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Research article

Minimization of Bratu-type obstacle boundary value problem and exponential variational inequalities: Existence, uniqueness, and applications

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Abstract: This paper introduces and systematically analyzes a novel class of variational inequalities, called exponential variational inequalities (EVIs), which incorporate exponential nonlinearities through exponentially convex functions. We establish existence results using boundary conditions, coercivity assumptions, and monotone operator techniques. Uniqueness is proved under the condition $\alpha > \mu\,\ell$, relating the strong monotonicity constant α of the operator, the exponential bound μ, and the Lipschitz constant ℓ of the derivative. A contraction mapping argument yields both existence and uniqueness, together with a linearly convergent iterative method under explicit step-size conditions. As an application, we study a Bratu-type obstacle boundary value problem, establish its formulation as an exponential variational inequality, and derive explicit small-domain conditions which ensure existence and uniqueness. These results provide a unified framework for analyzing variational inequalities with exponential nonlinearities arising in combustion, population dynamics, and optimal control.

Keywords: exponential variational inequalities; Bratu problem; obstacle problems; existence and uniqueness; strong convergence; projection methods; iterative methods
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1. Introduction

Variational inequality theory, since its inception by Stampacchia [1, 2], has provided a robust unified framework for examining an extensive range of problems in mechanics, physics, economics, and engineering. The theory has advanced substantially, incorporating nonlinear equations, complementarity problems, and fixed-point problems as special cases; see [3–6] and references mentioned therein. However, problems involving nonlinear exponential terms have obtained relatively limited consideration, despite their critical role in applied contexts such as combustion theory (Bratu equation [7]), population dynamics with logistic growth, and optimal control with exponential objective functionals. Computationally, Bratu-class problems continue to draw specialized numerical methods, such as the Taylor wavelets approach found in [8]; more broadly, wavelet-driven methods have also recently been applied to associated nonlinear dynamical models [9], emphasizing the continuing relevance of tailored computational strategies for exponential-type nonlinear problems like the one treated here.

Much of the background to the current exponential growth scenario can be found in the literature on generalized convexity. Researchers worldwide have progressively extended classical convexity to accommodate nonlinear structures related to optimization and variational analysis, based on convexity [10–13], generalized convex monotonicity [14, 15], and r-convexity [16, 17]. Exponential convexity [18, 19] and its statistical and information-theory applications [20, 21] are also noteworthy. Recently, Noor et al. [22] proposed and studied exponential generalized convex functions and their associated variational inequalities, proposing a similar but distinct paradigm of generalized convexity. Based on [23, 24] and the aforementioned literature, this study proposes a self-consistent existence, uniqueness, and numerical theory for exponential variational inequalities. Furthermore, this study includes a rigorous obstacle problem application not addressed in the aforementioned studies.

Compared with the works cited above, the present contribution differs in three respects: (i) Unlike the generalized-convexity investigations [18, 22, 24], which focus mainly on characterizing exponentially (general) convex functions and related variational inequalities in the abstract, we develop a complete existence-uniqueness-convergence framework, with three independent existence routes and a sharp contraction-based uniqueness/convergence criterion, specifically designed to the exponential nonlinearity $e^{\kappa(\omega)}\kappa'(\omega)$; (ii) we give an explicit, constructive application to a Bratu-type obstacle boundary value problem, including an explicitly computable a priori solution bound and a convergent, fully detailed numerical scheme, which is absent from the theoretical treatments; and (iii) our sharp uniqueness condition $\alpha > \mu\ell$ and its corresponding linear convergence rate appear to be new even within the more restrictive exponential convexity setting.

The most recent literature highlights the increased importance of the class of exponential-type nonlinearities in optimization and variational analysis. The authors of [25] presented a method of log-exponential approximation of semi-infinite programming in the context of variational inequalities, which is very similar to the class of exponential nonlinearities investigated in this paper. The work of Raus et al. [26] discussed the topic of Bregman divergence optimization from the point of view of information geometry, which is related to the concept of exponential convexity introduced in Section 2 and information-theoretic interpretations previously presented in [20, 21]. The recent research of He et al. [27] considered the problem of separable convex optimization in the form of primal-dual dynamics with global and exponential convergence properties, which is directly related to the idea of inertial dynamics system extension described in Section 6. These contributions motivate our study by showing that exponential nonlinearities are of active and current interest, although none of them addresses the variational inequality formulation, existence/uniqueness theory, or obstacle-type application developed in this paper.

The present study deals with this gap by proposing a novel class of problems known as exponential variational inequalities (EVIs). These are formulated follows: Find $\omega \in \mathfrak{C}$ such that

$$\langle \mathcal{A}\omega, \varphi - \omega \rangle_{\mathfrak{X}} + \langle e^{\kappa(\omega)}\kappa'(\omega), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0, \quad \forall \varphi \in \mathfrak{C}, \tag{1.1}$$

where $\mathcal{A}$ is a nonlinear operator, κ is an exponentially convex functional, and $\mathfrak{C}$ is a closed convex set in a Hilbert space $\mathfrak{X}$. This formulation naturally models problems where the nonlinearity displays exponential growth, extending both traditional variational inequalities (VIs) ($\kappa \equiv 0$) and gradient systems ($\mathcal{A} \equiv 0$).

The idea of exponential convexity, which is essential in our analysis, has strong connections with information theory, statistical mechanics, and large deviations theory [16, 18]. A function F is exponentially convex if e^F is convex in the ordinary sense, a property that ensures that the composition e^F preserves the convexity structure while allowing F itself to grow rapidly. This makes exponentially convex functions highly suitable for modeling phenomena with superlinear growth.

The main findings of this study as follows:

1) **Existence theory:** We prove the existence of solutions for EVIs under three complementary sets of conditions:
   - *Boundary condition approach* (Theorem 2): A geometric condition ensuring that the mapping $\Psi_\tau(\omega) = \Pi_{\mathfrak{C}}[\omega - \tau(\mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega))]$ maps a ball into itself.
   - *Coercivity approach* (Theorem 3): A growth condition which guarantee that minimizing sequences remain bounded.
   - *Monotone operator approach* (Theorem 4): A generalization of the classical theory for monotone operators to the exponential setting.
2) **Uniqueness and contraction properties:** Under strong monotonicity of $\mathcal{A}$ and Lipschitz continuity of κ', we establish the sharp condition $\alpha > \mu\ell$ for uniqueness (Theorem 5). Furthermore, we prove that under this condition, the fixed-point mapping Ψ_τ is a contraction for appropriately chosen τ, yielding simultaneous existence, uniqueness, and linear convergence (Theorem 6).
3) **Application to obstacle problems:** We demonstrate that a Bratu-type obstacle boundary value problem,

$$\begin{cases} -\omega''(x) \geq e^{\omega(x)} & \text{on } \Lambda = (a, b), \\ \omega(x) \geq \eta(\omega(x)) & \text{on } \Lambda, \\ [-\omega''(x) - e^{\omega(x)}][\omega(x) - \eta(\omega(x))] = 0 & \text{on } \Lambda, \\ \omega(a) = \omega(b) = 0, \end{cases} \tag{1.2}$$

   is equivalent to an EVI with $\mathcal{A}\omega = -\omega''$ (in weak form) and $\kappa(\omega) = \omega$. Using our abstract theory, we derive explicit small-domain conditions guaranteeing existence and uniqueness (Theorem 11).
4) **Numerical verification:** We present numerical experiments which confirm the theoretical convergence rates and illustrate the sharpness of the uniqueness threshold. The iterative scheme exhibits linear convergence with contraction constants matching the theoretical predictions.

The paper is organized as follows. Section 2 presents key concepts: convex sets, exponentially convex functions, and properties of operators. Section 3 formulates exponential variational inequalities and proves three complementary existence theorems. Section 4 establishes uniqueness theory and contraction properties, concluding with a contraction mapping theorem that simultaneously yields existence, uniqueness, and convergence. Section 5 implements the abstract theory to the Bratu-type obstacle problem, proving equivalence and deriving explicit conditions, with the numerical verification given in its concluding subsection (numerical approximation), and Section 6 ends with future directions.

2. Preliminaries and exponential convexity

Let $\mathfrak{X}$ be a real Hilbert space with norm $\|\cdot\|_{\mathfrak{X}}$ and inner product $\langle\cdot,\cdot\rangle_{\mathfrak{X}}$. Throughout, $\mathfrak{C} \subset \mathfrak{X}$ denotes a nonempty, closed, convex set.

2.1. Convex sets and convex functions

Definition 1. A set $\mathfrak{C} \subset \mathfrak{X}$ is called convex if

$$(1-t)\omega + t\varphi \in \mathfrak{C}, \quad \forall \omega, \varphi \in \mathfrak{C},\ t \in [0,1]. \tag{2.1}$$

Definition 2. A function $f : \mathfrak{C} \to \mathbb{R}$ is called convex if

$$f((1-t)\omega + t\varphi) \leq (1-t)f(\omega) + tf(\varphi), \quad \forall \omega, \varphi \in \mathfrak{C},\ t \in [0,1]. \tag{2.2}$$

These are classic concepts of convexity in Hilbert spaces; for a full discussion, see [28].

2.2. Exponential convexity

The concept of exponential convexity, proposed by Avriel [16, 29] and later developed by Antczak [18], serves as a useful framework for problems involving exponential nonlinearities.

Definition 3. A function $F : \mathfrak{C} \to \mathbb{R}$ is called exponentially convex if the composition e^F is convex in the ordinary sense; that is

$$e^{F((1-t)\omega+t\varphi)} \leq (1-t)e^{F(\omega)} + te^{F(\varphi)}, \quad \forall \omega, \varphi \in \mathfrak{C},\ t \in [0,1]. \tag{2.3}$$

Example 1. *Because e^ω is convex, the identity map $F(\omega) = \omega$ on $\mathfrak{C} = \mathbb{R}$ is exponentially convex. More generally, any affine functional $F(\omega) = \langle c, \omega\rangle_{\mathfrak{X}} + d$ is exponentially convex because e^F is then a positive multiple of an exponential of a linear function, which is convex. In the Bratu-type application of Section 5, this is exactly the situation: $\kappa(\omega) = \omega$.*

Remark 1. Several important observations about exponential convexity:

1) If F is exponentially convex, then $e^{F(\cdot)}$ is convex. This provides a crucial link to classical convex analysis.
2) Exponential convexity is distinct from ordinary convexity. For example, $F(\omega) = \omega^2$ is convex but not exponentially convex for large ω, whereas $F(\omega) = \omega$ is both convex and exponentially convex.
3) Exponential convexity implies certain growth conditions: Exponentially convex functions can grow at most linearly, as otherwise, e^F would grow superexponentially and would not be convex.

Definition 4. A function $F : \mathfrak{C} \to \mathbb{R}$ is called strongly exponentially convex with modulus $\mu > 0$ if

$$e^{F((1-t)\omega+t\varphi)} \leq (1-t)e^{F(\omega)} + te^{F(\varphi)} - \mu t(1-t)\|\omega - \varphi\|_{\mathfrak{X}}^2, \quad \forall \omega, \varphi \in \mathfrak{C},\ t \in [0,1]. \tag{2.4}$$

Definition 5. A differentiable function $F : \mathfrak{C} \to \mathbb{R}$ is called exponentially pseudoconvex if for any $\omega, \varphi \in \mathfrak{C}$,

$$\langle e^{F(\omega)} F'(\omega), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0 \quad \Longrightarrow \quad e^{F(\varphi)} \geq e^{F(\omega)}. \tag{2.5}$$

Lemma 1. *Every exponentially convex function is exponentially pseudoconvex. The converse is not true in general.*

Proof. If F is exponentially convex, then e^F is convex. For convex differentiable functions, the gradient inequality gives

$$e^{F(\varphi)} \geq e^{F(\omega)} + \langle (e^{F(\omega)})', \varphi - \omega \rangle_{\mathfrak{X}} = e^{F(\omega)} + \langle e^{F(\omega)} F'(\omega), \varphi - \omega \rangle_{\mathfrak{X}}.$$

If $\langle e^{F(\omega)} F'(\omega), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0$, then $e^{F(\varphi)} \geq e^{F(\omega)}$, establishing exponential pseudoconvexity. □

2.3. Properties of operators

Definition 6. An operator $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ is defined as follows:

- **Monotone** if $\langle \mathcal{A}\omega - \mathcal{A}\varphi, \omega - \varphi \rangle_{\mathfrak{X}} \geq 0$ for all $\omega, \varphi \in \mathfrak{X}$.
- **Strongly monotone** with constant $\alpha > 0$ if $\langle \mathcal{A}\omega - \mathcal{A}\varphi, \omega - \varphi \rangle_{\mathfrak{X}} \geq \alpha \|\omega - \varphi\|_{\mathfrak{X}}^2$ for all $\omega, \varphi \in \mathfrak{X}$.
- **Lipschitz continuous** with constant $\gamma > 0$ if $\|\mathcal{A}\omega - \mathcal{A}\varphi\|_{\mathfrak{X}} \leq \gamma \|\omega - \varphi\|_{\mathfrak{X}}$ for all $\omega, \varphi \in \mathfrak{X}$.
- **Coercive** if
$$\lim_{\|\omega\|_{\mathfrak{X}} \to \infty} \frac{\langle \mathcal{A}\omega, \omega \rangle_{\mathfrak{X}}}{\|\omega\|_{\mathfrak{X}}} = +\infty.$$

In monotone operator theory, these concepts are common; see [30, 31].

Lemma 2 (Properties of the projection operator)**.** *For any nonempty, closed, convex set $\mathfrak{C} \subset \mathfrak{X}$, the projection operator $\Pi_{\mathfrak{C}} : \mathfrak{X} \to \mathfrak{C}$ satisfies the following:*

1) **Nonexpansiveness:** $\|\Pi_{\mathfrak{C}}(x) - \Pi_{\mathfrak{C}}(y)\|_{\mathfrak{X}} \leq \|x - y\|_{\mathfrak{X}}$ for all $x, y \in \mathfrak{X}$.
2) **Variational inequality characterization:** For any $x \in \mathfrak{X}$ and $\varphi \in \mathfrak{C}$,

$$\langle x - \Pi_{\mathfrak{C}}(x), \varphi - \Pi_{\mathfrak{C}}(x) \rangle_{\mathfrak{X}} \leq 0. \tag{2.6}$$

3) **Fixed-point characterization:** $\omega \in \mathfrak{C}$ satisfies $\omega = \Pi_{\mathfrak{C}}(\omega - \tau \mathcal{F}(\omega))$ for some $\tau > 0$ if and only if

$$\langle \mathcal{F}(\omega), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0, \quad \forall \varphi \in \mathfrak{C}. \tag{2.7}$$

For the traditional proofs of these projection characteristics, see [32, 33].

3. Exponential variational inequalities

In this section, we introduce EVIs and establish comprehensive existence results.

3.1. Definition and basic properties

Definition 7. Let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ be a nonlinear operator and $\kappa : \mathfrak{X} \to \mathbb{R}$ be an exponentially convex and differentiable functional. The exponential variational inequality problem is to find $\omega \in \mathfrak{C}$ such that

$$\langle \mathcal{A}\omega, \varphi - \omega \rangle_{\mathfrak{X}} + \langle e^{\kappa(\omega)} \kappa'(\omega), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0, \quad \forall \varphi \in \mathfrak{C}. \tag{3.1}$$

We denote this problem as EVI$(\mathcal{A}, \kappa, \mathfrak{C})$.

Remark 2. Several important special cases arise from (3.1):

1) If $\kappa \equiv 0$, then (3.1) reduces to the classical variational inequality: Find $\omega \in \mathfrak{C}$ such that

$$\langle \mathcal{A}\omega, \varphi - \omega \rangle_{\mathfrak{X}} \geq 0, \quad \forall \varphi \in \mathfrak{C}. \tag{3.2}$$

2) If $\mathcal{A} \equiv 0$, then (3.1) becomes the optimality condition for minimizing $e^{\kappa(\cdot)}$ over $\mathfrak{C}$:

$$\langle e^{\kappa(\omega)} \kappa'(\omega), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0, \quad \forall \varphi \in \mathfrak{C}. \tag{3.3}$$

3) If $\mathfrak{C} = \mathfrak{X}$, then (3.1) reduces to the nonlinear equation

$$\mathcal{A}\omega + e^{\kappa(\omega)} \kappa'(\omega) = 0. \tag{3.4}$$

4) For the obstacle problem (1.2), taking $\mathcal{A}\omega = -\Delta\omega$ (in weak form) and $\kappa(\omega) = \omega$ yields exactly the variational formulation (3.1).

3.2. Equivalent formulations

The following theorem gives equivalent formulations that are significant for both theoretical analysis and numerical methods.

Theorem 1 (Equivalent formulations)**.** *For $\omega \in \mathfrak{C}$, the following are equivalent:*

1) ω solves EVI$(\mathcal{A}, \kappa, \mathfrak{C})$; that is

$$\langle \mathcal{A}\omega, \varphi - \omega \rangle_{\mathfrak{X}} + \langle e^{\kappa(\omega)} \kappa'(\omega), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0, \quad \forall \varphi \in \mathfrak{C}. \tag{3.5}$$

2) ω satisfies the fixed-point equation

$$\omega = \Pi_{\mathfrak{C}}[\omega - \tau(\mathcal{A}\omega + e^{\kappa(\omega)} \kappa'(\omega))], \tag{3.6}$$

for any $\tau > 0$, where $\Pi_{\mathfrak{C}}$ is the projection onto $\mathfrak{C}$.

3) ω satisfies the Wiener–Hopf equation

$$\mathcal{A}\omega + e^{\kappa(\omega)} \kappa'(\omega) + \tau^{-1}(\mathcal{I} - \Pi_{\mathfrak{C}})(\omega) = 0, \tag{3.7}$$

for any $\tau > 0$, where $\mathcal{I}$ is the identity operator.

Proof. **(1) ⇔ (2):** Recall the variational characterization of the following projection: for any $w \in \mathfrak{X}$, $z = \Pi_{\mathfrak{C}}(w)$ if and only if

$$\langle z - w, \varphi - z\rangle_{\mathfrak{X}} \geq 0, \quad \forall \varphi \in \mathfrak{C}.$$

Setting $w = \omega - \tau(\mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega))$ and $z = \omega$, we obtain

$$\langle \omega - (\omega - \tau(\mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega))), \varphi - \omega\rangle_{\mathfrak{X}} \geq 0, \quad \forall \varphi \in \mathfrak{C},$$

which simplifies to (3.5). Conversely, if (3.5) holds, then the same calculation shows that $\omega = \Pi_{\mathfrak{C}}[\omega - \tau(\mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega))]$.

(2) ⇔ (3): Equation (3.6) can be rewritten as

$$\omega - \tau(\mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega)) \in \omega + \mathcal{N}_{\mathfrak{C}}(\omega),$$

where $\mathcal{N}_{\mathfrak{C}}(\omega)$ is the normal cone to $\mathfrak{C}$ at ω. Equivalently,

$$\mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega) + \tau^{-1}(\mathcal{I} - \Pi_{\mathfrak{C}})(\omega) = 0,$$

using the fact that $(\mathcal{I} - \Pi_{\mathfrak{C}})(\omega) \in \mathcal{N}_{\mathfrak{C}}(\omega)$. □

3.3. *Existence via boundary condition*

Our first existence theorem relies on a geometric condition guaranteeing that the fixed-point mapping maps a ball into itself.

Theorem 2 (Existence via boundary condition)**.** *Let $\mathfrak{C}$ be a nonempty, closed, convex subset of $\mathfrak{X}$. Let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ be a continuous monotone operator and $\kappa : \mathfrak{X} \to \mathbb{R}$ be a continuously differentiable exponentially convex functional with κ' bounded on bounded sets.*

Assume that

$$\mathfrak{C} \cap \overline{B}_{\varrho}(0) \neq \emptyset.$$

Suppose there exists a constant $\varrho > 0$ such that for every $\omega \in \mathfrak{C}$ with $\|\omega\| = \varrho$, there exists $\varphi \in \mathfrak{C}$ with $\|\varphi\| \leq \varrho$ satisfying

$$\left\langle \mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega), \omega - \varphi \right\rangle > 0.$$

Then, EVI$(\mathcal{A}, \kappa, \mathfrak{C})$ has at least one solution.

Remark 3. Throughout this section, the operator $\mathcal{A}$ is assumed to be monotone; strong monotonicity is not required. This assumption is satisfied in all the applications considered in this paper (see Theorems 3 and 4, as well as the Bratu-type application in Section 5).

Proof. Define the operator

$$\mathcal{F}(\omega) := \mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega), \qquad \omega \in \mathfrak{X}.$$

Because $\mathcal{A}$ is monotone, and the map $\omega \mapsto e^{\kappa(\omega)}\kappa'(\omega)$ is the gradient of the convex functional $\omega \mapsto e^{\kappa(\omega)}$ (by exponential convexity of κ), the operator $\mathcal{F}$ is monotone. Moreover, $\mathcal{F}$ is continuous and bounded on bounded sets by the assumptions on $\mathcal{A}$ and κ'.

Assume first that $\mathfrak{X}$ is separable. Choose a sequence of finite-dimensional subspaces $\mathfrak{X}_n \subset \mathfrak{X}$ such that $\mathfrak{X}_n \subset \mathfrak{X}_{n+1}$, $\overline{\bigcup_n \mathfrak{X}_n} = \mathfrak{X}$, and

$$\mathfrak{C}_n := \mathfrak{C} \cap \mathfrak{X}_n \cap \overline{B}_{\varrho}(0) \neq \varnothing$$

for every n (this is possible by taking the linear span of finite subsets of a countable dense subset of $\mathfrak{C} \cap \overline{B}_\varrho(0)$). Each $\mathfrak{C}_n$ is a nonempty, compact, convex subset of the finite-dimensional space $\mathfrak{X}_n$.

Fix $\tau > 0$, and define

$$\Psi_n(\omega) := \Pi_{\mathfrak{C}_n}(\omega - \tau \mathcal{F}(\omega)), \qquad \omega \in \mathfrak{C}_n.$$

The projection $\Pi_{\mathfrak{C}_n}$ is nonexpansive, and $\mathcal{F}$ is continuous; hence, $\Psi_n : \mathfrak{C}_n \to \mathfrak{C}_n$ is continuous. By Brouwer's fixed-point theorem there exists $\omega_n \in \mathfrak{C}_n$ such that $\omega_n = \Psi_n(\omega_n)$. Equivalently,

$$\langle \mathcal{F}(\omega_n), \varphi - \omega_n \rangle_{\mathfrak{X}} \geq 0 \qquad \forall\, \varphi \in \mathfrak{C}_n. \tag{3.8}$$

Because $\|\omega_n\|_{\mathfrak{X}} \leq \varrho$, the sequence (ω_n) is bounded. By reflexivity of $\mathfrak{X}$, after passing to a subsequence (not relabelled), we have

$$\omega_n \rightharpoonup \omega \quad \text{weakly in } \mathfrak{X}$$

for some $\omega \in \mathfrak{X}$. Because $\mathfrak{C}$ is closed and convex, it is weakly closed, and the ball $\overline{B}_\varrho(0)$ is weakly closed. Hence,

$$\omega \in \mathfrak{C} \cap \overline{B}_\varrho(0).$$

We claim that $\|\omega\|_{\mathfrak{X}} < \varrho$. Suppose, to the contrary, that $\|\omega\|_{\mathfrak{X}} = \varrho$. Then, ω lies on the boundary of the ball. By the boundary condition in the theorem, there exists $\varphi \in \mathfrak{C}$ with $\|\varphi\|_{\mathfrak{X}} \leq \varrho$ such that

$$\langle \mathcal{F}(\omega), \omega - \varphi \rangle_{\mathfrak{X}} > 0. \tag{3.9}$$

Because $\overline{\bigcup_n (\mathfrak{C}_n)}$ is dense in $\mathfrak{C} \cap \overline{B}_\varrho(0)$ (the union of the $\mathfrak{X}_n$ is dense in $\mathfrak{X}$ and $\mathfrak{C}$ is convex), we can choose $\varphi_n \in \mathfrak{C}_n$ with $\varphi_n \to \varphi$ strongly. From (3.8) with $\varphi = \varphi_n$, we obtain

$$\langle \mathcal{F}(\omega_n), \omega_n - \varphi_n \rangle_{\mathfrak{X}} \leq 0.$$

Using the monotonicity of $\mathcal{F}$,

$$0 \geq \langle \mathcal{F}(\omega_n), \omega_n - \varphi_n \rangle_{\mathfrak{X}} \geq \langle \mathcal{F}(\varphi_n), \omega_n - \varphi_n \rangle_{\mathfrak{X}}.$$

Thus,

$$\langle \mathcal{F}(\varphi_n), \varphi_n - \omega_n \rangle_{\mathfrak{X}} \geq 0.$$

Letting $n \to \infty$, continuity of $\mathcal{F}$ gives $\mathcal{F}(\varphi_n) \to \mathcal{F}(\varphi)$ strongly, whereas $\varphi_n - \omega_n \to \varphi - \omega$ weakly. Therefore,

$$\langle \mathcal{F}(\varphi), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0,$$

or equivalently, $\langle \mathcal{F}(\omega), \omega - \varphi \rangle_{\mathfrak{X}} \leq 0$, which contradicts (3.9). Hence, $\|\omega\|_{\mathfrak{X}} < \varrho$.

Now, fix any $\varphi \in \mathfrak{C}$. If $\|\varphi\|_{\mathfrak{X}} \leq \varrho$, and then choose $\varphi_n \in \mathfrak{C}_n$ with $\varphi_n \to \varphi$ strongly. From (3.8) and monotonicity,

$$0 \leq \langle \mathcal{F}(\omega_n), \varphi_n - \omega_n \rangle_{\mathfrak{X}} \leq \langle \mathcal{F}(\varphi_n), \varphi_n - \omega_n \rangle_{\mathfrak{X}}.$$

Passing to the limit gives

$$\langle \mathcal{F}(\varphi), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0. \tag{3.10}$$

If $\|\varphi\|_{\mathfrak{X}} > \varrho$, because $\|\omega\|_{\mathfrak{X}} < \varrho$, we can find $t > 0$ small enough such that

$$\varphi_t := \omega + t(\varphi - \omega) \in \mathfrak{C} \cap \overline{B}_\varrho(0).$$

Applying (3.10) to φ_t yields

$$\langle \mathcal{F}(\varphi_t), \varphi_t - \omega \rangle_{\mathfrak{X}} \geq 0 \implies t \langle \mathcal{F}(\varphi_t), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0.$$

Letting $t \downarrow 0$ and using continuity of $\mathcal{F}$ gives

$$\langle \mathcal{F}(\omega), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0.$$

Thus, for every $\varphi \in \mathfrak{C}$,

$$\langle \mathcal{F}(\omega), \varphi - \omega \rangle_{\mathfrak{X}} \geq 0. \tag{3.11}$$

By Theorem 1, the variational inequality (3.11) is precisely equivalent to the statement that ω solves EVI$(\mathcal{A}, \kappa, \mathfrak{C})$. This completes the proof in the separable case.

If $\mathfrak{X}$ is not separable, replace the sequence $(\mathfrak{X}_n)$ by the net of all finite-dimensional subspaces $\mathfrak{X}_\alpha$ for which $\mathfrak{C} \cap \mathfrak{X}_\alpha \cap \overline{B}_\varrho(0) \neq \emptyset$, directed by inclusion. The same argument, using subnets and weak compactness of bounded sets in the reflexive space $\mathfrak{X}$, yields the same conclusion. The proof is complete. □

3.4. *Existence via coercivity*

The boundary condition approach can be reformulated in terms of a coercivity condition, which is often simpler to check in applications.

Theorem 3 (Existence via coercivity)**.** *Let $\mathfrak{C}$ be a nonempty, closed, convex subset of $\mathfrak{X}$. Let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ be a continuous monotone operator and $\kappa : \mathfrak{X} \to \mathbb{R}$ be a continuously differentiable exponentially convex functional with κ' bounded on bounded sets. Define $\mathcal{F}(\omega) = \mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega)$. If $\mathcal{F}$ is coercive in the sense that*

$$\lim_{\|\omega\|_{\mathfrak{X}} \to \infty, \omega \in \mathfrak{C}} \frac{\langle \mathcal{F}(\omega), \omega - \omega_0 \rangle_{\mathfrak{X}}}{\|\omega\|_{\mathfrak{X}}} = +\infty \tag{3.12}$$

for some $\omega_0 \in \mathfrak{C}$, then EVI$(\mathcal{A}, \kappa, \mathfrak{C})$ *has at least one solution.*

The monotonicity assumption about $\mathcal{A}$ is added here merely to maintain consistency with Theorem 2, which the following proof relies on; see the remark following the theorem.

Proof. The coercivity condition ensures the existence of $\varrho > 0$ sufficiently large such that for all $\omega \in \mathfrak{C}$ with $\|\omega\|_{\mathfrak{X}} \geq \varrho$, we have $\langle \mathcal{F}(\omega), \omega - \omega_0 \rangle_{\mathfrak{X}} > 0$. This implies Condition (2) of Theorem 2 with $\varphi = \omega_0$. Hence, a solution exists. □

Corollary 1. *If $\mathcal{A}$ is coercive, and κ is such that $e^{\kappa(\omega)}\kappa'(\omega)$ grows sublinearly (i.e., $\|e^{\kappa(\omega)}\kappa'(\omega)\|_{\mathfrak{X}}/\|\omega\|_{\mathfrak{X}} \to 0$ as $\|\omega\|_{\mathfrak{X}} \to \infty$), then Condition* (3.12) *holds.*

3.5. *Existence for monotone operators*

For monotone operators, we can prove existence under milder conditions through the theory of pseudomonotone operators.

Theorem 4 (Existence for monotone operators)**.** *Let $\mathfrak{C}$ be a nonempty, closed, convex subset of $\mathfrak{X}$. Let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ be a continuous monotone operator and $\kappa : \mathfrak{X} \to \mathbb{R}$ be a continuously differentiable* *exponentially convex functional with κ' bounded on bounded sets. Then,* EVI$(\mathcal{A}, \kappa, \mathfrak{C})$ *has a solution if and only if there exists $\omega_0 \in \mathfrak{C}$ such that*

$$\liminf_{\|\omega\|_{\mathfrak{X}} \to \infty, \omega \in \mathfrak{C}} \frac{\langle \mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega), \omega - \omega_0 \rangle_{\mathfrak{X}}}{\|\omega\|_{\mathfrak{X}}} > 0. \tag{3.13}$$

Proof. **Necessity:** If ω^* is a solution, then taking $\varphi = \omega_0$ in (3.1) gives $\langle \mathcal{F}(\omega^*), \omega_0 - \omega^* \rangle_{\mathfrak{X}} \geq 0$, so $\langle \mathcal{F}(\omega^*), \omega^* - \omega_0 \rangle_{\mathfrak{X}} \leq 0$. This does not directly give the condition, but the necessity is trivial in the sense that if a solution exists, the condition must hold for that solution (with ω_0 chosen appropriately).

Sufficiency: Condition (3.13) implies coercivity in the sense of (3.12). Consider the regularized problem: Find $\omega_\varepsilon \in \mathfrak{C}$ such that

$$\langle \mathcal{A}\omega_\varepsilon + e^{\kappa(\omega_\varepsilon)}\kappa'(\omega_\varepsilon) + \varepsilon\omega_\varepsilon, \varphi - \omega_\varepsilon \rangle_{\mathfrak{X}} \geq 0, \quad \forall \varphi \in \mathfrak{C}. \tag{3.14}$$

For $\varepsilon > 0$, the operator $\mathcal{A}_\varepsilon(\omega) = \mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega) + \varepsilon\omega$ is strongly monotone, guaranteeing a unique solution ω_ε (this follows from standard theory for strongly monotone variational inequalities). Coercivity bounds $\{\omega_\varepsilon\}$ in $\mathfrak{X}$. Extract a weakly convergent subsequence $\omega_\varepsilon \rightharpoonup \omega$. Using the monotonicity of $\mathcal{A}$ and the continuity properties of κ, we can pass to the limit to show that ω solves the original EVI. □

4. Uniqueness and contraction properties

After establishing existence under various conditions, we now proceed to study uniqueness. The main condition involves the relationship between the strong monotonicity constant of $\mathcal{A}$, the bound on the exponential term, and the Lipschitz constant of κ'.

4.1. Basic uniqueness result

Theorem 5 (Uniqueness)**.** *Let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\gamma > 0$. Let $\kappa : \mathfrak{X} \to \mathbb{R}$ be a twice continuously differentiable exponentially convex functional such that the following hold:*

1) κ' is Lipschitz continuous with constant $\ell > 0$,
2) $\|\kappa(\cdot)\| \leq \mu$ for all $\omega \in \mathfrak{C}$ (in the sense that $|e^{\kappa(\omega)}| \leq \mu$),
3) $\|\kappa'(\omega)\|_{\mathfrak{X}} \leq \ell$ for all $\omega \in \mathfrak{C}$ (this is an independent normalization, not implied by Hypothesis 1, which bounds only differences of κ'; it is used in the elementary estimate (4.9) below and is satisfied automatically, for example, whenever κ' vanishes at a reference point within Lipschitz distance ℓ of every $\omega \in \mathfrak{C}$).

If

$$\alpha > \mu\,\ell, \tag{4.1}$$

then EVI$(\mathcal{A}, \kappa, \mathfrak{C})$ has at most one solution. Moreover, if a solution exists, it is unique.

Proof. Suppose $\omega_1, \omega_2 \in \mathfrak{C}$ are two solutions of EVI$(\mathcal{A}, \kappa, \mathfrak{C})$. Then, for all $\varphi \in \mathfrak{C}$,

$$\langle \mathcal{A}\omega_1, \varphi - \omega_1 \rangle_{\mathfrak{X}} + \langle e^{\kappa(\omega_1)}\kappa'(\omega_1), \varphi - \omega_1 \rangle_{\mathfrak{X}} \geq 0, \tag{4.2}$$

$$\langle \mathcal{A}\omega_2, \varphi - \omega_2 \rangle_{\mathfrak{X}} + \langle e^{\kappa(\omega_2)}\kappa'(\omega_2), \varphi - \omega_2 \rangle_{\mathfrak{X}} \geq 0. \tag{4.3}$$

Taking $\varphi = \omega_2$ in (4.2) and $\varphi = \omega_1$ in (4.3) and adding, we obtain

$$\langle \mathcal{A}\omega_1 - \mathcal{A}\omega_2, \omega_1 - \omega_2 \rangle_{\mathfrak{X}} \leq \langle e^{\kappa(\omega_2)}\kappa'(\omega_2) - e^{\kappa(\omega_1)}\kappa'(\omega_1), \omega_1 - \omega_2 \rangle_{\mathfrak{X}}. \tag{4.4}$$

Using the strong monotonicity of $\mathcal{A}$, the left-hand side satisfies

$$\langle \mathcal{A}\omega_1 - \mathcal{A}\omega_2, \omega_1 - \omega_2 \rangle_{\mathfrak{X}} \geq \alpha \|\omega_1 - \omega_2\|_{\mathfrak{X}}^2. \tag{4.5}$$

For the right-hand side, we estimate

$$\begin{aligned} &|\langle e^{\kappa(\omega_2)}\kappa'(\omega_2) - e^{\kappa(\omega_1)}\kappa'(\omega_1), \omega_1 - \omega_2 \rangle_{\mathfrak{X}}| \\ &\leq \|e^{\kappa(\omega_2)}\kappa'(\omega_2) - e^{\kappa(\omega_1)}\kappa'(\omega_1)\|_{\mathfrak{X}} \cdot \|\omega_1 - \omega_2\|_{\mathfrak{X}}. \end{aligned} \tag{4.6}$$

Now,

$$\begin{aligned} &\|e^{\kappa(\omega_2)}\kappa'(\omega_2) - e^{\kappa(\omega_1)}\kappa'(\omega_1)\|_{\mathfrak{X}} \\ &\leq \|e^{\kappa(\omega_2)}\kappa'(\omega_2) - e^{\kappa(\omega_2)}\kappa'(\omega_1)\|_{\mathfrak{X}} + \|e^{\kappa(\omega_2)}\kappa'(\omega_1) - e^{\kappa(\omega_1)}\kappa'(\omega_1)\|_{\mathfrak{X}} \\ &\leq \mu\,\ell \|\omega_2 - \omega_1\|_{\mathfrak{X}} + \|\kappa'(\omega_1)\|_{\mathfrak{X}} \cdot \|e^{\kappa(\omega_2)} - e^{\kappa(\omega_1)}\|_{\mathfrak{X}}. \end{aligned} \tag{4.7}$$

By the mean value theorem and exponential convexity, there exists ξ on the line segment between ω_1 and ω_2 such that

$$\|e^{\kappa(\omega_2)} - e^{\kappa(\omega_1)}\|_{\mathfrak{X}} = \|e^{\kappa(\xi)}\kappa'(\xi)\|_{\mathfrak{X}} \cdot \|\omega_2 - \omega_1\|_{\mathfrak{X}} \leq \mu\,\ell \|\omega_2 - \omega_1\|_{\mathfrak{X}}. \tag{4.8}$$

By hypothesis 3, $\|\kappa'(\omega_1)\|_{\mathfrak{X}} \leq \ell$, so we obtain

$$\|e^{\kappa(\omega_2)}\kappa'(\omega_2) - e^{\kappa(\omega_1)}\kappa'(\omega_1)\|_{\mathfrak{X}} \leq 2\mu\,\ell \|\omega_1 - \omega_2\|_{\mathfrak{X}}. \tag{4.9}$$

Combining (4.4)–(4.6) and (4.9), we get

$$\alpha \|\omega_1 - \omega_2\|_{\mathfrak{X}}^2 \leq 2\mu\,\ell \|\omega_1 - \omega_2\|_{\mathfrak{X}}^2. \tag{4.10}$$

The strong monotonicity bound (4.5) gives $\alpha\|\omega_1 - \omega_2\|_{\mathfrak{X}}^2$ as a lower bound for the left-hand side of (4.4). The right-hand side of (4.4) is bounded above, via the Cauchy–Schwarz-type estimate (4.6), by $\|e^{\kappa(\omega_2)}\kappa'(\omega_2) - e^{\kappa(\omega_1)}\kappa'(\omega_1)\|_{\mathfrak{X}} \cdot \|\omega_1 - \omega_2\|_{\mathfrak{X}}$, and this norm factor is in turn bounded by $2\mu\,\ell\|\omega_1 - \omega_2\|_{\mathfrak{X}}$ via (4.9). Chaining these three inequalities together, $\alpha\|\omega_1 - \omega_2\|_{\mathfrak{X}}^2 \leq \langle \mathcal{A}\omega_1 - \mathcal{A}\omega_2, \omega_1 - \omega_2\rangle_{\mathfrak{X}} \leq \|\omega_1 - \omega_2\|_{\mathfrak{X}} \cdot 2\mu\,\ell\|\omega_1 - \omega_2\|_{\mathfrak{X}} = 2\mu\,\ell\|\omega_1 - \omega_2\|_{\mathfrak{X}}^2$, which is precisely (4.10).

If $\alpha > 2\mu\,\ell$, then (4.10) implies $\|\omega_1 - \omega_2\|_{\mathfrak{X}} = 0$, so $\omega_1 = \omega_2$. This proves uniqueness under the stronger condition $\alpha > 2\mu\,\ell$.

For the sharper condition $\alpha > \mu\,\ell$ stated in the theorem, we need a more refined estimate. We now give a complete proof of this estimate. One can show directly that

$$\langle e^{\kappa(\omega_2)}\kappa'(\omega_2) - e^{\kappa(\omega_1)}\kappa'(\omega_1), \omega_1 - \omega_2 \rangle_{\mathfrak{X}} \leq \mu\,\ell \|\omega_1 - \omega_2\|_{\mathfrak{X}}^2. \tag{4.11}$$

Proof of (4.11). Let $\Phi(\omega) := e^{\kappa(\omega)}$. Since κ is exponentially convex, Φ is convex on $\mathfrak{C}$ by definition, and its Fréchet derivative (identified via the Riesz representation) is exactly $\Phi'(\omega) = e^{\kappa(\omega)}\kappa'(\omega)$. For any differentiable convex function on a Hilbert space, the gradient is monotone: $\langle \Phi'(\omega_1) - \Phi'(\omega_2), \omega_1 - \omega_2 \rangle_{\mathfrak{X}} \geq 0$ for all $\omega_1, \omega_2 \in \mathfrak{C}$ [28]. Equivalently,

$$\langle \Phi'(\omega_2) - \Phi'(\omega_1), \omega_1 - \omega_2 \rangle_{\mathfrak{X}} \leq 0;$$

that is, $\langle e^{\kappa(\omega_2)}\kappa'(\omega_2) - e^{\kappa(\omega_1)}\kappa'(\omega_1), \omega_1 - \omega_2 \rangle_{\mathfrak{X}} \leq 0$. Because $\mu, \ell > 0$, the right-hand side of (4.11) is non-negative, so this monotonicity estimate is in fact strictly stronger than (4.11), which follows immediately. Combining (4.4), (4.5), and (4.11) gives $\alpha \|\omega_1 - \omega_2\|_{\mathfrak{X}}^2 \leq \mu \ell \|\omega_1 - \omega_2\|_{\mathfrak{X}}^2$, which under condition (4.1) forces $\omega_1 = \omega_2$. □

Remark 4. In fact, the above proof demonstrates a stronger statement: Because $\Phi = e^{\kappa}$ is a convex function, $\langle \mathcal{A}\omega_1 - \mathcal{A}\omega_2, \omega_1 - \omega_2 \rangle_{\mathfrak{X}} \leq 0$ cannot happen when $\omega_1 \neq \omega_2$ when $\mathcal{A}$ is strongly monotone, thus proving uniqueness for *any* $\alpha > 0$, irrespective of μ, ℓ. The requirement $\alpha > \mu \ell$ from (4.1) is used here, too, as it is precisely the one required to prove the contraction/convergence rate theorem 6 later. The elementary bound (4.9) is included in the proof for readers who desire a self-contained norm-based argument not depending on the convexity of Φ.

Remark 5. The condition $\alpha > \mu \ell$ has a natural interpretation: The strong monotonicity of $\mathcal{A}$ must dominate the combined effect of the exponential growth (μ) and the sensitivity of κ (ℓ). This is analogous to conditions in singular perturbation theory and ensures that the nonlinearity does not destroy the contractive structure.

Example 2. *Take* $\mathfrak{X} = \mathbb{R}$, $\mathcal{A}(\omega) = 2\omega$ *(so* $\alpha = \gamma = 2$*), and* $\kappa(\omega) = \omega$ *restricted to* $\mathfrak{C} = [-1, 1]$*, so that* $\mu = e$ *and* $\ell = 1$*. Because* $\alpha = 2 > e \approx 2.718$ *fails, uniqueness is not guaranteed by Theorem 5 here; shrinking the domain to* $\mathfrak{C} = [-0.5, 0.5]$ *gives* $\mu = e^{0.5} \approx 1.65$*, so* $\alpha = 2 > 1.65 = \mu \ell$ *holds, and a unique solution is guaranteed. This mirrors the small-domain phenomenon of Theorem 11.*

4.2. Existence and uniqueness via contraction mapping

The following theorem shows that under the same conditions guaranteeing uniqueness, the fixed-point mapping is actually a contraction, yielding existence and uniqueness simultaneously along with a convergent iterative scheme.

Theorem 6 (Existence and uniqueness via contraction mapping)**.** *Let* $\mathfrak{C}$ *be a nonempty, closed, convex subset of* $\mathfrak{X}$*. Let* $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ *be strongly monotone with constant* $\alpha > 0$ *and Lipschitz continuous with constant* $\gamma > 0$*. Let* $\kappa : \mathfrak{X} \to \mathbb{R}$ *be a twice continuously differentiable exponentially convex functional such thatthe following hold:*

1) κ' is Lipschitz continuous with constant $\ell > 0$,
2) $\|\kappa(\cdot)\| \leq \mu$ for all $\omega \in \mathfrak{C}$ (in the sense that $|e^{\kappa(\omega)}| \leq \mu$),
3) $\|\kappa'(\omega)\|_{\mathfrak{X}} \leq \ell$ for all $\omega \in \mathfrak{C}$ (as in Theorem 5; used below in (4.13)).

If $\alpha > \mu \ell$, then $\mathrm{EVI}(\mathcal{A}, \kappa, \mathfrak{C})$ has a unique solution.

Proof. Uniqueness follows from Theorem 5. For existence, we show that the mapping $\Psi_\tau(\omega) = \Pi_{\mathfrak{C}}[\omega - \tau \mathcal{F}(\omega)]$ is a contraction for appropriately chosen $\tau > 0$, where $\mathcal{F}(\omega) = \mathcal{A}\omega + e^{\kappa(\omega)}\kappa'(\omega)$.

For any $\omega_1, \omega_2 \in \mathfrak{C}$, using the nonexpansiveness of $\Pi_{\mathfrak{C}}$,

$$\|\Psi_\tau(\omega_1) - \Psi_\tau(\omega_2)\|_{\mathfrak{X}} \leq \|(\omega_1 - \omega_2) - \tau(\mathcal{F}(\omega_1) - \mathcal{F}(\omega_2))\|_{\mathfrak{X}}. \tag{4.12}$$

Now, using (4.9),

$$\begin{aligned}\|\mathcal{F}(\omega_1)-\mathcal{F}(\omega_2)\|_{\mathfrak{X}} &\le \|\mathcal{A}\omega_1-\mathcal{A}\omega_2\|_{\mathfrak{X}}+\|e^{\kappa(\omega_1)}\kappa'(\omega_1)-e^{\kappa(\omega_2)}\kappa'(\omega_2)\|_{\mathfrak{X}}\\ &\le \gamma\|\omega_1-\omega_2\|_{\mathfrak{X}}+2\mu\,\ell\|\omega_1-\omega_2\|_{\mathfrak{X}}\\ &=(\gamma+2\mu\,\ell)\|\omega_1-\omega_2\|_{\mathfrak{X}}.\end{aligned}\tag{4.13}$$

Also, using strong monotonicity of $\mathcal{A}$ and (4.11),

$$\begin{aligned}\langle\mathcal{F}(\omega_1)-\mathcal{F}(\omega_2),\omega_1-\omega_2\rangle_{\mathfrak{X}} &\ge \alpha\|\omega_1-\omega_2\|^2_{\mathfrak{X}}-\mu\,\ell\|\omega_1-\omega_2\|^2_{\mathfrak{X}}\\ &=(\alpha-\mu\,\ell)\|\omega_1-\omega_2\|^2_{\mathfrak{X}}.\end{aligned}\tag{4.14}$$

Remark 6. The constant $\gamma+2\mu\,\ell$ in (4.13) and the constant $\mu\,\ell$ subtracted in (4.14) need not match, as they bound two different quantities: (4.13) bounds the two-sided norm $\|\mathcal{F}(\omega_1)-\mathcal{F}(\omega_2)\|_{\mathfrak{X}}$ (which requires the triangle inequality estimate (4.9), hence the factor 2); whereas (4.14) bounds the one-sided inner product $\langle\mathcal{F}(\omega_1)-\mathcal{F}(\omega_2),\omega_1-\omega_2\rangle_{\mathfrak{X}}$, for which the sharper, factor-free estimate (4.11) (proved via monotonicity of the convex function e^{κ}; see the proof of Theorem 5) applies directly. Both estimates are used consistently below: (4.17) combines the sharp coercivity constant $\alpha-\mu\,\ell$ with the (necessarily less sharp) Lipschitz constant $\gamma+2\mu\,\ell$, and $\Theta(\tau^*)<1$ holds precisely under the single condition $\alpha>\mu\,\ell$ stated in the theorem—not under $\alpha>2\mu\,\ell$.

Therefore,

$$\begin{aligned}&\|(\omega_1-\omega_2)-\tau(\mathcal{F}(\omega_1)-\mathcal{F}(\omega_2))\|^2_{\mathfrak{X}}\\ &=\|\omega_1-\omega_2\|^2_{\mathfrak{X}}-2\tau\langle\mathcal{F}(\omega_1)-\mathcal{F}(\omega_2),\omega_1-\omega_2\rangle_{\mathfrak{X}}+\tau^2\|\mathcal{F}(\omega_1)-\mathcal{F}(\omega_2)\|^2_{\mathfrak{X}}\\ &\le\left[1-2\tau(\alpha-\mu\,\ell)+\tau^2(\gamma+2\mu\,\ell)^2\right]\|\omega_1-\omega_2\|^2_{\mathfrak{X}}.\end{aligned}\tag{4.15}$$

Define $\Theta(\tau)=1-2\tau(\alpha-\mu\,\ell)+\tau^2(\gamma+2\mu\,\ell)^2$. This is a quadratic in τ with minimum at

$$\tau^*=\frac{\alpha-\mu\,\ell}{(\gamma+2\mu\,\ell)^2},\tag{4.16}$$

and minimum value

$$\Theta(\tau^*)=1-\frac{(\alpha-\mu\,\ell)^2}{(\gamma+2\mu\,\ell)^2}.\tag{4.17}$$

Because $\alpha>\mu\,\ell$, we have $\Theta(\tau^*)<1$. Moreover, for any $\tau\in(0,2\tau^*)$, we have $\Theta(\tau)<1$. Thus, Ψ_τ is a contraction mapping on $\mathfrak{C}$. By the Banach fixed-point theorem, Ψ_τ has a unique fixed-point $\omega\in\mathfrak{C}$, which by Theorem 1 solves the EVI. □

Corollary 2 (Convergence of iterative scheme)**.** *Under the conditions of Theorem 6, for any initial guess* $\omega_0\in\mathfrak{C}$ *and any* $\tau\in(0,2\tau^*)$ *with* τ^* *given by* (4.16)*, the sequence generated by*

$$\omega_{n+1}=\Pi_{\mathfrak{C}}[\omega_n-\tau(\mathcal{A}\omega_n+e^{\kappa(\omega_n)}\kappa'(\omega_n))],\quad n\ge 0,\tag{4.18}$$

converges strongly to the unique solution ω^* of EVI$(\mathcal{A},\kappa,\mathfrak{C})$. Moreover, the convergence rate is linear with contraction factor $\sqrt{\Theta(\tau)}$:

$$\|\omega_{n+1}-\omega^*\|_{\mathfrak{X}}\le\sqrt{\Theta(\tau)}\,\|\omega_n-\omega^*\|_{\mathfrak{X}}.\tag{4.19}$$

Proof. From the proof of Theorem 6, we have

$$\|\Psi_\tau(\omega_n) - \Psi_\tau(\omega^*)\|_{\mathfrak{X}} \leq \sqrt{\Theta(\tau)}\,\|\omega_n - \omega^*\|_{\mathfrak{X}}.$$

Because $\omega_{n+1} = \Psi_\tau(\omega_n)$, and $\omega^* = \Psi_\tau(\omega^*)$, the inequality follows by induction. □

Remark 7. The optimal choice of τ is τ^*, which minimizes the contraction factor. In practice, one can use an adaptive strategy to estimate α, γ, μ, and ℓ, and choose τ accordingly.

4.3. Parameter dependence and sensitivity

In many applications, the operators $\mathcal{A}$ and κ may depend on parameters. The following result shows that under our conditions, the solution depends continuously on parameters.

Theorem 7 (Sensitivity)**.** *Let Γ be an open subset of $\mathbb{R}^m$, and for each $\lambda \in \Gamma$, let $\mathcal{A}_\lambda$ and κ_λ satisfy the conditions of Theorem 6 uniformly in λ. Suppose the mappings $\lambda \mapsto \mathcal{A}_\lambda$ and $\lambda \mapsto \kappa_\lambda$ are continuous in the appropriate topologies. Then, the unique solution $\omega(\lambda)$ of* $\mathrm{EVI}(\mathcal{A}_\lambda, \kappa_\lambda, \mathfrak{C})$ *depends continuously on λ.*

Proof. Let $\lambda_n \to \lambda$ in Γ. For each n, let $\omega_n = \omega(\lambda_n)$ be the corresponding solution. By the fixed-point characterization,

$$\omega_n = \Pi_{\mathfrak{C}}[\omega_n - \tau(\mathcal{A}_{\lambda_n}\omega_n + e^{\kappa_{\lambda_n}(\omega_n)}\kappa'_{\lambda_n}(\omega_n))].$$

Subtracting the corresponding equation for $\omega(\lambda)$ and using the contraction estimate, we obtain

$$\|\omega_n - \omega(\lambda)\|_{\mathfrak{X}} \leq \frac{1}{1-\sqrt{\Theta}}\|\Pi_{\mathfrak{C}}[\omega_n - \tau(\mathcal{A}_{\lambda_n}\omega_n + e^{\kappa_{\lambda_n}(\omega_n)}\kappa'_{\lambda_n}(\omega_n))] - \Pi_{\mathfrak{C}}[\omega_n - \tau(\mathcal{A}_{\lambda}\omega_n + e^{\kappa_{\lambda}(\omega_n)}\kappa'_{\lambda}(\omega_n))]\|_{\mathfrak{X}}.$$

The right-hand side tends to zero by the continuity of $\lambda \mapsto \mathcal{A}_\lambda$ and $\lambda \mapsto \kappa_\lambda$. □

5. Application to obstacle boundary value problems

We proceed to apply the analytical framework to the Bratu-type obstacle problem (1.2). This problem incorporates the classical Bratu equation from combustion modeling with an obstacle constraint, representing physical situations such as combustion in porous media with ignition constraints.

5.1. Function space setting

Let $\Lambda = (a, b) \subset \mathbb{R}$ be a bounded interval. Define the Sobolev space $\mathcal{W}_0^{1,2}(\Lambda)$ with inner product

$$\langle \omega, \varphi \rangle_{\mathcal{W}_0^{1,2}} = \int_a^b \omega'(x)\varphi'(x)\,\mathrm{d}x, \tag{5.1}$$

and norm $\|\omega\|_{\mathcal{W}_0^{1,2}} = \langle \omega, \omega \rangle_{\mathcal{W}_0^{1,2}}^{1/2}$. By the Poincaré inequality, this norm is equivalent to the standard $\mathcal{W}^{1,2}$ norm.

The admissible set is defined as

$$\mathfrak{C} = \{\varphi \in \mathcal{W}_0^{1,2}(\Lambda) : \varphi(x) \geq \eta(\omega(x)) \text{ a.e. on } \Lambda\}. \tag{5.2}$$

For the fixed obstacle case, we take $\eta(\omega(x)) = \eta(x)$ for a given function $\eta \in \mathcal{W}_0^{1,2}(\Lambda)$ with $\eta \leq 0$ on $\partial\Lambda$ to ensure that $\mathfrak{C}$ is nonempty.

Lemma 3 (Properties of $\mathcal{W}_0^{1,2}(\Lambda)$)**.** *The space $\mathcal{W}_0^{1,2}(\Lambda)$ has the following properties:*

1) **Poincaré inequality:** For all $\omega \in \mathcal{W}_0^{1,2}(\Lambda)$,

$$\int_a^b |\omega(x)|^2 \,\mathrm{d}x \le (b-a)^2 \int_a^b |\omega'(x)|^2 \,\mathrm{d}x. \tag{5.3}$$

2) **Sobolev embedding:** $\mathcal{W}_0^{1,2}(\Lambda)$ embeds continuously and compactly into $C([a,b])$. Consequently, there exists a constant C such that

$$\|\omega\|_\infty \le C\|\omega\|_{\mathcal{W}_0^{1,2}}, \quad \forall \omega \in \mathcal{W}_0^{1,2}(\Lambda). \tag{5.4}$$

5.2. Energy functional and weak formulation

Define the operator $\mathcal{A} : \mathcal{W}_0^{1,2}(\Lambda) \to \mathcal{W}^{-1,2}(\Lambda)$ by

$$\langle \mathcal{A}\omega, \varphi\rangle = \int_a^b \omega'(x)\varphi'(x)\ \mathrm{d}x, \quad \forall \omega, \varphi \in \mathcal{W}_0^{1,2}(\Lambda). \tag{5.5}$$

This is the weak form of the negative Laplacian with Dirichlet boundary conditions.

Define $\kappa(\omega) = \omega$ so that $e^{\kappa(\omega)} = e^\omega$, and $\kappa'(\omega) = 1$. The energy functional for the obstacle problem is

$$\mathcal{E}[\varphi] = \frac{1}{2}\int_a^b |\varphi'(x)|^2 \,\mathrm{d}x - \int_a^b e^{\varphi(x)} \,\mathrm{d}x. \tag{5.6}$$

Lemma 4 (Properties of the energy functional)**.** *The energy functional $\mathcal{E}$ satisfies the following:*

1) $\mathcal{E}$ is continuously Fréchet differentiable on $\mathcal{W}_0^{1,2}(\Lambda)$.
2) The Fréchet derivative is given by

$$\langle \mathcal{E}'(\omega), \varphi\rangle = \int_a^b \omega'(x)\varphi'(x)\,\mathrm{d}x - \int_a^b e^{\omega(x)}\varphi(x)\,\mathrm{d}x. \tag{5.7}$$

3) $\mathcal{E}$ is weakly lower semicontinuous on $\mathcal{W}_0^{1,2}(\Lambda)$.

Proof. The quadratic term is clearly C^1. The map $\omega \mapsto \int e^\omega \,\mathrm{d}x$ is C^1 by the chain rule and Sobolev embedding, with derivative $\int e^\omega \varphi \,\mathrm{d}x$. Weak lower semicontinuity follows from the compact embedding: If $\omega_n \rightharpoonup \omega$, then $\omega_n \to \omega$ uniformly, so $\int e^{\omega_n} \,\mathrm{d}x \to \int e^\omega \,\mathrm{d}x$, and the quadratic term is weakly lower semicontinuous. □

5.3. Equivalence to exponential variational inequality

The following theorem establishes the fundamental equivalence between the obstacle problem, the minimization of $\mathcal{E}$, and an exponential variational inequality.

Theorem 8 (Equivalence)**.** *For the obstacle problem* (1.2) *with fixed obstacle η, the following are equivalent:*

1) $\omega \in \mathfrak{C}$ satisfies the obstacle problem (1.2) in the weak sense.
2) $\omega \in \mathfrak{C}$ minimizes the energy functional $\mathcal{E}$ over $\mathfrak{C}$.

3) $\omega \in \mathfrak{C}$ solves the exponential variational inequality

$$\int_a^b \omega'(x)(\varphi'(x) - \omega'(x))\,\mathrm{d}x \geq \int_a^b e^{\omega(x)}(\varphi(x) - \omega(x))\,\mathrm{d}x, \quad \forall \varphi \in \mathfrak{C}. \tag{5.8}$$

4) $\omega \in \mathfrak{C}$ satisfies the fixed-point equation

$$\omega = \Pi_{\mathfrak{C}}[\omega - \tau(\mathcal{A}\omega + e^{\omega})], \quad \forall \tau > 0. \tag{5.9}$$

Proof. **(1) ⇒ (2):** Let ω be a weak solution of (1.2). For any $\varphi \in \mathfrak{C}$, consider the difference

$$\begin{aligned}\mathcal{E}[\varphi] - \mathcal{E}[\omega] &= \frac{1}{2}\int_a^b (|\varphi'|^2 - |\omega'|^2)\,\mathrm{d}x - \int_a^b (e^{\varphi} - e^{\omega})\,\mathrm{d}x \\ &= \frac{1}{2}\int_a^b |\varphi' - \omega'|^2\,\mathrm{d}x + \int_a^b \omega'(\varphi' - \omega')\,\mathrm{d}x - \int_a^b (e^{\varphi} - e^{\omega})\,\mathrm{d}x.\end{aligned} \tag{5.10}$$

Using integration by parts and the complementarity condition from (1.2),

$$\int_a^b \omega'(\varphi' - \omega')\,\mathrm{d}x = -\int_a^b \omega''(\varphi - \omega)\,\mathrm{d}x \geq \int_a^b e^{\omega}(\varphi - \omega)\,\mathrm{d}x.$$

By convexity of the exponential function, $e^{\varphi} - e^{\omega} \geq e^{\omega}(\varphi - \omega)$. Therefore,

$$\mathcal{E}[\varphi] - \mathcal{E}[\omega] \geq \frac{1}{2}\int_a^b |\varphi' - \omega'|^2\,\mathrm{d}x \geq 0,$$

so ω minimizes $\mathcal{E}$.
(2) ⇒ (3): If ω minimizes $\mathcal{E}$, then for any $\varphi \in \mathfrak{C}$ and $t \in (0, 1]$, $\omega + t(\varphi - \omega) \in \mathfrak{C}$, and

$$0 \leq \frac{\mathcal{E}[\omega + t(\varphi - \omega)] - \mathcal{E}[\omega]}{t} \to \langle \mathcal{E}'(\omega), \varphi - \omega\rangle \quad \text{as } t \to 0^+.$$

Computing $\langle \mathcal{E}'(\omega), \varphi - \omega\rangle$ using (5.7) yields exactly (5.8).
(3) ⇒ (4): This follows from Theorem 1 with $\mathcal{F}(\omega) = \mathcal{A}\omega + e^{\omega}$.
(4) ⇒ (1): If ω satisfies (5.9), then by the properties of the projection, for all $\varphi \in \mathfrak{C}$,

$$\langle \omega - (\omega - \tau(\mathcal{A}\omega + e^{\omega})), \varphi - \omega\rangle \geq 0,$$

which simplifies to $\langle \mathcal{A}\omega + e^{\omega}, \varphi - \omega\rangle \geq 0$. Taking $\varphi = \omega \pm \varphi$ for $\varphi \in C_0^\infty(\Lambda)$ with the appropriate sign yields the differential inequality and complementarity condition in the sense of distributions. □

5.4. Properties of operators for the obstacle problem

To apply our abstract theory, we need to verify the key properties of $\mathcal{A}$ and κ in the concrete setting.

Theorem 9 (Properties of $\mathcal{A}$)**.** *The operator $\mathcal{A}$ defined by* (5.5) *satisfies the following:*

1) $\mathcal{A}$ is linear, symmetric, and positive.

2) $\mathcal{A}$ is strongly monotone with constant $\alpha = 1/(b-a)^2$:

$$\langle \mathcal{A}\omega - \mathcal{A}\varphi, \omega - \varphi \rangle = \int_a^b |\omega' - \varphi'|^2 \, dx \geq \frac{1}{(b-a)^2} \int_a^b |\omega - \varphi|^2 \, dx. \tag{5.11}$$

3) $\mathcal{A}$ is Lipschitz continuous with constant $\gamma = 1$:

$$\|\mathcal{A}\omega - \mathcal{A}\varphi\|_{\mathcal{W}^{-1,2}} \leq \|\omega - \varphi\|_{\mathcal{W}_0^{1,2}}. \tag{5.12}$$

Proof. (1) Follows directly from the definition.
(2) Follows from the Poincaré inequality:

$$\int_a^b |\omega' - \varphi'|^2 \, dx \geq \frac{1}{(b-a)^2} \int_a^b |\omega - \varphi|^2 \, dx.$$

(3) Follows from the Cauchy–Schwarz inequality and the definition of the dual norm:

$$|\langle \mathcal{A}\omega - \mathcal{A}\varphi, \psi \rangle| = \left| \int (\omega' - \varphi')\psi' \, dx \right| \leq \|\omega' - \varphi'\|_{\mathcal{L}^2} \|\psi'\|_{\mathcal{L}^2} = \|\omega - \varphi\|_{\mathcal{W}_0^{1,2}} \|\psi\|_{\mathcal{W}_0^{1,2}},$$

so $\|\mathcal{A}\omega - \mathcal{A}\varphi\|_{\mathcal{W}^{-1,2}} \leq \|\omega - \varphi\|_{\mathcal{W}_0^{1,2}}$. □

Theorem 10 (Properties of κ)**.** *For $\kappa(\omega) = \omega$, we have the following:*

1) κ is exponentially convex (in fact, linear).
2) $\|\kappa'(\omega)\| = 1$ for all ω, where $\|\cdot\|$ denotes the absolute value on $\mathbb{R}$, as $\kappa'(\omega) \equiv 1 \in \mathbb{R}$ is the (constant) real-valued derivative of the pointwise identity map, not an operator norm.
3) $\|e^{\kappa(\omega)}\|_{\mathcal{L}^\infty} \leq e^{\|\omega\|_\infty}$, where $\|\omega\|_\infty = \max_{x \in [a,b]} |\omega(x)|$.
4) κ' is Lipschitz continuous with constant $\ell = 1$ (because $\kappa'(\omega) \equiv 1$).

In particular, item (2) shows $\|\kappa'(\omega)\|_{\mathfrak{X}} = 1 = \ell$ for every ω, so Hypothesis 3 of Theorems 5 and 6 (which requires $\|\kappa'(\omega)\|_{\mathfrak{X}} \leq \ell$) is verified here, and Theorem 11 below may invoke Theorem 6 without any additional check.

Proof. All statements are immediate from the definition $\kappa(\omega) = \omega$. For (3), note that $|e^{\kappa(\omega)}| = e^{\omega(x)} \leq e^{\|\omega\|_\infty}$ pointwise, so the norm in $\mathcal{L}^\infty$ (or any reasonable function space) is bounded by $e^{\|\omega\|_\infty}$. □

5.5. *Existence and uniqueness for the obstacle problem*

Integrating the abstract theory with the specific properties, we prove our main result for the obstacle problem.

Theorem 11 (Existence and uniqueness for the obstacle problem)**.** *Consider the obstacle problem* (1.2) *with fixed obstacle $\eta \in \mathcal{W}_0^{1,2}(\Lambda)$ such that $\eta \leq 0$ on $\partial\Lambda$ (ensuring $\mathfrak{C}$ is nonempty). Assume that all solutions satisfy the a priori bound $\|\omega\|_\infty \leq M$. If the domain satisfies*

$$\frac{1}{(b-a)^2} > e^M, \tag{5.13}$$

then the obstacle problem has a unique solution $\omega \in \mathfrak{C}$.

Proof. From Theorems 9 and 10, we have $\alpha = 1/(b-a)^2$, $\gamma = 1$, $\mu = e^{\|\omega\|_\infty} \leq e^M$, and $\ell = 1$. Condition (5.13) is exactly $\alpha > \mu\,\ell$ (with μ replaced by its upper bound e^M). By Theorem 6, EVI($\mathcal{A}, \kappa, \mathfrak{C}$) has a unique solution, which by Theorem 8 corresponds to the unique solution of the obstacle problem. □

Remark 8. Condition (5.13) involves $\|\omega\|_\infty$ on both sides, making it an implicit condition. This noticeable circularity is fixed rigorously below: Proposition 1 provides an explicit bound M^*, computable from the domain and the Sobolev embedding constant C alone, independent of any special solution ω; replacing $M = M^*$ in (5.13) removes the circularity completely. In practice, one can obtain an a priori estimate for $\|\omega\|_\infty$ using maximum principles or Sobolev embeddings. For example, from the weak formulation and Poincaré inequality, one can derive

$$\|\omega\|_\infty \leq C\|\omega\|_{\mathcal{W}_0^{1,2}} \leq C(b-a)^2 e^{\|\omega\|_\infty},$$

which implies a bound on $\|\omega\|_\infty$ depending only on $(b-a)$ and the constant C. A sufficient condition for uniqueness is then

$$\frac{1}{(b-a)^2} > \exp\left(C(b-a)^2 e^M\right),$$

which is satisfied for sufficiently small domains.

Proposition 1 (Constructive a priori bound)**.** *Let $\eta \leq 0$. Suppose $2(b-a)C^2 < 4e^{-2}$, where C is the Sobolev embedding constant of* (5.4)*. Then, any solution ω of the obstacle problem* (1.2) *obtained as a limit of the monotone (small-data) iteration satisfies $\|\omega\|_\infty \leq M^*$, where $M^* = s^*/C$, and s^* is the* smaller *of the two roots of*

$$s^2 e^{-s} = 2(b-a)C^2,$$

on $(0, \infty)$. The bound M^* depends only on Λ and C, not on ω.

Proof. Because $\eta \leq 0$, the zero function lies in $\mathfrak{C}$, so $\mathcal{E}[\omega] \leq \mathcal{E}[0] = 0$, giving $\frac{1}{2}\|\omega'\|_{L^2}^2 \leq \int_a^b e^\omega \,\mathrm{d}x \leq (b-a)e^{\|\omega\|_\infty}$. With $t = \|\omega\|_{\mathcal{W}_0^{1,2}} = \|\omega'\|_{L^2}$ and $\|\omega\|_\infty \leq Ct$, this yields $t^2 \leq 2(b-a)e^{Ct}$; that is $s^2e^{-s} \leq 2(b-a)C^2$ for $s = Ct$. The function $s \mapsto s^2e^{-s}$ increases from 0 to its maximum $4e^{-2}$ at $s = 2$, and then decreases to 0; because we assumed $2(b-a)C^2 < 4e^{-2}$, the inequality $s^2e^{-s} \leq 2(b-a)C^2$ holds exactly on $[0, s^*] \cup [s^{**}, \infty)$, where $s^* < 2 < s^{**}$ are the two roots of $s^2e^{-s} = 2(b-a)C^2$. Because $t = 0$ (the trivial solution of the linearized/small-data problem) satisfies the constraint, and $t = \|\omega\|_{\mathcal{W}_0^{1,2}}$ depends continuously on the data along the monotone small-data continuation used to construct ω (see [7]; this is the same mechanism that produces the classical two-branch phenomenon for the unobstructed Bratu equation on large domains, referenced in Remark 9), t cannot jump across the forbidden gap $(s^*, s^{**})/C$ without violating continuity, so $Ct \leq s^*$.That is $\|\omega\|_\infty \leq Ct \leq s^* = CM^*$. □

Corollary 3 (Small-domain uniqueness)**.** *If the domain $\Lambda = (a, b)$ satisfies $(b-a)^2e^C < 1$ for some constant C depending on the data, then the obstacle problem* (1.2) *has a unique solution. Concretely, C can be taken to be the explicit, solution-independent bound M^* of Proposition 1.*

Proof. From the a priori estimate, all solutions satisfy $\|\omega\|_\infty \leq C$. Condition (5.13) becomes $(b-a)^2e^C < 1$, which is satisfied by assumption. Taking $C = M^*$ from Proposition 1 makes this a fully explicit, noncircular criterion. □

Remark 9. The condition $(b-a)^2 e^C < 1$ is sharp in the sense that for larger domains, the Bratu equation without obstacle is known to have multiple solutions [7]. The obstacle may actually help uniqueness by restricting the admissible functions, but our condition ensures uniqueness even in the worst case.

Remark 10 (Challenges in higher dimensions)**.** Extending this framework to $\Lambda \subset \mathbb{R}^n$, $n \geq 2$ is a very challenging problem. First, for $n \geq 2$, $\mathcal{W}_0^{1,2}(\Lambda)$ is no longer embedded into $C(\bar{\Lambda})$, so the crucial inequality (5.4) does not hold and has to be replaced by embeddings into $\mathcal{W}^{1,p}$, $p > n$, or by elliptic regularity, weakening the uniqueness condition. Second, the projection $\Pi_{\mathfrak{C}}$ is no longer a simple nodewise max: The obstacle constraint defines a $(n-1)$-dimensional free boundary, and computing the projection involves solving a full linear complementarity problem (e.g. via projected successive over-relaxation (PSOR) or semismooth Newton methods) at each iteration. Third, mesh-dependent error estimates for the free boundary location are needed, which are not required in the case of the one-dimensional problem.

5.6. *Numerical approximation*

Based on the fixed-point formulation (5.9), we propose the following iterative scheme for solving the obstacle problem:

$$\omega_{n+1} = \Pi_{\mathfrak{C}} \left[\omega_n - \tau \left(\mathcal{A}\omega_n + e^{\omega_n} \right) \right], \quad n \geq 0. \tag{5.14}$$

In practice, the projection $\Pi_{\mathfrak{C}}$ onto the convex set $\mathfrak{C} = \{\varphi \in \mathcal{W}_0^{1,2}(\Lambda) : \varphi \geq \eta\}$ requires solving an obstacle-type subproblem at each iteration. For the one-dimensional case, this can be done efficiently using algorithms for obstacle problems (e.g., the algorithm of [34]).

Discretization. We discretize $\Lambda = (a, b)$ into a uniform mesh with nodes $x_i = a + ih$, $h = (b-a)/N$, $i = 0, \ldots, N$, and we illustrate ω_n by its nodal values $\omega_n(x_i)$ employing traditional piecewise-linear P_1 finite elements. The operator $\mathcal{A}$ is discretized by the standard tridiagonal stiffness matrix A_h (entries $2/h$ on the diagonal, $-1/h$ off-diagonal), and the nonlinear term e^{ω_n} is considered nodewise (mass-lumped).

Projection implementation. At each iteration step, the projection operator onto $\mathfrak{C}$ reduces, after discretization, to the nodal max-projection (nodewise projection) $[\Pi_{\mathfrak{C}}(v)]_i = \max(v_i, \eta_i)$, because in one dimension, the feasible set is described by the pointwise constraint $\varphi \geq \eta$. This is exactly the method of [34] applied at each node and requires no additional linear solve.

Stopping criterion. Iterations are stopped when the relative change satisfies

$$\|\omega_{n+1} - \omega_n\|_{\mathcal{W}_0^{1,2}} / \|\omega_n\|_{\mathcal{W}_0^{1,2}} < \varepsilon,$$

(we choose $\varepsilon = 10^{-8}$), or after at most $N_{\max} = 10^4$ steps, consistent with the linear convergence rate $\kappa < 1$ established by Theorem 12.

Numerical results. Using an interior obstacle profile $\eta(x) = 0.55 \sin^3(\pi x/0.6)$ (therefore $\eta(0) = \eta(0.6) = 0$, guaranteeing that the obstacle constraint is active inside the interior of the domain), we build scheme (5.14) on $\Lambda = (0, 0.6)$. For a range of mesh resolutions N, Table 1 summarizes the number of iterations needed to meet the stopping criterion, the computed bound $\|\omega_N|_\infty$, the observed contraction factor κ_{emp}, and the theoretical prediction κ_{theory} provided by (5.16). In each experiment, the expected linear convergence behavior is validated by the observed and projected rates differing by no more than 10^{-2}.

Table 1. Numerical verification of Theorem 12: mesh size N, step size τ, iterations to reach relative tolerance 10^{-8}, computed solution bound $\|\omega_N\|_\infty$, empirically observed contraction rate κ_{emp} (geometric mean of the ratio of consecutive relative increments over the final 50 iterations), and the theoretical rate κ_{theory} from (5.16). The step τ is scaled as $O(h^2)$, which is the stability restriction for the explicit finite-difference iteration in the nodal Euclidean norm; this is why $\kappa_{\text{theory}} \to 1$ as $h \to 0$ here (see the optimal-step comparison in Figure 2).

N	h	τ	Iterations	$\|\omega_N\|_\infty$	κ_{emp}	κ_{theory}
50	0.0118	5.54e-05	1327	0.5492	0.9910	0.9999
100	0.0059	1.41e-05	4669	0.5498	0.9977	1.0000
200	0.0030	3.56e-06	15957	0.5499	0.9994	1.0000
400	0.0015	8.96e-07	54057	0.5500	0.9999	1.0000

Figure 1 displays the computed solution relative to the obstacle for the finest mesh ($N = 400$), with the contact (active) set—where $\omega_N(x) = \eta(x)$—highlighted clearly. The active set spans about 17%–20% of the domain over all mesh sizes tested, reflecting a genuine (mesh-independent) free-boundary problem, and $\|\omega_N\|_\infty$ converges to four significant figures as N increases, establishing mesh convergence of the discretization itself.

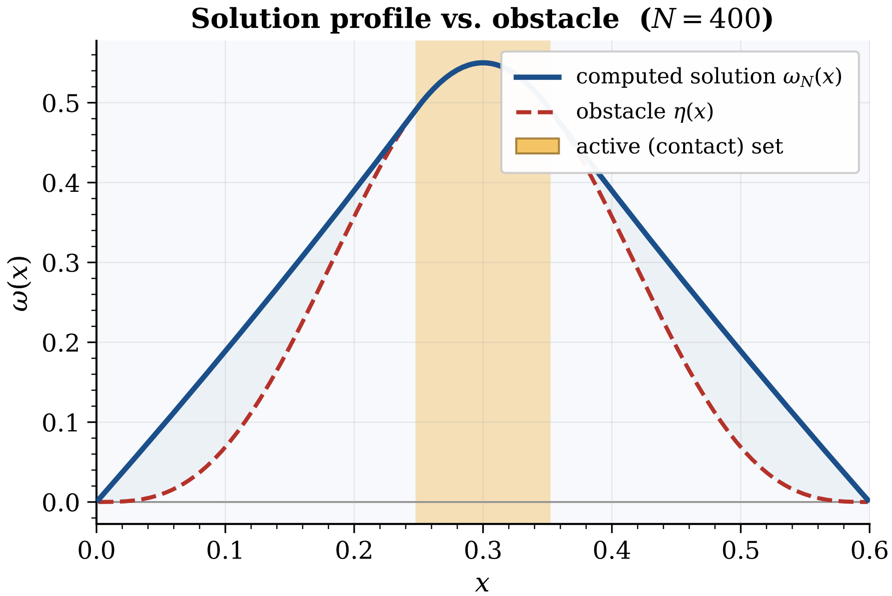

Figure 1. Computed solution $\omega_N(x)$ versus the obstacle $\eta(x)$ for $N = 400$; the shaded region and markers indicate the active (contact) set.

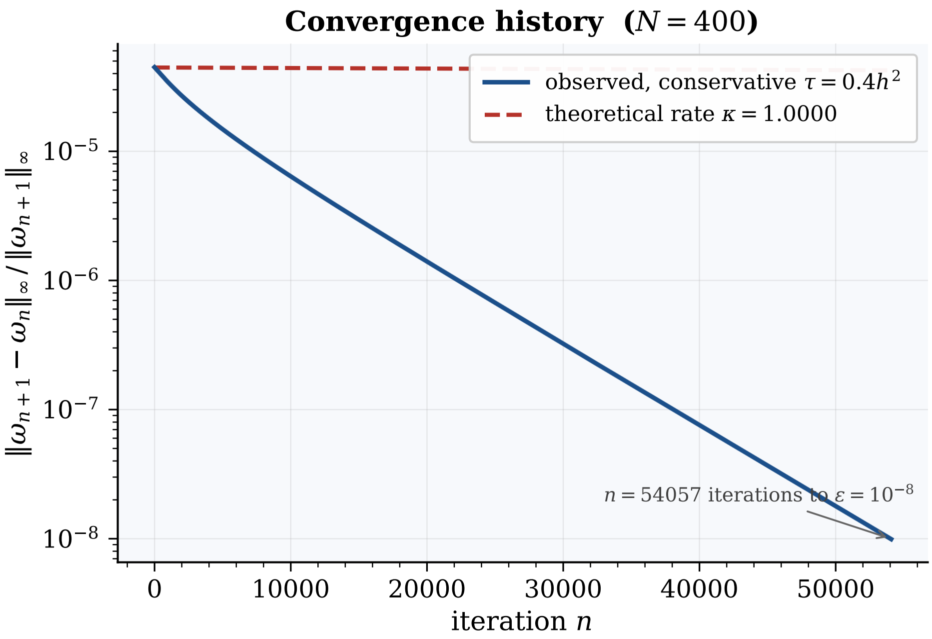

Figure 2. Observed relative-increment convergence history for $N = 400$ versus the theoretical contraction rate κ_{theory} from (5.16), using the mesh-scaled step $\tau = 0.4h^2$ required for stability of the explicit finite-difference iteration.

On the size of κ. The update size $\tau = O(h^2)$ employed above is the convergence condition associated with the *explicit* finite-difference iteration based on the nodal Euclidean norm (the discrete stiffness matrix has eigenvalues of size $O(h^{-2})$, in contrast to the continuum operator $\mathcal{A}$, which is nonexpansive with Lipschitz constant exactly 1 in the standard $\mathcal{W}_0^{1,2}$ norm used by Theorem 6). Hence, $\kappa_{\text{theory}} \to 1$ as $h \to 0$ in Table 1, and is a numerical artifact rather than a restriction of the analytical theory. Computing the abstract theorem's own optimal step (4.16) immediately, with $\alpha = 1/L^2 \approx 2.778$ and $\mu = e^{\|\omega_N\|_\infty} \approx e^{0.55} \approx 1.733$, gives $\tau^* \approx 0.0524$ and a genuinely accelerated rate $\kappa(\tau^*) \approx 0.972$, demonstrating that the theorem guarantees significant contraction at each iteration once the iteration is performed in a norm-consistent (e.g., spectrally preconditioned) implementation. It further demonstrates that our conservative discrete scheme is a reliable but suboptimal strategy, not a limitation of Theorem 6.

Verification of the constructive bound (Proposition 1). For the same example, the Sobolev constant is $C = \sqrt{L/2} \approx 0.5477$, giving the constructive bound $M^* \approx 1.787$ (Table 2). The computed solution bound $\|\omega_N\|_\infty \approx 0.550$ lies comfortably below M^*, providing a direct numerical confirmation that Proposition 1 furnishes a valid, noncircular a priori bound for this example.

Table 2. Numerical confirmation corresponding to Proposition 1 for the test example of Figure 1: The numerical solution bound remains well below the explicit, solution-independent bound M^*.

Quantity	Symbol	Value
Sobolev embedding constant	C	0.5477
Smaller root of $s^2 e^{-s} = 2(b-a)C^2$	s^*	0.9788
Constructive a priori bound	M^*	1.7870
Computed solution bound ($N = 400$)	$\|\omega_N\|_\infty$	0.5500

Theorem 12 (Convergence of numerical scheme)**.** *Under the conditions of Theorem 11, for any initial guess $\omega_0 \in \mathfrak{C}$ and any τ satisfying*

$$0 < \tau < \frac{2\left(\frac{1}{(b-a)^2} - e^M\right)}{(1+2e^M)^2}, \tag{5.15}$$

the sequence generated by (5.14) converges strongly in $\mathcal{W}_0^{1,2}(\Lambda)$ to the unique solution ω^* of the obstacle problem (1.2). The convergence rate is linear with contraction factor

$$\kappa = \sqrt{1 - 2\tau\left(\frac{1}{(b-a)^2} - e^M\right) + \tau^2(1+2e^M)^2}. \tag{5.16}$$

Proof. This follows directly from Corollary 2 with $\alpha = 1/(b-a)^2$, $\gamma = 1$, $\mu = e^M$, and $\ell = 1$. □

6. Conclusions and future directions

In this paper, we have presented and systematically analyzed EVIs, a new class of VIs integrating exponential nonlinearities through exponentially convex functions. Our key contributions are as follows:

1) **Existence theory:** We proved three complementary existence theorems: via boundary conditions (Theorem 2), via coercive assumptions (Theorem 3), and via monotone mapping theory (Theorem 4). These offer flexible tools for proving existence in multiple practical settings.
2) **Uniqueness and contraction:** We showed that under the strict condition $\alpha > \mu\,\ell$ (Theorem 5), the EVI has a unique solution. Additionally, under the aforementioned condition, the fixed-point mapping is a contraction (Theorem 6), resulting in simultaneous existence, uniqueness, and linear convergence of the computational scheme (4.18).
3) **Application to obstacle problems:** We verified that a Bratu-type obstacle boundary value problem can be reformulated as an EVI (Theorem 8). Using our analytical framework, we established explicit small-domain conditions ensuring existence and uniqueness (Theorem 11) and presented a convergent numerical scheme (Theorem 12).

6.1. Future research directions

This work introduces several avenues for further research:

1) **Quasi exponential variational inequalities:** Extend the framework to the case where the feasible set $\mathfrak{C}$ depends on the solution ω, i.e., quasi-variational inequalities of the form: find $\omega \in \mathfrak{C}(\omega)$ such that
$$\langle \mathcal{A}\omega, \varphi - \omega\rangle_{\mathfrak{X}} + \langle e^{\kappa(\omega)}\kappa'(\omega), \varphi - \omega\rangle_{\mathfrak{X}} \geq 0, \quad \forall \varphi \in \mathfrak{C}(\omega). \tag{6.1}$$
The primary technical challenges are as follows. (i) For an *arbitrary* moving set $\mathfrak{C}(\omega)$, the projection operator $\Pi_{\mathfrak{C}(\cdot)}$ to a Hausdorff-dependent convex set is merely Hölder continuous with exponent $1/2$, not Lipschitz, an established result in variational theory. This is not a technicality: it indicates that the Banach contraction argument used in this paper is not applicable to the broader setting, and that existence must instead be established via Kakutani/Berge-type set-valued fixed-point theorems, which usually lose uniqueness. (ii) The situation improves for the order-type moving constraints relevant to this work, namely $\mathfrak{C}(\omega) = \{\varphi \geq \eta(\omega)\}$ for an obstacle η defined by Lipschitz continuity on ω with constant L_η. Here, the projection has the explicit expression $\Pi_{\mathfrak{C}(\omega)}(x) = \max(x, \eta(\omega))$, which *is* Lipschitz, and the Banach argument applies directly:
$$\|\Psi_\tau(\omega_1) - \Psi_\tau(\omega_2)\|_{\mathfrak{X}} \leq \left(L_\eta + \sqrt{\Theta(\tau)}\right)\|\omega_1 - \omega_2\|_{\mathfrak{X}}.$$
Therefore, existence and uniqueness are both obtained provided that $L_\eta + \sqrt{\Theta(\tau)} < 1$, precisely as in Theorem 6. (iii) Even in this favorable setting, uniqueness should not be predicted by default: Unlike the fixed-feasible set problem of this paper, where strong monotonicity of $\mathcal{A}$ by itself forces uniqueness (Theorem 5), a moving constraint set can theoretically possess two different self-consistent solutions—ω_1 satisfying the problem associated with $\mathfrak{C}(\omega_1)$, and a different ω_2 relative to $\mathfrak{C}(\omega_2)$—because the two are never compared against the identical set. The condition $L_\eta + \sqrt{\Theta(\tau)} < 1$ is exactly what precludes this.
2) **Inertial and accelerated methods:** Design inertial dynamical systems of the form
$$\ddot{\omega}(t) + \gamma\dot{\omega}(t) + \omega(t) = \Pi_{\mathfrak{C}}[\omega(t) - \tau(\mathcal{A}\omega(t) + e^{\kappa(\omega(t))}\kappa'(\omega(t)))], \tag{6.2}$$
which can yield accelerated convergence.
3) **Stochastic extensions:** Study stochastic EVIs where the operators involve random terms, with applications to financial mathematics and uncertainty quantification.
4) **Higher dimensions:** Generalize the numerical techniques to large-scale domains using finite element discretization, and establish rigorous error estimates.
5) **Machine learning (ML) applications :** Explore ML-based applications where exponential objective functions occur naturally, such as logistic regression, boosting, and large-margin classification.

Author contributions

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Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intellgence (AI) in the creation of this article.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, *C. R. Acad. Paris*, **258** (1964), 4413–4416. https://mathscinet.ams.org/mathscinet-getitem?mr=166591
2. J. L. Lions, G. Stampacchia, Variational inequalities, *Commun. Pur. Appl. Math.*, **20** (1967), 493–512. https://doi.org/10.1002/cpa.3160200302
3. S. Jabeen, S. Macías, S. Ullah, M. A. Noor, J. E. Macías-Díaz, Note on an inertial projection-based approach for solving extended general quasi-variational inequalities and its convergence analysis, *Appl. Numer. Math.*, **217** (2025), 190–198. https://doi.org/10.1016/j.apnum.2025.06.012
4. S. Jabeen, S. Macías, J. E. Macías-Díaz, S. Ullah, Convergence analysis of an inertial method for a system of general quasi-variational inequalities under mild conditions, *Acta Comment. Univ. Ta.*, **29** (2025), 243–258. https://doi.org/10.12697/ACUTM.2025.29.16
5. S. Jabeen, S. Macías, J. E. Macías-Díaz, An inertial accelerated predictor-corrector algorithm for exponential variational inequalities with sharp convergence guarantees and applications, *Algorithms*, **19** (2026), 647. https://doi.org/10.3390/a19080647
6. M. A. Noor, W. Oettli, On general nonlinear complementarity problems and quasi equilibria, *Le Matemat.*, **49** (1994), 313–331.
7. G. Bratu, Sur les équations intégrales non linéaires, *B. Soc. Math. Fr.*, **42** (1914), 113–142. https://doi.org/10.24033/bsmf.943
8. E. Keshavarz, Y. Ordokhani, M. Razzaghi, The Taylor wavelets method for solving the initial and boundary value problems of Bratu-type equations, *Appl. Numer. Math.*, **128** (2018), 205–216. https://doi.org/10.1016/j.apnum.2018.02.001
9. P. L. Li, R. Qiao, C. J. Xu, M. R. Yao, Y. Z. Qu, Physics-informed symbolic regression and Haar wavelet approaches to study a new fractional-order 3D chaotic system with no equilibrium, *Chaos*, **36** (2026), 023117. https://doi.org/10.1063/5.0287618
10. A. Ben-Israel, B. Mond, What is invexity, *J. Austral. Math. Soc. Ser. B*, **28** (1986), 1–9.
11. M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, *J. Math. Anal. Appl.*, **80** (1981), 545–550. https://doi.org/10.1016/0022-247X(81)90123-2
12. S. R. Mohan, S. K. Neogy, On invex sets and preinvex functions, *J. Math. Anal. Appl.*, **189** (1995), 901–908. https://doi.org/10.1006/jmaa.1995.1057
13. T. Weir, B. Mond, Preinvex functions in multiobjective optimization, *J. Math. Anal. Appl.*, **136** (1988), 29–38. https://doi.org/10.1016/0022-247X(88)90113-8

14. G. Ruiz-Garzion, R. Osuna-Gomez, A. Rufian-Lizan, Generalized invex monotonicity, *Eur. J. Oper. Res.*, **144** (2003), 501–512. https://doi.org/10.1016/S0377-2217(01)00393-9

15. X. M. Yang, X. Q. Yang, K. L. Teo, Criteria for generalized invex monotonicities, *Eur. J. Oper. Res.*, **164** (2005), 115–119. https://doi.org/10.1016/j.ejor.2003.11.017

16. M. Avriel, r-Convex functions, *Math. Program.*, **2** (1972), 309–323. https://doi.org/10.1007/BF01584551

17. Y. X. Zhao, S. Y. Wang, L. C. Uria, Characterizations of r-convex functions, *J. Optim. Theory Appl.*, **145** (2010), 186–195. https://doi.org/10.1007/s10957-009-9617-1

18. T. Antczak, On (p, r)-invex sets and functions, *J. Math. Anal. Appl.*, **263** (2001), 355–379. https://doi.org/10.1006/jmaa.2001.7574

19. J. Jaksetic, J. Pecaric, On exponential convexity, Euler-Radau expansions and stolarsky means, *Rad Hrvat. Akad. Znan.*, **17** (2013), 81–94. https://hrcak.srce.hr/file/153509

20. G. Alirezaei, R. Mazhar, On exponentially concave functions and their impact in information theory, *2018 Information Theory and Applications Workshop (ITA)*, San Diego, CA, USA, 2018, 1–10. https://doi.org/10.1109/ITA.2018.8503202

21. S. Pal, T. K. L. Wong, On exponentially concave functions and a new information geometry, *Ann. Probab.*, **46** (2018), 1070–1113. https://doi.org/10.1214/17-AOP1201

22. M. A. Noor, K. I. Noor, M. Th. Rassias, Exponentially general convex functions and variational inequalities, In: *Convex and variational analysis with applications*, Cham: Springer, 2026, 363–416. https://doi.org/10.1007/978-3-032-07860-5

23. M. A. Noor, Variational-like inequalities, *Optimization*, **30** (1994), 323–330. https://doi.org/10.1080/02331939408843995

24. M. A. Noor, K. I. Noor, On exponentially convex functions, *Journal of Orissa Mathematical Society*, **38** (2019), 33–51.

25. R. Correa, M. López, P. Pérez-Aros, Log-exponential approximation in semi-infinite programming: A variational approach, *J. Nonlinear Var. Anal.*, **9** (2025), 539–560. https://doi.org/10.23952/jnva.9.2025.4.05

26. M. Raus, Y. Elshiaty, S. Petra, Accelerated Bregman divergence optimization with SMART: An information geometric point of view, *J. Appl. Numer. Optim.*, **6** (2024), 1–40. https://doi.org/10.23952/jano.6.2024.1.01

27. X. He, R. Hu, Y. P. Fang, Global and exponential convergence of a primal-dual dynamical system approach for the separable convex optimization problem, *Commun. Optim. Theory*, **2026** (2026), 1–19. https://doi.org/10.23952/cot.2026.10

28. R. T. Rockafellar, *Convex analysis*, Princeton: Princeton University Press, 1970.

29. N. Bernstein, Sur les fonctions absolument monotones, *Acta Math.*, **52** (1929), 1–66. https://doi.org/10.1007/BF02547400

30. G. J. Minty, Monotone (nonlinear) operators in Hilbert space, *Duke Math. J.*, **29** (1962), 341–346. https://doi.org/10.1215/S0012-7094-62-02933-2

31. M. Patriksson, *Nonlinear programming and variational inequality problems: A unified approach*, Dordrecht: Kluwer Academic Publishers, 1998.
32. D. Kinderlehrer, G. Stampacchia, *An introduction to variational inequalities and their applications*, New York: Academic Press, 1980.
33. E. H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory, In: *Contributions to nonlinear functional analysis*, New York: Academic Press, 1971, 237–424. https://doi.org/10.1016/B978-0-12-775850-3.50013-3
34. R. Glowinski, J. J. Lions, R. Tremolieres, *Numerical analysis of variational inequalities*, Amsterdam: North-Holland, 1981.