\}(p) = (p^r)^m(p^r)^{-\alpha}X_a(p) = (p^r)^{-(\alpha-m)}X_a(p).$$

The last quantity is, by Theorem 1, the transform of ${}_aI^{\alpha-m,r}x$. Transform injectivity yields (4.13). □

Corollary 3. *Let $0 < \beta < \alpha$. Under the hypotheses of the previous results,*

$${}_aD^{\beta,r}{}_aI^{\alpha,r}x = {}_aI^{\alpha-\beta,r}x. \tag{4.14}$$

Proof. Let $n = n_\beta$. From Definition 5 and the semigroup property,

$${}_aD^{\beta,r}{}_aI^{\alpha,r}x = D^{n,r}{}_aI^{n-\beta,r}{}_aI^{\alpha,r}x = D^{n,r}{}_aI^{n+\alpha-\beta,r}x.$$

Since $n + \alpha - \beta > n$, Proposition 4 applies and gives

$$D^{n,r}{}_aI^{n+\alpha-\beta,r}x = {}_aI^{\alpha-\beta,r}x.$$

□

Theorem 3. *Let $n = n_\alpha$ and set*

$$f(\vartheta) = {}_aI^{n-\alpha,r}x(\vartheta).$$

Assume that $D^{k,r}f(a)$ exists for $k = 0, \ldots, n-1$. Then

$${}_aI^{\alpha,r}{}_aD^{\alpha,r}x(\vartheta) = x(\vartheta) - \sum_{k=0}^{n-1}\frac{D^{k,r}f(a)}{\Gamma^r(\alpha-n+k+1)}\xi_a(\vartheta)^{\alpha-n+k}. \tag{4.15}$$

Thus ${}_aD^{\alpha,r}$ is a left inverse of ${}_aI^{\alpha,r}$, but ${}_aI^{\alpha,r}$ is a right inverse only after the fractional initial traces have been removed.

Proof. By Theorem 2, we have

$$\mathcal{L}_a^r\{{}_aD^{\alpha,r}x\}(p) = (p^r)^\alpha X_a(p) - \sum_{k=0}^{n-1}(p^r)^{n-1-k}D^{k,r}f(a).$$

Applying ${}_aI^{\alpha,r}$ corresponds, by Theorem 1, to multiplication by $(p^r)^{-\alpha}$. Therefore

$$\begin{aligned}\mathcal{L}_a^r\{{}_aI^{\alpha,r}{}_aD^{\alpha,r}x\}(p) &= X_a(p) - \sum_{k=0}^{n-1}(p^r)^{n-1-k-\alpha}D^{k,r}f(a)\\ &= X_a(p) - \sum_{k=0}^{n-1}\frac{D^{k,r}f(a)}{(p^r)^{\alpha-n+k+1}}.\end{aligned}$$

Now Lemma 1 gives

$$\mathcal{L}_a^{r,-1}\left\{\frac{1}{(p^r)^\nu}\right\} = \frac{\xi_a^{\nu-1}}{\Gamma^r(\nu)}, \qquad \nu > 0.$$

Taking $\nu = \alpha - n + k + 1$ in every term gives exactly (4.15). The formula also displays the reason for the RL initial conditions: the missing information is carried by the traces of ${}_aI^{n-\alpha,r}x$, rather than directly by the point values of x. □

Remark 2. *When* $0 < \alpha < 1$, $n = 1$ *and* (4.15) *reduces to*

$${}_aI^{\alpha,r}{}_aD^{\alpha,r}x(\vartheta) = x(\vartheta) - \frac{({}_aI^{1-\alpha,r}x)(a)}{\Gamma^r(\alpha)}\xi_a(\vartheta)^{\alpha-1}.$$

4.2. A local Mittag–Leffler function associated with Γ^r

Definition 6. *For* $\alpha > 0$, $\beta > 0$, *and* $z \in \mathbb{C}$, *define*

$$E^{(r)}_{\alpha,\beta}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma^r(\alpha k+\beta)} = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1 + r(\alpha k+\beta-1))}. \tag{4.16}$$

Using (3.12)*, this may equivalently be written with the denominator* $\Gamma(1 + r(\alpha k + \beta - 1))$*. For* $r = 1$*, it is the classical two-parameter Mittag–Leffler function* $E_{\alpha,\beta}$*. For* $\alpha = \beta = 1$*, it is consistent with Yang's* E_r *series after writing the argument in the local coordinate.*

Lemma 3. *Suppose the series can be transformed term by term and* $|z| < |(p^r)^\alpha|$*. Then*

$$\mathcal{L}_a^r\left\{\xi_a(\vartheta)^{\beta-1}E^{(r)}_{\alpha,\beta}(z\xi_a(\vartheta)^\alpha)\right\}(p) = \frac{(p^r)^{\alpha-\beta}}{(p^r)^\alpha - z}. \tag{4.17}$$

Proof. By Definition 6, we have

$$\xi_a^{\beta-1}E^{(r)}_{\alpha,\beta}(z\xi_a^\alpha) = \sum_{k=0}^{\infty}\frac{z^k\xi_a^{\alpha k+\beta-1}}{\Gamma^r(\alpha k+\beta)}.$$

Using (3.11) term by term, we get

$$\mathcal{L}_a^r\{\xi_a^{\beta-1}E^{(r)}_{\alpha,\beta}(z\xi_a^\alpha)\} = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma^r(\alpha k+\beta)}\frac{\Gamma^r(\alpha k+\beta)}{(p^r)^{\alpha k+\beta}}$$

$$
\begin{aligned}
&= \frac{1}{(p^r)^\beta} \sum_{k=0}^{\infty} \left(\frac{z}{(p^r)^\alpha} \right)^k \\
&= \frac{1}{(p^r)^\beta} \frac{1}{1 - z/(p^r)^\alpha} \\
&= \frac{(p^r)^{\alpha-\beta}}{(p^r)^\alpha - z},
\end{aligned}
$$

which proves (4.17). □

Remark 3. *The denominator in* (4.16) *is*

$$\Gamma^r(\alpha k + \beta) = \Gamma(1 + r(\alpha k + \beta - 1)).$$

For $\alpha > 0$ and $r > 0$, this gamma factor grows super-polynomially with k. Consequently the defining power series is entirely in z for the parameter range used here. At $r = 1$, one recovers the classical $E_{\alpha,\beta}$ function. If $\alpha = \beta = 1$, then

$$E_{1,1}^{(r)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + rk)} = E_r(z),$$

which is Yang's one-parameter local Mittag–Leffler function.

Lemma 4. *Let $\nu > 0$. Under the same termwise-transform assumptions as Lemma 3,*

$$\mathcal{L}_a^r \left\{ \xi_a^{\alpha+\nu-1} E_{\alpha,\alpha+\nu}^{(r)}(z\xi_a^\alpha) \right\}(p) = \frac{(p^r)^{-\nu}}{(p^r)^\alpha - z}. \tag{4.18}$$

Proof. Apply Lemma 3 with $\beta = \alpha + \nu$. Then

$$(p^r)^{\alpha-(\alpha+\nu)} = (p^r)^{-\nu},$$

and (4.18) follows immediately. This simple shifted identity will be used to evaluate power-law forcing terms without leaving the transform framework. □

Theorem 4 (RL eigenkernel)**.** *Let $0 < \alpha < 1$ and define*

$$\Phi_{\alpha,z}(\vartheta) = \xi_a(\vartheta)^{\alpha-1} E_{\alpha,\alpha}^{(r)}(z\xi_a(\vartheta)^\alpha).$$

Then $\Phi_{\alpha,z}$ satisfies

$${}_aD^{\alpha,r}\Phi_{\alpha,z} = z\Phi_{\alpha,z}, \qquad ({}_aI^{1-\alpha,r}\Phi_{\alpha,z})(a) = 1. \tag{4.19}$$

Proof. By Lemma 3 with $\beta = \alpha$,

$$\mathcal{L}_a^r\{\Phi_{\alpha,z}\}(p) = \frac{1}{(p^r)^\alpha - z}.$$

Next, Theorem 1 gives

$$\mathcal{L}_a^r\{{}_aI^{1-\alpha,r}\Phi_{\alpha,z}\}(p) = \frac{(p^r)^{\alpha-1}}{(p^r)^\alpha - z}.$$

By Lemma 3 with $\beta = 1$, the inverse transform is

$$ {}_aI^{1-\alpha,r}\Phi_{\alpha,z} = E^{(r)}_{\alpha,1}(z\xi_a^{\alpha}). $$

At $\vartheta = a$, $\xi_a(a) = 0$ and the series begins with the unit term, so the RL trace equals one. The RL derivative transform (4.11) now yields

$$ \begin{aligned} \mathcal{L}^r_a\{{}_aD^{\alpha,r}\Phi_{\alpha,z}\}(p) &= \frac{(p^r)^{\alpha}}{(p^r)^{\alpha} - z} - 1 \\ &= \frac{z}{(p^r)^{\alpha} - z} = z\mathcal{L}^r_a\{\Phi_{\alpha,z}\}(p). \end{aligned} $$

Transform injectivity proves the eigenvalue equation. □

4.3. Application

Theorem 5 (RL linear initial-value problem)**.** *Let* $0 < \alpha < 1$ *and consider*

$$ {}_aD^{\alpha,r}x(\vartheta) - zx(\vartheta) = y(\vartheta), \qquad ({}_aI^{1-\alpha,r}x)(a) = x_a. \tag{4.20} $$

Then, under the transformability assumptions above,

$$ \begin{aligned} x(\vartheta) &= x_a\,\xi_a(\vartheta)^{\alpha-1}E^{(r)}_{\alpha,\alpha}(z\xi_a(\vartheta)^{\alpha}) \\ &\quad + \frac{1}{\Gamma(1+r)}\int_a^{\vartheta}(\vartheta^r - s^r)^{\alpha-1}E^{(r)}_{\alpha,\alpha}(z(\vartheta^r - s^r)^{\alpha})\,y(s)(ds)^r. \end{aligned} \tag{4.21} $$

Proof. We give the transform argument in detail because it shows precisely where the RL initial datum enters. Set

$$ X_a(p) = \mathcal{L}^r_a\{x\}(p), \qquad Y_a(p) = \mathcal{L}^r_a\{y\}(p). $$

Since $0 < \alpha < 1$, Theorem 2 gives

$$ \mathcal{L}^r_a\{{}_aD^{\alpha,r}x\}(p) = (p^r)^{\alpha}X_a(p) - ({}_aI^{1-\alpha,r}x)(a). $$

Using the prescribed fractional trace $({}_aI^{1-\alpha,r}x)(a) = x_a$ in (4.20), we obtain

$$ (p^r)^{\alpha}X_a(p) - x_a - zX_a(p) = Y_a(p). $$

Collecting the unknown transform yields

$$ ((p^r)^{\alpha} - z)X_a(p) = x_a + Y_a(p), $$

and therefore

$$ X_a(p) = \frac{x_a}{(p^r)^{\alpha} - z} + \frac{Y_a(p)}{(p^r)^{\alpha} - z}. \tag{4.22} $$

For the first term, Lemma 3 with $\beta = \alpha$ gives

$$ \mathcal{L}^{r,-1}_a\left\{\frac{1}{(p^r)^{\alpha} - z}\right\} = \xi_a^{\alpha-1}E^{(r)}_{\alpha,\alpha}(z\xi_a^{\alpha}). $$

For the second term, the same inverse transform acts as a causal kernel. By the local convolution theorem,

$$\mathcal{L}_a^{r,-1}\left\{\frac{Y_a(p)}{(p^r)^\alpha - z}\right\}(\vartheta) = \frac{1}{\Gamma(1+r)}\int_a^\vartheta (\vartheta^r - s^r)^{\alpha-1} E_{\alpha,\alpha}^{(r)}(z(\vartheta^r - s^r)^\alpha)\, y(s)(ds)^r.$$

Combining the two inverse transforms proves (4.21). □

Remark 4. *The first term in* (4.21) *is not* $x_a E_{\alpha,1}^{(r)}(z\xi_a^\alpha)$. *That latter kernel belongs to the Caputo initial condition* $x(a) = x_a$. *Keeping these two kernels distinct is necessary already in the classical limit* $r = 1$.

5. Caputo local fractional operators

Definition 7. *Let* $n = n_\alpha$. *The left and right Caputo local fractional derivatives are*

$$ {}_a^C D^{\alpha,r} x = {}_a I^{n-\alpha,r} D^{n,r} x, \tag{5.1}$$

$$ {}^C D_b^{\alpha,r} x = I_b^{n-\alpha,r} (-1)^n D^{n,r} x. \tag{5.2}$$

For $0 < \alpha < 1$,

$$ {}_a^C D^{\alpha,r} x(\vartheta) = \frac{1}{\Gamma^r(1-\alpha)\Gamma(1+r)} \int_a^\vartheta (\vartheta^r - s^r)^{-\alpha} D^r x(s)(ds)^r. \tag{5.3}$$

For $r = 1$, (5.1) and (5.2) reduce to the classical Caputo derivatives. For $\alpha = 1$, they reduce to the basic Yang local derivative.

5.1. Detailed consequences of the Caputo definition

The main practical difference between the RL and Caputo constructions is the location of the repeated local derivative. In the Caputo operator, the integer/local derivatives are taken first and the fractional integral is applied afterward. As a result, constants are annihilated and the transform formula involves the physically familiar pointwise traces $D^{k,r}x(a)$.

Proposition 5. *Let* $m \in \mathbb{N}$ *and assume* $D^{k,r}x(a)$ *exists for* $0 \le k \le m-1$. *Then*

$$ {}_a I^{m,r} D^{m,r} x(\vartheta) = x(\vartheta) - \sum_{k=0}^{m-1} \frac{D^{k,r}x(a)}{\Gamma^r(k+1)} \xi_a(\vartheta)^k. \tag{5.4}$$

Proof. We prove the result through the Yang–Laplace transform. Lemma 2 gives

$$\mathcal{L}_a^r\{D^{m,r}x\}(p) = (p^r)^m X_a(p) - \sum_{k=0}^{m-1} (p^r)^{m-1-k} D^{k,r}x(a).$$

Applying ${}_a I^{m,r}$ multiplies this expression by $(p^r)^{-m}$, hence

$$\mathcal{L}_a^r\{{}_a I^{m,r} D^{m,r} x\}(p) = X_a(p) - \sum_{k=0}^{m-1} \frac{D^{k,r}x(a)}{(p^r)^{k+1}}.$$

By (3.11),

$$\mathcal{L}_a^r\left\{\frac{\xi_a^k}{\Gamma^r(k+1)}\right\}(p) = \frac{1}{(p^r)^{k+1}}.$$

Inverting term by term proves (5.4). For $r = 1$, this becomes the usual Taylor-polynomial correction formula $I^m x^{(m)} = x - \sum_{k=0}^{m-1} x^{(k)}(a)(\vartheta - a)^k/k!$. □

Corollary 4 (The first-order local fundamental formula)**.** *For $m = 1$,*

$${}_aI^{1,r}D^r x(\vartheta) = x(\vartheta) - x(a).$$

Thus the local integral and derivative satisfy the expected endpoint-corrected fundamental relation.

Remark 5. *Formula* (5.6) *means that the Caputo derivative ignores the local Taylor modes of order $0, \ldots, n-1$. In particular, for $0 < \alpha < 1$, only the constant mode is removed. This explains simultaneously why ${}_a^C D^{\alpha,r} c = 0$ and why the Caputo transform contains $x(a)$ instead of the RL fractional trace $({}_aI^{1-\alpha,r}x)(a)$.*

Proposition 6. *Let $n = n_\alpha$ and assume $\beta > n$. Then*

$${}_a^C D^{\alpha,r} \xi_a(\vartheta)^{\beta-1} = \frac{\Gamma^r(\beta)}{\Gamma^r(\beta-\alpha)} \xi_a(\vartheta)^{\beta-\alpha-1}. \tag{5.5}$$

Moreover ${}_a^C D^{\alpha,r} c = 0$ for every constant c.

Proof. Let $f(\vartheta) = \xi_a(\vartheta)^{\beta-1}$. Because $\beta > n$, the first n local Taylor traces of f that occur in the Caputo transform vanish at a. Hence Lemma 5 reduces to

$$\mathcal{L}_a^r\{{}_a^C D^{\alpha,r} f\}(p) = (p^r)^\alpha F_a(p).$$

Using (3.11),

$$F_a(p) = \frac{\Gamma^r(\beta)}{(p^r)^\beta},$$

so

$$\mathcal{L}_a^r\{{}_a^C D^{\alpha,r} f\}(p) = \frac{\Gamma^r(\beta)}{(p^r)^{\beta-\alpha}}.$$

Again by (3.11),

$$\mathcal{L}_a^r\left\{\frac{\Gamma^r(\beta)}{\Gamma^r(\beta-\alpha)} \xi_a^{\beta-\alpha-1}\right\}(p) = \frac{\Gamma^r(\beta)}{(p^r)^{\beta-\alpha}}.$$

The power formula follows by injectivity. For a constant c, $D^{n,r}c = 0$ for every $n \geq 1$, and Definition 7 immediately yields ${}_a^C D^{\alpha,r} c = 0$. □

Theorem 6. *Let $n = n_\alpha$ and assume that the local derivatives $D^{k,r}x(a)$ exist for $k = 0, \ldots, n-1$. Then*

$${}_aI^{\alpha,r}\,{}_a^C D^{\alpha,r} x(\vartheta) = x(\vartheta) - \sum_{k=0}^{n-1} \frac{D^{k,r}x(a)}{\Gamma^r(k+1)} \xi_a(\vartheta)^k. \tag{5.6}$$

Proof. By the local Taylor representation associated with repeated Yang derivatives,

$$ {}_aI^{n,r}D^{n,r}x = x - \sum_{k=0}^{n-1} \frac{D^{k,r}x(a)}{\Gamma^r(k+1)}\xi_a^k. $$

Using the semigroup property,

$$ {}_aI^{\alpha,r}{}^C_aD^{\alpha,r}x = {}_aI^{\alpha,r}{}_aI^{n-\alpha,r}D^{n,r}x = {}_aI^{n,r}D^{n,r}x, $$

which gives (5.6). □

Lemma 5 (Caputo Yang–Laplace transform)**.** *Let $n = n_\alpha$. If the required transforms exist, then*

$$ \mathcal{L}_a^r\{{}^C_aD^{\alpha,r}x\}(p) = (p^r)^\alpha X_a(p) - \sum_{k=0}^{n-1}(p^r)^{\alpha-1-k}D^{k,r}x(a). \tag{5.7} $$

Proof. From Definition 7 and Theorem 1, we get

$$ \mathcal{L}_a^r\{{}^C_aD^{\alpha,r}x\} = (p^r)^{-(n-\alpha)}\mathcal{L}_a^r\{D^{n,r}x\}. $$

Substitution of Lemma 2 yields (5.7). □

Proposition 7 (RL–Caputo relation)**.** *Under the assumptions of Theorem 6, then*

$$ {}^C_aD^{\alpha,r}x(\vartheta) = {}_aD^{\alpha,r}\left[x(\vartheta) - \sum_{k=0}^{n-1}\frac{D^{k,r}x(a)}{\Gamma^r(k+1)}\xi_a(\vartheta)^k\right]. \tag{5.8} $$

Equivalently, when the power terms are admissible,

$$ {}^C_aD^{\alpha,r}x = {}_aD^{\alpha,r}x - \sum_{k=0}^{n-1}\frac{D^{k,r}x(a)}{\Gamma^r(k+1-\alpha)}\xi_a^{k-\alpha}. \tag{5.9} $$

Proof. Apply ${}_aD^{\alpha,r}$ to both sides of (5.6) and use the left-inverse property. Formula (5.9) follows from the power rule. □

Lemma 6. *For $0 < \alpha \le 1$ and admissible z,*

$$ {}^C_aD^{\alpha,r}E^{(r)}_{\alpha,1}(z\xi_a(\vartheta)^\alpha) = zE^{(r)}_{\alpha,1}(z\xi_a(\vartheta)^\alpha). \tag{5.10} $$

Proof. The $k = 0$ term in (4.16) is constant and its Caputo derivative is zero. For $k \ge 1$, Proposition 6, the factor $\Gamma^r(\alpha k + 1)$, after which the index shift $j = k - 1$ yields (5.10). □

5.2. Application

Theorem 7. *Let $n-1<\alpha\le n$ with $n\in\mathbb{N}$, and consider*

$$ {}_a^C D^{\alpha,r}x(\vartheta)-zx(\vartheta)=y(\vartheta),\qquad D^{k,r}x(a)=x_k,\quad k=0,\dots,n-1. \tag{5.11}$$

Then, under the transformability assumptions of Lemma 5,

$$\begin{aligned} x(\vartheta)=&\sum_{k=0}^{n-1} x_k\,\xi_a(\vartheta)^k E^{(r)}_{\alpha,k+1}(z\xi_a(\vartheta)^\alpha)\\ &+\frac{1}{\Gamma(1+r)}\int_a^\vartheta(\vartheta^r-s^r)^{\alpha-1}E^{(r)}_{\alpha,\alpha}(z(\vartheta^r-s^r)^\alpha)\,y(s)(ds)^r. \end{aligned}\tag{5.12}$$

The solution is unique in every class on which the Yang–Laplace transform is injective.

Proof. Applying Lemma 5 to (5.11), we obtain

$$(p^r)^\alpha X_a(p)-\sum_{k=0}^{n-1}(p^r)^{\alpha-1-k}x_k-zX_a(p)=Y_a(p).$$

Therefore

$$X_a(p)=\sum_{k=0}^{n-1}x_k\frac{(p^r)^{\alpha-k-1}}{(p^r)^\alpha-z}+\frac{Y_a(p)}{(p^r)^\alpha-z}. \tag{5.13}$$

For each k, Lemma 3 with $\beta=k+1$ gives

$$\mathcal{L}_a^r\left\{\xi_a^k E^{(r)}_{\alpha,k+1}(z\xi_a^\alpha)\right\}(p)=\frac{(p^r)^{\alpha-k-1}}{(p^r)^\alpha-z}.$$

The forcing term is inverted by the local convolution theorem exactly as in Theorem 5. This proves (5.12). For uniqueness, take the difference of two solutions satisfying the same initial data. Its transform $W_a(p)$ satisfies

$$((p^r)^\alpha-z)W_a(p)=0.$$

Hence $W_a=0$ in the transform region and injectivity gives $w=0$. □

Remark 6. *For $0<\alpha\le 1$, $n=1$ and Theorem 7 reduces exactly to the one-data Caputo formula used below. For $1<\alpha\le 2$, two initial quantities are required, namely $x(a)$, and $D^r x(a)$, and the homogeneous part becomes*

$$x_0E^{(r)}_{\alpha,1}(z\xi_a^\alpha)+x_1\xi_aE^{(r)}_{\alpha,2}(z\xi_a^\alpha).$$

This parallels the classical Caputo theory, with the ordinary displacement $\vartheta-a$ replaced by the local coordinate $\xi_a(\vartheta)$ and the classical gamma coefficients replaced by Γ^r.

6. Examples

The graphical implementation now follows the Cantor-mass interpretation described above. A smooth curve obtained by evaluating ϑ^r as an ordinary real power on a uniform Euclidean grid is useful only as a classical extension; it does not display the fractal support used in Yang's local fractional examples. For a Cantor realization, we instead evaluate the closed-form kernels with

$$\xi_a(\vartheta) = \vartheta^r - a^r \quad \rightsquigarrow \quad \Xi_{C,a}(\vartheta) = F_C(\vartheta) - F_C(a). \tag{6.1}$$

For example, when $a = 0$, the Caputo relaxation kernel is displayed as

$$x_C(\vartheta) = x_0 E^{(r)}_{\alpha,1}(-\lambda[F_C(\vartheta)]^\alpha), \qquad \vartheta \in C, \tag{6.2}$$

rather than by treating $\vartheta^{r\alpha}$ as an ordinary power on all of $[0, 1]$. Notice that the exponent r is not applied a second time to F_C: the Cantor mass already represents the local coordinate of fractal order r. The parameter r continues to enter the local Gamma coefficients of $E^{(r)}_{\alpha,\beta}$ and determines the geometry of the support.

For the ternary Cantor set, the numerical construction is obtained from Bernoulli digits $B_{\ell j} \in \{0, 1\}$:

$$\vartheta_j = \sum_{\ell=1}^{K} 2\,3^{-\ell} B_{\ell j}, \tag{6.3}$$

$$F_C(\vartheta_j) \approx \sum_{\ell=1}^{K} 2^{-\ell} B_{\ell j}. \tag{6.4}$$

This digit map is the finite-level form of (3.1) and (3.2). Yang's MATLAB-style visualizations often use a sorted Cantor sample together with a monotonically increasing mass coordinate and the command `stairs`; the latter is a plotting device, while the fractal structure itself comes from the Cantor support and the Cantor mass. See Chapter 1, Section 1.1.2 of [7] for the original relaxation examples and their figures.

To compare several values of the local parameter r, we use the same idea on a two-branch self-similar Cantor set with the contraction factor

$$c_r = 2^{-1/r}, \qquad 2c_r^r = 1. \tag{6.5}$$

The geometric points and their equal-weight Cantor mass are generated by

$$\vartheta_j^{(r)} = \sum_{\ell=1}^{K} (1 - c_r) c_r^{\ell-1} B_{\ell j}, \tag{6.6}$$

$$F_r(\vartheta_j^{(r)}) \approx \sum_{\ell=1}^{K} 2^{-\ell} B_{\ell j}. \tag{6.7}$$

For $r = r_C = \log 2/\log 3$, one has $c_r = 1/3$, so the geometric construction reduces exactly to the ternary Cantor set. The left panel of each figure fixes this Yang case and varies α. The right panel fixes α and varies r, thereby changing the support dimension while retaining the corresponding Cantor mass coordinate. In addition, every right-hand panel now contains the limiting case $r = 1$. In the case where $c_1 = 1/2$, the two branches fill the whole interval $[0, 1]$, $\mathbb{R}^r$ reduces to the ordinary real setting, and the corresponding solution is drawn on a uniform Euclidean grid as a smooth dashed curve. Thus each figure displays directly the transition from Cantor-supported nondifferentiable profiles for $r < 1$ to the classical smooth profile for $r = 1$. A fixed random seed is used in the supplied MATLAB program so that every figure is reproducible.

Recent papers follow the same Cantor-supported modeling philosophy. Wang and Shi [12] plotted exact local-fractional modified BBM solutions on Cantor sets, Li and Wang [13] constructed nondifferentiable heat-conduction solutions using Mittag–Leffler functions on Cantor sets and Yang's nondifferentiable transformation, Wang [14] displayed the transmission-line dynamics specifically for $r = \log 2/\log 3$, and Du et al. [15] graphed exact local-fractional KdV solutions on Cantor sets using MATLAB. These references show that the use of the Cantor support is not merely a historical illustration but remains part of current local-fractional applications.

Example 1. *Consider the local fractional relaxation problem*

$$ {}_0^C D^{\alpha,r} x(\vartheta) + \lambda x(\vartheta) = 0, \qquad x(0) = x_0. $$

Its solution is

$$ x(\vartheta) = x_0 E^{(r)}_{\alpha,1}(-\lambda(\vartheta^r)^\alpha). \tag{6.8} $$

At $r = 1$, this is the standard Caputo relaxation law. For $r < 1$, the same memory order α is expressed in the local coordinate ϑ^r and through the local gamma function in Definition 3. For the graphical illustration, we take

$$ x_0 = 1, \qquad \lambda = 1. $$

The corresponding profiles are shown in Figure 1.

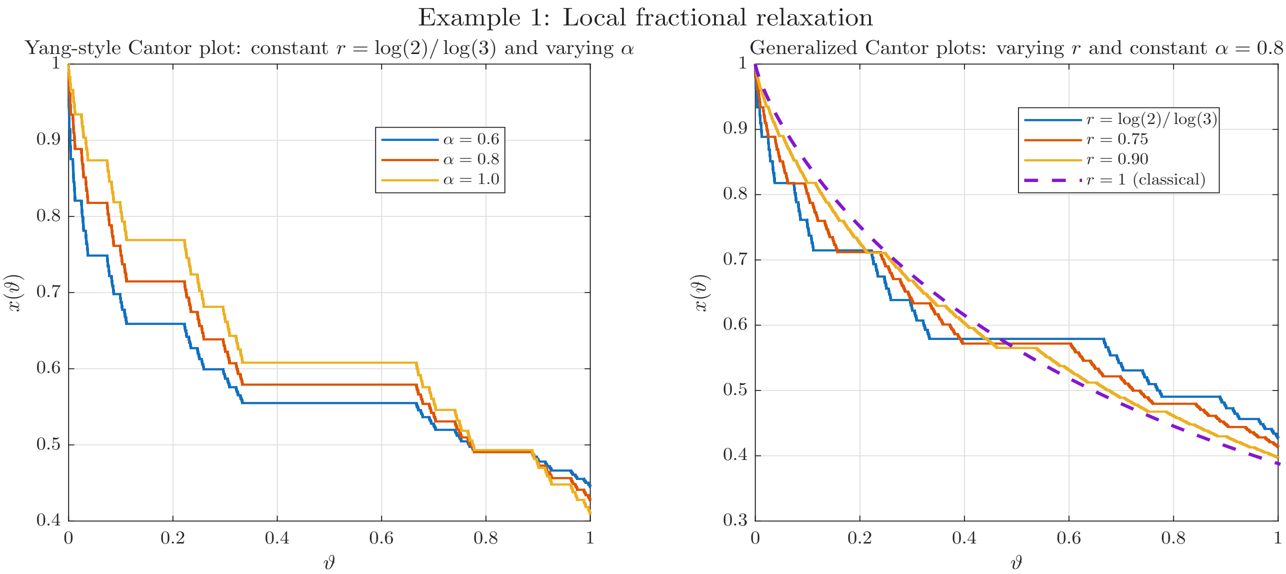

Figure 1. Yang-style profiles for Example 1. Left: $r = \log 2/\log 3$ and $\alpha \in \{0.6, 0.8, 1.0\}$. Right: $\alpha = 0.8$ and $r \in \{\log 2/\log 3, 0.75, 0.90, 1\}$. The staircase profiles correspond to Cantor-supported cases $r < 1$, whereas the dashed $r = 1$ curve gives the classical smooth limit.

Example 2. *Consider*

$$ {}_0^C D^{\alpha,r} x + \lambda x = q, \qquad x(0) = x_0. $$

Since $\mathcal{L}_0^r\{1\} = 1/p^r$, Theorem 7 or Lemma 3 gives

$$ x(\vartheta) = x_0 E_{\alpha,1}^{(r)}(-\lambda\vartheta^{r\alpha}) + q\,\vartheta^{r\alpha} E_{\alpha,\alpha+1}^{(r)}(-\lambda\vartheta^{r\alpha}). \tag{6.9} $$

This example shows explicitly how a nonzero source enters through the second Mittag–Leffler parameter. For the graphical illustration, we take $\lambda = 1, q = 1, x_0 = 0$, where $x_0 = 0$ isolates the effect of the constant source. The corresponding profiles are shown in Figure 2.

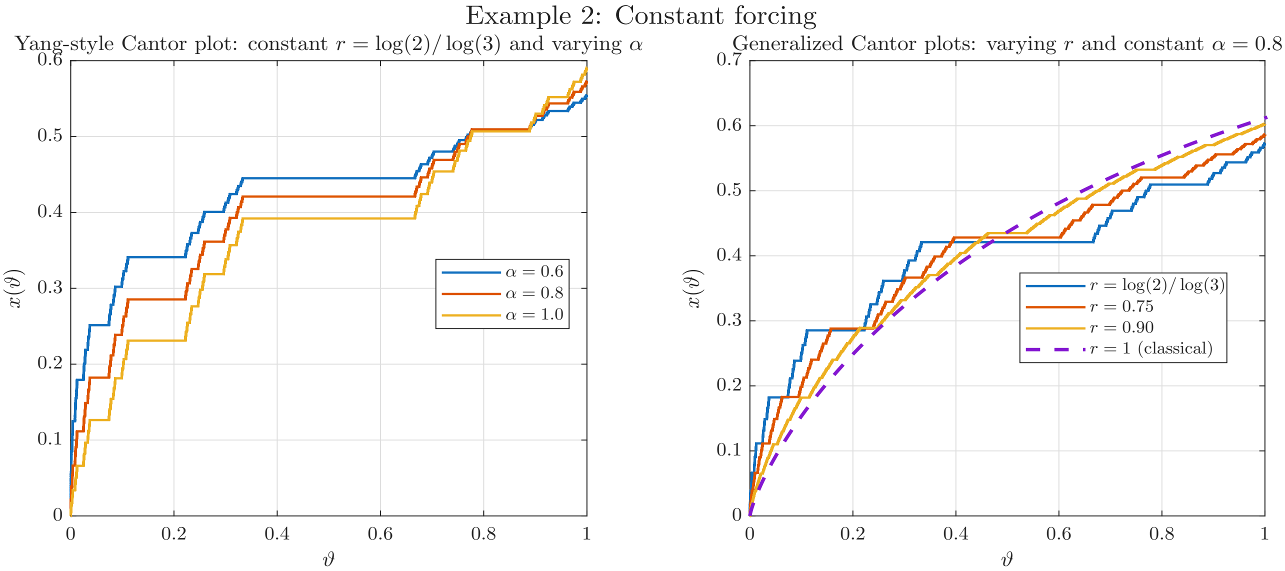

Figure 2. Yang-style profiles for Example 2, with $x_0 = 0$ in the plot to isolate the constant source. Left: $r = \log 2/\log 3$ and $\alpha \in \{0.6, 0.8, 1.0\}$. Right: $\alpha = 0.8$ and $r \in \{\log 2/\log 3, 0.75, 0.90, 1\}$; the dashed $r = 1$ curve is the classical smooth limit.

Example 3. *Let $1 < \alpha \le 2$ and consider*

$$ {}_0^C D^{\alpha,r} x(\vartheta) + \lambda x(\vartheta) = 0, \qquad x(0) = x_0, \qquad D^r x(0) = x_1. $$

Theorem 7 with $n = 2$ and $z = -\lambda$ yields

$$ x(\vartheta) = x_0 E_{\alpha,1}^{(r)}(-\lambda\vartheta^{r\alpha}) + x_1 \vartheta^r E_{\alpha,2}^{(r)}(-\lambda\vartheta^{r\alpha}). \tag{6.10} $$

When $r = 1$, this is precisely the classical two-data Caputo solution. When $\alpha = 2$ and $r = 1$, the Mittag–Leffler functions reduce to the familiar trigonometric representation of the second-order equation $x'' + \lambda x = 0$.

For the graphical illustration, we take

$$ x_0 = 1, \qquad \lambda = 1, \qquad x_1 = 0.5. $$

The corresponding profiles are shown in Figure 3.

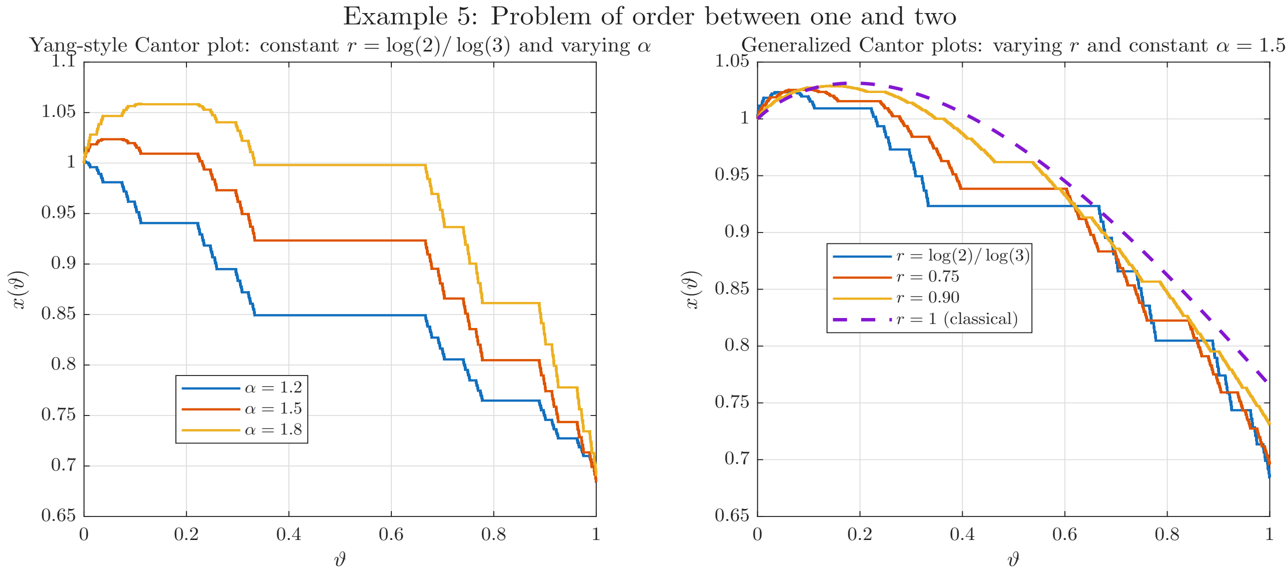

Figure 3. Yang-style profiles for Example 3. Left: $r = \log 2/\log 3$ and $\alpha \in \{1.2, 1.5, 1.8\}$. Right: $\alpha = 1.5$ and $r \in \{\log 2/\log 3, 0.75, 0.90, 1\}$. The dashed $r = 1$ curve shows the classical two-data Caputo limit.

The scalar examples also illustrate how the two parameters should be interpreted in applications. Holding α fixed while changing r modifies the local geometric scaling.

7. Conclusions

We have developed a two-parameter Riemann–Liouville/Caputo construction within Yang's local fractional calculus. The parameter $r \in (0, 1]$ describes the local/fractal geometry associated with the formal set $\mathbb{R}^r$, whereas $\alpha > 0$ is an independent order governing the RL/Caputo-type history operator. This separation is essential for the interpretation of the model: Yang's elementary derivative D^r is local, while the memory effect is introduced through the history kernel $(\vartheta^r - s^r)^{\alpha-1}$ in the proposed integral operators. In this sense, the present framework extends Yang's basic local calculus rather than replacing it, and the classical real setting is recovered when $r = 1$.

The analytical development establishes the local power rules, Yang-Laplace transform formulas, semigroup and composition properties, and relation between the RL and Caputo versions. The local gamma function is retained in the integral form of Definition 3; its computational representation obtained from the Yang–Laplace transform is then used consistently in the generalized local Mittag–Leffler functions. The transform analysis also makes the difference between the natural initial data transparent: the homogeneous RL kernel contains $\xi_a^{\alpha-1} E^{(r)}_{\alpha,\alpha}$, whereas the Caputo kernel contains $E^{(r)}_{\alpha,1}$. Three representative linear problems were considered in the Applications section. Their Cantor-mass visualizations follow Yang's graphical philosophy and illustrate separately the influence of α and r. In particular, the revised figures also include the limiting case where $r = 1$; the fractal staircase profiles for $r < 1$ are thereby compared directly with the corresponding smooth classical solution on $[0, 1]$.

Several directions suggested by the reviewers remain open. Boundary-value and spectral problems should be developed for the new operators, together with coupled systems in which different components may have different local parameters or memory orders. From the numerical point of view, a discrete local fractional calculus, stable iterative solvers, efficient convolution algorithms, and rigorous truncation/error estimates for local Mittag–Leffler evaluations are needed for nonlinear and multidimensional models. Recent two-grid, graded-mesh ADI, and high-order schemes for classical fractional equations indicate useful strategies for stability and accuracy [16–18]. Equally important is the preservation of qualitative physical structures. Motivated in particular by the discrete-maximum-principle-preserving finite-volume methodology in [19], future local-fractal algorithms should be designed to preserve positivity, solution bounds, maximum principles, and, whenever appropriate, discrete energy laws. Numerical tests on genuine fractal geometries, parameter identification for r and α, and extensions to nonlinear local fractional partial differential equations are further natural topics for future investigation.

Author contributions

Kawther K. Alarfaj and Jawaher A. Alzurayq: Formal analysis, Software, Funding, Software. Wasim Raza and Lakhlifa Sadek: Conceptualization, Formal analysis, Software, Methodology, Writing–original draft, Writing–review and editing. All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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