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Research article

The 3-adic valuations of Stirling numbers of the first kind

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Abstract: Let v_3 denote the usual 3-adic valuation, and let $s(n,k)$ be the unsigned Stirling number of the first kind. In this paper, for $a \in \{1,2\}$, we determine the values of $v_3(s(a3^n,k))$ for all $1 \le k \le a3^n$. More precisely, for each admissible pair (m,k), we obtain an explicit formula for $v_3(s(a3^n, a3^m - k))$. This proves the case $p = 3$ of a conjecture of Hong and Qiu proposed in 2020. As applications, we derive formulas near the diagonal, comparison results for adjacent orders, sharp upper bounds for the families $v_3(s(3^n,k))$ and $v_3(s(2\cdot 3^n,k))$, and partial confirmations of conjectures of Lengyel and of Leonetti and Sanna.

Keywords: 3-adic valuation; 3-adic analysis; Stirling number of the first kind; the m-th Stirling number of the first kind; elementary symmetric function
Mathematics Subject Classification: 11A07, 11B73

1. Introduction

1.1. Background and motivation

For integers $n \ge 1$ and $0 \le k \le n$, we write $s(n,k)$ for the unsigned Stirling number of the first kind and $S(n,k)$ for the Stirling number of the second kind. These two families are characterized by

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} s(n,k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1),$$

respectively. Equivalently, $s(n,k)$ counts permutations of an n-element set by their number of cycles, while $S(n,k)$ counts partitions of such a set into k nonempty blocks.

Throughout this paper, p denotes a prime. The arithmetic properties of Stirling numbers have been studied extensively. For Stirling numbers of the second kind, divisibility properties were studied in [1–3]. In particular, the 2-adic valuations of $S(n,k)$ were studied in [4–6], with further results in [7, 8]. The 2-adic valuations of differences of Stirling numbers of the second kind were considered in [9–11]. For general primes, the p-adic valuations of $S(n,k)$ were studied in [12].

For Stirling numbers of the first kind, the corresponding valuation problem is less completely understood. Stirling numbers of the first kind are closely related to generalized harmonic sums. For $0 \le k \le n$, define

$$H(n,0) := 1, \qquad H(n,k) := \sum_{1\le i_1<\cdots<i_k\le n} \frac{1}{i_1\cdots i_k}.$$

Then, $H(n,k)$ is the kth elementary symmetric function of $1, 1/2, \ldots, 1/n$. The identity

$$s(n+1,k+1) = n!H(n,k) \tag{1.1}$$

links Stirling numbers of the first kind with these reciprocal elementary symmetric functions; see [13].

The arithmetic properties of harmonic sums have been studied for a long time; see, for example, [14–16]. The p-adic study of harmonic numbers was developed further by Eswarathasan and Levine and by Boyd [17, 18]. Later, related questions were considered for generalized harmonic sums and reciprocal elementary symmetric functions [19–21]. Further results on p-adic properties and integrality can be found in [22–24] and [25, 26]. More recently, the relation with Wolstenholme primes and the conjectures of Eswarathasan–Levine and Boyd has also been studied [27, 28]. In view of (1.1), these results are also closely related to the p-adic properties of Stirling numbers of the first kind.

We write v_p for the usual p-adic valuation. Thus, for a nonzero integer n, $v_p(n)$ is the largest integer r such that $p^r \mid n$, and we set $v_p(0) = \infty$. For a rational number $\frac{n_1}{n_2}$, we define $v_p(\frac{n_1}{n_2}) = v_p(n_1) - v_p(n_2)$, where $n_1, n_2 \in \mathbb{Z}$ and $n_2 \neq 0$. By Legendre's formula, $v_p(n!) = \frac{n-d_p(n)}{p-1}$, where $d_p(n)$ is the sum of the base-p digits of n. Hence, (1.1) shows that the study of $v_p(H(n,k))$ is equivalent to that of $v_p(s(n+1,k+1))$.

1.2. Known results and conjectures

Next, we place the present work in the context of several earlier results and conjectures. Lengyel proved in [29] that, for every fixed positive integer k, the sequence $v_p(s(n,k))$ tends to infinity as $n \to \infty$. More precisely, for each pair (k, p), there exist constants $c = c(k, p) > 0$ and $n_0 = n_0(k, p)$ such that $v_p(s(n,k)) \ge cn$ whenever $n \ge n_0$. Together with (1.1), this gives corresponding lower estimates for $v_p(H(n,k))$. Leonetti and Sanna proposed the following conjecture and confirmed it for some special cases.

Conjecture 1.1 ([13, Conjecture 1.1]). *For any prime number p and any integer $k \ge 1$, there exists a constant $c = c(p,k) > 0$ such that*

$$v_p(H(n,k)) < -c\log n$$

for all sufficiently large integers n.

Several more explicit formulas are known in special situations. Komatsu and Young [30] considered integers of the form $n = kp^r+m$, where $0 \le m < p^r$, and obtained $v_p(s(n+1,k+1)) = v_p(n!)-v_p(k!)-kr$. For powers of 2, Qiu and Hong [31] determined $v_2(s(2^n,k))$ for the full range $1 \le k \le 2^n$. In a subsequent work, Hong and Qiu [32] proposed a general conjectural formula for $v_p(s(ap^n, ap^m - k))$. To state this, let $\langle k\rangle$ be the integer satisfying $0 \le \langle k\rangle \le p-2$ and $k \equiv \langle k\rangle \pmod{p-1}$. Take $\epsilon_k := 0$ if k is even, and $\epsilon_k := 1$ if k is odd. For a real number x, let $\lfloor x\rfloor$ denote the greatest integer not exceeding x, and let B_n be the nth Bernoulli number. Hong and Qiu proposed the following conjecture.

Conjecture 1.2 ([32, Conjecture 5.1]). *Let p be an odd prime. Let a, n, m, k be positive integers such that $1 \le a \le p-1$, $1 \le m \le n$, and $2 \le k \le ap^n - 2$. Then, each of the following is true:*

(i) *If $2 \le k \le a(p-1)p^{m-1} + 1 < ap^m$, then*

$$v_p(s(ap^n, ap^m - k)) = \frac{a}{p-1}(p^n - p^m) - (n-m)(ap^m - k) + m + (m + v_p(k))\frac{1-(-1)^k}{2} + T_k,$$

where

$$T_k := \begin{cases} -1 - v_p(\lfloor \frac{k}{2} \rfloor), & \text{if } k \equiv \frac{1-(-1)^k}{2} \pmod{p-1}; \\ v_p(B_{2\lfloor \frac{\langle k \rangle}{2} \rfloor}), & \text{if } k \not\equiv \frac{1-(-1)^k}{2} \pmod{p-1}. \end{cases}$$

(ii) *If $a \ge 4$ and $a(p-1) + 2 \le k \le ap - 2$, then*

$$v_p(s(ap^n, ap - k)) \ge \frac{a}{p-1}(p^n - p) - (n-1)(ap-k) + a + k - ap.$$

Hong and Qiu also presented results on $v_p(s(ap, k))$ and $v_p(s(ap^n, ap^n - k))$ when $p \ge 5$ in [32].

The results above indicate that the p-adic valuations of Stirling numbers of the first kind remain far from fully understood, especially in explicit form. In this paper, we focus on the case $p = 3$. Some isolated 3-adic formulas were previously known. For example, Lengyel [29] proved that $v_3(s(3^n, 2)) = \frac{1}{2}(3^n + 3) - 2n$, $v_3(s(3^n, 3)) = \frac{1}{2}(3^n + 3) - 3n$, and $v_3(s(2\cdot 3^n, 2)) = 3^n - 2n - 1$. Komatsu and Young [30], using a p-adic Newton polygon argument, proved that $v_3(s(3^n, 3^m)) = \frac{1}{2}(3^n - 3^m) - 3^m(n-m)$ and $v_3(s(2\cdot 3^n, 2\cdot 3^m)) = 3^n - 3^m - 2\cdot 3^m(n-m)$. Our result gives a uniform formula that covers these cases as special instances. For $a \in \{1, 2\}$, our aim is to determine $v_3(s(a3^n, i))$ for every $1 \le i \le a3^n$. The two boundary values follow from the standard identities for Stirling numbers of the first kind; see, for example, [33]. Indeed,

$$v_3(s(a3^n, a3^n)) = v_3(1) = 0 \quad \text{and} \quad v_3(s(a3^n, a3^n - 1)) = v_3\Big(\binom{a3^n}{2}\Big) = v_3\Big(\frac{a3^n(a3^n - 1)}{2}\Big) = n.$$

Thus, it remains to understand the remaining indices. If $1 \le i \le a3^n - 2$, then i can be written in the form $i = a3^m - k$ with $(m, k) \in T_{a,n}$, where

$$T_{a,n} := \{(m,k) : 1 \le m \le n,\ 2 \le k \le 2a3^{m-1} + 1,\ k < a3^m, m, k \in \mathbb{Z}\}.$$

1.3. Main results

We now state the main results of this paper. The first result gives an explicit formula for $v_3(s(a3^n, a3^m - k))$ with $a \in \{1, 2\}$ and $(m, k) \in T_{a,n}$.

Theorem 1.3. *Let $a \in \{1, 2\}$, and let n, m, k be positive integers such that $1 \le m \le n$, $2 \le k \le 2a3^{m-1} + 1$ and $k < a3^m$. Then,*

$$v_3(s(a3^n, a3^m - k)) = \frac{a}{2}(3^n - 3^m) - (n-m)(a3^m - k) + m - 1 - v_3\Big(\Big\lfloor \frac{k}{2} \Big\rfloor\Big) + (m + v_3(k))\frac{1-(-1)^k}{2}. \quad (1.2)$$

Since $p - 1 = 2$ when $p = 3$, every integer k satisfies $k \equiv \frac{1-(-1)^k}{2} \pmod 2$. Hence, Theorem 1.3 confirms the first part of Conjecture 1.2 in the case $p = 3$; the second part of that conjecture is then vacuous. Moreover, Lengyel proved in [29] that, for any prime p, any integer $a \geq 1$ with $(a, p) = 1$, and any even $k \geq 2$ with the condition

$$\text{there exists } n_1 > 3\log_p k + \log_p a \text{ such that } v_p(s(ap^{n_1}, ap^{n_1} - k)) < n_1 \tag{1.3}$$

one has $v_p(s(ap^{n+1}, ap^{n+1} - k)) = v_p(s(ap^n, ap^n - k)) + 1$ for all $n \geq n_1$. The same conclusion also holds for $k = 1$ with $n_1 = 1$. Lengyel further expected that (1.3) holds for all even integers $k \geq 2$. For odd integers $k \geq 3$, he proposed the following conjecture.

Conjecture 1.4 ([29, Conjecture 3.1]). *Let p be a prime, let $a \geq 1$ be an integer with $(a, p) = 1$, and let $k \geq 3$ be odd. Then, there exists $n_1 = n_1(p, k)$ such that, for all $n \geq n_1$,*

$$v_p(s(ap^{n+1}, ap^{n+1} - k)) = v_p(s(ap^n, ap^n - k)) + 2.$$

By taking $m = n$ in Theorem 1.3, we obtain the following explicit formula near the diagonal.

Corollary 1.5. *Let $a \in \{1, 2\}$ and let $n, k \in \mathbb{Z}^+$ with $2 \leq k \leq 2a3^{n-1} + 1$ and $k < a3^n$. Then,*

$$v_3(s(a3^n, a3^n - k)) = \begin{cases} n - 1 - v_3(k), & \text{if } 2 \mid k, \\ 2n - 1 + v_3(k) - v_3(k-1), & \text{if } 2 \nmid k. \end{cases}$$

For even k, Corollary 1.5 verifies Lengyel's expected condition (1.3) in the cases $p = 3$ and $a \in \{1, 2\}$, provided that n is sufficiently large. For odd k, it gives the predicted eventual increment in Lengyel's conjecture in the same 3-adic setting.

The main formula also yields a comparison between adjacent orders. Similar phenomena are known in the 2-adic case: Hong, Zhao, and Zhao [8] proved an adjacent-order identity for Stirling numbers of the second kind, and Qiu and Hong [31] obtained the corresponding identity $v_2(s(2^n + 1, k + 1)) = v_2(s(2^n, k))$ for Stirling numbers of the first kind. The next theorem gives the corresponding 3-adic analogue, with an additional parity distinction.

Theorem 1.6. *Let $a \in \{1, 2\}$ and let $n, k \in \mathbb{Z}^+$ with $k \leq a3^n$. Then,*

$$\begin{aligned} v_3(s(a3^n + 1, k + 1)) &= v_3(s(a3^n, k)), && \text{if } 2 \mid (k - a), \\ v_3(s(a3^n + 1, k + 1)) &\geq v_3(s(a3^n, k + 1)) + n, && \text{if } 2 \nmid (k - a). \end{aligned} \tag{1.4}$$

Another consequence concerns the maximal possible valuation in each family. While the 2-adic situation has a particularly simple upper bound, the 3-adic case separates according to whether the order is 3^n or $2 \cdot 3^n$.

Theorem 1.7. *Let $n, k \in \mathbb{Z}^+$ with $k \leq 3^n$. Then,*

$$v_3(s(3^n, k)) \leq \begin{cases} v_3(s(3^n, 1)) = \dfrac{1}{2}(3^n - 2n - 1), & \text{if } n \geq 3, \\ v_3(s(3^2, 6)) = 4, & \text{if } n = 2, \\ v_3(s(3, 2)) = 1, & \text{if } n = 1. \end{cases} \tag{1.5}$$

Theorem 1.8. *Let $n, k \in \mathbb{Z}^+$ with $k \leq 2 \cdot 3^n$. Then,*

$$v_3(s(2 \cdot 3^n, k)) \leq \begin{cases} v_3(s(2 \cdot 3^n, 1)) = 3^n - n - 1, & \text{if } n \geq 2, \\ v_3(s(2 \cdot 3, 3)) = 2, & \text{if } n = 1. \end{cases} \tag{1.6}$$

Finally, by translating the preceding estimates through (1.1), we obtain the following consequence for the elementary symmetric functions $H(n,k)$, which partially confirms Conjecture 1.1.

Corollary 1.9. *Let $a \in \{1,2\}$. Let n and k be positive integers such that $n \geq 3$, $k \leq a3^n$ and $2 \mid (k-a)$. Then, $v_3(H(a3^n,k)) \leq -n$.*

Thus, our results provide evidence for conjectures of Hong and Qiu, Lengyel, and Leonetti and Sanna in the 3-adic setting.

1.4. Proof strategy and organization

We briefly describe the main strategy of the proof of Theorem 1.3. The main difference from the 2-adic case lies in the structure of the parameter range. For powers of 2, one deals with the single family $s(2^n,k)$. In the present 3-adic setting, the two families $s(3^n,k)$ and $s(2\cdot 3^n,k)$ have to be treated simultaneously. In addition, the valuation formula contains an essential parity correction through $\frac{1-(-1)^k}{2}$ and $v_3(\lfloor k/2\rfloor)$, which is responsible for the finer case decomposition in the proof.

The proof proceeds by induction on n. We first show that the odd values of k follow from the even values. Thus, it remains to establish the formula for even k. Under the inductive hypothesis for the family $s(3^n,3^m-k)$, we derive the corresponding formula for $v_3(s(2\cdot 3^n,2\cdot 3^m-k))$. This is the first key step. We then use this formula, together with the convolution identity for Stirling numbers of the first kind and estimates for the mth Stirling numbers, to obtain the formula for $v_3(s(3^{n+1},3^m-k))$. This completes the induction.

The remainder of this paper is organized as follows. Section 2 collects the auxiliary results needed in what follows, including the reduction from odd values of k to even values of k and comparison estimates for the mth Stirling numbers of the first kind. In Section 3, we establish the key inductive step from the order 3^n to the order $2\cdot 3^n$. Section 4 completes the induction by passing from the order $2\cdot 3^n$ to the order 3^{n+1}, thereby proving Theorem 1.3. Section 5 applies the main formula to prove Theorems 1.6–1.8 and Corollary 1.9. Section 6 concludes the paper with a brief discussion of the restriction to the prime 3 and the obstruction to a direct extension of the present method to arbitrary odd primes.

2. Preliminaries and reductions

This section collects the auxiliary results used in the proof of Theorem 1.3. We first record the basic identities for ordinary and shifted Stirling numbers of the first kind. We then prove a reduction showing that the odd case follows from the even case. Finally, we derive the comparison estimates for shifted Stirling numbers needed in the induction.

2.1. Basic identities

We begin with two standard identities for Stirling numbers of the first kind. The first one relates neighboring coefficients in a fixed row.

Lemma 2.1 ([34]). *Let n and k be positive integers. If $n+k$ is odd, then*

$$s(n,k) = \frac{1}{2}\sum_{i=k+1}^{n}(-1)^{n-i}s(n,i)\binom{i-1}{i-k}n^{i-k}.$$

This identity will be used to relate neighboring values of the parameter k and to reduce the odd case to the even case.

We then recall the mth Stirling numbers of the first kind. These numbers may be regarded as shifted analogues of the ordinary Stirling numbers of the first kind and will be used to compare Stirling numbers with shifted arguments in the inductive step. They are defined by (see [31])

$$(x+m)(x+m+1)\cdots(x+m+n-1)=\sum_{k=0}^{n}s_m(n,k)x^k$$

with $n\geq 1$, and m and k being nonnegative integers.

Lemma 2.2 ([31]). *Let $n\geq 1$, and let m and k be nonnegative integers. Then,*

$$s(m+n,k)=\sum_{i=0}^{k}s(m,i)s_m(n,k-i)\quad \textit{and}\quad s_m(n,k)=\sum_{i=k}^{n}s(n,i)\binom{i}{i-k}m^{i-k}.$$

The first identity is the convolution formula obtained from the product decomposition of the defining polynomials, while the second expresses the shifted coefficients in terms of the ordinary Stirling numbers.

2.2. *Reduction from odd k to even k*

The aim of this subsection is to show that the odd case of formula (1.2) is a consequence of the even case. We first establish a valuation relation between the adjacent Stirling numbers $s(a3^n,a3^n-2t)$ and $s(a3^n,a3^n-2t-1)$, under the assumption that (1.2) holds for the relevant even indices. We then apply this relation to obtain formula (1.2) for odd k.

Lemma 2.3. *Let $a\in\{1,2\}$, and let n be a positive integer. Assume that (1.2) holds for all integers m and even k with $(m,k)\in T_{a,n}$. Then, for every integer t satisfying $0\leq t\leq \frac{a3^n-2}{2}$, we have*

$$v_3(s(a3^n,a3^n-2t-1))=v_3(s(a3^n,a3^n-2t))+v_3(2t+1)+n. \tag{2.1}$$

Proof. We prove (2.1) by induction on t. The case $t=0$ is immediate, since

$$v_3(s(a3^n,a3^n-1))=n=v_3(s(a3^n,a3^n))+v_3(1)+n.$$

Now, let $1\leq t\leq \frac{a3^n-2}{2}$. Assume that (2.1) holds for any nonnegative integer e with $e\leq t-1$. We show that (2.1) is still true for t.

The idea is to isolate the principal contribution to $s(a3^n,a3^n-2t-1)$. By Lemma 2.1, we have

$$\begin{aligned} s(a3^n,a3^n-2t-1)&=\frac{1}{2}\sum_{i=a3^n-2t}^{a3^n}(-1)^{a3^n-i}s(a3^n,i)\binom{i-1}{i-a3^n+2t+1}(a3^n)^{i-a3^n+2t+1}\\ &=\frac{a3^n}{2}s(a3^n,a3^n-2t)(a3^n-2t-1)+\frac{a3^n}{2}L, \end{aligned}\tag{2.2}$$

where

$$L=\sum_{i=a3^n-2t+1}^{a3^n}(-1)^{a3^n-i}\binom{i-1}{i-a3^n+2t+1}L_i$$

with $L_i := s(a3^n, i)(a3^n)^{i-a3^n+2t}$.

Thus, the first term on the right-hand side of (2.2) is the desired principal term. It remains to show that the term involving L has strictly larger 3-adic valuation. We shall prove the following estimate:

$$v_3(L_i) \geq v_3(s(a3^n, a3^n - 2t)) + v_3(2t + 1) + 1 \tag{2.3}$$

for every integer i with $a3^n - 2t + 1 \leq i \leq a3^n$. If (2.3) holds, then the isosceles triangle principle (see, for example, [35]) gives

$$v_3(L) \geq v_3(s(a3^n, a3^n - 2t)) + v_3(2t + 1) + 1 = v_3(s(a3^n, a3^n - 2t)) + v_3(a3^n - 2t - 1) + 1 \tag{2.4}$$

since $3 \leq 2t + 1 \leq a3^n - 1$. Note that $v_3(\frac{a3^n}{2}) = n$. Hence, (2.1) follows immediately from (2.2)–(2.4). So, to finish the proof of Lemma 2.3, it remains to prove (2.3). This will be done in what follows.

We first reduce the estimate for odd indices i to the corresponding estimate for even indices. Every integer i with $a3^n - 2t + 1 \leq i \leq a3^n$ can be written in one of the two forms

$$i = a3^n - 2t + 2r \quad \text{or} \quad i = a3^n - 2t + 2r - 1$$

for some integer r with $1 \leq r \leq t$. By the inductive hypothesis, for $1 \leq r \leq t$,

$$\begin{aligned} v_3(s(a3^n, a3^n - 2t + 2r - 1)) &= v_3(s(a3^n, a3^n - 2(t - r) - 1)) \\ &= v_3(s(a3^n, a3^n - 2(t - r))) + v_3(2(t - r) + 1) + n. \end{aligned} \tag{2.5}$$

Hence, it is enough to prove that

$$v_3(L_{a3^n-2t+2r}) = v_3(s(a3^n, a3^n - 2t + 2r)) + 2rn \geq v_3(s(a3^n, a3^n - 2t)) + v_3(2t + 1) + 1 \tag{2.6}$$

for every $1 \leq r \leq t$. Indeed, once (2.6) holds, (2.5) gives

$$\begin{aligned} v_3(L_{a3^n-2t+2r-1}) &= v_3(s(a3^n, a3^n - 2t + 2r - 1)) + (2r - 1)n \\ &= v_3(s(a3^n, a3^n - 2t + 2r)) + 2rn + v_3(2(t - r) + 1) \\ &\geq v_3(s(a3^n, a3^n - 2t)) + v_3(2t + 1) + 1. \end{aligned}$$

Thus, (2.3) follows from (2.6).

We now prove (2.6). Since $1 \leq t \leq \frac{a3^n-2}{2}$, we have $2 \leq a3^n - 2t \leq a3^n - 2$. Write $a3^n - 2t = a3^m - k$, where $(m, k) \in T_{a,n}$ and k is even. Note that $2t \leq a3^n - 2$ implies $k \leq a3^m - 2$. Together with $n \geq m$ and $2 \leq k \leq 2a3^{m-1}$, we obtain

$$v_3(2t + 1) = v_3(a3^n - a3^m + k + 1) = v_3(k + 1) \leq m.$$

By the assumed even case of (1.2), we get

$$v_3(s(a3^n, a3^m - k)) = \frac{a}{2}(3^n - 3^m) - (n - m)(a3^m - k) + m - 1 - v_3(k). \tag{2.7}$$

It remains to compare $v_3(L_{a3^n-2t+2r})$ with $v_3(s(a3^n, a3^n - 2t)) + v_3(2t + 1)$. Note that $a3^n - 2t + 2 \leq a3^n - 2t + 2r \leq a3^n$. Consider the following two cases.

If $a3^n - 2t + 2r = a3^n$, then $2r = 2t = a3^n - a3^m + k$. It follows from (2.7) that

$$\begin{aligned}
&v_3(L_{a3^n}) - v_3(s(a3^n, a3^n - 2t)) - v_3(2t+1) = 2rn - v_3(s(a3^n, a3^m - k)) - v_3(2t+1) \\
&= n(a3^n - a3^m + k) - \frac{a}{2}(3^n - 3^m) + (n-m)(a3^m - k) - m + 1 + v_3(k) - v_3(2t+1) \\
&= \Big(n - \frac{1}{2}\Big)a3^n - \Big(m - \frac{1}{2}\Big)a3^m + (k-1)m + v_3(k) - v_3(2t+1) + 1 \\
&\geq (k-1)m - v_3(2t+1) + 1 \geq m - m + 1 = 1.
\end{aligned}$$

Thus, (2.6) holds in this case.

If $a3^n - 2t + 2 \leq a3^n - 2t + 2r \leq a3^n - 2$, then we can write $a3^n - 2t + 2r = a3^l - j$ for $(l, j) \in T_{a,n}$ with $2 \mid j$. Then, $m \leq l \leq n$, and if $l = m$, then $k - j \geq 2$. By (2.7) and the assumed even case of (1.2), we obtain

$$\begin{aligned}
&v_3(L_{a3^n-2t+2r}) - v_3(s(a3^n, a3^n - 2t)) - v_3(2t+1) \\
&= v_3(s(a3^n, a3^l - j)) + n(a3^l - j - a3^m + k) - v_3(s(a3^n, a3^m - k)) - v_3(2t+1) \\
&= \Big(l - \frac{1}{2}\Big)a3^l - \Big(m - \frac{1}{2}\Big)a3^m - lj + l + mk - m + v_3(k) - v_3(j) - v_3(2t+1) =: D_l.
\end{aligned}$$

It remains to prove $D_l \geq 1$ for $m \leq l \leq n$. If $l = m$, then $2 \leq j \leq 2a3^{m-1}$, and so $v_3(j) \leq m - 1$. Thus,

$$D_m = m(k-j) + v_3(k) - v_3(j) - v_3(2t+1) \geq 2m - (m-1) - m = 1.$$

If $m + 1 \leq l \leq n$, then $2 \leq j \leq 2a3^{l-1}$ and

$$\begin{aligned}
D_l &\geq \Big(l - \frac{1}{2}\Big)a3^l - \Big(m - \frac{1}{2}\Big)a3^m - 2la3^{l-1} + l + m - (l-1) - m \\
&= \Big(l - \frac{3}{2}\Big)a3^{l-1} - \Big(m - \frac{1}{2}\Big)a3^m + 1 \geq 1.
\end{aligned}$$

Thus, (2.6) holds in all cases. Consequently, (2.3) is proved, and the inductive step follows. This completes the proof of Lemma 2.3. □

Next, we show how Lemma 2.3 converts the even case of (1.2) into the odd case. If k is odd, then $k - 1$ is even, and the index $a3^m - k$ can be written in the form $a3^n - 2t - 1$. Lemma 2.3 then relates its valuation to that of the adjacent even index $a3^m - k + 1$.

Lemma 2.4. *Let $a \in \{1, 2\}$, and let n be a positive integer. Assume that* (1.2) *holds for every pair $(m, k) \in T_{a,n}$ with k even. Then,* (1.2) *also holds for every pair $(m, k) \in T_{a,n}$ with k odd.*

Proof. Let $(m, k) \in T_{a,n}$ with k odd. Then, $3 \leq k \leq 2a3^{m-1} + 1$, $k < a3^m$, and $k - 1$ is even. Hence, $2 \leq k - 1 \leq 2a3^{m-1}$, and so $(m, k-1) \in T_{a,n}$. By the assumed even case of (1.2), we have

$$v_3(s(a3^n, a3^m - (k-1))) = \frac{a}{2}(3^n - 3^m) - (n-m)(a3^m - (k-1)) + m - 1 - v_3\Big(\frac{k-1}{2}\Big). \tag{2.8}$$

Note that $a3^m - k = a3^n - 2 \cdot \frac{a3^n - a3^m + k - 1}{2} - 1$. It follows from Lemma 2.3 that

$$v_3(s(a3^n, a3^m - k)) = v_3(s(a3^n, a3^m - k + 1)) + v_3(a3^n - a3^m + k) + n$$

$$= v_3(s(a3^n, a3^m - k + 1)) + v_3(k) + n \tag{2.9}$$

since $n \geq m$ and $3 \leq k < a3^m$. By (2.8), (2.9), and $\lfloor \frac{k}{2} \rfloor = \frac{k-1}{2}$, one derives that

$$\begin{aligned} v_3(s(a3^n, a3^m - k)) &= \frac{a}{2}(3^n - 3^m) - (n-m)(a3^m - (k-1)) + m - 1 - v_3\Big(\frac{k-1}{2}\Big) + v_3(k) + n \\ &= \frac{a}{2}(3^n - 3^m) - (n-m)(a3^m - k) - n + m + m - 1 - v_3\Big(\Big\lfloor \frac{k}{2} \Big\rfloor\Big) + v_3(k) + n \\ &= \frac{a}{2}(3^n - 3^m) - (n-m)(a3^m - k) + m - 1 - v_3\Big(\Big\lfloor \frac{k}{2} \Big\rfloor\Big) + m + v_3(k). \end{aligned}$$

Hence, (1.2) holds for k. This finishes the proof of Lemma 2.4. □

The preceding lemma completes the reduction. Lemma 2.3 expresses the valuation of the odd-indexed term in terms of the adjacent even-indexed term, while Lemma 2.4 verifies that substituting the even formula gives exactly the odd formula. Therefore, in the proof of Theorem 1.3, it remains only to establish formula (1.2) for even values of k.

2.3. Auxiliary estimates under the inductive hypothesis

In this subsection, we derive the estimates for the 3^n-th shifted Stirling numbers needed in the induction. The following lemmas compare the shifted coefficients with the ordinary Stirling numbers at level n. They are stated in conditional form and will be used only under the relevant induction hypotheses in Sections 3 and 4.

Lemma 2.5. *Let $a \in \{1, 2\}$, and let $n, t \in \mathbb{Z}^+$ with $t \leq a3^n$. Assume that* (1.2) *holds for this value of a. If* $2 \mid (t - a)$*, then*

$$v_3(s_{3^n}(a3^n, t)) = v_3(s(a3^n, t)) \quad \textit{and} \quad v_3(s_{3^n}(a3^n, t) - s(a3^n, t)) \geq v_3(s(a3^n, t)) + 2. \tag{2.10}$$

If $2 \nmid (t - a)$*, then*

$$v_3(s_{3^n}(a3^n, t)) \geq v_3(s(a3^n, t + 1)) + n. \tag{2.11}$$

Proof. The proof is based on the expansion in Lemma 2.2. We first treat the boundary values $t = a3^n$ and $t = a3^n - 1$. Then, for $1 \leq t \leq a3^n - 2$, we write $t = a3^m - k$ with $(m, k) \in T_{a,n}$, and distinguish the two cases according to the parity of $t - a$.

If $t = a3^n$, then $2 \mid (a3^n - a)$ and $s_{3^n}(a3^n, a3^n) = 1 = s(a3^n, a3^n)$. Hence, (2.10) holds.

If $t = a3^n - 1$, then $2 \nmid (t - a)$, and by Lemma 2.2 we have

$$s_{3^n}(a3^n, a3^n - 1) = s(a3^n, a3^n - 1) + s(a3^n, a3^n)3^n\binom{a3^n}{1} = s(a3^n, a3^n - 1) + a3^{2n}.$$

Thus, $v_3(s_{3^n}(a3^n, a3^n - 1)) = v_3(s(a3^n, a3^n - 1)) = n = v_3(s(a3^n, a3^n)) + n$, so (2.11) holds when $t = a3^n - 1$.

It remains to consider the case $1 \leq t \leq a3^n - 2$. Then, write $t = a3^m - k$ with $(m, k) \in T_{a,n}$. By Lemma 2.2,

$$s_{3^n}(a3^n, t) = s(a3^n, t) + \sum_{i=t+1}^{a3^n} \binom{i}{i-t} L_i \tag{2.12}$$

with $L_i := s(a3^n, i)3^{n(i-t)}$. We now distinguish two cases according to the parity of $t-a$.

Case 1: $2 \mid (t-a)$.

In this case, we prove (2.10). Since $2 \mid (t-a)$, we have $2 \mid k$, and hence $2 \le k \le 2a3^{m-1}$. By (1.2),

$$v_3(s(a3^n, a3^m - k)) = \frac{a}{2}(3^n - 3^m) - (n-m)(a3^m - k) + m - 1 - v_3(k). \tag{2.13}$$

By (2.12), it is enough to show that the summation term in (2.12) has 3-adic valuation of at least $v_3(s(a3^n, t)) + 2$. Equivalently, it suffices to prove that

$$v_3(L_i) \ge v_3(s(a3^n, t)) + 2 \tag{2.14}$$

for every i with $t+1 \le i \le a3^n$.

We now prove (2.14). First, let $i \in \{a3^n - 1, a3^n\}$. Then,

$$v_3(L_i) = n(a3^n - t) = n(a3^n - a3^m + k).$$

Therefore, by (2.13), $1 \le m \le n$, and $k \ge 2$, we obtain

$$\begin{aligned} v_3(L_i) - v_3(s(a3^n, t)) &= n(a3^n - a3^m + k) - v_3(s(a3^n, a3^m - k)) \\ &= \Big(n - \frac{1}{2}\Big)a3^n - \Big(m - \frac{1}{2}\Big)a3^m + (k-1)m + 1 + v_3(k) \ge m + 1 \ge 2. \end{aligned}$$

Hence, (2.14) holds for $i \in \{a3^n - 1, a3^n\}$.

Next, let $t+1 \le i \le a3^n - 2$. Write $i = a3^l - j$ with $(l, j) \in T_{a,n}$. Then, $l \ge m$ and $2 \le j \le k-1$ if $l = m$. By (1.2) and (2.13), we get

$$\begin{aligned} & v_3(L_i) - v_3(s(a3^n, t)) \\ &= v_3(s(a3^n, a3^l - j)) + n(a3^l - j - a3^m + k) - v_3(s(a3^n, a3^m - k)) \\ &= \frac{a}{2}(3^n - 3^l) - (n-l)(a3^l - j) + l - 1 - v_3\Big(\Big\lfloor \frac{j}{2} \Big\rfloor\Big) + (l + v_3(j))\epsilon_j \\ &\quad + n(a3^l - j - a3^m + k) - \frac{a}{2}(3^n - 3^m) + (n-m)(a3^m - k) - m + 1 + v_3(k) \\ &= \Big(l - \frac{1}{2}\Big)a3^l - \Big(m - \frac{1}{2}\Big)a3^m + l - lj + mk - m + v_3(k) - v_3\Big(\Big\lfloor \frac{j}{2} \Big\rfloor\Big) + (l + v_3(j))\epsilon_j =: D_j. \end{aligned}$$

It remains to prove $D_j \ge 2$ for $2 \le j \le 2a3^{l-1} + 1$.

If j is even, then $2 \le j \le 2a3^{l-1}$, and hence $v_3(j) \le l - 1$. In this case,

$$D_j = \Big(l - \frac{1}{2}\Big)a3^l - \Big(m - \frac{1}{2}\Big)a3^m - l(j-1) + m(k-1) + v_3(k) - v_3(j).$$

If $l = m$, then $k - j \ge 2$ since $k - j \ge 1$ and both k and j are even. Thus,

$$D_j = m(k-j) + v_3(k) - v_3(j) \ge 2m - (m-1) = m + 1 \ge 2.$$

If $l \ge m+1$, then

$$D_j \ge \Big(l - \frac{1}{2}\Big)a3^l - \Big(m - \frac{1}{2}\Big)a3^m - l(2a3^{l-1} - 1) + m - (l-1)$$

$$= \Big(l - \frac{3}{2}\Big)a3^{l-1} - \Big(m - \frac{1}{2}\Big)a3^m + m + 1 \geq m + 1 \geq 2.$$

Therefore, $D_j \geq 2$ holds when j is even.

If j is odd, then $3 \leq j \leq 2a3^{l-1} + 1$ and $j < a3^l$. Note that $j - 1$ is even and $2 \leq j - 1 \leq 2a3^{l-1}$. Using the even case just proved, we obtain

$$D_j = \Big(l - \frac{1}{2}\Big)a3^l - \Big(m - \frac{1}{2}\Big)a3^m - l(j-1) + m(k-1) + v_3(k) - v_3(j-1) + l + v_3(j) = D_{j-1} + v_3(j) \geq 2.$$

Thus, $D_j \geq 2$ also holds when j is odd.

Consequently, (2.14) holds for all i with $t + 1 \leq i \leq a3^n$. Hence, the summation term in (2.12) has 3-adic valuation at least $v_3(s(a3^n, t)) + 2$. This proves (2.10).

Case 2: $2 \nmid (t - a)$.

In this case, we have $2 \nmid k$, $3 \leq k \leq 2a3^{m-1}+1$ and $k < a3^m$. Since $2 \nmid (t-a)$, we have $2 \mid (a3^n - t - 1)$. Thus, $t = a3^n - 2 \cdot \frac{a3^n-t-1}{2} - 1$. By Lemma 2.3,

$$v_3(s(a3^n, t)) = v_3(s(a3^n, t+1)) + v_3(a3^n - t) + n.$$

In particular,

$$v_3(s(a3^n, t)) \geq v_3(s(a3^n, t+1)) + n. \tag{2.15}$$

From (2.12), we write

$$s_{3^n}(a3^n, t) = s(a3^n, t) + (t+1)3^n s(a3^n, t+1) + \sum_{i=t+2}^{a3^n} \binom{i}{i-t} L_i. \tag{2.16}$$

Since $2 \mid (t + 1 - a)$, the estimate proved in Case 1, applied to $t + 1$, gives

$$v_3(L_i) = v_3(s(a3^n, i)3^{n(i-t)}) = v_3(s(a3^n, i)3^{n(i-(t+1))}) + n \geq v_3(s(a3^n, t+1)) + n + 2 \tag{2.17}$$

for every $t + 2 \leq i \leq a3^n$. Therefore, (2.11) follows from (2.15)–(2.17).

The proof of Lemma 2.5 is complete. □

In Sections 3 and 4, (2.10) will be used to replace the 3^n-th Stirling numbers by the corresponding ordinary Stirling numbers up to higher-order error terms, while (2.11) will be used to control the opposite parity cases. We shall also use the following adjacent-index estimate under the same inductive assumption.

Lemma 2.6. *Let $a \in \{1, 2\}$ be fixed, and assume that (1.2) holds for this value of a. Let $n, t \in \mathbb{Z}^+$ with $3 \leq t \leq a3^n$. If $2 \mid (t - a)$, then*

$$v_3(s(a3^n, t)) - v_3(s(a3^n, t-2)) \geq -2n + 2. \tag{2.18}$$

Proof. We first consider the boundary case $t = a3^n$. By (1.2) with $m = n$ and $k = 2$, we have $v_3(s(a3^n, a3^n - 2)) = n - 1$. Since $v_3(s(a3^n, a3^n)) = 0$, it follows that

$$v_3(s(a3^n, t)) - v_3(s(a3^n, t-2)) = v_3(s(a3^n, a3^n)) - v_3(s(a3^n, a3^n-2)) = -n + 1 \geq -2n + 2.$$

Thus, (2.18) holds in this case.

Now assume that $3 \le t \le a3^n - 2$. Write $t = a3^l - j$, where $1 \le l \le n$ and $2 \le j \le 2a3^{l-1}$. Since $2 \mid (t - a)$, we have $2 \mid j$. Then, by the assumed formula (1.2), we obtain

$$v_3(s(a3^n, t)) = v_3(s(a3^n, a3^l - j)) = \frac{a}{2}(3^n - 3^l) - (n - l)(a3^l - j) + l - 1 - v_3(j).$$

Since $t - 2 = a3^l - j - 2 = a3^l - (j + 2)$, we distinguish two cases.

If $j + 2 \le 2a3^{l-1}$, then

$$v_3(s(a3^n, t - 2)) = v_3(s(a3^n, a3^l - (j + 2))) = \frac{a}{2}(3^n - 3^l) - (n - l)(a3^l - (j + 2)) + l - 1 - v_3(j + 2).$$

Hence,

$$\begin{aligned} v_3(s(a3^n, t)) - v_3(s(a3^n, t - 2)) &= -2(n - l) + v_3(j + 2) - v_3(j) \\ &\ge -2n + 2l - (l - 1) \ge -2n + l + 1 \ge -2n + 2. \end{aligned}$$

Therefore, (2.18) holds in this case.

If $j + 2 \ge 2a3^{l-1} + 2$, then $j = 2a3^{l-1}$, and so $t - 2 = a3^{l-1} - 2$. Since $t - 2 \ge 1$, we have $l \ge 2$. Thus

$$v_3(s(a3^n, t - 2)) = v_3(s(a3^n, a3^{l-1} - 2)) = \frac{a}{2}(3^n - 3^{l-1}) - (n - l + 1)(a3^{l-1} - 2) + l - 2$$

and

$$v_3(s(a3^n, t)) = v_3(s(a3^n, a3^l - 2a3^{l-1})) = \frac{a}{2}(3^n - 3^l) - (n - l)a3^{l-1}.$$

Therefore,

$$v_3(s(a3^n, t)) - v_3(s(a3^n, t - 2)) = -2 - 2(n - l) - l + 2 = -2n + l \ge -2n + 2.$$

Thus, (2.18) holds in this case. This completes the proof of Lemma 2.6. □

3. An inductive step: from $v_3(s(3^n, 3^m - k))$ to $v_3(s(2 \cdot 3^n, 2 \cdot 3^m - k))$

In this section, we establish the first key inductive step in the proof of Theorem 1.3. More precisely, we show that formula (1.2) for $a = 1$ implies the corresponding formula for $a = 2$. For convenience, throughout this paper, define

$$s^{(1)}_{n,m,k} := s(3^n, 3^m - k), \quad s^{(1)*}_{n,m,k} := s_{3^n}(3^n, 3^m - k),$$

$$s^{(2)}_{n,m,k} := s(2 \cdot 3^n, 2 \cdot 3^m - k), \quad s^{(2)*}_{n,m,k} := s_{3^n}(2 \cdot 3^n, 2 \cdot 3^m - k).$$

We first outline the proof strategy. The aim of this section is to derive the formula for $s^{(2)}_{n,m,k}$ from the corresponding formula $s^{(1)}_{n,m,k}$. By Lemma 2.4, it is enough to treat the case where k is even. For such k, we use the convolution formula from Lemma 2.2 to decompose $s^{(2)}_{n,m,k}$ into three parts, $s^{(2)}_{n,m,k} = s_1 + s_2 + s_3$. The set defining s_1 contains the terms expected to give the exact valuation, while s_2 and s_3 are remainder sums.

For each fixed pair (m, k), we denote by W the value predicted by formula (1.2); this is made explicit in (3.1) below. The proof is then reduced to showing $v_3(s_1) = W$, $v_3(s_2) \geq W + 1$, and $v_3(s_3) \geq W + 2$. The principal part s_1 is handled by replacing the 3^n-Stirling numbers $s_{3^n}(3^n, \cdot)$ with the corresponding ordinary Stirling numbers $s(3^n, \cdot)$. By Lemma 2.5, the replacement error has 3-adic valuation at least two larger than the predicted leading value, and hence it cannot affect the final 3-adic valuation. The sums s_2 and s_3 are then controlled by separating the relevant indices according to parity and applying the auxiliary estimates from Section 2.

We prove the following lemma.

Lemma 3.1. *If (1.2) holds for $a = 1$, then (1.2) also holds for $a = 2$.*

Proof. Let n be a positive integer. Suppose that (1.2) holds for $a = 1$. Then, Lemmas 2.3 and 2.5 are applicable with $a = 1$. By Lemma 2.4, it suffices to prove (1.2) holds for all the integers m and even k with $1 \leq m \leq n$ and $2 \leq k \leq 4 \cdot 3^{m-1}$.

Let $m, k \in \mathbb{Z}$ with $1 \leq m \leq n$, $2 \leq k \leq 4 \cdot 3^{m-1}$ and $2 \mid k$. Since k is even, proving (1.2) for $a = 2$ is equivalent to proving the following identity:

$$v_3(s^{(2)}_{n,m,k}) = 3^n - 3^m - (n - m)(2 \cdot 3^m - k) + m - 1 - v_3(k) =: W. \tag{3.1}$$

Setting $(m, n, k) \mapsto (3^n, 3^n, 2 \cdot 3^m - k)$ in Lemma 2.2 gives us that

$$s^{(2)}_{n,m,k} = \sum_{i=1}^{2\cdot 3^m - k} s(3^n, i) s_{3^n}(3^n, 2 \cdot 3^m - k - i).$$

Now, let $g_{1,k}(i) := s(3^n, i) s_{3^n}(3^n, 2 \cdot 3^m - k - i)$. Define

$$s_1 := \sum_{i\in A} g_{1,k}(i), \quad s_2 := \sum_{i\in B} g_{1,k}(i), \quad s_3 := \sum_{i\in C} g_{1,k}(i),$$

where

$$A := \begin{cases} \{3^m, 3^m - k\}, & \text{if } 2 \leq k \leq 2 \cdot 3^{m-1}, \\ \{3^{m-1}, 2 \cdot 3^m - k - 3^{m-1}\}, & \text{if } m \geq 2 \text{ and } 2 \cdot 3^{m-1} + 2 \leq k \leq 4 \cdot 3^{m-1} - 2, \\ \{3^{m-1}\}, & \text{if } k = 4 \cdot 3^{m-1} \end{cases}$$

and

$$B := ([1, 2 \cdot 3^m - k] \cap (1 + 2\mathbb{Z})) \setminus A, \quad C := [1, 2 \cdot 3^m - k] \cap (2\mathbb{Z}).$$

Then,

$$s^{(2)}_{n,m,k} = s_1 + s_2 + s_3.$$

It remains to prove the following three estimates:

$$v_3(s_1) = W, \qquad v_3(s_2) \geq W + 1, \qquad v_3(s_3) \geq W + 2.$$

Estimate for the principal part s_1.

We prove that $v_3(s_1) = W$. The proof is divided according to the three possible forms of A. In each case, we replace the terms involving $s_{3^n}(3^n, \cdot)$ by the corresponding terms involving $s(3^n, \cdot)$ plus error terms. More precisely, we write $s_1 = M + E$, where M is the principal contribution obtained from the ordinary Stirling numbers $s(3^n, \cdot)$, and E is the total error term arising from this replacement. By Lemma 2.5, the error term always satisfies $v_3(E) \geq W + 2$. Therefore, it remains only to verify that $v_3(M) = W$. Then, $v_3(s_1) = W$ follows in each case.

Case 1: $2 \leq k \leq 2 \cdot 3^{m-1}$.

In this case, one has $A = \{3^m, 3^m - k\}$. Hence,

$$s_1 = g_{1,k}(3^m) + g_{1,k}(3^m - k) = s^{(1)}_{n,m,0} \cdot s^{(1)*}_{n,m,k} + s^{(1)}_{n,m,k} \cdot s^{(1)*}_{n,m,0}.$$

By Lemma 2.2, we may write $s^{(1)*}_{n,m,k} = s^{(1)}_{n,m,k} + L_1$ and $s^{(1)*}_{n,m,0} = s^{(1)}_{n,m,0} + L_2$ with $L_1, L_2 \in \mathbb{Z}$. Since Lemma 2.5 holds for $a = 1$ and k is even, we have $v_3(L_1) \geq v_3(s^{(1)}_{n,m,k}) + 2$ and $v_3(L_2) \geq v_3(s^{(1)}_{n,m,0}) + 2$.

Thus, $s_1 = M + E$, where

$$M := 2s^{(1)}_{n,m,0} \cdot s^{(1)}_{n,m,k}, \quad E := s^{(1)}_{n,m,0} L_1 + s^{(1)}_{n,m,k} L_2.$$

It remains to compute $v_3(M)$. Note that $v_3(s^{(1)}_{n,m,0}) = \frac{1}{2}(3^n - 3^m) - (n-m)3^m$. Indeed, this is clear when $m = n$, and for $1 \leq m \leq n-1$ it follows from $v_3(s^{(1)}_{n,m,0}) = v_3(s^{(1)}_{n,m+1,2\cdot 3^m}) = \frac{1}{2}(3^n - 3^m) - (n-m)3^m$. Therefore, by (1.2) for $a = 1$, we obtain

$$\begin{aligned} v_3(M) &= v_3(2s^{(1)}_{n,m,0} \cdot s^{(1)}_{n,m,k}) = v_3(s^{(1)}_{n,m,0}) + v_3(s^{(1)}_{n,m,k}) = v_3(g_{1,k}(3^m)) = v_3(g_{1,k}(3^m - k)) \\ &= \frac{1}{2}(3^n - 3^m) - (n-m)3^m + \frac{1}{2}(3^n - 3^m) - (n-m)(3^m - k) + m - 1 - v_3(k) \\ &= 3^n - 3^m - (n-m)(2 \cdot 3^m - k) + m - 1 - v_3(k) = W. \end{aligned}$$

Since $s_1 = M + E$, $v_3(M) = W$, and $v_3(E) \geq W + 2$, we have $v_3(s_1) = W$.

Case 2: $m \geq 2$ *and* $2 + 2 \cdot 3^{m-1} \leq k \leq 4 \cdot 3^{m-1} - 2$.

In this case, $A = \{3^{m-1}, 2 \cdot 3^m - k - 3^{m-1}\}$. Note that $2 \cdot 3^m - k - 3^{m-1} = 3^m - (k - 2 \cdot 3^{m-1})$ where $2 \leq k - 2 \cdot 3^{m-1} \leq 2 \cdot 3^{m-1} - 2$, which implies that $v_3(k - 2 \cdot 3^{m-1}) = v_3(k)$. Hence,

$$s_1 = g_{1,k}(3^{m-1}) + g_{1,k}(2 \cdot 3^m - k - 3^{m-1}) = s^{(1)}_{n,m-1,0} \cdot s^{(1)*}_{n,m,k-2\cdot 3^{m-1}} + s^{(1)}_{n,m,k-2\cdot 3^{m-1}} \cdot s^{(1)*}_{n,m-1,0}.$$

Again, write $s^{(1)*}_{n,m,k-2\cdot 3^{m-1}} = s^{(1)}_{n,m,k-2\cdot 3^{m-1}} + L_3$ and $s^{(1)*}_{n,m-1,0} = s^{(1)}_{n,m-1,0} + L_4$ with $L_3, L_4 \in \mathbb{Z}$. Then, $v_3(L_3) \geq v_3(s^{(1)}_{n,m,k-2\cdot 3^{m-1}}) + 2$ and $v_3(L_4) \geq v_3(s^{(1)}_{n,m-1,0}) + 2$. Thus, $s_1 = M + E$, where

$$M := 2s^{(1)}_{n,m-1,0} \cdot s^{(1)}_{n,m,k-2\cdot 3^{m-1}}, \quad E := s^{(1)}_{n,m-1,0} L_3 + s^{(1)}_{n,m,k-2\cdot 3^{m-1}} L_4.$$

By (1.2) for $a = 1$, we have

$$\begin{aligned} v_3(M) &= v_3(2s^{(1)}_{n,m-1,0} \cdot s^{(1)}_{n,m,k-2\cdot 3^{m-1}}) = v_3(s^{(1)}_{n,m,2\cdot 3^{m-1}}) + v_3(s^{(1)}_{n,m,k-2\cdot 3^{m-1}}) \\ &= v_3(g_{1,k}(3^{m-1})) = v_3(g_{1,k}(2 \cdot 3^m - k - 3^{m-1})) \\ &= \frac{1}{2}(3^n - 3^m) - (n-m)(3^m - 2 \cdot 3^{m-1}) \\ &\quad + \frac{1}{2}(3^n - 3^m) - (n-m)(3^m - (k - 2 \cdot 3^{m-1})) + m - 1 - v_3(k - 2 \cdot 3^{m-1}) \end{aligned}$$

$$= 3^n - 3^m - (n-m)(2\cdot 3^m - k) + m - 1 - v_3(k) = W.$$

Since $s_1 = M + E$, $v_3(M) = W$ and $v_3(E) \geq W + 2$, we have $v_3(s_1) = W$.

Case 3: $k = 4\cdot 3^{m-1}$.

In this case, $A = \{3^{m-1}\}$ and

$$s_1 = g_{1,k}(3^{m-1}) = s^{(1)}_{n,m-1,0} \cdot s^{(1)*}_{n,m-1,0} = (s^{(1)}_{n,m-1,0})^2 + s^{(1)}_{n,m-1,0} L_4$$

with $v_3(L_4) \geq v_3(s^{(1)}_{n,m-1,0}) + 2$. Therefore, we have $s_1 = M + E$, where

$$M := (s^{(1)}_{n,m-1,0})^2, \quad E := s^{(1)}_{n,m-1,0} L_4$$

and

$$\begin{aligned} v_3(M) &= v_3(g_{1,k}(3^{m-1})) = v_3((s^{(1)}_{n,m-1,0})^2) = 2v_3(s^{(1)}_{n,m,2\cdot 3^{m-1}}) = 3^n - 3^m - 2(n-m)(3^m - 2\cdot 3^{m-1}) \\ &= 3^n - 3^m - (n-m)(2\cdot 3^m - 4\cdot 3^{m-1}) + m - 1 - v_3(4\cdot 3^{m-1}) = W. \end{aligned}$$

Since $s_1 = M + E$, $v_3(M) = W$, and $v_3(E) \geq W + 2$, we have $v_3(s_1) = W$.

The three cases above exhaust all possible forms of A. Therefore, $v_3(s_1) = W$. This proves the estimate for the principal part s_1.

Estimate for the odd-index part s_2

We next estimate the odd-index part s_2. The purpose of this part is twofold: First, to prove $v_3(s_2) \geq W + 1$; and second, to identify the possible terms in s_2 whose 3-adic valuation is exactly $W + 1$.

If $m = 1$, then $B = \varnothing$, hence $s_2 = 0$. Thus, we may assume that $2 \leq m \leq n$. For $i \in B$, if either $i > 3^n$ or $2\cdot 3^m - k - i > 3^n$, then $g_{1,k}(i) = 0$. Therefore, it suffices to consider $B' = B \cap [2\cdot 3^m - k - 3^n, 3^n]$. If $B' = \varnothing$, then $s_2 = 0$, and there is nothing to prove.

Assume henceforth that $B' \neq \varnothing$. We decompose B' as $B' := B_1 \cup B_2 \cup B_3$, where

$$B_1 := \Big[1, 3^m - \frac{k}{2} - 1\Big] \cap B', \quad B_2 := \Big[3^m - \frac{k}{2} + 1, 2\cdot 3^m - k\Big] \cap B', \quad B_3 := \Big\{3^m - \frac{k}{2}\Big\} \cap B'.$$

Since at least one of B_1, B_2, and B_3 is nonempty, one obtains that

$$s_2 = \sum_{i\in B'} g_{1,k}(i) = \Big(\sum_{i\in B_1} + \sum_{i\in B_2} + \sum_{i\in B_3}\Big) g_{1,k}(i).$$

The central term B_3.

We first estimate the possible central term B_3. If $B_3 \neq \varnothing$, then let $i \in B_3$. So, $i = 3^m - \frac{k}{2}$. Since i is odd, one has $4 \mid k$. Thus,

$$\begin{aligned} v_3(g_{1,k}(3^m - \frac{k}{2})) &= v_3(s^{(1)}_{n,m,\frac{k}{2}} \cdot s^{(1)*}_{n,m,\frac{k}{2}}) = v_3((s^{(1)}_{n,m,\frac{k}{2}})^2) \\ &= 3^n - 3^m - (n-m)(2\cdot 3^m - k) + 2(m - 1 - v_3(k)) = W + m - 1 - v_3(k). \end{aligned}$$

By $2 \leq k \leq 4\cdot 3^{m-1}$ and $4 \mid k$, one has $v_3(k) \leq m - 1$. Thus, $v_3(g_{1,k}(3^m - \frac{k}{2})) \geq W$ with equality if and only if $k = 4\cdot 3^{m-1}$. If $k = 4\cdot 3^{m-1}$, then $i = 3^{m-1}$, which contradicts the definition of B when

$k = 4 \cdot 3^{m-1}$. Therefore, $v_3(g_{1,k}(3^m - \frac{k}{2})) \geq W + 1$. Further, $v_3(g_{1,k}(3^m - \frac{k}{2})) = W + 1$ holds if and only if $k = 4 \cdot 3^{m-2}$ or $k = 8 \cdot 3^{m-2}$, i.e.,

$$v_3(g_{1,4\cdot 3^{m-2}}(3^m - 2 \cdot 3^{m-2})) = v_3(s^{(1)}_{n,m,2\cdot 3^{m-2}} \cdot s^{(1)*}_{n,m,2\cdot 3^{m-2}}) = v_3((s^{(1)}_{n,m,2\cdot 3^{m-2}})^2) = W + 1, \tag{3.2}$$

$$v_3(g_{1,8\cdot 3^{m-2}}(3^m - 4 \cdot 3^{m-2})) = v_3(s^{(1)}_{n,m,4\cdot 3^{m-2}} \cdot s^{(1)*}_{n,m,4\cdot 3^{m-2}}) = v_3((s^{(1)}_{n,m,4\cdot 3^{m-2}})^2) = W + 1. \tag{3.3}$$

Thus, every term indexed by B_3 has 3-adic valuation at least $W + 1$. Moreover, equality can occur only in the case listed in (3.2) and (3.3). All other terms indexed by B_3 have valuation at least $W + 2$.

The paired terms B_1 and B_2.

We now consider the terms indexed by B_1 and B_2. The two parts are symmetric with respect to the involution $i \mapsto 2 \cdot 3^m - k - i$. Indeed, $i \in B_1$ if and only if $2 \cdot 3^m - k - i \in B_2$. Moreover, by Lemma 2.5, the two corresponding terms have the same 3-adic valuation:

$$\begin{aligned} v_3(g_{1,k}(i)) &= v_3(s(3^n, i)s_{3^n}(3^n, 2 \cdot 3^m - k - i)) = v_3(s(3^n, i)s(3^n, 2 \cdot 3^m - k - i)) \\ &= v_3(s(3^n, 2 \cdot 3^m - k - i)s_{3^n}(3^n, i)) = v_3(g_{1,k}(2 \cdot 3^m - k - i)). \end{aligned}$$

Hence, it is enough to prove $v_3(g_{1,k}(i)) \geq W + 1$ holds for $i \in B_1$. This will be done in what follows.

Let $i \in B_1$. Then, $1 \leq i \leq 3^m - \frac{k}{2} - 1 \leq 3^m - 2$. Write $i = 3^{l_1} - j_1$ with $(l_1, j_1) \in T_{1,m}$ and $2 \mid j_1$. Note that $\frac{k}{2} + 1 \leq j_1 \leq 2 \cdot 3^{m-1}$ if $l_1 = m$. Since $1 \leq l_1 \leq m \leq n$, by (1.2) for $a = 1$,

$$v_3(s^{(1)}_{n,l_1,j_1}) = \frac{1}{2}(3^n - 3^{l_1}) - (n - l_1)(3^{l_1} - j_1) + l_1 - 1 - v_3(j_1). \tag{3.4}$$

Note that $2\cdot 3^m - k - i \in B_2$ and $3^{m-1} + 1 \leq 3^m - \frac{k}{2} + 1 \leq 2\cdot 3^m - k - i \leq \min\{2\cdot 3^m - k - 1, 3^n\} \leq \min\{2\cdot 3^m - 3, 3^n\}$.

If $2 \cdot 3^m - k - i = 3^n$, then we must have $m = n$ and $i = 2 \cdot 3^m - k - 3^n = 3^n - k = 3^m - k$. It implies that $2 + 2 \cdot 3^{m-1} \leq k \leq 3^m - 1$ since $i \geq 1$ and $i \neq 3^m - k$ when $2 \leq k \leq 2 \cdot 3^{m-1}$. Hence, $1 \leq i \leq 3^{m-1} - 2 = 3^{n-1} - 2$, which implies that $l_1 \leq n - 1$. Note that $2 \leq j_1 \leq 2 \cdot 3^{l_1 - 1} \leq 2 \cdot 3^{n-2}$ and $i = 3^{l_1} - j_1 = 3^n - k$ imply that $v_3(k) = v_3(j_1)$. From (3.4), $s^{(1)*}_{n,n,0} = 1$ and $W = n - 1 - v_3(k)$ when $m = n$, we deduce that

$$\begin{aligned} v_3(g_{1,k}(i)) - W &= v_3(s(3^n, i)) - W = v_3(s^{(1)}_{n,l_1,j_1}) - W \\ &= \frac{1}{2}(3^n - 3^{l_1}) - (n - l_1)3^{l_1} + (n - l_1)(j_1 - 1) \geq \frac{1}{2}(3^n - 3^{l_1}) - (n - l_1)(3^{l_1} - 1) \geq 1 \end{aligned}$$

with the equality if and only if $l_1 = n - 1$ and $j_1 = 2$. So, for $k = 2 \cdot 3^{n-1} + 2$, one has

$$v_3(g_{1,k}(3^{n-1} - 2)) = v_3(g_{1,k}(3^n)) = v_3(s^{(1)}_{n,n-1,2} \cdot s^{(1)}_{n,n,0}) = W + 1. \tag{3.5}$$

For any other integer k in this case, we have $v_3(g_{1,k}(i)) \geq W + 2$.

If $2 \cdot 3^m - k - i \leq 3^n - 2$, then write $2 \cdot 3^m - k - i = 3^{l_2} - j_2$ with $(l_2, j_2) \in T_{1,n}$ and $2 \mid j_2$. Since $3^{m-1} + 1 \leq 2 \cdot 3^m - k - i \leq \min\{2 \cdot 3^m - 3, 3^n\}$, we have $m \leq l_2 \leq m + 1$. Also note that $2 \leq j_2 \leq \frac{k}{2} - 1$ if $l_2 = m$ and $3^m + 3 \leq j_2 \leq 2 \cdot 3^m$ if $l_2 = m + 1$. By (1.2) for $a = 1$,

$$v_3(s^{(1)}_{n,l_2,j_2}) = \frac{1}{2}(3^n - 3^{l_2}) - (n - l_2)(3^{l_2} - j_2) + l_2 - 1 - v_3(j_2).$$

Hence, together with (3.4) one derives that

$$\begin{aligned}v_3(g_{1,k}(i)) - W &= v_3(s(3^n, i)) + v_3(s(3^n, 2\cdot 3^m - k - i)) - W = v_3(s^{(1)}_{n,l_1,j_1}) + v_3(s^{(1)}_{n,l_2,j_2}) - W \\ &= 3^m - \frac{1}{2}(3^{l_1} + 3^{l_2}) - (m - l_1)(3^{l_1} - j_1) - (m - l_2)(3^{l_2} - j_2) \\ &\quad + l_1 + l_2 - m - 1 + v_3(k) - v_3(j_1) - v_3(j_2) =: D_{l_1,l_2}.\end{aligned}$$

Since $2\cdot 3^m - k = 3^{l_1} - j_1 + 3^{l_2} - j_2$ with $2 \le k \le 4\cdot 3^{m-1}$, $2 \le j_1 \le 2\cdot 3^{l_1-1}$, and $2 \le j_2 \le 2\cdot 3^{l_2-1}$, we get that $v_3(k) - v_3(j_1) - v_3(j_2) \ge -\max\{v_3(j_1), v_3(j_2)\}$. In what follows, we prove that $D_{l_1,l_2} \ge 1$. Since $m \le l_2 \le m+1$, there are two cases to consider.

The case $l_2 = m$.

In this case, we have $2 \le j_2 \le \frac{k}{2} - 1 \le 2\cdot 3^{m-1} - 1$. Since $2 \mid j_2$, one derives that $2 \le j_2 \le 2\cdot 3^{m-1} - 2$ and $v_3(j_2) \le m-2$. Note that

$$D_{l_1,m} = \frac{1}{2}(3^m - 3^{l_1}) - (m - l_1)(3^{l_1} - j_1) + l_1 - 1 + v_3(k) - v_3(j_1) - v_3(j_2).$$

If $1 \le l_1 \le m-1$, then $2 \le j_1 \le 2\cdot 3^{m-2}$. Thus, $v_3(j_1) \le m-2$, and one has

$$D_{l_1,m} \ge \frac{1}{2}(3^m - 3^{l_1}) - (m - l_1)(3^{l_1} - j_1) + l_1 - 1 - (m-2) \ge \frac{1}{2}(3^m - 3^{l_1}) - (m - l_1)(3^{l_1} - 1) + 1 \ge 2.$$

So, $v_3(g_{1,k}(i)) \ge W + 2$ is proved when $1 \le l_1 \le m-1$ and $l_2 = m$.

If $l_1 = m$, then $\frac{k}{2} + 1 \le j_1 \le 2\cdot 3^{m-1}$. Note that $j_1 + j_2 = k$ and $2 \le j_2 \le j_1 - 2$. This implies that $6 \le k \le 4\cdot 3^{m-1} - 2$. Since $i \ne 3^{m-1}$ when $m \ge 2$ and $2 + 2\cdot 3^{m-1} \le k \le 4\cdot 3^{m-1} - 2$, we have $4 \le j_1 \le 2\cdot 3^{m-1} - 2$. Thus, $v_3(j_1) \le m-2$ and

$$D_{m,m} = m - 1 + v_3(k) - v_3(j_1) - v_3(j_2) \ge m - 1 - (m-2) = 1,$$

where the equality holds if and only if j_1 (or j_2) equals $2\cdot 3^{m-2}$ or $4\cdot 3^{m-2}$ and $v_3(j_2) = v_3(k)$ (or $v_3(j_1) = v_3(k)$). Hence, $v_3(g_{1,k}(i)) \ge W + 1$ is proved when $l_1 = l_2 = m$. Moreover, one notes that if $m \ge 3$ and $2\cdot 3^{m-2} + 2 \le k \le 4\cdot 3^{m-2} - 2$, then

$$v_3(g_{1,k}(3^m - 2\cdot 3^{m-2})) = v_3(g_{1,k}(3^m - (k - 2\cdot 3^{m-2}))) = v_3(s^{(1)}_{n,m,2\cdot 3^{m-2}} \cdot s^{(1)}_{n,m,k-2\cdot 3^{m-2}}) = W + 1. \qquad (3.6)$$

If $m \ge 3$ and $4\cdot 3^{m-2} + 2 \le k \le 8\cdot 3^{m-2} - 2$ with $k \ne 2\cdot 3^{m-1}$, then (3.6) holds and

$$v_3(g_{1,k}(3^m - 4\cdot 3^{m-2})) = v_3(g_{1,k}(3^m - (k - 4\cdot 3^{m-2}))) = v_3(s^{(1)}_{n,m,4\cdot 3^{m-2}} \cdot s^{(1)}_{n,m,k-4\cdot 3^{m-2}}) = W + 1. \qquad (3.7)$$

If $m \ge 3$ and $8\cdot 3^{m-2} + 2 \le k \le 10\cdot 3^{m-2} - 2$, then (3.7) holds.

For any other integer k in this case, we have $v_3(g_{1,k}(i)) \ge W + 2$.

The case $l_2 = m+1$.

In this case, we have $3^m + 3 \le j_2 \le 2\cdot 3^m$. Note that

$$D_{l_1,m+1} = \frac{1}{2}(3^m - 3^{l_1}) - (m - l_1)(3^{l_1} - j_1) + 2\cdot 3^m - j_2 + l_1 + v_3(k) - v_3(j_1) - v_3(j_2).$$

If $l_1 = m$, then by $2 \cdot 3^m - k - i = 3^{m+1} - j_2 \geq 3^m$, one knows that $i = 3^m - j_1 \leq 3^m - k$. Thus, $k \leq j_1 \leq 2 \cdot 3^{m-1}$. By the definition of B, $i \neq 3^m - k$. Hence, $j_2 \leq 2 \cdot 3^m - 2$ and

$$D_{m,m+1} = 2 \cdot 3^m - j_2 + m + \nu_3(k) - \nu_3(j_1) - \nu_3(j_2) \geq 2.$$

So, $\nu_3(g_{1,k}(i)) \geq W + 2$ is proved when $l_1 = m$ and $l_2 = m + 1$.

If $1 \leq l_1 \leq m - 1$, then

$$\begin{aligned} D_{l_1,m+1} &\geq \frac{1}{2}(3^m - 3^{l_1}) - (m - l_1)(3^{l_1} - j_1) + 2 \cdot 3^m - 2 \cdot 3^m + l_1 - m \\ &= \frac{1}{2}(3^m - 3^{l_1}) - (m - l_1)(3^{l_1} - j_1 + 1) \geq \frac{1}{2}(3^m - 3^{l_1}) - (m - l_1)(3^{l_1} - 1) \geq 1, \end{aligned}$$

where the equality holds if and only if $j_2 = 2 \cdot 3^m$, $l_1 = m - 1$, $j_1 = 2$, i.e., $i = 3^m - k = 3^{m-1} - 2$. Hence, $\nu_3(g_{1,k}(i)) \geq W + 1$ is proved when $1 \leq l_1 \leq m - 1$ and $l_2 = m + 1$. Moreover, if $k = 2 \cdot 3^{m-1} + 2$, then

$$\nu_3(g_{1,k}(3^m - k)) = \nu_3(g_{1,k}(3^m)) = \nu_3(s^{(1)}_{n,m-1,2} \cdot s^{(1)}_{n,m,0}) = W + 1. \tag{3.8}$$

For any other integer k in this case, we have $\nu_3(g_{1,k}(i)) \geq W + 2$.

Consequently, every term indexed by $B_1 \cup B_2$ has 3-adic valuation at least $W + 1$. All possible terms with $\nu_3(g_{1,k}(i)) = W + 1$ are precisely those listed in (3.5)–(3.8). All other terms indexed by $B_1 \cup B_2$ have 3-adic valuation at least $W + 2$.

Combining the estimates for B_3 and $B_1 \cup B_2$, we obtain $\nu_3(g_{1,k}(i)) \geq W + 1$ for all $i \in B'$. Moreover, all possible terms with 3-adic valuation exactly $W + 1$ are listed in (3.2)–(3.8). Since $g_{1,k}(i) = 0$ for $i \in B \setminus B'$, it follows that $\nu_3(s_2) \geq W + 1$. This proves the estimate for the odd-index part s_2, together with the description of the possible $W + 1$-terms.

Estimate for the even-index part s_3.

It remains to estimate the contribution from even indices. We prove that $\nu_3(s_3) \geq W + 2$.

Let $i \in C$. Then, $2 \mid i$. If either $i \geq 3^n + 1$ or $2 \cdot 3^m - k - i \geq 3^n + 1$, then $g_{1,k}(i) = 0$. Hence, it suffices to consider $C' := C \cap [2 \cdot 3^m - k - 3^n, 3^n]$. Equivalently, $C' = C \cap [2 \cdot 3^m - k - 3^n + 1, 3^n - 1]$, since C consists of even integers. Note that $C' \neq \emptyset$, since either $2 \cdot 3^m - k \in C'$ or $3^n - 1 \in C'$.

For $i \in C'$, both i and $2 \cdot 3^m - k - i$ are even. By Lemmas 2.3 and 2.5, we have

$$\nu_3(s(3^n, i)) = \nu_3(s(3^n, i + 1)) + \nu_3(3^n - i) + n \tag{3.9}$$

and

$$\nu_3(s_{3^n}(3^n, 2 \cdot 3^m - k - i)) \geq \nu_3(s(3^n, 2 \cdot 3^m - k - i + 1)) + n. \tag{3.10}$$

Moreover, since i is even and $3 \leq i + 1 \leq 3^n$, the inductive hypothesis and Lemma 2.6 imply

$$\nu_3(s(3^n, i + 1)) - \nu_3(s(3^n, i - 1)) \geq -2n + 2. \tag{3.11}$$

Then, it follows from (3.9)–(3.11) that

$$\begin{aligned} \nu_3(g_{1,k}(i)) &= \nu_3(s(3^n, i)) + \nu_3(s_{3^n}(3^n, 2 \cdot 3^m - k - i)) \\ &\geq \nu_3(s(3^n, i + 1)) + \nu_3(s(3^n, 2 \cdot 3^m - k - i + 1)) + 2n \end{aligned}$$

$$\geq v_3(s(3^n, i-1)) + v_3(s(3^n, 2\cdot 3^m - k - (i-1))) + 2$$
$$= v_3(s(3^n, i-1)) + v_3(s_{3^n}(3^n, 2\cdot 3^m - k - (i-1))) + 2 = v_3(g_{1,k}(i-1)) + 2. \tag{3.12}$$

Since $i-1$ is odd and $2 \leq i \leq 2\cdot 3^m - k$, one has $i-1 \in A \cup B$. By the estimates for s_1 and s_2, we know that $v_3(g_{1,k}(i-1)) \geq W$. It follows from (3.12) that

$$v_3(g_{1,k}(i)) \geq v_3(g_{1,k}(i-1)) + 2 \geq W + 2.$$

Therefore, $v_3(g_{1,k}(i)) \geq W+2$ for $i \in C'$, and consequently $v_3(s_3) \geq W+2$. This proves the estimate for the even-index part s_3.

Combining the estimates for s_1, s_2, and s_3, we obtain $v_3(s^{(2)}_{n,m,k}) = W$. Therefore, formula (1.2) holds for $a = 2$ whenever it holds for $a = 1$. This completes the proof of Lemma 3.1. □

The following decompositions will be used in Section 4 to isolate the possible leading terms in the passage from $2\cdot 3^n$ to 3^{n+1}.

Remark 3.2. Let $n, m, k \in \mathbb{Z}$ such that $1 \leq m \leq n$ and $2 \leq k \leq 4\cdot 3^{m-1}$ with $2 \mid k$. If (1.2) holds for $a = 1$, then from the proof of Lemma 3.1, we can conclude that

$$s^{(2)}_{n,m,k} = \bar{s}_1 + \bar{s}_2 + \bar{s}_3$$

with $\bar{s}_1, \bar{s}_2, \bar{s}_3 \in \mathbb{Z}$, $v_3(\bar{s}_1) = v_3(s^{(2)}_{n,m,k}) = W$, $v_3(\bar{s}_2) \geq W+1$, and $v_3(\bar{s}_3) \geq W+2$, where

$$\bar{s}_1 = \begin{cases} 2s^{(1)}_{n,m,0}\cdot s^{(1)}_{n,m,k}, & \text{if } 2 \leq k \leq 2\cdot 3^{m-1}, \\ 2s^{(1)}_{n,m-1,0}\cdot s^{(1)}_{n,m,k-2\cdot 3^{m-1}}, & \text{if } k \in [2\cdot 3^{m-1}+2, 4\cdot 3^{m-1}-2] \text{ with } m \geq 2, \\ (s^{(1)}_{n,m-1,0})^2, & \text{if } k = 4\cdot 3^{m-1}. \end{cases}$$

If $m = 2$ and $k = 2\cdot 3^{m-1} + 2\cdot 3^{m-2}$, then by (3.3), (3.5), and (3.8), one has

$$\bar{s}_2 = 2s^{(1)}_{n,m-1,2}\cdot s^{(1)}_{n,m,0} + (s^{(1)}_{n,m,4\cdot 3^{m-2}})^2.$$

If $m \geq 2$ and $k = 4\cdot 3^{m-2}$, then by (3.2),

$$\bar{s}_2 = (s^{(1)}_{n,m,2\cdot 3^{m-2}})^2.$$

Assume now that $m \geq 3$. If $k \in [2\cdot 3^{m-2}+2, 4\cdot 3^{m-2}-2]$, then by (3.6),

$$\bar{s}_2 = 2s^{(1)}_{n,m,2\cdot 3^{m-2}}\cdot s^{(1)}_{n,m,k-2\cdot 3^{m-2}}.$$

If $k \in [4\cdot 3^{m-2}+2, 8\cdot 3^{m-2}-2]$ with $k \notin \{2\cdot 3^{m-1}, 2\cdot 3^{m-1}+2\}$, then by (3.6) and (3.7),

$$\bar{s}_2 = 2s^{(1)}_{n,m,2\cdot 3^{m-2}}\cdot s^{(1)}_{n,m,k-2\cdot 3^{m-2}} + 2s^{(1)}_{n,m,4\cdot 3^{m-2}}\cdot s^{(1)}_{n,m,k-4\cdot 3^{m-2}}.$$

If $k = 2\cdot 3^{m-1} + 2$, then by (3.5)–(3.8),

$$\bar{s}_2 = 2s^{(1)}_{n,m-1,2}\cdot s^{(1)}_{n,m,0} + 2s^{(1)}_{n,m,2\cdot 3^{m-2}}\cdot s^{(1)}_{n,m,k-2\cdot 3^{m-2}} + 2s^{(1)}_{n,m,4\cdot 3^{m-2}}\cdot s^{(1)}_{n,m,k-4\cdot 3^{m-2}}.$$

If $k = 8\cdot 3^{m-2}$, then by (3.3),

$$\bar{s}_2 = (s^{(1)}_{n,m,4\cdot 3^{m-2}})^2.$$

If $k \in [8\cdot 3^{m-2}+2, 10\cdot 3^{m-2}-2]$, then by (3.7),

$$\bar{s}_2 = 2s^{(1)}_{n,m,4\cdot 3^{m-2}}\cdot s^{(1)}_{n,m,k-4\cdot 3^{m-2}}.$$

For any other k, one has $\bar{s}_2 = 0$.

4. From $2 \cdot 3^n$ to 3^{n+1}: proof of Theorem 1.3

In this section, we complete the proof of Theorem 1.3. By Lemma 3.1, it suffices to prove formula (1.2) for $a = 1$. We shall prove this by induction on n.

We briefly describe the structure of the inductive step. Assuming that Theorem 1.3 holds for n, Lemma 3.1 gives the required formula for the family $s^{(2)}_{n,m,k}$. The remaining task is to derive the formula for $s^{(1)}_{n+1,m,k}$. We do this by applying the convolution formula of Lemma 2.2, which expresses $s^{(1)}_{n+1,m,k}$ as a sum of products involving $s(3^n, \cdot)$ and $s_{3^n}(2 \cdot 3^n, \cdot)$.

After reducing to even k, we denote by $W_{m,k}$ the value predicted by the formula. The proof then proceeds by estimating the summands $g_{2,k}(i) = s(3^n, i)s_{3^n}(2\cdot 3^n, 3^m - k - i)$. Even-index terms are shown to have 3-adic valuation at least $W_{m,k} + 1$, and therefore cannot contribute to the leading valuation. Odd-index terms may have 3-adic valuation as low as $W_{m,k} - 1$, so we isolate a special subset A of odd indices containing all possible leading contributions. The key point is then to prove $v_3(\sum_{i\in A} g_{2,k}(i)) = W_{m,k}$. Once this is established, all remaining summands have strictly larger 3-adic valuation, and the desired value of $v_3(s^{(1)}_{n+1,m,k})$ follows.

4.1. Induction setup

We first set up the inductive step. The case $n = 1$ is immediate. Indeed, then $m = 1$ and $k = 2$, and $v_3(s(3, 3 - 2)) = v_3(s(3, 1)) = v_3(2) = 0$. Thus, Theorem 1.3 holds for $n = 1$.

Assume now that Theorem 1.3 holds for n. We prove it for $n + 1$. This is equivalent to showing that, for all integers m and k satisfying $1 \le m \le n + 1$, $2 \le k \le 2 \cdot 3^{m-1} + 1$, and $k < 3^m$, one has

$$v_3(s^{(1)}_{n+1,m,k}) = \frac{1}{2}(3^{n+1} - 3^m) - (n + 1 - m)(3^m - k) + m - 1 - v_3\Big(\Big\lfloor \frac{k}{2} \Big\rfloor\Big) + (m + v_3(k))\frac{1 - (-1)^k}{2}. \quad (4.1)$$

The case $m = 1$ is also immediate, since then $k = 2$ and

$$v_3(s^{(1)}_{n+1,1,2}) = v_3(s(3^{n+1}, 1)) = v_3((3^{n+1} - 1)!) = \frac{1}{2}(3^{n+1} - 2n - 3).$$

Hence, we may assume that $2 \le m \le n+1$. Moreover, by Lemma 2.4, it suffices to prove (4.1) for even k. Thus, throughout the rest of this section, we assume that $2 \le m \le n + 1$, $2 \le k \le 2 \cdot 3^{m-1}$, and k is even. It remains to prove

$$v_3(s^{(1)}_{n+1,m,k}) = \frac{1}{2}(3^{n+1} - 3^m) - (n + 1 - m)(3^m - k) + m - 1 - v_3(k) =: W_{m,k}. \quad (4.2)$$

4.2. The convolution decomposition and estimates for $g_{2,k}(i)$

In this subsection, we decompose $s^{(1)}_{n+1,m,k}$ into a sum of terms $g_{2,k}(i)$ and establish valuation estimates for these terms according to the parity of i. By Lemma 2.2, with the substitution $(m, n, k) \mapsto (3^n, 2 \cdot 3^n, 3^m - k)$, we have

$$s^{(1)}_{n+1,m,k} = \sum_{i=1}^{3^m-k} s(3^n, i)s_{3^n}(2 \cdot 3^n, 3^m - k - i).$$

Let $i \in \mathbb{Z}$ and $1 \le i \le 3^m - k$. Define $g_{2,k}(i) := s(3^n, i)s_{3^n}(2 \cdot 3^n, 3^m - k - i)$. Then,

$$s^{(1)}_{n+1,m,k} = \sum_{i=1}^{3^m-k} g_{2,k}(i). \quad (4.3)$$

The proof of (4.2) is based on the following strategy. We first establish two estimates for the summands $g_{2,k}(i)$:

$$\begin{aligned} v_3(g_{2,k}(i)) &\geq W_{m,k} - 1 \qquad \text{if } 2 \nmid i, \\ v_3(g_{2,k}(i)) &\geq W_{m,k} + 1 \qquad \text{if } 2 \mid i. \end{aligned} \tag{4.4}$$

The second estimate in (4.4) shows that even-index terms cannot affect the leading valuation. The first estimate in (4.4) shows that possible low-valuation terms among odd indices can only have valuation $W_{m,k} - 1$ or $W_{m,k}$. Accordingly, define

$$\begin{aligned} A_1 &:= \{i \in [1, 3^m - k] \cap (1 + 2\mathbb{Z}) : v_3(g_{2,k}(i)) = W_{m,k} - 1\}, \\ A_2 &:= \{i \in [1, 3^m - k] \cap (1 + 2\mathbb{Z}) : v_3(g_{2,k}(i)) = W_{m,k}\}, \quad A := A_1 \cup A_2. \end{aligned}$$

We also define the two remainder sets

$$B := \big([1, 3^m - k] \cap (1 + 2\mathbb{Z})\big) \setminus A, \qquad C := [1, 3^m - k] \cap 2\mathbb{Z}.$$

Thus, the index set $[1, 3^m - k]$ is the disjoint union of A, B, and C. Set

$$S_A := \sum_{i \in A} g_{2,k}(i), \qquad S_B := \sum_{i \in B} g_{2,k}(i), \qquad S_C := \sum_{i \in C} g_{2,k}(i).$$

Once (4.4) is proved, the definition of A will imply that

$$v_3(S_B) \geq W_{m,k} + 1, \qquad v_3(S_C) \geq W_{m,k} + 1.$$

Thus, the only remaining leading contribution can come from S_A. The key estimate is

$$v_3(S_A) = W_{m,k}. \tag{4.5}$$

Therefore, to complete the inductive step, it remains to prove (4.4) and (4.5).

Odd indices.

Now we prove the first estimate in (4.4). Let i be an odd integer with $1 \leq i \leq 3^m - k$. If $i > 3^n$ or $3^m - k - i > 2 \cdot 3^n$, then $g_{2,k}(i) = 0$, and there is nothing to prove. Hence, we may assume that $3^m - k - 2 \cdot 3^n \leq i \leq 3^n$.

We first consider the boundary case $i = 3^m - k$. In this case, $3^m - k - i = 0$, and therefore

$$v_3(s_{3^n}(2 \cdot 3^n, 3^m - k - i)) = v_3(s_{3^n}(2 \cdot 3^n, 0)) = v_3\Big(\frac{(3^{n+1} - 1)!}{(3^n - 1)!}\Big) = 3^n - 1. \tag{4.6}$$

If $i = 3^m - k = 3^n$, then by $2 \leq m \leq n + 1$ and $2 \leq k \leq 2 \cdot 3^{m-1}$, we get that $m = n + 1$ and $k = 2 \cdot 3^n$. Since $s(3^n, 3^n) = 1$ and $W_{m,k} = W_{n+1, 2 \cdot 3^n} = 0$, one then deduces that

$$v_3(g_{2,k}(i)) = v_3(g_{2,2\cdot 3^n}(3^n)) = 3^n - 1 \geq 2 = W_{m,k} + 2$$

as required by (4.4). If $i = 3^m - k \leq 3^n - 2$, then $m \leq n$. By the inductive hypothesis and (4.6), we obtain

$$v_3(g_{2,k}(i)) - W_{m,k} = v_3(s^{(1)}_{n,m,k}) + v_3(s_{3^n}(2 \cdot 3^n, 0)) - W_{m,k} = 3^m - k - 1 \geq 3^{m-1} - 1 \geq 2.$$

So, the first estimate in (4.4) holds when $i = 3^m - k$.

It remains to consider the case $1 \le i \le 3^m - k - 2$. Since i is odd and k is even, the integer $3^m - k - i$ is even, and $2 \le 3^m - k - i < 2 \cdot 3^m - 2$. Write

$$i = 3^{l_1} - j_1, \quad 3^m - k - i = 2 \cdot 3^{l_2} - j_2,$$

where $(l_1, j_1) \in T_{1,m}$, $(l_2, j_2) \in T_{2,m}$ with $2 \mid j_1$ and $2 \mid j_2$. Since $2 \le k \le 2 \cdot 3^{m-1}$, we have $3^{m-1} \le 3^m - k \le 3^m - 2$. In particular, the case $l_1 = l_2 = m$ cannot occur. Indeed, if $l_1 = l_2 = m$, then $3^m - 2 \ge 3^m - k = i + 3^m - k - i = 3^m - j_1 + 2 \cdot 3^m - j_2 \ge 3^m - 2 \cdot 3^{m-1} + 2 \cdot 3^m - 4 \cdot 3^{m-1} = 3^m$, which is a contradiction. Since $2 \mid (3^m - k - i)$ and $2 \le 3^m - k - i \le 2 \cdot 3^n$, Lemma 2.5 also gives us that

$$v_3(g_{2,k}(i)) = v_3(s(3^n, i)) + v_3(s_{3^n}(2 \cdot 3^n, 3^m - k - i)) = v_3(s(3^n, i)) + v_3(s(2 \cdot 3^n, 3^m - k - i)). \tag{4.7}$$

We first consider the boundary case $i = 3^{l_1} - j_1 = 3^n$. Then, $l_1 = m = n + 1$ and $j_1 = 2 \cdot 3^{l_1 - 1} = 2 \cdot 3^n$ since $l_1 \le m \le n + 1$ and $2 \le j_1 \le 2 \cdot 3^{l_1 - 1}$. Hence, $1 \le l_2 \le m - 1 = n$. Moreover, $2 \cdot 3^{l_2} - j_2 = 3^m - k - i = 2 \cdot 3^n - k$. Since $2 \le j_2 \le 4 \cdot 3^{l_2 - 1}$, we have $2 \le k \le 2 \cdot 3^n - 2$, and consequently $v_3(j_2) = v_3(k)$. By the inductive hypothesis and (4.7), we derive that

$$\begin{aligned} v_3(g_{2,k}(i)) - W_{m,k} &= v_3(g_{2,k}(3^n)) - W_{n+1,k} = v_3(s^{(1)}_{n,n,0}) + v_3(s^{(2)}_{n,l_2,j_2}) - W_{n+1,k} \\ &= 3^n - 3^{l_2} - (n - l_2)(2 \cdot 3^{l_2} - j_2 + 1) - 1 =: D_{l_2}. \end{aligned}$$

If $l_2 = n$, then $2 \le j_2 = k \le 4 \cdot 3^{n-1}$ and $D_n = -1$. If $n \ge 2$ and $1 \le l_2 \le n - 1$, then

$$D_{l_2} \ge 3^n - 3^{l_2} - (n - l_2)(2 \cdot 3^{l_2} - 1) - 1 \ge 3^n - 3^{n-1} - (2 \cdot 3^{n-1} - 1) - 1 = 0,$$

where the equality holds if and only if $j_2 = 2$ and $l_2 = n - 1$. Hence, the first estimate in (4.4) holds when $i = 3^n$. Moreover, in this boundary case, the possible equality cases are as follows:

$$\begin{aligned} &v_3(g_{2,k}(i)) = W_{m,k} - 1 \ \text{ if } \ 2 \le k \le 4 \cdot 3^{n-1} = 4 \cdot 3^{m-2} \ \text{ and } \ i = 3^n = 3^{m-1}; \\ &v_3(g_{2,k}(i)) = W_{m,k} \ \text{ if } \ k = 4 \cdot 3^{n-1} + 2 = 4 \cdot 3^{m-2} + 2 \ \text{ and } \ i = 3^n = 3^{m-1}, \ m \ge 3. \end{aligned}$$

In all other cases with $i = 3^n = 3^{m-1}$, one has $v_3(g_{2,k}(i)) \ge W_{m,k} + 1$.

Next, we consider the other boundary case $3^m - k - i = 2 \cdot 3^{l_2} - j_2 = 2 \cdot 3^n$. Then, $l_2 = m = n + 1$ and $j_2 = 4 \cdot 3^{l_2 - 1} = 4 \cdot 3^n$. Hence, $1 \le l_1 \le m - 1 = n$. Moreover, $3^{l_1} - j_1 = i = 3^n - k$. Thus, $2 \le k \le 3^n - 1$, and consequently $v_3(j_1) = v_3(k)$. By the inductive hypothesis and (4.7), we have

$$\begin{aligned} v_3(g_{2,k}(i)) - W_{m,k} &= v_3(g_{2,k}(3^n - k)) - W_{n+1,k} = v_3(s^{(1)}_{n,l_1,j_1}) + v_3(s^{(2)}_{n,n,0}) - W_{n+1,k} \\ &= \frac{1}{2}(3^n - 3^{l_1}) - (n - l_1)(3^{l_1} - j_1 + 1) - 1 =: D_{l_1}. \end{aligned}$$

If $l_1 = n$, then $2 \le j_1 = k \le 2 \cdot 3^{n-1}$ and $D_n = -1$. If $n \ge 2$ and $1 \le l_1 \le n - 1$, then

$$D_{l_1} \ge \frac{1}{2}(3^n - 3^{l_1}) - (n - l_1)(3^{l_1} - 1) - 1 \ge \frac{1}{2}(3^n - 3^{n-1}) - (3^{n-1} - 1) - 1 = 0,$$

where the equality holds if and only if $j_1 = 2$ and $l_1 = n - 1$. Hence, (4.4) holds when $3^m - k - i = 2 \cdot 3^n$. Moreover, in this boundary case, the possible equality cases are as follows:

$$v_3(g_{2,k}(i)) = W_{m,k} - 1 \ \text{ if } \ 2 \le k \le 2 \cdot 3^{n-1} = 2 \cdot 3^{m-2} \ \text{ and } \ i = 3^n - k = 3^{m-1} - k;$$

$$v_3(g_{2,k}(i)) = W_{m,k} \text{ if } k = 2\cdot 3^{n-1}+2 = 2\cdot 3^{m-2}+2 \text{ and } i = 3^n - k = 3^{m-1}-k,\ m \geq 3.$$

In all other cases with $i = 3^n - k = 3^{m-1} - k$, one has $v_3(g_{2,k}(i)) \geq W_{m,k} + 1$.

It remains to consider the case $i \leq 3^n - 2$ and $3^m - k - i \leq 2\cdot 3^n - 2$. Since $i = 3^{l_1} - j_1$ and $3^m - k - i = 2\cdot 3^{l_2} - j_2$, we have $l_1 \leq n$ and $l_2 \leq n$. By the induction assumption and (4.7), we obtain

$$\begin{aligned} v_3(g_{2,k}(i)) - W_{m,k} &= v_3(s^{(1)}_{n,l_1,j_1}) + v_3(s^{(2)}_{n,l_2,j_2}) - W_{m,k} \\ &= \frac{1}{2}(3^m - 3^{l_1}) - 3^{l_2} + (3^{l_1} - j_1)(l_1 + 1 - m) + (2\cdot 3^{l_2} - j_2)(l_2 + 1 - m) \\ &\quad + l_1 + l_2 - m - 1 + v_3(k) - v_3(j_1) - v_3(j_2) =: D_{l_1,l_2}. \end{aligned} \tag{4.8}$$

Therefore, it remains to prove that

$$D_{l_1,l_2} \geq -1. \tag{4.9}$$

Note that $3^m - k = i + 3^m - k - i = 3^{l_1} - j_1 + 2\cdot 3^{l_2} - j_2$ with $2 \leq k \leq 2\cdot 3^{m-1}$, $2 \leq j_1 \leq 2\cdot 3^{l_1-1}$, and $2 \leq j_2 \leq 4\cdot 3^{l_2-1}$ implies that

$$v_3(k) - v_3(j_1) - v_3(j_2) \geq -\max\{v_3(j_1), v_3(j_2)\}.$$

We prove (4.9) by considering the possible values of l_1. Since the case $l_1 = l_2 = m$ has already been excluded and $1 \leq l_1 \leq m$, it remains to consider the following three cases:

$$l_1 = m, \quad l_1 = m-1, \quad 1 \leq l_1 \leq m-2.$$

Case 1: $l_1 = m$.

In this case, we have $2 \leq j_1 \leq 2\cdot 3^{m-1}$ and $1 \leq l_2 \leq m-1$. Hence, $v_3(j_1) \leq m-1$ and $v_3(j_2) \leq l_2 - 1 \leq m-2$. Moreover,

$$D_{m,l_2} = 3^m - j_1 - 3^{l_2} + (2\cdot 3^{l_2} - j_2)(l_2 + 1 - m) + l_2 - 1 + v_3(k) - v_3(j_1) - v_3(j_2).$$

We distinguish two subcases according to whether j_1 attains its maximal value $2\cdot 3^{m-1}$.

Subcase 1.1: $j_1 = 2\cdot 3^{m-1}$.

In this case, $i = 3^{m-1}$. We further distinguish the value of l_2.

If $l_2 = m-1$, then $k = j_2 \leq 4\cdot 3^{m-2}$. A direct calculation gives

$$D_{m,m-1} = m - 1 - 1 - (m-1) = -1.$$

If $1 \leq l_2 \leq m-2$, then $m \geq 3$, $i = 3^{m-1}$ and $3^m - k - i = 2\cdot 3^{m-1} - k = 2\cdot 3^{l_2} - j_2$. Thus, $4\cdot 3^{m-2} + 2 \leq k \leq 2\cdot 3^{m-1} - 2$, and consequently $v_3(k) = v_3(j_2)$. It follows that

$$D_{m,l_2} = 3^{m-1} - 3^{l_2}(2(m - l_2 - 1) + 1) + (j_2 - 1)(m - l_2 - 1) - 1 \geq j_2 - 2 \geq 0.$$

Moreover, the equality holds if and only if $l_2 = m-2$ and $j_2 = 2$.

Subcase 1.2: $2 \leq j_1 \leq 2\cdot 3^{m-1} - 2$.

In this case, $v_3(j_1) \le m-2$, and hence

$$\begin{aligned} D_{m,l_2} &\ge 3^{m-1} + 2 - 3^{l_2} + (2\cdot 3^{l_2} - j_2)(l_2+1-m) + l_2 - 1 - (m-2) \\ &= 3^{m-1} - 3^{l_2}(2(m-l_2-1)+1) + (j_2-1)(m-l_2-1) + 2 \ge 2. \end{aligned}$$

Combining the two subcases, we have proved (4.9) in the case $l_1 = m$. Moreover, the equality cases in Case 1 are as follows.

$$\begin{aligned} &v_3(g_{2,k}(i)) = W_{m,k} - 1 \;\text{ if }\; 2 \le k \le 4\cdot 3^{m-2} \;\text{ and }\; i = 3^{m-1}; \\ &v_3(g_{2,k}(i)) = W_{m,k} \;\text{ if }\; k = 4\cdot 3^{m-2}+2 \;\text{ and }\; i = 3^{m-1},\; m \ge 3. \end{aligned}$$

In all remaining cases covered by Case 1, we have $v_3(g_{2,k}(i)) \ge W_{m,k} + 1$.

Case 2: $l_1 = m-1$.

In this case, we have $2 \le j_1 \le 2\cdot 3^{m-2}$, and hence $v_3(j_1) \le m-2$. Moreover,

$$D_{m-1,l_2} = 3^{m-1} - 3^{l_2} + (2\cdot 3^{l_2} - j_2)(l_2+1-m) + l_2 - 2 + v_3(k) - v_3(j_1) - v_3(j_2).$$

We distinguish three subcases according to the value of l_2, where $1 \le l_2 \le m$.

Subcase 2.1: $l_2 = m$.

In this case, $2 \le j_2 \le 4\cdot 3^{m-1}$ and

$$D_{m-1,m} = 4\cdot 3^{m-1} - j_2 + m - 2 + v_3(k) - v_3(j_1) - v_3(j_2).$$

If $j_2 = 4\cdot 3^{m-1}$, then $i = 3^{m-1} - k$, and so $k = j_1 \le 2\cdot 3^{m-2}$. Hence,

$$D_{m-1,m} = m - 2 - (m-1) = -1.$$

If $2 \le j_2 \le 4\cdot 3^{m-1} - 2$, then $v_3(j_2) \le m-1$ and

$$D_{m-1,m} \ge 4\cdot 3^{m-1} - (4\cdot 3^{m-1} - 2) + m - 2 - (m-1) = 1.$$

Hence, (4.9) holds in this subcase. The only exceptional case occurs when $j_2 = 4\cdot 3^{m-1}$, therefore

$$v_3(g_{2,k}(i)) = W_{m,k} - 1 \quad\text{if}\quad 2 \le k \le 2\cdot 3^{m-2} \quad\text{and}\quad i = 3^{m-1} - k.$$

In all remaining cases of Subcase 2.1, $D_{m-1,m} \ge 1$, hence $v_3(g_{2,k}(i)) \ge W_{m,k} + 1$.

Subcase 2.2: $l_2 = m-1$.

In this case, $2 \le j_2 \le 4\cdot 3^{m-2}$. Note that in this case, $j_1 + j_2 = k$ and

$$D_{m-1,m-1} = m - 3 + v_3(k) - v_3(j_1) - v_3(j_2).$$

We first identify the cases in which $D_{m-1,m-1} = m-3-(m-2) = -1$. This happens only in the following situations:

$$\begin{aligned} &j_1 = 2\cdot 3^{m-2}, \quad v_3(k) = v_3(j_2) = v_3(k - 2\cdot 3^{m-2}); \\ &j_2 = 2\cdot 3^{m-2}, \quad v_3(k) = v_3(j_1) = v_3(k - 2\cdot 3^{m-2}); \end{aligned}$$

$$j_2 = 4 \cdot 3^{m-2}, \quad v_3(k) = v_3(j_1) = v_3(k - 4 \cdot 3^{m-2}).$$

In each of these cases,

$$D_{m-1,m-1} = m - 3 - (m - 2) = -1.$$

In all remaining cases, we have $D_{m-1,m-1} \geq 0$. The equality $D_{m-1,m-1} = 0$ occurs only in the following exceptional cases:

$$j_1 = 2 \cdot 3^{m-2}, \quad j_2 = 4 \cdot 3^{m-2};$$

or, when $m \geq 3$, in one of the following cases:

$$\begin{aligned}
&j_1 = 2 \cdot 3^{m-3} \text{ or } j_2 = 2 \cdot 3^{m-3}, \quad v_3(k) = v_3(k - 2 \cdot 3^{m-3});\\
&j_1 = 4 \cdot 3^{m-3} \text{ or } j_2 = 4 \cdot 3^{m-3}, \quad v_3(k) = v_3(k - 4 \cdot 3^{m-3});\\
&j_2 = 8 \cdot 3^{m-3}, \quad v_3(k) = v_3(k - 8 \cdot 3^{m-3});\\
&j_2 = 10 \cdot 3^{m-3}, \quad v_3(k) = v_3(k - 10 \cdot 3^{m-3}).
\end{aligned}$$

Thus, (4.9) is proved in this subcase.

The equality cases above yield the following possibilities for $v_3(g_{2,k}(i))$.

First, $v_3(g_{2,k}(i)) = W_{m,k} - 1$ in the following cases:

$$\begin{aligned}
&2 \cdot 3^{m-2} + 2 \leq k \leq 2 \cdot 3^{m-1} - 2, && i = 3^{m-1} - 2 \cdot 3^{m-2};\\
&2 \cdot 3^{m-2} + 2 \leq k \leq 4 \cdot 3^{m-2}, && i = 3^{m-1} - (k - 2 \cdot 3^{m-2});\\
&4 \cdot 3^{m-2} + 2 \leq k \leq 2 \cdot 3^{m-1} - 2, && i = 3^{m-1} - (k - 4 \cdot 3^{m-2}).
\end{aligned}$$

Second, $v_3(g_{2,k}(i)) = W_{m,k}$ if

$$k = 2 \cdot 3^{m-1}, \quad i = 3^{m-2};$$

or, when $m \geq 3$, in one of the following cases:

$$\begin{aligned}
&2 \cdot 3^{m-3} + 2 \leq k \leq 14 \cdot 3^{m-3} - 2,\ k \notin \{2 \cdot 3^{m-2}, 8 \cdot 3^{m-3}, 4 \cdot 3^{m-2}\}, && i = 3^{m-1} - 2 \cdot 3^{m-3};\\
&2 \cdot 3^{m-3} + 2 \leq k \leq 8 \cdot 3^{m-3} - 2,\ k \neq 2 \cdot 3^{m-2}, && i = 3^{m-1} - (k - 2 \cdot 3^{m-3});\\
&4 \cdot 3^{m-3} + 2 \leq k \leq 16 \cdot 3^{m-3} - 2,\ k \notin \{2 \cdot 3^{m-2}, 10 \cdot 3^{m-3}, 4 \cdot 3^{m-2}\}, && i = 3^{m-1} - 4 \cdot 3^{m-3};\\
&4 \cdot 3^{m-3} + 2 \leq k \leq 10 \cdot 3^{m-3} - 2,\ k \neq 2 \cdot 3^{m-2}, && i = 3^{m-1} - (k - 4 \cdot 3^{m-3});\\
&8 \cdot 3^{m-3} + 2 \leq k \leq 14 \cdot 3^{m-3} - 2,\ k \neq 4 \cdot 3^{m-2}, && i = 3^{m-1} - (k - 8 \cdot 3^{m-3});\\
&10 \cdot 3^{m-3} + 2 \leq k \leq 16 \cdot 3^{m-3} - 2,\ k \neq 4 \cdot 3^{m-2}, && i = 3^{m-1} - (k - 10 \cdot 3^{m-3}).
\end{aligned}$$

For other i and k in this case, we have $v_3(g_{2,k}(i)) \geq W_{m,k} + 1$.

Subcase 2.3: $m \geq 3$ and $1 \leq l_2 \leq m - 2$.

In this case, $2 \leq j_2 \leq 4 \cdot 3^{m-3}$, and hence $v_3(j_2) \leq m - 3$.

If $j_1 = 2 \cdot 3^{m-2}$, then $i = 3^{m-2}$. Moreover, $3^m - k - i = 2 \cdot 3^{m-1} - k = 2 \cdot 3^{l_2} - j_2$, which implies $v_3(k) = v_3(j_2)$. Therefore,

$$D_{m-1,l_2} = 3^{m-1} - 3^{l_2}(2(m - l_2 - 1) + 1) + (j_2 - 1)(m - l_2 - 1) - 1 \geq 0.$$

The equality in the last inequality can occur only when $l_2 = m - 2$ and $j_2 = 2$. In that case, $k = 3^m - 3^{m-2} - (2 \cdot 3^{m-2} - 2) = 2 \cdot 3^{m-1} + 2$, which contradicts the assumption $2 \leq k \leq 2 \cdot 3^{m-1}$. Hence, $D_{m-1,l_2} \geq 1$ when $j_1 = 2 \cdot 3^{m-2}$.

If $2 \leq j_1 \leq 2 \cdot 3^{m-2} - 2$, then $v_3(j_1) \leq m-3$. Thus,

$$\begin{aligned} D_{m-1,l_2} &\geq 3^{m-1} - 3^{l_2}(2(m-l_2-1)+1) + j_2(m-l_2-1) + l_2 - 2 - (m-3) \\ &= 3^{m-1} - 3^{l_2}(2(m-l_2-1)+1) + (j_2-1)(m-l_2-1) \geq 1. \end{aligned}$$

Therefore, $D_{m-1,l_2} \geq 1$ throughout Subcase 2.3. Consequently, $v_3(g_{2,k}(i)) \geq W_{m,k} + 1$ in this subcase.

Combining Subcases 2.1 to 2.3, we have $D_{m-1,l_2} \geq -1$ in Case 2. The equality cases in Case 2 were identified in Subcases 2.1 and 2.2. In Subcase 2.3, all terms have valuation at least $W_{m,k} + 1$.

Case 3: $m \geq 3$ and $1 \leq l_1 \leq m-2$.

We first note that $l_2 \geq m-1$. Indeed, if $l_2 \leq m-2$, then $3^{m-1} \leq 3^m - k \leq 3^{m-2} - 2 + 2 \cdot 3^{m-2} - 2 = 3^{m-1} - 4$, which is a contradiction. Hence, $l_2 = m-1$ or $l_2 = m$. Consider the following two subcases.

Subcase 3.1: $l_2 = m-1$.

In this case, $2 \leq j_2 \leq 4 \cdot 3^{m-2}$. We first show that $2 \leq j_2 \leq 4 \cdot 3^{m-2} - 2$. Indeed, if $j_2 = 4 \cdot 3^{m-2}$, then $3^{m-1} \leq 3^m - k = 3^{l_1} - j_1 + 2 \cdot 3^{m-1} - 4 \cdot 3^{m-2} \leq 3^{m-1} - 2$, which is a contradiction. Hence, $2 \leq j_2 \leq 4 \cdot 3^{m-2} - 2$, and so $v_3(j_2) \leq m-2$. Therefore,

$$\begin{aligned} D_{l_1,m-1} &= \frac{1}{2}(3^{m-1} - 3^{l_1}) + (3^{l_1} - j_1)(l_1 + 1 - m) + l_1 - 2 + v_3(k) - v_3(j_1) - v_3(j_2) \\ &\geq \frac{3^{m-1}}{2} - \frac{3^{l_1}}{2}(2(m-l_1-1)+1) + j_1(m-l_1-1) + l_1 - 2 - (m-2) \\ &= \frac{3^{m-1}}{2} - \frac{3^{l_1}}{2}(2(m-l_1-1)+1) + (j_1-1)(m-l_1-1) - 1 \geq 0. \end{aligned}$$

The equality in the last inequality occurs if and only if

$$l_1 = m-2, \quad j_1 = 2, \quad j_2 = 2 \cdot 3^{m-2}.$$

Thus, $D_{l_1,m-1} \geq 0$ holds in this subcase.

The equality case in Subcase 3.1 gives $v_3(g_{2,k}(i)) = W_{m,k}$ only when

$$k = 4 \cdot 3^{m-2} + 2, \quad i = 3^{m-2} - 2 = 3^{m-1} - (k - 2 \cdot 3^{m-2}), \quad m \geq 3.$$

In all other cases of Subcase 3.1, we have $v_3(g_{2,k}(i)) \geq W_{m,k} + 1$.

Subcase 3.2: $l_2 = m$.

In this case, $2 \leq j_2 \leq 4 \cdot 3^{m-1}$ and

$$\begin{aligned} D_{l_1,m} &= \frac{1}{2}(3^{m+1} - 3^{l_1}) + (3^{l_1} - j_1)(l_1 + 1 - m) - j_2 + l_1 - 1 + v_3(k) - v_3(j_1) - v_3(j_2) \\ &\geq \frac{1}{2}(3^{m+1} - 3^{l_1}) + (3^{l_1} - j_1)(l_1 + 1 - m) - 4 \cdot 3^{m-1} + l_1 - 1 - (m-1) \\ &= \frac{1}{2}(3^{m-1} - 3^{l_1}) + (3^{l_1} - j_1)(l_1 + 1 - m) + l_1 - m \\ &= \frac{3^{m-1}}{2} - \frac{3^{l_1}}{2}(2(m-l_1-1)+1) + (j_1-1)(m-l_1-1) - 1 \geq 0. \end{aligned}$$

The equality in the last inequality occurs if and only if

$$l_1 = m-2, \quad j_1 = 2, \quad j_2 = 4 \cdot 3^{m-1}.$$

Thus, $D_{l_1,m} \geq 0$ holds in this subcase.

The equality case in Subcase 3.2 gives $v_3(g_{2,k}(i)) = W_{m,k}$ only when

$$k = 2 \cdot 3^{m-2} + 2, \quad i = 3^{m-2} - 2 = 3^{m-1} - k, \quad m \geq 3.$$

In all other cases of Subcase 3.2, we have $v_3(g_{2,k}(i)) \geq W_{m,k} + 1$.

From Subcases 3.1 and 3.2, we obtain $D_{l_1,l_2} \geq 0$ throughout Case 3. Hence, all terms covered by Case 3 have 3-adic valuation at least $W_{m,k}$. The equality cases have been identified in Subcases 3.1 and 3.2; in all other cases, the 3-adic valuation of $g_{2,k}(i)$ is at least $W_{m,k} + 1$.

Combining Cases 1 to 3, we have proved that $D_{l_1,l_2} \geq -1$ in all remaining cases. Together with (4.8), this gives $v_3(g_{2,k}(i)) \geq W_{m,k} - 1$ for every odd integer i with $1 \leq i \leq 3^m - k$. Hence, the first estimate in (4.4) is proved.

Even indices.

We now prove the second estimate in (4.4). Let i be an even integer with $1 \leq i \leq 3^m - k$. Then, $2 \leq i \leq 3^m - k - 1$. If $i > 3^n$ or $3^m - k - i > 2 \cdot 3^n$, then $g_{2,k}(i) = 0$, and there is nothing to prove. Hence, we may assume that $2 \leq i \leq 3^n - 1$ and $1 \leq 3^m - k - i \leq 2 \cdot 3^n$.

Since $i - 1$ is odd and $1 \leq i - 1 \leq 3^m - k - 2$, the first estimate in (4.4) gives

$$v_3(g_{2,k}(i - 1)) \geq W_{m,k} - 1. \tag{4.10}$$

On the other hand, since i is even and $2 \leq i \leq 3^n - 1$, and since $3^m - k - i$ is odd with $1 \leq 3^m - k - i \leq 2 \cdot 3^n$, the estimates (3.9), Lemma 2.5, (3.11), and (4.7) imply

$$\begin{aligned} v_3(g_{2,k}(i)) &= v_3(s(3^n, i)) + v_3(s_{3^n}(2 \cdot 3^n, 3^m - k - i)) \\ &\geq v_3(s(3^n, i + 1)) + n + v_3(s(2 \cdot 3^n, 3^m - k - i + 1)) + n \\ &\geq v_3(s(3^n, i - 1)) + v_3(s(2 \cdot 3^n, 3^m - k - (i - 1))) + 2 = v_3(g_{2,k}(i - 1)) + 2. \end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we obtain $v_3(g_{2,k}(i)) \geq W_{m,k} + 1$. This proves the second estimate in (4.4).

4.3. The leading contribution

The goal of this subsection is to identify the leading terms and to prove that their total contribution has 3-adic valuation $W_{m,k}$. More precisely, we shall determine the set A introduced above and prove $v_3(S_A) = W_{m,k}$, which is precisely (4.5).

Recall that

$$A = A_1 \cup A_2, \qquad S_A = \sum_{i \in A} g_{2,k}(i),$$

where A_1 and A_2 were defined by the conditions

$$v_3(g_{2,k}(i)) = W_{m,k} - 1 \quad \text{and} \quad v_3(g_{2,k}(i)) = W_{m,k},$$

respectively. Thus, A consists exactly of the odd-index terms whose valuations may affect the leading valuation.

We shall prove (4.5) in several steps. First, since the sets A_1 and A_2 depend on the value of k, we decompose the possible values of k and give an explicit description of A_1 and A_2 in each range. Next, we replace $g_{2,k}(i)$ by a corresponding term involving ordinary Stirling numbers and reduce the proof of (4.5) to an equivalent estimate for a modified sum S'_A. Finally, we prove this estimate case by case, treating the exceptional cases first and then the remaining ranges of k.

Decomposition of the parameter range and description of A_1, A_2.

Since the sets A_1 and A_2 depend on the value of k, we now decompose the even parameter range. Define

$$K_1 = [2,\ 2\cdot 3^{m-2}] \cap 2\mathbb{Z}, \qquad K_3 = \{4\cdot 3^{m-2}\}, \qquad K_5 = \{2\cdot 3^{m-1}\},$$

and, for $m \geq 3$,

$$K_2 = [2 + 2\cdot 3^{m-2},\ 4\cdot 3^{m-2} - 2] \cap 2\mathbb{Z},$$
$$K_4 = [2 + 4\cdot 3^{m-2},\ 2\cdot 3^{m-1} - 2] \cap 2\mathbb{Z}.$$

These sets are pairwise disjoint and their union is the full admissible even range

$$[2,\ 2\cdot 3^{m-1}] \cap 2\mathbb{Z},$$

with empty intervals omitted when m is small.

The ordinary ranges K_1, K_2, and K_4 will be further divided into subranges. These refinements are used only to describe the possible leading indices in A_1 and A_2.

For K_1, set

$$K_{11} = [2 + 2\cdot 3^{m-3},\ 4\cdot 3^{m-3} - 2] \cap 2\mathbb{Z},$$
$$K_{12} = \{4\cdot 3^{m-3}\},$$
$$K_{13} = [2 + 4\cdot 3^{m-3},\ 2\cdot 3^{m-2} - 2] \cap 2\mathbb{Z},$$

where K_{12} is defined for $m \geq 3$, and K_{11}, K_{13} are defined for $m \geq 4$. We then set

$$K_{14} = K_1 \setminus \bigcup_{j=1}^{3} K_{1j}.$$

For K_2, set

$$K_{21} = \{2 + 2\cdot 3^{m-2}\},$$
$$K_{22} = [4 + 2\cdot 3^{m-2},\ 8\cdot 3^{m-3} - 2] \cap 2\mathbb{Z},$$
$$K_{23} = \{8\cdot 3^{m-3}\},$$
$$K_{24} = [2 + 8\cdot 3^{m-3},\ 10\cdot 3^{m-3} - 2] \cap 2\mathbb{Z},$$
$$K_{25} = \{10\cdot 3^{m-3}\},$$
$$K_{26} = [2 + 10\cdot 3^{m-3},\ 4\cdot 3^{m-2} - 2] \cap 2\mathbb{Z}.$$

Here, K_{21}, K_{22}, K_{24}, and K_{26} are defined for $m \geq 4$; for $m = 3$, only the singleton cases K_{23} and K_{25} occur.

For K_4, set

$$K_{41} = \{2 + 4\cdot 3^{m-2}\},$$
$$K_{42} = [4 + 4\cdot 3^{m-2},\ 14\cdot 3^{m-3} - 2] \cap 2\mathbb{Z},$$
$$K_{43} = \{14\cdot 3^{m-3}\},$$
$$K_{44} = [2 + 14\cdot 3^{m-3},\ 16\cdot 3^{m-3} - 2] \cap 2\mathbb{Z},$$

where K_{41}, K_{42}, and K_{44} are defined for $m \geq 4$. Finally, set

$$K_{45} = K_4 \setminus \bigcup_{j=1}^{4} K_{4j}.$$

Thus, the subranges K_{1j}, K_{2j}, and K_{4j} form disjoint decompositions of K_1, K_2, and K_4, respectively, after empty sets are omitted.

We now give the corresponding descriptions of A_1 and A_2. We first record the two exceptional cases. If $k \in K_5$, then

$$A_1 = \varnothing, \qquad A_2 = \{3^{m-2}\}.$$

If $k \in K_3$, then

$$A_1 = \{3^{m-2}, 3^{m-1}\}, \qquad A_2 = \varnothing.$$

It remains to describe A_1 and A_2 on the three ordinary ranges. For an integer a and a finite set T of integers, write

$$a - T := \{a - t : \ t \in T\}.$$

For $k \in K_1 \cup K_2 \cup K_4$, the sets A_1 and A_2 have the form

$$A_1 = 3^{m-1} - T_1(k), \qquad A_2 = 3^{m-1} - T_2(k),$$

where $T_1(k)$ and $T_2(k)$ are given as follows.

If $k \in K_1$, then

$$T_1(k) = \{0, k\},$$

and

$$T_2(k) = \begin{cases} \{2 \cdot 3^{m-3},\ k - 2 \cdot 3^{m-3}\}, & k \in K_{11}, \\ \{2 \cdot 3^{m-3}\}, & k \in K_{12}, \\ \{2 \cdot 3^{m-3},\ k - 2 \cdot 3^{m-3}, 4 \cdot 3^{m-3},\ k - 4 \cdot 3^{m-3}\}, & k \in K_{13}, \\ \varnothing, & k \in K_{14}. \end{cases}$$

If $k \in K_2$, then

$$T_1(k) = \{0,\ 2 \cdot 3^{m-2},\ k - 2 \cdot 3^{m-2}\},$$

and

$$T_2(k) = \begin{cases} \{k,\ 2 \cdot 3^{m-3},\ k - 2 \cdot 3^{m-3}, 4 \cdot 3^{m-3},\ k - 4 \cdot 3^{m-3}\}, & k \in K_{21}, \\ \{2 \cdot 3^{m-3},\ k - 2 \cdot 3^{m-3}, 4 \cdot 3^{m-3},\ k - 4 \cdot 3^{m-3}\}, & k \in K_{22}, \\ \{k,\ 4 \cdot 3^{m-3}\}, & m = 3,\ k \in K_{23}, \\ \{4 \cdot 3^{m-3}\}, & m \geq 4,\ k \in K_{23}, \\ \{2 \cdot 3^{m-3},\ 4 \cdot 3^{m-3}, k - 4 \cdot 3^{m-3},\ k - 8 \cdot 3^{m-3}\}, & k \in K_{24}, \\ \{2 \cdot 3^{m-3}\}, & k \in K_{25}, \\ \{2 \cdot 3^{m-3},\ 4 \cdot 3^{m-3}, k - 8 \cdot 3^{m-3},\ k - 10 \cdot 3^{m-3}\}, & k \in K_{26}. \end{cases}$$

If $k \in K_4$, then

$$T_1(k) = \{2 \cdot 3^{m-2},\ k - 4 \cdot 3^{m-2}\},$$

and

$$T_2(k) = \begin{cases} \{0,\ k-2\cdot 3^{m-2},\ 2\cdot 3^{m-3},\ 4\cdot 3^{m-3}, k-8\cdot 3^{m-3},\ k-10\cdot 3^{m-3}\}, & k \in K_{41}, \\ \{2\cdot 3^{m-3},\ 4\cdot 3^{m-3}, k-8\cdot 3^{m-3},\ k-10\cdot 3^{m-3}\}, & k \in K_{42}, \\ \{0,\ k-2\cdot 3^{m-2},\ 4\cdot 3^{m-3}\}, & m=3,\ k \in K_{43}, \\ \{4\cdot 3^{m-3}\}, & m \geq 4,\ k \in K_{43}, \\ \{4\cdot 3^{m-3},\ k-10\cdot 3^{m-3}\}, & k \in K_{44}, \\ \varnothing, & k \in K_{45}. \end{cases}$$

The description above identifies all odd indices that may contribute to the leading valuation. We first deal with the immediate case K_5.

The immediate case K_5.

Suppose that $k \in K_5$. Then, $k = 2\cdot 3^{m-1}$, $A_1 = \varnothing$, and $A_2 = \{3^{m-2}\}$. Hence,

$$v_3\Big(\sum_{i\in A} g_{2,k}(i)\Big) = v_3\Big(\sum_{i\in A_2} g_{2,k}(i)\Big) = v_3(g_{2,2\cdot 3^{m-1}}(3^{m-2})) = W_{m,k}.$$

Thus, (4.5) holds when $k \in K_5$.

Reduction to ordinary Stirling numbers.

It remains to consider $k \in \bigcup_{j=1}^{4} K_j$. For $i \in A$, we have $2 \leq 3^m - k - i \leq 2\cdot 3^n$. By Lemma 2.5, applied with $a = 2$, we have

$$g_{2,k}(i) = g'_{2,k}(i) + O(3^{v_3(g'_{2,k}(i))+2}),$$

where $O(3^r)$ denotes an integer with 3-adic valuation at least r, and

$$g'_{2,k}(i) := s(3^n, i)s(2\cdot 3^n, 3^m - k - i).$$

Since $i \in A = A_1 \cup A_2$, this gives

$$g_{2,k}(i) = g'_{2,k}(i) + O(3^{W_{m,k}+1}).$$

Consequently,

$$v_3(g'_{2,k}(i)) = v_3(g_{2,k}(i)) = \begin{cases} W_{m,k} - 1, & i \in A_1, \\ W_{m,k}, & i \in A_2. \end{cases}$$

Define

$$S'_A := \sum_{i\in A} g'_{2,k}(i).$$

Then, $S_A = S'_A + O(3^{W_{m,k}+1})$. Therefore, to prove (4.5), it suffices to prove

$$v_3(S'_A) = v_3\Big(\sum_{i\in A} s(3^n, i)s(2\cdot 3^n, 3^m - k - i)\Big) = W_{m,k}. \tag{4.12}$$

The exceptional case K_3.

We first prove (4.12) for $k \in K_3$. In this case, $k = 4 \cdot 3^{m-2}$, $A_1 = \{3^{m-2}, 3^{m-1}\}$, and $A_2 = \varnothing$. Thus,

$$S'_A = \sum_{i \in A_1} g'_{2,k}(i) = g'_{2,k}(3^{m-2}) + g'_{2,k}(3^{m-1}) = s^{(1)}_{n,m-2,0} \cdot s^{(2)}_{n,m-1,2\cdot 3^{m-2}} + s^{(1)}_{n,m-1,0} \cdot s^{(2)}_{n,m-1,4\cdot 3^{m-2}}. \tag{4.13}$$

In Remark 3.2, by setting $(n, m, k) \mapsto (n, m-1, 2 \cdot 3^{m-2})$ and $(n, m, k) \mapsto (n, m-1, 4 \cdot 3^{m-2})$, respectively, we obtain that

$$s^{(2)}_{n,m-1,2\cdot 3^{m-2}} = 2s^{(1)}_{n,m-1,0} \cdot s^{(1)}_{n,m-1,2\cdot 3^{m-2}} + L_1 = 2s^{(1)}_{n,m-1,0} \cdot s^{(1)}_{n,m-2,0} + L_1 \text{ and } s^{(2)}_{n,m-1,4\cdot 3^{m-2}} = (s^{(1)}_{n,m-2,0})^2 + L_2, \tag{4.14}$$

where $v_3(L_1) \geq v_3(s^{(1)}_{n,m-1,0} \cdot s^{(1)}_{n,m-2,0}) + 2$ and $v_3(L_2) \geq v_3((s^{(1)}_{n,m-2,0})^2) + 2$. By (4.13) and (4.14), we have

$$S'_A = 3s^{(1)}_{n,m-1,0}(s^{(1)}_{n,m-2,0})^2 + s^{(1)}_{n,m-2,0}L_1 + s^{(1)}_{n,m-1,0}L_2. \tag{4.15}$$

By the definition of A_1 together with (4.13) and (4.14), we deduce that

$$v_3(g'_{2,k}(3^{m-2})) = v_3(g'_{2,k}(3^{m-1})) = W_{m,k} - 1 = v_3(s^{(1)}_{n,m-1,0}(s^{(1)}_{n,m-2,0})^2) \tag{4.16}$$

and $v_3(s^{(1)}_{n,m-2,0}L_1 + s^{(1)}_{n,m-1,0}L_2) \geq v_3(s^{(1)}_{n,m-1,0}(s^{(1)}_{n,m-2,0})^2) + 2 = W_{m,k} + 1$. Thus, (4.15) and (4.16) give

$$v_3(S'_A) = v_3(3s^{(1)}_{n,m-1,0}(s^{(1)}_{n,m-2,0})^2) = W_{m,k}.$$

Hence, (4.12) is proved when $k \in K_3$.

The remaining cases.

It remains to prove (4.12) for $k \in K_1 \cup K_2 \cup K_4$. In these cases, the terms indexed by A_1 produce the principal contribution, while the terms indexed by A_2 must be combined with certain secondary terms arising from the expansion in Remark 3.2.

To keep the formulas readable, we introduce the following abbreviations. For $a_1, a_2, a_3 \in \mathbb{Z}$, let

$$\begin{aligned} S_{a_1,a_2} &:= s^{(1)}_{n,m-1,a_1} \cdot s^{(1)}_{n,m-1,a_2}, & S_{a_1,a_2,a_3} &:= s^{(1)}_{n,m-1,a_1} \cdot s^{(1)}_{n,m-1,a_2} \cdot s^{(1)}_{n,m-1,a_3}, \\ S'_{a_1,a_2} &:= s^{(1)}_{n,m-2,a_1} \cdot s^{(1)}_{n,m-1,a_2}, & S'_{a_1,a_2,a_3} &:= s^{(1)}_{n,m-2,a_1} \cdot s^{(1)}_{n,m-1,a_2} \cdot s^{(1)}_{n,m-1,a_3}, \\ S''_{a_1,a_2} &:= s^{(1)}_{n,m-2,a_1} \cdot s^{(1)}_{n,m-2,a_2}, & S''_{a_1,a_2,a_3} &:= s^{(1)}_{n,m-2,a_1} \cdot s^{(1)}_{n,m-2,a_2} \cdot s^{(1)}_{n,m-1,a_3}. \end{aligned}$$

We now treat the cases $k \in K_1, k \in K_2$, and $k \in K_4$, separately.

Case 1: $k \in K_1$.

In this case, $2 \leq k \leq 2 \cdot 3^{m-2}$ and $A_1 = 3^{m-1} - \{0, k\}$. Hence,

$$g'_{2,k}(3^{m-1}) = s^{(1)}_{n,m-1,0} \cdot s^{(2)}_{n,m-1,k}, \qquad g'_{2,k}(3^{m-1} - k) = s^{(1)}_{n,m-1,k} \cdot s^{(2)}_{n,m-1,0}. \tag{4.17}$$

Setting $(n, m, k) \mapsto (n, m-1, k)$ in Remark 3.2, we get

$$s^{(2)}_{n,m-1,k} = 2s^{(1)}_{n,m-1,0} \cdot s^{(1)}_{n,m-1,k} + L_{11} + L_{12} = 2S_{0,k} + L_{11} + L_{12}, \tag{4.18}$$

where $L_{11}, L_{12} \in \mathbb{Z}$, $v_3(L_{11}) \geq v_3(S_{0,k}) + 2$, and $v_3(L_{12}) \geq v_3(S_{0,k}) + 1$ with

$$L_{12} = \begin{cases} 2s^{(1)}_{n,m-1,2\cdot 3^{m-3}} \cdot s^{(1)}_{n,m-1,k-2\cdot 3^{m-3}}, & k \in K_{11}, \\ (s^{(1)}_{n,m-1,2\cdot 3^{m-3}})^2, & k \in K_{12}, \\ 2s^{(1)}_{n,m-1,2\cdot 3^{m-3}} \cdot s^{(1)}_{n,m-1,k-2\cdot 3^{m-3}} + 2s^{(1)}_{n,m-1,4\cdot 3^{m-3}} \cdot s^{(1)}_{n,m-1,k-4\cdot 3^{m-3}}, & k \in K_{13}, \\ 0, & k \in K_{14}. \end{cases}$$

Moreover, since $s^{(2)}_{n,m-1,0} = s^{(1)}_{n,m-1,0} = 1$ if $m = n+1$, and $s^{(2)}_{n,m-1,0} = s^{(2)}_{n,m,4\cdot 3^{m-1}}$ if $m \leq n$, Remark 3.2 gives

$$s^{(2)}_{n,m-1,0} = (s^{(1)}_{n,m-1,0})^2 + L_{13} = S_{0,0} + L_{13}, \tag{4.19}$$

where $v_3(L_{13}) \geq v_3(S_{0,0}) + 2$.

Substituting (4.18) and (4.19) into (4.17), we obtain

$$\sum_{i \in A_1} g'_{2,k}(i) = g'_{2,k}(3^{m-1}) + g'_{2,k}(3^{m-1} - k) = S_{11} + S_{12} + S_{13}, \tag{4.20}$$

where

$$S_{11} := 3(s^{(1)}_{n,m-1,0})^2 s^{(1)}_{n,m-1,k} = 3S_{0,0,k}, \quad S_{12} := s^{(1)}_{n,m-1,0} L_{12}, \quad S_{13} := s^{(1)}_{n,m-1,0} L_{11} + s^{(1)}_{n,m-1,k} L_{13}.$$

By the definition of A_1 together with the second identity in (4.17) and (4.19), we have

$$v_3(g'_{2,k}(3^{m-1} - k)) = W_{m,k} - 1 = v_3(S_{0,0,k}). \tag{4.21}$$

So, we have

$$v_3(S_{11}) = v_3(S_{0,0,k}) + 1 = W_{m,k} \ \text{ and } \ v_3(S_{13}) \geq v_3(S_{0,0,k}) + 2 = W_{m,k} + 1. \tag{4.22}$$

It remains to deal with S_{12}. We first observe that

$$S_{12} = 0 \quad \Longleftrightarrow \quad k \in K_{14} \quad \Longleftrightarrow \quad A_2 = \varnothing.$$

If $k \in K_{14}$, then $S_{12} = 0$ and $A_2 = \varnothing$. Hence, by (4.20) and (4.22),

$$v_3(S'_A) = v_3\Big(\sum_{i \in A} g'_{2,k}(i)\Big) = v_3\Big(\sum_{i \in A_1} g'_{2,k}(i)\Big) = v_3(S_{11}) = W_{m,k}.$$

Thus, (4.12) holds when $k \in K_{14}$.

It remains to consider $k \in K_{11} \cup K_{12} \cup K_{13}$. In this case, $S_{12} \neq 0$ and $A_2 \neq \varnothing$. We claim that

$$v_3\Big(S_{12} + \sum_{i \in A_2} g'_{2,k}(i)\Big) \geq W_{m,k} + 1. \tag{4.23}$$

Assuming (4.23), it follows from (4.20) and (4.22) that

$$v_3(S'_A) = v_3\Big(\sum_{i \in A} g'_{2,k}(i)\Big) = v_3\Big(S_{11} + S_{12} + S_{13} + \sum_{i \in A_2} g'_{2,k}(i)\Big) = v_3(S_{11}) = W_{m,k}.$$

Thus, (4.12) holds in Case 1 once (4.23) is proved.

We now prove (4.23). We do this according to the three possible forms of A_2, corresponding to $k \in K_{11}$, $k \in K_{12}$, and $k \in K_{13}$. In each subcase, we combine S_{12} with the terms indexed by A_2 and show that the resulting sum has 3-adic valuation at least $W_{m,k} + 1$.

Subcase 1.1: $k \in K_{11}$.

In this subcase, $m \geq 4$ and $A_2 = 3^{m-1} - \{2 \cdot 3^{m-3}, k - 2 \cdot 3^{m-3}\}$. Moreover,

$$S_{12} = s^{(1)}_{n,m-1,0} L_{12} = 2S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}}. \tag{4.24}$$

Since $2 \leq k - 2 \cdot 3^{m-3} \leq 2 \cdot 3^{m-3} - 2$, Remark 3.2 gives

$$\begin{aligned} g'_{2,k}(3^{m-1} - 2 \cdot 3^{m-3}) &= s^{(1)}_{n,m-1,2\cdot 3^{m-3}} \cdot s^{(2)}_{n,m-1,k-2\cdot 3^{m-3}} = 2S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}} + L_{111}, \\ g'_{2,k}(3^{m-1} - (k - 2 \cdot 3^{m-3})) &= s^{(1)}_{n,m-1,k-2\cdot 3^{m-3}} \cdot s^{(2)}_{n,m-1,2\cdot 3^{m-3}} = 2S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}} + L_{112}, \end{aligned} \tag{4.25}$$

where $\min\{v_3(L_{111}), v_3(L_{112})\} \geq v_3(S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}}) + 1$.

Since $A_2 = 3^{m-1} - \{2 \cdot 3^{m-3}, k - 2 \cdot 3^{m-3}\}$ and

$$v_3(g'_{2,k}(3^{m-1} - 2 \cdot 3^{m-3})) = W_{m,k} = v_3(S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}}), \tag{4.26}$$

from (4.24)–(4.26), we derive that

$$v_3\Big(S_{12} + \sum_{i\in A_2} g'_{2,k}(i)\Big) = v_3(6S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}} + L_{111} + L_{112}) \geq v_3(S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}}) + 1 = W_{m,k} + 1.$$

Thus, (4.23) is proved in this subcase.

Subcase 1.2: $k \in K_{12}$.

In this subcase, $m \geq 3$ and $k = 4 \cdot 3^{m-3}$. We have

$$S_{12} = s^{(1)}_{n,m-1,0} L_{12} = s^{(1)}_{n,m-1,0} (s^{(1)}_{n,m-1,2\cdot 3^{m-3}})^2 = S_{0,2\cdot 3^{m-3},2\cdot 3^{m-3}} \tag{4.27}$$

and $A_2 = \{3^{m-1} - 2 \cdot 3^{m-3}\}$. It follows from Remark 3.2 that

$$g'_{2,k}(3^{m-1} - 2 \cdot 3^{m-3}) = s^{(1)}_{n,m-1,2\cdot 3^{m-3}} \cdot s^{(2)}_{n,m-1,2\cdot 3^{m-3}} = 2S_{0,2\cdot 3^{m-3},2\cdot 3^{m-3}} + L_{121}, \tag{4.28}$$

where $v_3(L_{121}) \geq v_3(S_{0,2\cdot 3^{m-3},2\cdot 3^{m-3}}) + 1$. Therefore, by (4.27) and (4.28), we obtain

$$\begin{aligned} v_3\Big(S_{12} + \sum_{i\in A_2} g'_{2,k}(i)\Big) &= v_3(3S_{0,2\cdot 3^{m-3},2\cdot 3^{m-3}} + L_{121}) \geq v_3(S_{0,2\cdot 3^{m-3},2\cdot 3^{m-3}}) + 1 \\ &= v_3(g'_{2,k}(3^{m-1} - 2 \cdot 3^{m-3})) + 1 = W_{m,k} + 1. \end{aligned}$$

Thus, (4.23) is proved in this subcase.

Subcase 1.3: $k \in K_{13}$.

In this subcase, $m \geq 4$ and $2 + 4 \cdot 3^{m-3} \leq k \leq 2 \cdot 3^{m-2} - 2$. Moreover,

$$S_{12} = s^{(1)}_{n,m-1,0} L_{12} = 2S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}} + 2S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}} \tag{4.29}$$

and $A_2 = 3^{m-1} - \{2\cdot 3^{m-3}, k-2\cdot 3^{m-3}, 4\cdot 3^{m-3}, k-4\cdot 3^{m-3}\}$. The two terms corresponding to $2\cdot 3^{m-3}$ and $k-2\cdot 3^{m-3}$ are handled as in Subcase 1.1. Since $k-2\cdot 3^{m-3}\in K_1$, (4.25) and (4.26) remain valid.

For the remaining two terms, Remark 3.2 gives

$$g'_{2,k}(3^{m-1}-4\cdot 3^{m-3}) = s^{(1)}_{n,m-1,4\cdot 3^{m-3}}\cdot s^{(2)}_{n,m-1,k-4\cdot 3^{m-3}} = 2S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}} + L_{131}, \tag{4.30}$$

$$g'_{2,k}(3^{m-1}-(k-4\cdot 3^{m-3})) = s^{(1)}_{n,m-1,k-4\cdot 3^{m-3}}\cdot s^{(2)}_{n,m-1,4\cdot 3^{m-3}} = 2S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}} + L_{132}, \tag{4.31}$$

where $\min\{v_3(L_{131}), v_3(L_{132})\}\geq v_3(S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}})+1$. Note that

$$v_3(g'_{2,k}(3^{m-1}-4\cdot 3^{m-3})) = W_{m,k} = v_3(S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}}). \tag{4.32}$$

Combining (4.25), (4.26), and (4.29)–(4.32), we obtain

$$\begin{aligned} v_3\Big(S_{12}+\sum_{i\in A_2} g'_{2,k}(i)\Big) &= v_3(6S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}} + L_{111} + L_{112} + 6S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}} + L_{131} + L_{132}) \\ &\geq \min\{v_3(S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}}), v_3(S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}})\} + 1 = W_{m,k}+1. \end{aligned}$$

Thus, (4.23) is proved in this subcase.

Combining Subcases 1.1 to 1.3, we have proved (4.23). Hence, (4.12) holds in Case 1.

Case 2: $k\in K_2$.

In this case, $m\geq 3$, $A_1 = 3^{m-1} - \{0, 2\cdot 3^{m-2}, k-2\cdot 3^{m-2}\}$. We first record the three terms indexed by A_1. Recall from the first identity in (4.17) that

$$g'_{2,k}(3^{m-1}) = s^{(1)}_{n,m-1,0}\cdot s^{(2)}_{n,m-1,k}.$$

In addition, we have

$$g'_{2,k}(3^{m-1}-2\cdot 3^{m-2}) = g'_{2,k}(3^{m-2}) = s^{(1)}_{n,m-2,0}\cdot s^{(2)}_{n,m-1,k-2\cdot 3^{m-2}}, \tag{4.33}$$

$$g'_{2,k}(3^{m-1}-(k-2\cdot 3^{m-2})) = s^{(1)}_{n,m-1,k-2\cdot 3^{m-2}}\cdot s^{(2)}_{n,m-1,2\cdot 3^{m-2}}. \tag{4.34}$$

We now expand the three $s^{(2)}$-terms appearing above. First, setting $(n,m,k)\mapsto(n,m-1,2\cdot 3^{m-2})$ in Remark 3.2, we get

$$s^{(2)}_{n,m-1,2\cdot 3^{m-2}} = 2s^{(1)}_{n,m-1,0}\cdot s^{(1)}_{n,m-1,2\cdot 3^{m-2}} + L_{21} = 2s^{(1)}_{n,m-2,0}\cdot s^{(1)}_{n,m-1,0} + L_{21} = 2S'_{0,0} + L_{21}, \tag{4.35}$$

where $L_{21}\in\mathbb{Z}$ and $v_3(L_{21})\geq v_3(S'_{0,0})+2$.

Next, since $m\geq 3$ and $2+2\cdot 3^{m-2}\leq k\leq 4\cdot 3^{m-2}-2$, Remark 3.2 gives

$$s^{(2)}_{n,m-1,k} = 2s^{(1)}_{n,m-2,0}\cdot s^{(1)}_{n,m-1,k-2\cdot 3^{m-2}} + L_{22} + L_{23} = 2S'_{0,k-2\cdot 3^{m-2}} + L_{22} + L_{23}, \tag{4.36}$$

where $L_{22}, L_{23}\in\mathbb{Z}$, $v_3(L_{23})\geq v_3(S'_{0,k-2\cdot 3^{m-2}})+2$, and $v_3(L_{22})\geq v_3(S'_{0,k-2\cdot 3^{m-2}})+1$ with

$$L_{22} = \begin{cases} 2s^{(1)}_{n,m-2,2}s^{(1)}_{n,m-1,0} + 2s^{(1)}_{n,m-1,2\cdot 3^{m-3}}s^{(1)}_{n,m-1,k-2\cdot 3^{m-3}} + 2s^{(1)}_{n,m-1,4\cdot 3^{m-3}}s^{(1)}_{n,m-1,k-4\cdot 3^{m-3}}, & k\in K_{21},\\ 2s^{(1)}_{n,m-1,2\cdot 3^{m-3}}s^{(1)}_{n,m-1,k-2\cdot 3^{m-3}} + 2s^{(1)}_{n,m-1,4\cdot 3^{m-3}}s^{(1)}_{n,m-1,k-4\cdot 3^{m-3}}, & k\in K_{22},\\ 2s^{(1)}_{n,m-2,2}s^{(1)}_{n,m-1,0} + \left(s^{(1)}_{n,m-1,4\cdot 3^{m-3}}\right)^2, & k\in K_{23},\ m=3,\\ \left(s^{(1)}_{n,m-1,4\cdot 3^{m-3}}\right)^2, & k\in K_{23},\ m\geq 4,\\ 2s^{(1)}_{n,m-1,4\cdot 3^{m-3}}s^{(1)}_{n,m-1,k-4\cdot 3^{m-3}}, & k\in K_{24},\\ 0, & k\in K_{25}\cup K_{26}. \end{cases}$$

Finally, since $k - 2 \cdot 3^{m-2} \in K_1$, replacing k by $k - 2 \cdot 3^{m-2}$ in (4.18) gives

$$s^{(2)}_{n,m-1,k-2\cdot 3^{m-2}} = 2s^{(1)}_{n,m-1,0} \cdot s^{(1)}_{n,m-1,k-2\cdot 3^{m-2}} + L_{24} + L_{25} = 2S_{0,k-2\cdot 3^{m-2}} + L_{24} + L_{25}, \tag{4.37}$$

where $L_{24}, L_{25} \in \mathbb{Z}$, $v_3(L_{25}) \geq v_3(S_{0,k-2\cdot 3^{m-2}}) + 2$, and $v_3(L_{24}) \geq v_3(S_{0,k-2\cdot 3^{m-2}}) + 1$ with

$$L_{24} = \begin{cases} 2s^{(1)}_{n,m-1,2\cdot 3^{m-3}} s^{(1)}_{n,m-1,k-8\cdot 3^{m-3}}, & k \in K_{24}, \\ \left(s^{(1)}_{n,m-1,2\cdot 3^{m-3}}\right)^2, & k \in K_{25}, \\ 2s^{(1)}_{n,m-1,2\cdot 3^{m-3}} s^{(1)}_{n,m-1,k-8\cdot 3^{m-3}} + 2s^{(1)}_{n,m-1,4\cdot 3^{m-3}} s^{(1)}_{n,m-1,k-10\cdot 3^{m-3}}, & k \in K_{26}, \\ 0, & k \in K_{21} \cup K_{22} \cup K_{23}. \end{cases}$$

Substituting (4.35)–(4.37) into the first identity in (4.17), (4.33), and (4.34), we obtain

$$\sum_{i \in A_1} g'_{2,k}(i) = g'_{2,k}(3^{m-1}) + g'_{2,k}(3^{m-2}) + g'_{2,k}(3^{m-1} - (k - 2 \cdot 3^{m-2})) = S_{21} + S_{22} + S_{23}, \tag{4.38}$$

where

$$S_{21} := 6s^{(1)}_{n,m-2,0} \cdot s^{(1)}_{n,m-1,0} \cdot s^{(1)}_{n,m-1,k-2\cdot 3^{m-2}} = 6S'_{0,0,k-2\cdot 3^{m-2}},$$
$$S_{22} := s^{(1)}_{n,m-2,0} L_{24} + s^{(1)}_{n,m-1,0} L_{22}, \quad S_{23} := s^{(1)}_{n,m-2,0} L_{25} + s^{(1)}_{n,m-1,0} L_{23} + s^{(1)}_{n,m-1,k-2\cdot 3^{m-2}} L_{21}.$$

By the first identity in (4.17) and (4.36), we have

$$v_3(g'_{2,k}(3^{m-1})) = W_{m,k} - 1 = v_3(S'_{0,0,k-2\cdot 3^{m-2}}). \tag{4.39}$$

Therefore,

$$v_3(S_{21}) = v_3(S'_{0,0,k-2\cdot 3^{m-2}}) + 1 = W_{m,k} \tag{4.40}$$

and the bounds for L_{21}, L_{23}, and L_{25} imply

$$v_3(S_{23}) \geq v_3(S'_{0,0,k-2\cdot 3^{m-2}}) + 2 = W_{m,k} + 1. \tag{4.41}$$

It remains to control S_{22}. We claim that

$$v_3\Big(S_{22} + \sum_{i \in A_2} g'_{2,k}(i)\Big) \geq W_{m,k} + 1. \tag{4.42}$$

Assuming (4.42), it follows from (4.38), (4.40), and (4.41) that

$$\begin{aligned} v_3(S'_A) = v_3\Big(\sum_{i \in A} g'_{2,k}(i)\Big) &= v_3\Big(\sum_{i \in A_1} g'_{2,k}(i) + \sum_{i \in A_2} g'_{2,k}(i)\Big) \\ &= v_3\Big(S_{21} + S_{22} + S_{23} + \sum_{i \in A_2} g'_{2,k}(i)\Big) = v_3(S_{21}) = W_{m,k}. \end{aligned}$$

Thus, (4.12) holds in Case 2 once (4.42) is proved.

We now prove (4.42). The argument follows the same pattern as in Case 1: the secondary term S_{22} is combined with the contribution from A_2. Since A_2 has six possible forms, corresponding to $k \in K_{21}, \ldots, K_{26}$, we treat these cases separately. In each case, we prove that the combined contribution has 3-adic valuation at least $W_{m,k} + 1$.

Subcase 2.1: $k \in K_{21}$.

In this subcase, $k = 2 + 2 \cdot 3^{m-2}$, $m \geq 4$, $L_{22} \neq 0, L_{24} = 0$, and

$$A_2 = 3^{m-1} - \{k, 2 \cdot 3^{m-3}, k - 2 \cdot 3^{m-3}, 4 \cdot 3^{m-3}, k - 4 \cdot 3^{m-3}\}.$$

Moreover,

$$S_{22} = s^{(1)}_{n,m-1,0} L_{22} = 2S'_{2,0,0} + 2S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}} + 2S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}}. \tag{4.43}$$

Noting that $s^{(1)}_{n,m-1,k} = s^{(1)}_{n,m-2,2}$, the second identity in (4.17) and (4.19) remain valid, and we have

$$v_3(g'_{2,k}(3^{m-1} - k)) = W_{m,k} = v_3(S'_{2,0,0}). \tag{4.44}$$

Since $k - 2 \cdot 3^{m-3}, k - 4 \cdot 3^{m-3} \in K_1$, the estimates (4.25), (4.26), and (4.30)–(4.32) remain valid. Hence, applying the second identity in (4.17) together with (4.19), (4.25), (4.26), and (4.30)–(4.32), (4.43), and (4.44), we obtain

$$\begin{aligned} v_3\Big(S_{22} + \sum_{i \in A_2} g'_{2,k}(i)\Big) &= v_3(3S'_{2,0,0} + s^{(1)}_{n,m-2,2} L_{13} + 6S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}} + L_{111} + L_{112} \\ &\quad + 6S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}} + L_{131} + L_{132}) \\ &\geq \min\{v_3(S'_{2,0,0}), v_3(S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}}), v_3(S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}})\} + 1 = W_{m,k} + 1. \end{aligned}$$

Thus, (4.42) is proved in this subcase.

Subcase 2.2: $k \in K_{22}$.

In this subcase, $L_{22} \neq 0$, $L_{24} = 0$, and $A_2 = 3^{m-1} - \{2 \cdot 3^{m-3}, k - 2 \cdot 3^{m-3}, 4 \cdot 3^{m-3}, k - 4 \cdot 3^{m-3}\}$. Moreover,

$$S_{22} = s^{(1)}_{n,m-1,0} L_{22} = 2S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}} + 2S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}}. \tag{4.45}$$

Since $k - 2 \cdot 3^{m-3}$ and $k - 4 \cdot 3^{m-3} \in K_1$, (4.25), (4.26), and (4.30)–(4.32) remain valid. Combining them with (4.45), we get

$$\begin{aligned} v_3\Big(S_{22} + \sum_{i \in A_2} g'_{2,k}(i)\Big) &= v_3(6S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}} + L_{111} + L_{112} + 6S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}} + L_{131} + L_{132}) \\ &\geq \min\{v_3(S_{0,2\cdot 3^{m-3},k-2\cdot 3^{m-3}}), v_3(S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}})\} + 1 = W_{m,k} + 1. \end{aligned}$$

This proves (4.42) in this subcase.

Subcase 2.3: $k \in K_{23}$.

Here, $k = 8 \cdot 3^{m-3}$. We consider two cases according to whether $m = 3$ or $m \geq 4$.

If $m = 3$, then $A_2 = 3^{m-1} - \{k, 4 \cdot 3^{m-3}\}$ and

$$S_{22} = s^{(1)}_{n,m-1,0} L_{22} = 2S'_{2,0,0} + S_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}}. \tag{4.46}$$

Since $s^{(1)}_{n,m-1,k} = s^{(1)}_{n,m-2,2}$, as in Subcase 2.1, the second identity in (4.17) remains valid, as do (4.19) and (4.44). Moreover, by Remark 3.2,

$$g'_{2,k}(3^{m-1} - 4 \cdot 3^{m-3}) = s^{(1)}_{n,m-1,4\cdot 3^{m-3}} \cdot s^{(2)}_{n,m-1,4\cdot 3^{m-3}} = 2S_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}} + L_{231}, \tag{4.47}$$

where $v_3(L_{231}) \geq v_3(S_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}}) + 1$. Note that

$$v_3(g'_{2,k}(3^{m-1} - 4 \cdot 3^{m-3})) = W_{m,k} = v_3(S_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}}). \tag{4.48}$$

Thus, combining the second identity in (4.17) with (4.19), (4.44), and (4.46)–(4.48), we get

$$\begin{aligned} v_3\Big(S_{22} + \sum_{i\in A_2} g'_{2,k}(i)\Big) &= v_3(3S'_{2,0,0} + s^{(1)}_{n,m-2,2}L_{13} + 3S_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}} + L_{231}) \\ &\geq \min\{v_3(S'_{2,0,0}), v_3(S_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}})\} + 1 = W_{m,k} + 1. \end{aligned}$$

If $m \geq 4$, then $A_2 = \{3^{m-1} - 4 \cdot 3^{m-3}\}$ and

$$S_{22} = s^{(1)}_{n,m-1,0}L_{22} = s^{(1)}_{n,m-1,0}(s^{(1)}_{n,m-1,4\cdot 3^{m-3}})^2 = S_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}}. \tag{4.49}$$

Applying (4.47)–(4.49), we obtain

$$v_3\Big(S_{22} + \sum_{i\in A_2} g'_{2,k}(i)\Big) = v_3(3S_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}} + L_{231}) \geq v_3(S_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}}) + 1 = W_{m,k} + 1.$$

Thus, (4.42) is proved in this subcase.

Subcase 2.4: $k \in K_{24}$.

In this subcase, $m \geq 4$, $2 + 8 \cdot 3^{m-3} \leq k \leq 10 \cdot 3^{m-3} - 2$, and $A_2 = 3^{m-1} - \{2 \cdot 3^{m-3}, 4 \cdot 3^{m-3}, k - 4 \cdot 3^{m-3}, k - 8 \cdot 3^{m-3}\}$. Moreover,

$$S_{22} = s^{(1)}_{n,m-2,0}L_{24} + s^{(1)}_{n,m-1,0}L_{22} = 2S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}} + 2S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}}. \tag{4.50}$$

Since $k - 4 \cdot 3^{m-3} \in K_1$ and $k - 2 \cdot 3^{m-3}, 8 \cdot 3^{m-3} \in K_2$, (4.30) and (4.31) hold, and Remark 3.2 gives

$$\begin{aligned} g'_{2,k}(3^{m-1} - 2 \cdot 3^{m-3}) &= s^{(1)}_{n,m-1,2\cdot 3^{m-3}} \cdot s^{(2)}_{n,m-1,k-2\cdot 3^{m-3}} = 2S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}} + L_{241}, \\ g'_{2,k}(3^{m-1} - (k - 8 \cdot 3^{m-3})) &= s^{(1)}_{n,m-1,k-8\cdot 3^{m-3}} \cdot s^{(2)}_{n,m-1,8\cdot 3^{m-3}} = 2S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}} + L_{242}, \end{aligned} \tag{4.51}$$

where $\min\{v_3(L_{241}), v_3(L_{242})\} \geq v_3(S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}}) + 1$. Note that (4.32) holds and

$$v_3(g'_{2,k}(3^{m-1} - 2 \cdot 3^{m-3})) = W_{m,k} = v_3(S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}}). \tag{4.52}$$

Combining (4.30)–(4.32) and (4.50)–(4.52), we obtain

$$\begin{aligned} v_3\Big(S_{22} + \sum_{i\in A_2} g'_{2,k}(i)\Big) &= v_3(6S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}} + L_{131} + L_{132} + 6S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}} + L_{241} + L_{242}) \\ &\geq \min\{v_3(S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}}), v_3(S_{0,4\cdot 3^{m-3},k-4\cdot 3^{m-3}})\} + 1 = W_{m,k} + 1. \end{aligned}$$

Thus, (4.42) is proved in this subcase.

Subcase 2.5: $k \in K_{25}$.

In this subcase, $k = 10 \cdot 3^{m-3}$, $L_{22} = 0$, and $A_2 = \{3^{m-1} - 2 \cdot 3^{m-3}\}$. Moreover,

$$S_{22} = s^{(1)}_{n,m-2,0} L_{24} = s^{(1)}_{n,m-2,0} (s^{(1)}_{n,m-1,2\cdot 3^{m-3}})^2 = S'_{0,2\cdot 3^{m-3},2\cdot 3^{m-3}}. \tag{4.53}$$

Since $8 \cdot 3^{m-3} \in K_2$, Remark 3.2 gives

$$g'_{2,k}(3^{m-1} - 2 \cdot 3^{m-3}) = s^{(1)}_{n,m-1,2\cdot 3^{m-3}} \cdot s^{(2)}_{n,m-1,8\cdot 3^{m-3}} = 2S'_{0,2\cdot 3^{m-3},2\cdot 3^{m-3}} + L_{251}, \tag{4.54}$$

where $v_3(L_{251}) \geq v_3(S'_{0,2\cdot 3^{m-3},2\cdot 3^{m-3}}) + 1$.

Note that

$$v_3(g'_{2,k}(3^{m-1} - 2 \cdot 3^{m-3})) = W_{m,k} = v_3(S'_{0,2\cdot 3^{m-3},2\cdot 3^{m-3}}). \tag{4.55}$$

Then, it follows from (4.53)–(4.55) that

$$v_3\Big(S_{22} + \sum_{i\in A_2} g'_{2,k}(i)\Big) = v_3(3S'_{0,2\cdot 3^{m-3},2\cdot 3^{m-3}} + L_{251}) \geq v_3(S'_{0,2\cdot 3^{m-3},2\cdot 3^{m-3}}) + 1 = W_{m,k} + 1.$$

This proves (4.42) in this subcase.

Subcase 2.6: $k \in K_{26}$.

In this subcase, $m \geq 4$, $2 + 10 \cdot 3^{m-3} \leq k \leq 4 \cdot 3^{m-2} - 2$, $L_{22} = 0$, and

$$A_2 = 3^{m-1} - \{2 \cdot 3^{m-3}, 4 \cdot 3^{m-3}, k - 8 \cdot 3^{m-3}, k - 10 \cdot 3^{m-3}\}.$$

Moreover,

$$S_{22} = s^{(1)}_{n,m-2,0} L_{24} = 2S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}} + 2S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}}. \tag{4.56}$$

Since $k - 2 \cdot 3^{m-3}$, $k - 4 \cdot 3^{m-3}$, and $10 \cdot 3^{m-3} \in K_2$, (4.51) remains valid. By Remark 3.2, we have

$$\begin{aligned} g'_{2,k}(3^{m-1} - 4 \cdot 3^{m-3}) &= s^{(1)}_{n,m-1,4\cdot 3^{m-3}} \cdot s^{(2)}_{n,m-1,k-4\cdot 3^{m-3}} = 2S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}} + L_{261}, \\ g'_{2,k}(3^{m-1} - (k - 10 \cdot 3^{m-3})) &= s^{(1)}_{n,m-1,k-10\cdot 3^{m-3}} \cdot s^{(2)}_{n,m-1,10\cdot 3^{m-3}} = 2S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}} + L_{262}, \end{aligned} \tag{4.57}$$

where $\min\{v_3(L_{261}), v_3(L_{262})\} \geq v_3(S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}}) + 1$.

Note that (4.52) holds and we have

$$v_3(g'_{2,k}(3^{m-1} - 4 \cdot 3^{m-3})) = W_{m,k} = v_3(S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}}). \tag{4.58}$$

Applying (4.51), (4.52), and (4.56)–(4.58), we obtain

$$\begin{aligned} v_3\Big(S_{22} + \sum_{i\in A_2} g'_{2,k}(i)\Big) &= v_3(6S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}} + L_{241} + L_{242} + 6S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}} + L_{261} + L_{262}) \\ &\geq \min\{v_3(S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}}), v_3(S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}})\} + 1 = W_{m,k} + 1. \end{aligned}$$

Thus, (4.42) is proved in this subcase.

Combining Subcases 2.1 to 2.6, we prove (4.42). Hence, (4.12) holds in Case 2.

Case 3: $k \in K_4$.

In this case, $m \geq 3$ and $A_1 = 3^{m-1} - \{2 \cdot 3^{m-2}, k - 4 \cdot 3^{m-2}\}$. Thus, the contribution from A_1 consists of two terms. Recall from (4.33) that $g'_{2,k}(3^{m-2}) = s^{(1)}_{n,m-2,0} \cdot s^{(2)}_{n,m-1,k-2\cdot 3^{m-2}}$. Moreover,

$$g'_{2,k}(3^{m-1} - (k - 4 \cdot 3^{m-2})) = s^{(1)}_{n,m-1,k-4\cdot 3^{m-2}} \cdot s^{(2)}_{n,m-1,4\cdot 3^{m-2}}. \tag{4.59}$$

We now expand the two $s^{(2)}$-terms appearing above. Setting $(n, m, k) \mapsto (n, m-1, 4\cdot 3^{m-2})$ in Remark 3.2, we get

$$s^{(2)}_{n,m-1,4\cdot 3^{m-2}} = (s^{(1)}_{n,m-2,0})^2 + L_{41} = S''_{0,0} + L_{41}, \tag{4.60}$$

where $v_3(L_{41}) \geq v_3(S''_{0,0}) + 2$. Since $k - 2 \cdot 3^{m-2} \in K_2$, replacing k by $k - 2 \cdot 3^{m-2}$ in (4.36) gives

$$s^{(2)}_{n,m-1,k-2\cdot 3^{m-2}} = 2s^{(1)}_{n,m-2,0} \cdot s^{(1)}_{n,m-1,k-4\cdot 3^{m-2}} + L_{42} + L_{43} = 2S'_{0,k-4\cdot 3^{m-2}} + L_{42} + L_{43}, \tag{4.61}$$

where $v_3(L_{43}) \geq v_3(S'_{0,k-4\cdot 3^{m-2}}) + 2$ and $v_3(L_{42}) \geq v_3(S'_{0,k-4\cdot 3^{m-2}}) + 1$. Moreover,

$$L_{42} = \begin{cases} 2s^{(1)}_{n,m-2,2}s^{(1)}_{n,m-1,0} + 2s^{(1)}_{n,m-1,2\cdot 3^{m-3}}s^{(1)}_{n,m-1,k-8\cdot 3^{m-3}} + 2s^{(1)}_{n,m-1,4\cdot 3^{m-3}}s^{(1)}_{n,m-1,k-10\cdot 3^{m-3}}, & k \in K_{41}, \\ 2s^{(1)}_{n,m-1,2\cdot 3^{m-3}}s^{(1)}_{n,m-1,k-8\cdot 3^{m-3}} + 2s^{(1)}_{n,m-1,4\cdot 3^{m-3}}s^{(1)}_{n,m-1,k-10\cdot 3^{m-3}}, & k \in K_{42}, \\ 2s^{(1)}_{n,m-2,2}s^{(1)}_{n,m-1,0} + \left(s^{(1)}_{n,m-1,4\cdot 3^{m-3}}\right)^2, & k \in K_{43},\ m = 3, \\ \left(s^{(1)}_{n,m-1,4\cdot 3^{m-3}}\right)^2, & k \in K_{43},\ m \geq 4, \\ 2s^{(1)}_{n,m-1,4\cdot 3^{m-3}}s^{(1)}_{n,m-1,k-10\cdot 3^{m-3}}, & k \in K_{44}, \\ 0, & k \in K_{45}. \end{cases}$$

Substituting (4.59)–(4.61) into (4.33), we obtain

$$\sum_{i\in A_1} g'_{2,k}(i) = g'_{2,k}(3^{m-2}) + g'_{2,k}(3^{m-1} - (k - 4 \cdot 3^{m-2})) = S_{41} + S_{42} + S_{43}, \tag{4.62}$$

where

$$\begin{aligned} S_{41} &:= 3(s^{(1)}_{n,m-2,0})^2 s^{(1)}_{n,m-1,k-4\cdot 3^{m-2}} = 3S''_{0,0,k-4\cdot 3^{m-2}}, \\ S_{42} &:= s^{(1)}_{n,m-2,0}L_{42}, \qquad S_{43} := s^{(1)}_{n,m-2,0}L_{43} + s^{(1)}_{n,m-1,k-4\cdot 3^{m-2}}L_{41}. \end{aligned}$$

By the definition of A_1, together with (4.59) and (4.60), we have

$$v_3(g'_{2,k}(3^{m-1} - (k - 4 \cdot 3^{m-2}))) = W_{m,k} - 1 = v_3(S''_{0,0,k-4\cdot 3^{m-2}}). \tag{4.63}$$

Consequently,

$$v_3(S_{41}) = v_3(S''_{0,0,k-4\cdot 3^{m-2}}) + 1 = W_{m,k}, \tag{4.64}$$

and the bounds for L_{41} and L_{43} imply

$$v_3(S_{43}) \geq v_3(S''_{0,0,k-4\cdot 3^{m-2}}) + 2 = W_{m,k} + 1. \tag{4.65}$$

It remains to deal with S_{42}. We first observe that

$$S_{42} = 0 \quad \Longleftrightarrow \quad k \in K_{45} \quad \Longleftrightarrow \quad A_2 = \varnothing.$$

If $k \in K_{45}$, then $S_{42} = 0$ and $A_2 = \varnothing$. Hence, by (4.62), (4.64), and (4.65),

$$v_3(S'_A) = v_3\Big(\sum_{i\in A} g'_{2,k}(i)\Big) = v_3\Big(\sum_{i\in A_1} g'_{2,k}(i)\Big) = v_3(S_{41} + S_{43}) = v_3(S_{41}) = W_{m,k}.$$

Thus, (4.12) holds when $k \in K_{45}$.

It remains to consider $k \in \bigcup_{j=1}^{4} K_{4j}$. In this case, $S_{42} \neq 0$ and $A_2 \neq \varnothing$. We claim that

$$v_3\Big(S_{42} + \sum_{i\in A_2} g'_{2,k}(i)\Big) \geq W_{m,k} + 1. \tag{4.66}$$

Assuming (4.66), it follows from (4.62), (4.64), and (4.65) that

$$\begin{aligned} v_3(S'_A) &= v_3\Big(\sum_{i\in A} g'_{2,k}(i)\Big) = v_3\Big(\sum_{i\in A_1} g'_{2,k}(i) + \sum_{i\in A_2} g'_{2,k}(i)\Big) \\ &= v_3\Big(S_{41} + S_{42} + S_{43} + \sum_{i\in A_2} g'_{2,k}(i)\Big) = v_3(S_{41}) = W_{m,k}. \end{aligned}$$

Thus, (4.12) holds in Case 3 once (4.66) is proved.

We now prove (4.66). The argument follows the same pattern as in Case 1 and 2: the secondary term S_{42} is combined with the contribution from A_2. Since A_2 has four possible forms, corresponding to $k \in K_{41}, \ldots, K_{44}$, we treat these cases separately. In each case, we prove that the combined contribution has 3-adic valuation at least $W_{m,k} + 1$.

Subcase 3.1: $k \in K_{41}$.

In this subcase, $k = 2 + 4 \cdot 3^{m-2}$, $m \geq 4$, and

$$A_2 = 3^{m-1} - \{0, k - 2\cdot 3^{m-2}, 2\cdot 3^{m-3}, 4\cdot 3^{m-3}, k - 8\cdot 3^{m-3}, k - 10\cdot 3^{m-3}\}.$$

Moreover,

$$S_{42} = s^{(1)}_{n,m-2,0} L_{42} = 2S''_{0,2,0} + 2S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}} + 2S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}}. \tag{4.67}$$

Since $k = 2 + 4\cdot 3^{m-2}$, we have $s^{(2)}_{n,m-1,k} = s^{(2)}_{n,m-2,2}$. Thus, Remark 3.2 gives

$$\begin{aligned} g'_{2,k}(3^{m-1}) &= s^{(1)}_{n,m-1,0} \cdot s^{(2)}_{n,m-1,k} = 2S''_{0,2,0} + L_{411}, \\ g'_{2,k}(3^{m-1} - (k - 2\cdot 3^{m-2})) &= g'_{2,k}(3^{m-2} - 2) = s^{(1)}_{n,m-2,2} \cdot s^{(2)}_{n,m-1,2\cdot 3^{m-2}} = 2S''_{0,2,0} + L_{412}, \end{aligned} \tag{4.68}$$

where $\min\{v_3(L_{411}), v_3(L_{412})\} \geq v_3(S''_{0,2,0}) + 1$.

Since $k - 2\cdot 3^{m-3}$ and $k - 4\cdot 3^{m-3} \in K_2$, (4.51), (4.52), (4.57), and (4.58) hold, and we have

$$v_3(g'_{2,k}(3^{m-1})) = W_{m,k} = v_3(S''_{0,2,0}). \tag{4.69}$$

Combining (4.67)–(4.69) with (4.51), (4.52), (4.57), and (4.58), we obtain

$$\begin{aligned}v_3\Big(S_{42}+\sum_{i\in A_2} g'_{2,k}(i)\Big) &= v_3(6S''_{0,2,0}+L_{411}+L_{412}+6S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}}+L_{241}+L_{242}\\ &\qquad +6S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}}+L_{261}+L_{262})\\ &\geq \min\{v_3(S''_{0,2,0}),v_3(S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}}),v_3(S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}})\}+1=W_{m,k}+1.\end{aligned}$$

So, (4.66) is proved in this subcase.

Subcase 3.2: $k \in K_{42}$.

In this subcase, $m \geq 4$, $4+4\cdot 3^{m-2} \leq k \leq 14\cdot 3^{m-3}-2$, and

$$A_2 = 3^{m-1} - \{2\cdot 3^{m-3}, 4\cdot 3^{m-3}, k-8\cdot 3^{m-3}, k-10\cdot 3^{m-3}\}.$$

Moreover,

$$S_{42} = s^{(1)}_{n,m-2,0}L_{42} = 2S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}} + 2S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}}. \tag{4.70}$$

Since $k-2\cdot 3^{m-3}$ and $k-4\cdot 3^{m-3} \in K_2$, the estimates (4.51), (4.52), (4.57), and (4.58) apply. Combining these estimates with (4.70), we obtain that

$$\begin{aligned}v_3\Big(S_{42}+\sum_{i\in A_2} g'_{2,k}(i)\Big) &= v_3(6S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}}+L_{241}+L_{242}+6S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}}+L_{261}+L_{262})\\ &\geq \min\{v_3(S'_{0,2\cdot 3^{m-3},k-8\cdot 3^{m-3}}),v_3(S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}})\}+1=W_{m,k}+1.\end{aligned}$$

Thus, (4.66) is proved in this subcase.

Subcase 3.3: $k \in K_{43}$.

In this subcase, $k = 14\cdot 3^{m-3}$. We consider two cases according to whether $m=3$ or $m \geq 4$. If $m=3$, then $A_2 = 3^{m-1} - \{0, k-2\cdot 3^{m-2}, 4\cdot 3^{m-3}\}$ and

$$S_{42} = s^{(1)}_{n,m-2,0}L_{42} = 2S''_{0,2,0} + S'_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}}. \tag{4.71}$$

Since $m=3$ and $k=14\cdot 3^{m-3}$, one has $s^{(2)}_{n,m-1,k} = s^{(2)}_{n,m-2,2}$. Thus, (4.68) and (4.69) hold. It follows from Remark 3.2 that

$$g'_{2,k}(3^{m-1}-4\cdot 3^{m-3}) = s^{(1)}_{n,m-1,4\cdot 3^{m-3}} \cdot s^{(2)}_{n,m-1,10\cdot 3^{m-3}} = 2S'_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}} + L_{431}, \tag{4.72}$$

where $v_3(L_{431}) \geq v_3(S'_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}})+1$. Note that

$$v_3(g'_{2,k}(3^{m-1}-4\cdot 3^{m-3})) = W_{m,k} = v_3(S'_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}}). \tag{4.73}$$

Combining (4.71)–(4.73) with (4.68), and (4.69), we get

$$\begin{aligned}v_3\Big(S_{42}+\sum_{i\in A_2} g'_{2,k}(i)\Big) &= v_3(6S''_{0,2,0}+L_{411}+L_{412}+3S'_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}}+L_{431})\\ &\geq \min\{v_3(S''_{0,2,0}),v_3(S'_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}})\}+1=W_{m,k}+1.\end{aligned}$$

If $m \geq 4$, then $A_2 = 3^{m-1} - \{4 \cdot 3^{m-3}\}$ and

$$S_{42} = s^{(1)}_{n,m-2,0} L_{42} = s^{(1)}_{n,m-2,0} (s^{(1)}_{n,m-1,4\cdot 3^{m-3}})^2 = S'_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}}. \tag{4.74}$$

By (4.72)–(4.74), we obtain

$$v_3\Big(S_{42} + \sum_{i \in A_2} g'_{2,k}(i)\Big) = v_3(3S'_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}} + L_{431}) \geq v_3(S'_{0,4\cdot 3^{m-3},4\cdot 3^{m-3}}) + 1 = W_{m,k} + 1.$$

Thus, (4.66) is proved in this subcase.

Subcase 3.4: $k \in K_{44}$.

In this subcase, $m \geq 4$ and $A_2 = 3^{m-1} - \{4 \cdot 3^{m-3}, k - 10 \cdot 3^{m-3}\}$. Moreover,

$$S_{42} = s^{(1)}_{n,m-2,0} L_{42} = 2S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}}. \tag{4.75}$$

Since $m \geq 4$ and $2 + 14 \cdot 3^{m-3} \leq k \leq 16 \cdot 3^{m-3} - 2$, we have $k - 4 \cdot 3^{m-3} \in K_2$. Hence, the estimates (4.57) and (4.58) apply. Combining them with (4.75), we get

$$\begin{aligned} v_3\Big(S_{42} + \sum_{i \in A_2} g'_{2,k}(i)\Big) &= v_3(6S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}} + L_{261} + L_{262}) \\ &\geq v_3(S'_{0,4\cdot 3^{m-3},k-10\cdot 3^{m-3}}) + 1 = W_{m,k} + 1. \end{aligned}$$

Thus, (4.66) is proved in this subcase.

From Subcases 3.1 to 3.4, we have proved (4.66). Hence, (4.12) holds for $k \in K_4$.

Combining Cases 1 to 3, together with the previously treated cases $k \in K_3$ and $k \in K_5$, we obtain (4.12) for all admissible k. Therefore, (4.5) follows, and the analysis of the leading contribution is complete.

We now complete the inductive step. By (4.3) and the above decomposition,

$$s^{(1)}_{n+1,m,k} = S_A + S_B + S_C.$$

By (4.5), we have $v_3(S_A) = W_{m,k}$. Moreover, by (4.4) and the definition of A, we have $v_3(S_B) \geq W_{m,k} + 1$ and $v_3(S_C) \geq W_{m,k} + 1$. It follows that

$$v_3(s^{(1)}_{n+1,m,k}) = v_3(S_A + S_B + S_C) = W_{m,k}.$$

Thus, (4.2) holds. Hence, (1.2) holds for $a = 1$ and even k at level $n + 1$. By Lemma 2.4, it also holds for odd k. Therefore, (1.2) holds for $a = 1$ at level $n + 1$.

By induction, formula (1.2) holds for $a = 1$ for all positive integers n. Finally, Lemma 3.1 gives the corresponding formula for $a = 2$. This completes the proof of Theorem 1.3.

5. Consequences of the main formula

In this section we derive three consequences of the main formula. The first compares two adjacent orders, namely $a3^n + 1$ and $a3^n$. The second identifies the maximal 3-adic valuation among the families $\{s(3^n, k)\}$ and $\{s(2 \cdot 3^n, k)\}$. The final consequence translates these bounds into estimates for elementary symmetric functions, giving a partial verification of the conjecture of Leonetti and Sanna in the present 3-adic setting.

5.1. *Proof of Theorem 1.6*

We first prove Theorem 1.6, which compares $v_3(s(a3^n + 1, k + 1))$ with the 3-adic valuations of Stirling numbers of order $a3^n$.

Let $a \in \{1, 2\}$, and let $n, k \in \mathbb{Z}^+$ with $k \le a3^n$.

We first deal with the two boundary cases. If $k = a3^n$, then $2 \mid (k - a)$, and $v_3(s(a3^n + 1, a3^n + 1)) = v_3(1) = v_3(s(a3^n, a3^n)) = 0$. Thus, (1.4) holds in this case. If $k = a3^n - 1$, then $2 \nmid (k - a)$, and $v_3(s(a3^n + 1, a3^n)) = v_3(\binom{a3^n+1}{2}) = n$ and $v_3(s(a3^n, a3^n)) = 0$. Hence, (1.4) also holds for $k = a3^n - 1$. So, Theorem 1.6 holds when $k \in \{a3^n - 1, a3^n\}$.

It remains to consider $1 \le k \le a3^n - 2$. By the recurrence relation for Stirling numbers of the first kind (see, for example, [33]), we have

$$s(a3^n + 1, k + 1) = a3^n s(a3^n, k + 1) + s(a3^n, k). \tag{5.1}$$

We distinguish two cases according to the parity of $k - a$.

If $2 \mid (k - a)$, then $2 \nmid (k + 1 - a)$, and so we may write $k + 1 = a3^n - 2t - 1$, where $t \in \mathbb{Z}$ with $0 \le t \le \frac{a3^n-3}{2}$. By Lemma 2.3 and Lemma 2.6, we deduce that

$$\begin{aligned} v_3(a3^n s(a3^n, k + 1)) &= v_3(s(a3^n, a3^n - 2t - 1)) + n = v_3(s(a3^n, a3^n - 2t)) + v_3(2t + 1) + 2n \\ &= v_3(s(a3^n, k + 2)) + v_3(k + 1) + 2n \\ &\ge v_3(s(a3^n, k)) + v_3(k + 1) + 2 > v_3(s(a3^n, k)). \end{aligned} \tag{5.2}$$

From (5.1) and (5.2), we obtain

$$v_3(s(a3^n + 1, k + 1)) = v_3(s(a3^n, k)).$$

This proves (1.4) when $2 \mid (k - a)$.

If $2 \nmid (k - a)$, then we can write $k = a3^n - 2t' - 1$ with $t' \in \mathbb{Z}$ and $0 \le t' \le \frac{a3^n-2}{2}$. Again, by Lemma 2.3, we have

$$\begin{aligned} v_3(s(a3^n, k)) &= v_3(s(a3^n, a3^n - 2t' - 1)) = v_3(s(a3^n, a3^n - 2t')) + v_3(2t' + 1) + n \\ &= v_3(s(a3^n, k + 1)) + v_3(a3^n - k) + n \ge v_3(s(a3^n, k + 1)) + n. \end{aligned} \tag{5.3}$$

Hence, by (5.1) and (5.3) we obtain

$$v_3(s(a3^n + 1, k + 1)) \ge \min\{v_3(a3^n s(a3^n, k + 1)), v_3(s(a3^n, k))\} = v_3(s(a3^n, k + 1)) + n.$$

Thus, (1.4) holds when $2 \nmid (k - a)$. This finishes the proof of Theorem 1.6.

5.2. *Proofs of Theorems 1.7 and 1.8*

Next, we prove Theorems 1.7 and 1.8. Both arguments have the same structure: using Theorem 1.3, we compare $v_3(s(a3^n, k))$ with the 3-adic valuation at $k = 1$, and show that the latter gives the desired upper bound.

Proof of Theorem 1.7. For $n = 1$, we have $k \in \{1, 2, 3\}$. Since $s(3, 1) = 2$, $s(3, 2) = 3$, and $s(3, 3) = 1$, (1.5) is immediate.

Assume now that $n \geq 2$. The boundary cases are again immediate: $v_3(s(3^n, 3^n)) = 0$ and $v_3(s(3^n, 3^n - 1)) = n$. Hence, (1.5) holds when $k \in \{3^n - 1, 3^n\}$.

It remains to consider $1 \leq k \leq 3^n - 2$. Write $k = 3^l - j$, where $(l, j) \in T_{1,n}$. By Theorem 1.3,

$$v_3(s^{(1)}_{n,l,j}) = \frac{1}{2}(3^n - 3^l) - (n - l)(3^l - j) + l - 1 - v_3\Big(\Big\lfloor \frac{j}{2} \Big\rfloor\Big) + (l + v_3(j))\epsilon_j. \tag{5.4}$$

We first handle the exceptional case $n = 2$. If $l = 1$, then $j = 2$ and (5.4) gives $v_3(s^{(1)}_{n,l,j}) = 2 < 4$. If $l = 2$, then $2 \leq j \leq 7$, and (5.4) gives

$$v_3(s^{(1)}_{n,l,j}) = 1 - v_3\Big(\Big\lfloor \frac{j}{2} \Big\rfloor\Big) + (2 + v_3(j))\epsilon_j \leq 3 + v_3(j) \leq 4.$$

Therefore, (1.5) holds when $n = 2$.

We now assume $n \geq 3$. Since $1 \leq l \leq n$, $2 \leq j \leq 2 \cdot 3^{l-1} + 1$ and $j < 3^l$, we have $v_3(j) \leq l - 1$. By (5.4), we obtain

$$\begin{aligned} v_3(s(3^n, 1)) - v_3(s(3^n, k)) &= \frac{1}{2}(3^n - 2n - 1) - v_3(s^{(1)}_{n,l,j}) \\ &= \frac{1}{2}(3^l - 1) + (n - l)(3^l - j - 1) - 2l + 1 + v_3\Big(\Big\lfloor \frac{j}{2} \Big\rfloor\Big) - (l + v_3(j))\epsilon_j =: D^{(1)}_{l,j}. \end{aligned}$$

It remains to show that $D^{(1)}_{l,j} \geq 0$.

If j is even, then $2 \leq j \leq 2 \cdot 3^{l-1}$, and

$$\begin{aligned} D^{(1)}_{l,j} &= \frac{1}{2}(3^l - 1) + (n - l)(3^l - j - 1) - 2l + 1 + v_3(j) \\ &\geq \frac{1}{2}(3^l - 1) + (n - l)(3^{l-1} - 1) - 2l + 1 \geq \frac{1}{2}(3^l - 1) - 2l + 1 \geq 0. \end{aligned}$$

If j is odd, then $l \geq 2$, $3 \leq j \leq 2 \cdot 3^{l-1} + 1$. In this case,

$$D^{(1)}_{l,j} = \frac{1}{2}(3^l - 1) + (n - l)(3^l - j - 1) - 3l - v_3(j) + v_3(j - 1) + 1.$$

For $l = 2$, one checks directly that $D^{(1)}_{2,j} \geq 1$ holds for $j \in \{3, 5, 7\}$. For $l \geq 3$, we have

$$D^{(1)}_{l,j} \geq \frac{1}{2}(3^l - 1) + (n - l)(3^{l-1} - 2) - 3l - (l - 1) + 1 \geq \frac{1}{2}(3^l - 1) - 4l + 2 \geq 3.$$

Thus, $D^{(1)}_{l,j} \geq 0$ in all cases, hence (1.5) holds for $n \geq 3$. This completes the proof of Theorem 1.7. □

The proof of Theorem 1.8 is parallel, using the formula in Theorem 1.3 with $a = 2$. We give the details.

Proof of Theorem 1.8. First, the boundary cases follow from $v_3(s(2 \cdot 3^n, 2 \cdot 3^n)) = 0$ and $v_3(s(2 \cdot 3^n, 2 \cdot 3^n - 1)) = n$. Thus, (1.6) holds for $k \in \{2 \cdot 3^n - 1, 2 \cdot 3^n\}$.

Now let $1 \leq k \leq 2 \cdot 3^n - 2$. Write $k = 2 \cdot 3^l - j$ with $(l, j) \in T_{2,n}$. By Theorem 1.3,

$$v_3(s^{(2)}_{n,l,j}) = 3^n - 3^l - (n - l)(2 \cdot 3^l - j) + l - 1 - v_3\Big(\Big\lfloor \frac{j}{2} \Big\rfloor\Big) + (l + v_3(j))\epsilon_j. \tag{5.5}$$

We first handle the exceptional case $n = 1$. In this case, we have $l = 1$ and $2 \le j \le 5$. By (5.5),

$$v_3(s(2 \cdot 3^n, k)) = (1 + v_3(j))\epsilon_j - v_3\Big(\Big\lfloor \frac{j}{2} \Big\rfloor\Big) \le 2.$$

Hence, (1.6) holds when $n = 1$.

We now assume $n \ge 2$. By (5.5), we have

$$\begin{aligned} v_3(s(2 \cdot 3^n, 1)) - v_3(s(2 \cdot 3^n, k)) &= 3^n - n - 1 - v_3(s^{(2)}_{n,l,j}) \\ &= 3^l + (n-l)(2 \cdot 3^l - j - 1) - 2l + v_3\Big(\Big\lfloor \frac{j}{2} \Big\rfloor\Big) - (l + v_3(j))\epsilon_j =: D^{(2)}_{l,j}. \end{aligned}$$

It remains to show that $D^{(2)}_{l,j} \ge 0$.

If j is even, then $2 \le j \le 4 \cdot 3^{l-1}$ and

$$D^{(2)}_{l,j} = 3^l + (n-l)(2 \cdot 3^l - j - 1) - 2l + v_3(j) \ge (n-l)(2 \cdot 3^{l-1} - 1) + 3^l - 2l \ge 1.$$

Thus, (1.6) holds when j is even.

If j is odd, then $3 \le j \le 4 \cdot 3^{l-1} + 1$ and $v_3(j) \le l$, where $v_3(j) = l$ holds if and only if $j = 3^l$. In this case,

$$D^{(2)}_{l,j} = 3^l + (n-l)(2 \cdot 3^l - j - 1) - 3l - v_3(j) + v_3(j-1).$$

For $v_3(j) = l$, we have $j = 3^l$ and

$$D^{(2)}_{l,j} = (n-l)(3^l - 1) + 3^l - 4l \ge 1$$

since $1 \le l \le n$ and $n \ge 2$. For $v_3(j) \le l - 1$, we have

$$D^{(2)}_{l,j} \ge (n-l)(2 \cdot 3^{l-1} - 2) + 3^l - 4l + 1 \ge 0.$$

Thus, $D^{(2)}_{l,j} \ge 0$ in all cases, and (1.6) follows for $n \ge 2$. The proof of Theorem 1.8 is complete. □

5.3. *Proof of Corollary 1.9*

Finally, we prove Corollary 1.9. The point is to translate the valuation estimate for Stirling numbers of the first kind into one for the corresponding elementary symmetric functions.

Let $a \in \{1, 2\}$. By (1.1), we have

$$H(a3^n, k) = \frac{s(a3^n + 1, k + 1)}{(a3^n)!}. \tag{5.6}$$

Assume that $2 \mid (k - a)$, $n \ge 3$ and $1 \le k \le a3^n$. Then, by (5.6) and Theorems 1.6–1.8, we obtain

$$\begin{aligned} v_3(H(a3^n, k)) &= v_3(s(a3^n + 1, k + 1)) - v_3((a3^n)!) = v_3(s(a3^n, k)) - v_3((a3^n)!) \\ &\le v_3(s(a3^n, 1)) - v_3((a3^n)!) = v_3((a3^n - 1)!) - v_3((a3^n)!) = -v_3(a3^n) = -n. \end{aligned}$$

This proves Corollary 1.9. □

6. Conclusions

In this paper, we prove the 3-adic case of Conjecture 1.2 for the Stirling numbers of the first kind. More precisely, we obtain an explicit formula for $v_3(s(a3^n, k))$ for $a \in \{1, 2\}$ and $1 \leq k \leq a3^n$. As consequences, we derive a comparison formula for the adjacent family $s(a3^n + 1, k + 1)$ and determine the maximal 3-adic valuations in the ranges considered above.

For primes $p \geq 5$, Hong and Qiu [32] obtained partial results on $v_p(s(ap, k))$ and on the near-diagonal family $v_p(s(ap^n, ap^n - k))$. They also proposed Conjecture 1.2 for $v_p(s(ap^n, ap^m - k))$ in a wider range, but a complete result for $p > 3$ remains open.

The restriction to the prime 3 is essential to the present method. Indeed, when $p = 3$, the congruence class of k modulo $p-1$ is completely determined by the parity of k. In particular, every integer k satisfies $k \equiv \frac{1-(-1)^k}{2} \pmod 2$. Thus, the correction term T_k in Conjecture 1.2 always falls into its first case.

For primes $p \geq 5$, several nontrivial residue classes modulo $p - 1$ occur, and the Bernoulli-number term in Conjecture 1.2 can no longer be absorbed into a parity distinction. Consequently, a corresponding leading-term analysis would require additional congruence information and a substantially finer decomposition of the parameter range.

Author contributions

Min Qiu: Formal analysis, Writing–original draft; Zongbing Lin: Writing–original draft, Writing–review & editing; Long Chen: Writing–review & editing. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

AI-assisted tools were used only for language polishing. All mathematical results and proofs were developed and verified by the authors.

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Conflict of interest

The authors declare no conflict of interest.

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