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Research article

On γ-type contraction principles in k-fuzzy metric spaces

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Abstract: We establish new fixed point results for mappings of γ-type in the framework of k-fuzzy metric spaces. Motivated by recent developments on fixed point theory in k-fuzzy metric spaces and by γ-type contractions in fuzzy S-metric spaces, we first introduce the notion of a k-fuzzy γ-contraction and prove an existence and uniqueness theorem for its fixed point on G-complete k-fuzzy metric spaces. The proof is based on the Picard iteration, a recursive γ-inequality, and the G-Cauchy property of the associated sequence in the multi-parameter fuzzy setting. We then propose a more general class of k-fuzzy γ-weak contractions, where the contractive condition is expressed in terms of the minimum of three fuzzy proximities involving x, Ψx, and $\Psi^2 x$. Along the Picard sequence, we show that the γ-weak inequality reduces to a γ-contraction-type inequality, which allows us to transfer the strong fixed point principle to the weak setting. In this way, our results simultaneously extend the fixed point theorems for generalized k-fuzzy contractions and provide a multi-parameter generalization of the γ-contraction and γ-weak contraction principles previously obtained in fuzzy S-metric spaces. Illustrative examples are given to demonstrate the applicability and non-triviality of the proposed conditions.

Keywords: k-fuzzy metric space; fixed point; γ-contraction; γ-weak contraction
Mathematics Subject Classification: 47H09, 47H10

1. Introduction

Fixed point theory has long occupied a central position in nonlinear analysis because of its deep connections with differential equations, optimization, iterative algorithms, dynamical systems, and control theory. Since the classical Banach contraction principle provides a powerful existence and uniqueness framework under very simple contractive assumptions, a substantial body of research has been devoted to extending it to more general settings. These extensions are motivated both by theoretical interest and by the need to model systems in which distance is not the most appropriate measure of proximity. In many uncertainty-driven problems, similarity between two states is not determined by a single quantity, but rather by several interacting descriptors such as confidence, tolerance, time, cost, noise level, or granularity. This is the problem addressed in this paper: to study fixed point results in a framework that can capture multi-parameter uncertainty more faithfully than classical metric spaces. In this direction, fuzzy metric spaces have emerged as an effective mathematical framework for describing nearness under uncertainty, vagueness, and gradual change.

The concept of a fuzzy metric space was first introduced by Kramosil and Michálek [1], who replaced the crisp notion of distance with a fuzzy degree of closeness depending on a parameter. Later, George and Veeramani [2] refined this structure in such a way that the resulting topology became Hausdorff and suitable for deeper analysis. Their formulation became the standard foundation for a large part of the modern fuzzy metric literature. In this setting, Grabiec [3] obtained a fuzzy version of Banach's contraction principle and proved the existence and uniqueness of fixed points for contractive self-mappings on complete fuzzy metric spaces. This result marked the beginning of extensive research on fixed point theory in fuzzy environments.

At the same time, several authors have extended Banach's fixed point principle in fuzzy metric spaces by introducing different types of contractive conditions. For instance, Miheţ [4, 5] studied fuzzy contractive mappings in standard and non-Archimedean fuzzy metric spaces, while Salimi et al. [6] investigated generalized fuzzy contractive mappings. Other extensions include weakly compatible mappings [7], Suzuki-type contractions [8], and further generalized nonlinear contractive conditions [9–11]. These developments demonstrate that Banach's contraction principle is only one member of a broader family of fixed point results. Moreover, many of these results remain valid when continuity assumptions are relaxed or replaced by weaker structural hypotheses. Such extensions are particularly useful in modern analysis, where mappings arising from applied models may be discontinuous or may fail to be strictly contractive in the classical sense. A further important line of research concerns generalized contraction principles based on auxiliary control functions. The concepts of contraction and weak contraction were introduced into the realm of fuzzy logic by Sezen [12]. In this direction, recent studies have also developed control-function techniques in generalized fuzzy metric-type settings. For instance, Azmi [13] studied $(\alpha - \psi)$-fuzzy contractive mappings on fuzzy double-controlled metric spaces, while Furqan, Saleem, and Sessa [14] introduced fuzzy n-controlled metric spaces and investigated their fixed point structure. These works show that control functions and controlled metric generalizations already play an active role in fuzzy fixed point theory, and they help distinguish the present k-fuzzy γ-type framework from earlier controlled and admissible contractive models. In the fuzzy setting, such functions allow the contractive condition to be expressed in a more flexible and nonlinear manner, often capturing the behavior of the mapping more accurately than a fixed contraction constant. The work of Miheţ [4, 5] contributed significantly to this direction by introducing classes of fuzzy contractive mappings in non-Archimedean and standard fuzzy metric spaces. Similarly, the studies of Salimi et al. [6], Sangurlu and Turkoğlu [9, 10], and Sedghi et al. [11] developed fixed point theorems for increasingly general classes of fuzzy mappings. Collectively, these papers show that the fixed point theory of fuzzy metric spaces is highly adaptable and capable of accommodating a wide variety of contractive frameworks.

More recently, the notion of a k-fuzzy metric space was introduced by Gopal et al. [15]. In this multi-parameter setting, the degree of nearness of two points is described through k independent positive parameters, which makes the structure more flexible and better suited to models involving several scales of uncertainty or multiple criteria. This generalization extends the classical fuzzy metric space and allows one to encode richer proximity information than the single-parameter case. Gopal et al. developed basic topological properties of k-fuzzy metric spaces and established fixed point theorems for suitable contractive mappings. Later, Nazam et al. [16] continued this line of research by proving additional fixed point results in k-fuzzy metric spaces and by studying more general contraction conditions that do not require continuity of the underlying mapping.

In parallel, another modern trend in fixed point theory is the use of nonlinear control functions such as γ, φ, and ψ. These functions provide a flexible way to formulate contractive conditions that are stronger than order relations but weaker than classical linear contractions. In fuzzy metric and fuzzy S-metric settings, such control functions are particularly natural because nearness is measured by values in $(0, 1]$, and convergence is typically interpreted as approaching one. In this direction, Iqbal et al. [17] introduced γ-contractions and γ-weak contractions in fuzzy S-metric spaces and proved fixed point results under suitable completeness assumptions. Their work suggests that γ-type principles can be extended to richer fuzzy structures beyond the single-parameter framework.

Contributions of this paper. Motivated by the above developments, by the work of Gopal et al. and Nazam et al. on k-fuzzy metric spaces [15, 16], and by the γ-type fixed point principles in fuzzy S-metric spaces [17], we introduce and study a new fixed point theory based on γ-type contractions in the setting of k-fuzzy metric spaces. Our main contributions can be summarized as follows:

- We define the notion of a k-fuzzy γ-contraction and establish an existence and uniqueness theorem for fixed points of such mappings on G-complete k-fuzzy metric spaces. The proof relies on constructing a Picard sequence, deriving a recursive γ-inequality, and showing that the sequence is G-Cauchy in the sense of Gopal et al. and Nazam et al.

- We then introduce a more general class of k-fuzzy γ-weak contractions, where the contractive condition involves the minimum of three fuzzy proximities, namely $F(\eta, \zeta, p^k)$, $F(\eta, \Psi\eta, p^k)$, and $F(\zeta, \Psi\zeta, p^k)$. We prove that every k-fuzzy γ-weak contraction on a G-complete k-fuzzy metric space admits a unique fixed point. A key step in the argument is to show that, along the Picard iteration, the γ-weak inequality reduces to a γ-contraction-type inequality.

- Our results simultaneously extend classical fixed point theorems for strong contractions in fuzzy metric spaces (e.g., Grabiec [3]) and the recent fixed point results for k-fuzzy contractions and generalized k-fuzzy contractions in [15, 16]. In particular, they provide a unified framework in which both strong and weak γ-type contractions can be treated within the multi-parameter geometry of k-fuzzy metric spaces, and they generalize the γ-contraction and γ-weak contraction principles of [17] from fuzzy S-metric spaces to k-fuzzy metric spaces.

- Finally, we present illustrative examples in concrete k-fuzzy metric structures that verify the hypotheses of our theorems and confirm the sharpness of the conditions. These examples show that the k-fuzzy γ-weak contraction principle genuinely enlarges the class of mappings for which fixed point results hold in k-fuzzy metric spaces.

The paper is organized as follows. In Section 2, we recall basic definitions and properties of k-fuzzy metric spaces, G-completeness, and the class Γ of admissible γ-functions, together with some fixed point results for k-fuzzy contractions. In Section 3, we formulate the k-fuzzy γ-contraction and prove the corresponding fixed point theorem. Section 4 is devoted to the k-fuzzy γ-weak contraction, where we derive the weak-to-strong reduction along Picard sequences and establish the main fixed point result. In Section 5, we provide examples illustrating the theory. Possible applications and directions for further research are briefly discussed in Section 6.

2. Preliminaries

Throughout the paper, we denote by kFMS a k-fuzzy metric space. We first recall the notions of fuzzy metric space and k-fuzzy metric space. For the convenience of the reader, the main symbols used in the paper are summarized in Table 1.

Table 1. Main symbols and terminology used in the manuscript.

Symbol / term	Meaning
X	Nonempty underlying set.
F	k-fuzzy metric on X.
$*$	Continuous t-norm.
p^k	Abbreviation for $(p_1, p_2, \ldots, p_k) \in (0, \infty)^k$.
Γ	Class of admissible control functions $\gamma : (0, 1) \to \mathbb{R}$.
γ	Strictly increasing continuous control function with divergence at 1.
Ψ	Self-mapping on X.
kFMS	k-fuzzy metric space.

Definition 2.1 ([1]). *A continuous triangular norm (or continuous t-norm) is a binary operation* $* : [0, 1] \times [0, 1] \to [0, 1]$ *satisfying, for all* $a, b, c, d \in [0, 1]$,

(1) $a * b = b * a$ *(commutativity);*

(2) $(a * b) * c = a * (b * c)$ *(associativity);*

(3) $a * 1 = a$ *for all* $a \in [0, 1]$ *(identity law);*

(4) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *(monotonicity);*

(5) $*$ *is continuous (continuity).*

Definition 2.2 ([1]). *Let* X *be a nonempty set,* $*$ *a continuous t-norm, and* F *a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying, for all* $a, b, c \in X$ *and* $s, p > 0$,

(FM0) $F(a, b, 0) = 0$;

(FM1) $F(a, b, p) > 0$;

(FM2) $F(a, b, p) = 1$ *if and only if* $a = b$;

(FM3) $F(a, b, p) = F(b, a, p)$;

(FM4) $F(a, b, p) * F(b, c, s) \leq F(a, c, p + s)$;

(FM5) $F(a, b, \cdot) : (0, \infty) \to [0, 1]$ *is left-continuous.*

Then $(X, F, *)$ *is called a fuzzy metric space.*

Definition 2.3 ([2]). *A triple $(X, F, *)$ is called a fuzzy metric space in the sense of George and Veeramani if it satisfies (FM1)–(FM4) and*

(FM6) $F(a, b, \cdot) : (0, \infty) \to [0, 1]$ *is continuous*

for all $a, b \in X$.

We now recall the notion of a kFMS introduced in [15].

Definition 2.4 ([15]). *Let X be a nonempty set, $*$ a continuous t-norm, $k \in \mathbb{N}$, and F a fuzzy set on $X^2 \times (0, \infty)^k$. An ordered triple $(X, F, *)$ is called a kFMS if the following conditions hold for all $a, b, c \in X$, $p, s > 0$, and $(p_1, \ldots, p_k) \in (0, \infty)^k$:*

(KF1) $F(a, b, p_1, \ldots, p_k) > 0$;

(KF2) $F(a, b, p_1, \ldots, p_k) = 1$ *if and only if* $a = b$;

(KF3) $F(a, b, p_1, \ldots, p_k) = F(b, a, p_1, \ldots, p_k)$;

(KF4) for any $l \in \{1, \ldots, k\}$,

$$F(a, b, p_1, \ldots, p_{l-1}, p, p_{l+1}, \ldots, p_k) * F(b, c, p_1, \ldots, p_{l-1}, s, p_{l+1}, \ldots, p_k)$$

$$\leq F(a, c, p_1, \ldots, p_{l-1}, p + s, p_{l+1}, \ldots, p_k);$$

(KF5) $F(a, b, \cdot) : (0, \infty)^k \to [0, 1]$ *is continuous.*

Remark 2.1. *For $k = 1$, a kFMS reduces to a fuzzy metric space in the sense of George and Veeramani [2]. In this case, the notation $F(a, b, p^1)$ coincides with $F(a, b, p)$.*

For simplicity, we write

$$F(a, b, p_1, \ldots, p_k) = F(a, b, p^k),$$

where p^k denotes the k-tuple $(p_1, \ldots, p_k)$.

Definition 2.5 ([15]). *A kFMS $(X, F, *)$ is called an l-natural kFMS if there exists $l \in \{1, \ldots, k\}$ such that*

$$\lim_{p_l \to \infty} F(a, b, p_1, \ldots, p_k) = 1, \quad \forall\, a, b \in X.$$

Proposition 2.1 ([15]). *Let $(X, F, *)$ be a kFMS, $p, p_1, \ldots, p_k > 0$. If $p_l < p$ for some $l \in \{1, \ldots, k\}$, then*

$$F(a, b, p^k) \leq F(a, b, p_1, \ldots, p_{l-1}, p, p_{l+1}, \ldots, p_k), \quad \forall\, a, b \in X.$$

Remark 2.2 ([15]). *In a kFMS $(X, F, *)$, if*

$$F(a, b, p^k) > 1 - \varepsilon$$

for all $a, b \in X$, $p_1, \ldots, p_k > 0$ and $\varepsilon \in (0, 1)$, then for each $l \in \{1, \ldots, k\}$, there exists $p \in (0, p_l)$ such that

$$F(a, b, p_1, \ldots, p_{l-1}, p, p_{l+1}, \ldots, p_k) > 1 - \varepsilon.$$

Definition 2.6 ([15]). *Let $(X, F, *)$ be a kFMS. A sequence $\{a_n\}$ in X is said to converge to $a \in X$ if, for every $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that*

$$F(a_n, a, p^k) > 1 - \varepsilon, \quad \forall\, n \geq n_0,\ \forall\, p_1, \ldots, p_k > 0.$$

Lemma 2.1 ([15]). *Let $(X, F, *)$ be a kFMS. A sequence $\{a_n\}$ in X converges to $a \in X$ if and only if*

$$\lim_{n\to\infty} F(a_n, a, p^k) = 1, \quad \forall\, p_1, \ldots, p_k > 0.$$

Definition 2.7 ([15]). *Let $(X, F, *)$ be a kFMS and $\{a_n\}$ a sequence in X.*

(1) $\{a_n\}$ is called an F-Cauchy *sequence if, for every $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that*

$$F(a_n, a_m, p^k) > 1 - \varepsilon, \quad \forall\, n, m \geq n_0,\ \forall\, p_1, \ldots, p_k > 0.$$

(2) $\{a_n\}$ is called a G-Cauchy *sequence if*

$$\lim_{n\to\infty} F(a_n, a_{n+p}, p^k) = 1, \quad \forall\, p > 0,\ \forall\, p_1, \ldots, p_k > 0.$$

Since the notions of convergence and completeness in a kFMS are closely related to the induced topology, we now recall the corresponding open balls and the topology generated by the k-fuzzy metric.

Definition 2.8 ([15]). *Let $(X, F, *)$ be a kFMS, let $a \in X$, let $p_1, p_2, \ldots, p_k > 0$, and let $\varepsilon \in (0, 1)$. The open ball centered at a with radius ε and parameters $p^k = (p_1, p_2, \ldots, p_k)$ is defined by*

$$B(a, \varepsilon, p^k) = \{b \in X : F(a, b, p^k) > 1 - \varepsilon\}.$$

Definition 2.9 ([15]). *A subset $U \subseteq X$ is called open if for every $a \in U$, there exist $\varepsilon \in (0, 1)$ and $p_1, p_2, \ldots, p_k > 0$ such that*

$$B(a, \varepsilon, p^k) \subseteq U.$$

The family of all such open sets defines the topology induced by the k-fuzzy metric.

Remark 2.3. *This topology makes convergence and completeness compatible with the fuzzy nearness structure. In particular, a sequence converges to a if and only if its terms eventually enter every open ball centered at a.*

Example 2.1 ([16]). *Let $(\mathbb{R}, d)$ be a metric space, let $\circ$ be the product t-norm given by*

$$a \circ b = ab \qquad \text{or} \qquad a \circ b = \min\{a, b\},$$

let $\omega > 0$, and let k be a positive integer. Define a fuzzy set F on $\mathbb{R}^2 \times (0, \infty)^k$ by

$$F(a, b, p^k) = \frac{\omega \prod_{r=1}^{k} p_r}{\omega \prod_{r=1}^{k} p_r + d(a, b)}$$

for all $a, b \in \mathbb{R}$ and $p_r > 0$. Then F is a k-fuzzy metric on $\mathbb{R}$.

Now consider the sequence $\{Q_s\}$*, where*

$$Q_s = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{s}, \qquad s \in \mathbb{N}.$$

For every fixed $p \in \mathbb{N}$*, we have*

$$F(Q_{s+p}, Q_s, p^k) = \frac{\omega \prod_{r=1}^{k} p_r}{\omega \prod_{r=1}^{k} p_r + |Q_{s+p} - Q_s|} = \frac{\omega \prod_{r=1}^{k} p_r}{\omega \prod_{r=1}^{k} p_r + \frac{1}{s+1} + \cdots + \frac{1}{s+p}}.$$

Hence,

$$\lim_{s\to\infty} F(Q_{s+p}, Q_s, p^k) = 1$$

for every fixed $p \in \mathbb{N}$ *and every* $p^k \in (0,\infty)^k$*. Therefore,* $\{Q_s\}$ *is a G-Cauchy sequence.*

On the other hand, $\{Q_s\}$ *is not an F-Cauchy sequence. Indeed, if it were F-Cauchy, then*

$$F(Q_m, Q_s, p^k) = \frac{\omega \prod_{r=1}^{k} p_r}{\omega \prod_{r=1}^{k} p_r + |Q_m - Q_s|}$$

would imply that $\{Q_s\}$ *is Cauchy in the usual metric on* $\mathbb{R}$*. But the harmonic sequence of partial sums is not Cauchy in* $\mathbb{R}$*, since*

$$|Q_m - Q_s| = \sum_{j=s+1}^{m} \frac{1}{j}$$

does not remain small when m is much larger than s. Consequently, $\{Q_s\}$ *is not F-Cauchy.*

Remark 2.4. *This example shows that the notions of F-Cauchy and G-Cauchy sequences are genuinely different in general. Therefore, both notions are needed in the study of completeness and fixed point results in kFMS.*

Definition 2.10 ([15]). *Let* $(X, F, *)$ *be a kFMS.*

(1) $(X, F, *)$ *is called* F-complete *if every F-Cauchy sequence in X converges to some point of X.*

(2) $(X, F, *)$ *is called* G-complete *if every G-Cauchy sequence in X converges to some point of X.*

Definition 2.11 ([12]). *Let* Γ *denote the class of all functions* $\gamma : (0,1) \to \mathbb{R}$ *such that*

(1) γ *is strictly increasing;*

(2) γ *is continuous;*

(3) for every sequence $\{u_n\} \subset (0,1)$*,*

$$\lim_{n\to\infty} u_n = 1 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \gamma(u_n) = +\infty.$$

Remark 2.5. *Typical examples of functions in* Γ *are*

$$\gamma_1(u) = \frac{1}{1-u}, \qquad \gamma_2(u) = \frac{1}{1-u} + u, \qquad \gamma_3(u) = \frac{1}{1-u^2}$$

for all $u \in (0,1)$.

3. k-fuzzy γ-contractions

In this section, we introduce the notion of a k-fuzzy γ-contraction and establish the corresponding fixed point theorem.

Definition 3.1. *Let $(X, F, *)$ be a kFMS. A self-mapping $\Psi : X \to X$ is called a* k-fuzzy γ-contraction *if there exist a constant $\delta \in (0, 1)$ and a function $\gamma \in \Gamma$ such that, for all $\eta, \zeta \in X$ and all $p_1, p_2, \ldots, p_k > 0$,*

$$F(\Psi\eta, \Psi\zeta, p^k) < 1 \quad \Longrightarrow \quad \gamma(F(\Psi\eta, \Psi\zeta, p^k)) \geq \gamma(F(\eta, \zeta, p^k)) + \delta. \tag{3.1}$$

Remark 3.1. *From* (3.1)*, it is easy to conclude that every k-fuzzy γ-contraction is a contractive mapping in the fuzzy metric sense. Indeed, whenever $\Psi x \neq \Psi y$, the defining inequality implies*

$$\gamma(F(\Psi x, \Psi y, p^k)) > \gamma(F(x, y, p^k)),$$

and since γ is strictly increasing, we obtain

$$F(\Psi x, \Psi y, p^k) > F(x, y, p^k)$$

for all $x, y \in X$ and all $p^k \in (0, \infty)^k$ with $\Psi x \neq \Psi y$. Consequently, every k-fuzzy γ-contraction is a continuous mapping.

Theorem 3.1 (Existence and uniqueness for k-fuzzy γ-contraction)**.** *Let $(X, F, *)$ be a G-complete kFMS, and let $\Psi : X \to X$ be a k-fuzzy γ-contraction. Then Ψ has a unique fixed point $z \in X$.*

Proof. We divide the proof into four steps.

Step 1: Picard sequence and convergence of successive terms.

Let $\eta_0 \in X$ be arbitrary, and define the Picard sequence $\{\eta_n\}$ by

$$\eta_{n+1} = \Psi\eta_n, \qquad n = 0, 1, 2, \ldots. \tag{3.2}$$

If there exists $n_0 \in \mathbb{N}$ such that $\eta_{n_0} = \eta_{n_0+1}$, then η_{n_0} is a fixed point of Ψ, and the proof is complete. Hence, we may assume that

$$\eta_n \neq \eta_{n+1} \qquad \text{for all } n \geq 0. \tag{3.3}$$

By the definition of a k-fuzzy γ-contraction, we have

$$F(\Psi\eta_{n-1}, \Psi\eta_n, p^k) < 1 \quad \Longrightarrow \quad \gamma(F(\Psi\eta_{n-1}, \Psi\eta_n, p^k)) \geq \gamma(F(\eta_{n-1}, \eta_n, p^k)) + \delta \tag{3.4}$$

for every $n \geq 1$ and every $p^k \in (0, \infty)^k$.

Since $\eta_{n+1} = \Psi\eta_n$, (3.4) gives

$$\gamma(F(\eta_n, \eta_{n+1}, p^k)) \geq \gamma(F(\eta_{n-1}, \eta_n, p^k)) + \delta. \tag{3.5}$$

Repeating (3.5) inductively, we obtain

$$\begin{aligned}\gamma(F(\eta_n, \eta_{n+1}, p^k)) &\geq \gamma(F(\eta_{n-1}, \eta_n, p^k)) + \delta \\ &\geq \gamma(F(\eta_{n-2}, \eta_{n-1}, p^k)) + 2\delta\end{aligned}$$

$$\vdots$$
$$\geq \gamma(F(\eta_0, \eta_1, p^k)) + n\delta. \tag{3.6}$$

Letting $n \to \infty$ in (3.6), we obtain

$$\lim_{n\to\infty} \gamma(F(\eta_n, \eta_{n+1}, p^k)) = +\infty. \tag{3.7}$$

Since $\gamma \in \Gamma$, it follows that

$$\lim_{n\to\infty} F(\eta_n, \eta_{n+1}, p^k) = 1 \tag{3.8}$$

for all $p^k \in (0, \infty)^k$.

Step 2: The sequence $\{\eta_n\}$ is G-Cauchy.

Suppose, to the contrary, that $\{\eta_n\}$ is not G-Cauchy. Then there exist $\varepsilon \in (0, 1)$, a vector $p^k \in (0, \infty)^k$, and two sequences of integers $\{m(s)\}$ and $\{r(s)\}$ such that

$$m(s) > r(s) > s \qquad \text{and} \qquad F(\eta_{m(s)}, \eta_{r(s)}, p^k) \leq 1 - \varepsilon \tag{3.9}$$

for all $s \in \mathbb{N}$.

We may assume that $m(s)$ is the least integer exceeding $r(s)$ satisfying (3.9). Then

$$F(\eta_{m(s)-1}, \eta_{r(s)}, p^k) > 1 - \varepsilon. \tag{3.10}$$

Using the fuzzy triangle inequality (KF4), we get

$$\begin{aligned} F(\eta_{r(s)}, \eta_{m(s)}, p^k) &\geq F(\eta_{m(s)-1}, \eta_{m(s)}, p^k) * F(\eta_{m(s)-1}, \eta_{r(s)}, p^k) \\ &\geq F(\eta_{m(s)-1}, \eta_{m(s)}, p^k) * (1 - \varepsilon). \end{aligned} \tag{3.11}$$

Now, taking the limit as $s \to \infty$ in (3.11) and using (3.8), we obtain

$$\lim_{s\to\infty} F(\eta_{r(s)}, \eta_{m(s)}, p^k) = 1 - \varepsilon. \tag{3.12}$$

Next, applying (3.4) to the pair $(\eta_{m(s)-1}, \eta_{r(s)-1})$, we obtain

$$\gamma(F(\eta_{m(s)}, \eta_{r(s)}, p^k)) \geq \gamma(F(\eta_{m(s)-1}, \eta_{r(s)-1}, p^k)) + \delta. \tag{3.13}$$

Passing to the limit as $s \to \infty$ in (3.13) and using (3.12) together with the continuity of γ, we get

$$\gamma(1 - \varepsilon) \geq \gamma(1 - \varepsilon) + \delta, \tag{3.14}$$

which is impossible. Therefore, $\{\eta_n\}$ is a G-Cauchy sequence.

Since $(X, F, *)$ is G-complete, there exists $z \in X$ such that

$$\eta_n \to z. \tag{3.15}$$

Step 3: The limit point is a fixed point of Ψ.

We show that z is a fixed point of Ψ. Since $\eta_n \to z$, we have

$$F(\eta_n, z, p^k) \to 1 \qquad \text{as } n \to \infty \tag{3.16}$$

for every $p^k \in (0, \infty)^k$.

Assume, for contradiction, that $\Psi z \neq z$. Then

$$F(z, \Psi z, p^k) < 1. \tag{3.17}$$

Applying (3.4) to the pair (η_n, z), we obtain

$$\gamma\big(F(\eta_{n+1}, \Psi z, p^k)\big) \geq \gamma\big(F(\eta_n, z, p^k)\big) + \delta \tag{3.18}$$

for all $n \in \mathbb{N}$ such that $F(\eta_{n+1}, \Psi z, p^k) < 1$.

Now, by (3.16), we have

$$\lim_{n\to\infty} F(\eta_n, z, p^k) = 1.$$

Hence, taking the limit as $n \to \infty$ in (3.18) and using the continuity of γ, we obtain

$$\gamma\big(F(z, \Psi z, p^k)\big) \geq \gamma\big(F(z, \Psi z, p^k)\big) + \delta, \tag{3.19}$$

which is a contradiction. Therefore,

$$\Psi z = z. \tag{3.20}$$

Step 4: Uniqueness of the fixed point.

Let $z_1, z_2 \in X$ be two fixed points of Ψ. Suppose that $z_1 \neq z_2$. Then

$$\Psi z_1 = z_1 \qquad \text{and} \qquad \Psi z_2 = z_2.$$

Applying (3.4) to the pair (z_1, z_2), we get

$$\gamma\big(F(z_1, z_2, p^k)\big) = \gamma\big(F(\Psi z_1, \Psi z_2, p^k)\big) \geq \gamma\big(F(z_1, z_2, p^k)\big) + \delta, \tag{3.21}$$

which is impossible. Therefore, $z_1 = z_2$.

Hence, Ψ has a unique fixed point in X. □

Example 3.1. *Let $X = [0, 1]$, let $*$ be the product t-norm, and define*

$$F(x, y, p_1, p_2) = \frac{p_1 + p_2}{p_1 + p_2 + |x - y|} \tag{3.22}$$

for all $x, y \in X$ and all $p_1, p_2 > 0$. Then $(X, F, *)$ is a 2-fuzzy metric space.

Define

$$\gamma(u) = \frac{1}{1-u}, \qquad u \in (0, 1), \tag{3.23}$$

and let $\Psi : X \to X$ be given by

$$\Psi(x) = \frac{x}{2}. \tag{3.24}$$

We claim that Ψ is a 2-fuzzy γ-contraction.

Let $x, y \in X$ with $x \neq y$, and put

$$s = p_1 + p_2. \tag{3.25}$$

Then from (3.22)*, we obtain*

$$F(x, y, p_1, p_2) = \frac{s}{s + |x - y|}. \tag{3.26}$$

Also, using (3.24)*, we have*

$$F(\Psi x, \Psi y, p_1, p_2) = \frac{s}{s + \left|\frac{x}{2} - \frac{y}{2}\right|} = \frac{s}{s + \frac{|x-y|}{2}}. \tag{3.27}$$

Since $x \neq y$, we have $|x - y| > 0$, and thus

$$0 < F(\Psi x, \Psi y, p_1, p_2) < 1. \tag{3.28}$$

Now, by (3.23) *and* (3.27)*,*

$$\begin{aligned} \gamma(F(\Psi x, \Psi y, p_1, p_2)) &= \frac{1}{1 - \frac{s}{s + \frac{|x-y|}{2}}} \\ &= \frac{s + \frac{|x-y|}{2}}{\frac{|x-y|}{2}} \\ &= \frac{2s}{|x - y|} + 1. \end{aligned} \tag{3.29}$$

Similarly, by (3.23) *and* (3.26)*,*

$$\begin{aligned} \gamma(F(x, y, p_1, p_2)) &= \frac{1}{1 - \frac{s}{s+|x-y|}} \\ &= \frac{s + |x - y|}{|x - y|} \\ &= \frac{s}{|x - y|} + 1. \end{aligned} \tag{3.30}$$

Subtracting (3.30) *from* (3.29)*, we get*

$$\gamma(F(\Psi x, \Psi y, p_1, p_2)) - \gamma(F(x, y, p_1, p_2)) = \frac{s}{|x - y|}. \tag{3.31}$$

Since $s > 0$ and $|x - y| > 0$, it follows from (3.31) *that*

$$\gamma(F(\Psi x, \Psi y, p_1, p_2)) \geq \gamma\big(F(x, y, p_1, p_2)\big) + \delta \tag{3.32}$$

for any fixed constant

$$\delta \in \left(0, \frac{s}{|x - y|}\right). \tag{3.33}$$

Therefore, by (3.32)*, the mapping Ψ satisfies the defining condition of a 2-fuzzy γ-contraction.*

Hence, all the hypotheses of Theorem 3.1 are satisfied, and so Ψ has a unique fixed point in X. Finally, from (3.24)*, we have*

$$\Psi(x) = x \quad \Longleftrightarrow \quad \frac{x}{2} = x,$$

which gives $x = 0$. Therefore, the unique fixed point of Ψ is 0*.*

4. k-fuzzy γ-weak contractions

In this section, we introduce a weak version of the k-fuzzy γ-contraction.

Definition 4.1. *Let $(X, F, *)$ be a kFMS. A self-mapping $\Psi : X \to X$ is called a k-*fuzzy γ-weak contraction *if there exist a constant $\delta \in (0, 1)$ and a function $\gamma \in \Gamma$ such that, for all $\eta, \zeta \in X$ and all $p_1, p_2, \dots, p_k > 0$,*

$$F(\Psi\eta, \Psi\zeta, p^k) < 1 \Longrightarrow \gamma(F(\Psi\eta, \Psi\zeta, p^k)) \geq \gamma\Big(\min\{F(\eta, \zeta, p^k), F(\eta, \Psi\eta, p^k), F(\zeta, \Psi\zeta, p^k)\}\Big) + \delta. \quad (4.1)$$

Remark 4.1. *Every k-fuzzy γ-contraction is a k-fuzzy γ-weak contraction. However, the converse is not true in general, because the weak condition involves the minimum of three fuzzy proximities,*

$$F(x, y, p^k), \qquad F(x, \Psi x, p^k), \qquad F(y, \Psi y, p^k),$$

and therefore allows a wider class of mappings than the strong γ-contraction condition. Hence, the class of k-fuzzy γ-weak contractions strictly contains the class of k-fuzzy γ-contractions.

Example 4.1. *Let $X = [0, 1]$, let $*$ be the product t-norm, and define*

$$F(x, y, p_1, p_2) = \frac{p_1 + p_2}{p_1 + p_2 + |x - y|}$$

for all $x, y \in X$ and all $p_1, p_2 > 0$. Let

$$\gamma(u) = \frac{1}{1 - u^2}, \qquad u \in (0, 1),$$

and define $\Psi : X \to X$ by

$$\Psi(x) = \frac{x}{3}.$$

Then Ψ is a 2-fuzzy γ-weak contraction.

Indeed, for every $x, y \in X$ with $x \neq y$, putting $s = p_1 + p_2$, we have

$$F(x, y, p_1, p_2) = \frac{s}{s + |x - y|}, \qquad F(\Psi x, \Psi y, p_1, p_2) = \frac{s}{s + \frac{|x-y|}{3}}.$$

Hence,

$$F(\Psi x, \Psi y, p_1, p_2) > F(x, y, p_1, p_2),$$

and therefore

$$\gamma(F(\Psi x, \Psi y, p_1, p_2)) > \gamma(F(x, y, p_1, p_2)).$$

Hence, this example illustrates the weak contractive behavior of Ψ in the sense of the definition above.

On the other hand, Ψ does not satisfy the strong k-fuzzy γ-contraction condition in general, since the weak condition is verified through the minimum of three fuzzy proximities and may hold even when the direct strong inequality fails. Thus, this example shows that the class of k-fuzzy γ-weak contractions is strictly larger than the class of k-fuzzy γ-contractions.

This example illustrates that the weak condition may hold even when the strong contractive structure is not available, confirming that the class of k-fuzzy γ-weak contractions properly extends the class of k-fuzzy γ-contractions.

Theorem 4.1 (Existence and uniqueness for k-fuzzy γ-weak contraction)**.** *Let $(X, F, *)$ be a G-complete k-FMS, and let $\Psi : X \to X$ be a k-fuzzy γ-weak contraction. Then Ψ has a unique fixed point in X.*

Proof. We divide the proof into four steps.

Step 1: Picard sequence and convergence of successive terms.

Let $\eta_0 \in X$ be arbitrary, and define the Picard sequence $\{\eta_n\}$ by

$$\eta_{n+1} = \Psi\eta_n, \qquad n = 0, 1, 2, \ldots. \tag{4.2}$$

If there exists $n_0 \in \mathbb{N}$ such that $\eta_{n_0} = \eta_{n_0+1}$, then η_{n_0} is a fixed point of Ψ, and the proof is complete. Hence, we may assume that

$$\eta_n \neq \eta_{n+1} \qquad \text{for all } n \geq 0. \tag{4.3}$$

Since Ψ is a k-fuzzy γ-weak contraction, for every $n \geq 1$, we have

$$\gamma(F(\Psi\eta_{n-1}, \Psi\eta_n, p^k)) \geq \gamma\Big(\min\{F(\eta_{n-1}, \eta_n, p^k), F(\eta_{n-1}, \Psi\eta_{n-1}, p^k), F(\eta_n, \Psi\eta_n, p^k)\}\Big) + \delta \tag{4.4}$$

for all $p^k \in (0, \infty)^k$ such that $F(\Psi\eta_{n-1}, \Psi\eta_n, p^k) < 1$.

Using the Picard relation (4.2), we obtain

$$\gamma(F(\eta_n, \eta_{n+1}, p^k)) \geq \gamma\Big(\min\{F(\eta_{n-1}, \eta_n, p^k), F(\eta_{n-1}, \eta_n, p^k), F(\eta_n, \eta_{n+1}, p^k)\}\Big) + \delta. \tag{4.5}$$

Thus,

$$\gamma(F(\eta_n, \eta_{n+1}, p^k)) \geq \gamma\Big(\min\{F(\eta_{n-1}, \eta_n, p^k), F(\eta_n, \eta_{n+1}, p^k)\}\Big) + \delta. \tag{4.6}$$

If there exists $n \in \mathbb{N}$ such that

$$\min\{F(\eta_{n-1}, \eta_n, p^k), F(\eta_n, \eta_{n+1}, p^k)\} = F(\eta_n, \eta_{n+1}, p^k),$$

then (4.6) yields

$$\gamma(F(\eta_n, \eta_{n+1}, p^k)) \geq \gamma(F(\eta_n, \eta_{n+1}, p^k)) + \delta, \tag{4.7}$$

a contradiction. Therefore,

$$\min\{F(\eta_{n-1}, \eta_n, p^k), F(\eta_n, \eta_{n+1}, p^k)\} = F(\eta_{n-1}, \eta_n, p^k) \tag{4.8}$$

for all $n \geq 1$.

Consequently, from (4.6) and (4.8), we get

$$\gamma(F(\eta_n, \eta_{n+1}, p^k)) \geq \gamma(F(\eta_{n-1}, \eta_n, p^k)) + \delta \tag{4.9}$$

for all $n \geq 1$. Iterating (4.9), we obtain

$$\begin{aligned} \gamma(F(\eta_n, \eta_{n+1}, p^k)) &\geq \gamma(F(\eta_{n-1}, \eta_n, p^k)) + \delta \\ &\geq \gamma(F(\eta_{n-2}, \eta_{n-1}, p^k)) + 2\delta \end{aligned}$$

$$\vdots$$

$$\geq \gamma(F(\eta_0, \eta_1, p^k)) + n\delta. \tag{4.10}$$

Letting $n \to \infty$ in (4.10), we obtain

$$\lim_{n\to\infty} \gamma(F(\eta_n, \eta_{n+1}, p^k)) = +\infty. \tag{4.11}$$

Since $\gamma \in \Gamma$, it follows that

$$\lim_{n\to\infty} F(\eta_n, \eta_{n+1}, p^k) = 1 \tag{4.12}$$

for all $p^k \in (0, \infty)^k$.

Step 2: The sequence $\{\eta_n\}$ is G-Cauchy.

Suppose, to the contrary, that $\{\eta_n\}$ is not G-Cauchy. Then there exist $\varepsilon \in (0, 1)$, a vector $p^k \in (0, \infty)^k$, and sequences of integers $\{m(s)\}$ and $\{r(s)\}$ such that

$$m(s) > r(s) > s \qquad \text{and} \qquad F(\eta_{m(s)}, \eta_{r(s)}, p^k) \leq 1 - \varepsilon \tag{4.13}$$

for all $s \in \mathbb{N}$.

We may assume that $m(s)$ is the least integer greater than $r(s)$ satisfying (4.13). Then

$$F(\eta_{m(s)-1}, \eta_{r(s)}, p^k) > 1 - \varepsilon. \tag{4.14}$$

Using the fuzzy triangle inequality, we obtain

$$\begin{aligned} F(\eta_{r(s)}, \eta_{m(s)}, p^k) &\geq F(\eta_{m(s)-1}, \eta_{m(s)}, p^k) * F(\eta_{m(s)-1}, \eta_{r(s)}, p^k) \\ &\geq F(\eta_{m(s)-1}, \eta_{m(s)}, p^k) * (1 - \varepsilon). \end{aligned} \tag{4.15}$$

Taking the limit as $s \to \infty$ in (4.15) and using (4.12), we obtain

$$\lim_{s\to\infty} F(\eta_{r(s)}, \eta_{m(s)}, p^k) = 1 - \varepsilon. \tag{4.16}$$

Now, applying (4.4) to the pair $(\eta_{m(s)-1}, \eta_{r(s)-1})$, we get

$$\gamma(F(\eta_{m(s)}, \eta_{r(s)}, p^k)) \geq \gamma\Big(\min\{F(\eta_{m(s)-1}, \eta_{r(s)-1}, p^k), F(\eta_{m(s)-1}, \eta_{m(s)}, p^k), F(\eta_{r(s)-1}, \eta_{r(s)}, p^k)\}\Big) + \delta. \tag{4.17}$$

Passing to the limit as $s \to \infty$ and using (4.16) together with the continuity of γ, we obtain

$$\gamma(1 - \varepsilon) \geq \gamma(1 - \varepsilon) + \delta, \tag{4.18}$$

which is a contradiction. Therefore, $\{\eta_n\}$ is a G-Cauchy sequence.

Since $(X, F, *)$ is G-complete, there exists $z \in X$ such that

$$\eta_n \to z. \tag{4.19}$$

Step 3: The limit point is a fixed point of Ψ.

We now prove that z is a fixed point of Ψ. Since $\eta_n \to z$, we have

$$F(\eta_n, z, p^k) \to 1 \qquad \text{as } n \to \infty \tag{4.20}$$

for every $p^k \in (0,\infty)^k$.

Assume, for contradiction, that $\Psi z \neq z$. Then

$$F(z, \Psi z, p^k) < 1. \tag{4.21}$$

Applying (4.4) to the pair (η_n, z), we obtain

$$\gamma(F(\eta_{n+1}, \Psi z, p^k)) \geq \gamma\Big(\min\{F(\eta_n, z, p^k), F(\eta_n, \eta_{n+1}, p^k), F(z, \Psi z, p^k)\}\Big) + \delta \tag{4.22}$$

for all sufficiently large n such that $F(\eta_{n+1}, \Psi z, p^k) < 1$.

Now, by (4.20) and (4.12), we have

$$F(\eta_n, z, p^k) \to 1 \qquad \text{and} \qquad F(\eta_n, \eta_{n+1}, p^k) \to 1.$$

Hence, if $F(z, \Psi z, p^k) < 1$, then for sufficiently large n, the minimum in (4.22) becomes $F(z, \Psi z, p^k)$. Therefore,

$$\gamma(F(\eta_{n+1}, \Psi z, p^k)) \geq \gamma(F(z, \Psi z, p^k)) + \delta \tag{4.23}$$

for all sufficiently large n.

Taking the limit as $n \to \infty$ in (4.23) and using the continuity of γ, we obtain

$$\gamma(F(z, \Psi z, p^k)) \geq \gamma(F(z, \Psi z, p^k)) + \delta, \tag{4.24}$$

which is a contradiction. Hence,

$$\Psi z = z. \tag{4.25}$$

Step 4: Uniqueness of the fixed point.

Let $z_1, z_2 \in X$ be two fixed points of Ψ. Suppose that $z_1 \neq z_2$. Then

$$\Psi z_1 = z_1 \qquad \text{and} \qquad \Psi z_2 = z_2.$$

Applying (4.4) to the pair (z_1, z_2), we get

$$\begin{aligned}\gamma(F(z_1, z_2, p^k)) &= \gamma(F(\Psi z_1, \Psi z_2, p^k)) \\ &\geq \gamma\Big(\min\{F(z_1, z_2, p^k), F(z_1, \Psi z_1, p^k), F(z_2, \Psi z_2, p^k)\}\Big) + \delta \\ &= \gamma(F(z_1, z_2, p^k)) + \delta,\end{aligned} \tag{4.26}$$

which is impossible. Therefore, $z_1 = z_2$.

Hence, Ψ has a unique fixed point in X. □

Example 4.2. *Let $X = [0,1]$, let $*$ be the product t-norm, and let F be defined as in Example 3.1. Define*

$$\gamma(u) = \frac{1}{1-u^2}, \qquad u \in (0,1),$$

and let $\Psi : X \to X$ be given by

$$\Psi(x) = \frac{x}{3}.$$

We shall show that Ψ is a 2-fuzzy γ-weak contraction.

Let $x, y \in X$ with $x \neq y$, and put

$$s = p_1 + p_2.$$

Then, by the definition of F, we have

$$F(x, y, p_1, p_2) = \frac{s}{s + |x - y|}.$$

Also,

$$F(\Psi x, \Psi y, p_1, p_2) = \frac{s}{s + \left|\frac{x}{3} - \frac{y}{3}\right|} = \frac{s}{s + \frac{|x-y|}{3}}.$$

Since $x \neq y$, it follows that $|x - y| > 0$, and hence

$$0 < F(\Psi x, \Psi y, p_1, p_2) < 1.$$

Now, using the definition of γ, we obtain

$$\gamma(F(\Psi x, \Psi y, p_1, p_2)) = \frac{1}{1 - \left(\frac{s}{s + \frac{|x-y|}{3}}\right)^2},$$

$$\gamma(F(x, y, p_1, p_2)) = \frac{1}{1 - \left(\frac{s}{s+|x-y|}\right)^2}.$$

Because

$$\frac{s}{s + \frac{|x-y|}{3}} > \frac{s}{s + |x - y|},$$

we have

$$\gamma(F(\Psi x, \Psi y, p_1, p_2)) > \gamma(F(x, y, p_1, p_2)).$$

Therefore, there exists a constant $\delta \in (0, 1)$ such that

$$\gamma(F(\Psi x, \Psi y, p_1, p_2)) \geq \gamma(F(x, y, p_1, p_2)) + \delta.$$

Hence, Ψ satisfies the defining condition of a 2-fuzzy γ-weak contraction.

Therefore, all the hypotheses of Theorem 4.1 are satisfied, and so Ψ has a unique fixed point in X. Finally,

$$\Psi(x) = x \iff \frac{x}{3} = x,$$

which implies $x = 0$. Thus, the unique fixed point of Ψ is 0.

Remark 4.2. *Every k-fuzzy γ-contraction is a particular case of a k-fuzzy γ-weak contraction. Indeed, if $\Psi : X \to X$ satisfies*

$$\gamma(F_Y(\Psi\eta, \Psi\zeta, p^k)) \geq \gamma(F_Y(\eta, \zeta, p^k)) + \delta, \qquad \eta, \zeta \in X,$$

then

$$\min\{F_Y(\eta, \zeta, p^k), F_Y(\eta, \Psi\eta, p^k), F_Y(\zeta, \Psi\zeta, p^k)\} \leq F_Y(\eta, \zeta, p^k).$$

Since γ is strictly increasing, we have

$$\gamma\Big(\min\{F_Y(\eta, \zeta, p^k), F_Y(\eta, \Psi\eta, p^k), F_Y(\zeta, \Psi\zeta, p^k)\}\Big) \leq \gamma(F_Y(\eta, \zeta, p^k)).$$

Consequently,

$$\gamma(F_Y(\Psi\eta, \Psi\zeta, p^k)) \geq \gamma(F_Y(\eta, \zeta, p^k)) + \delta \geq \gamma\Big(\min\{F_Y(\eta, \zeta, p^k), F_Y(\eta, \Psi\eta, p^k), F_Y(\zeta, \Psi\zeta, p^k)\}\Big) + \delta.$$

Thus, Ψ satisfies the γ-weak contraction condition with the same δ.

5. Applications of the proposed k-fuzzy fixed point framework

We will present two illustrative applications of the proposed framework in uncertain dynamical settings. The aim is not to introduce new theoretical results, but to show how the notions of the k-fuzzy γ-contraction and k-fuzzy γ-weak contraction can be interpreted in practical iterative models. The examples are intentionally reduced-order and conceptual so that the emphasis remains on the mathematical structure of the update process rather than on large-scale simulation details.

Example 5.1 (Traffic-signal timing as a k-fuzzy γ-contraction)**.** *Let*

$$X = [10, 60]^2,$$

where each point

$$x = (t_1, t_2) \in X$$

represents a signal timing vector, with t_1 and t_2 denoting the green-light durations at two adjacent intersections. In practice, such timing variables are influenced by detector counts, queue estimates, and travel-time observations. Since these sources are not equally reliable, define the weighted discrepancy measure

$$D(x, y) = w_1|x_1 - y_1| + w_2|x_2 - y_2|,$$

where $w_1, w_2 > 0$ are reliability weights.

Fix $k \in \mathbb{N}$, and let

$$p^k = (p_1, \ldots, p_k) \in (0, \infty)^k.$$

Define

$$M(x, y, p^k) = \exp\left(-\sum_{i=1}^{k} \frac{D(x, y)}{p_i}\right), \qquad x, y \in X.$$

Then $(X, M, *)$, with the product t-norm, provides a natural k-fuzzy metric setting for the timing model.

Consider the reference timing plan

$$h = (30, 40) \in X,$$

and define

$$\Psi(x) = (1 - \alpha)x + \alpha h, \qquad 0 < \alpha < 1.$$

Then

$$D(\Psi x, \Psi y) = (1 - \alpha)D(x, y)$$

so that

$$M(\Psi x, \Psi y, p^k) = \exp\left(-\sum_{i=1}^{k} \frac{(1 - \alpha)D(x, y)}{p_i}\right).$$

Hence, the map Ψ reduces the discrepancy between timing vectors at each iteration and therefore may be viewed as a k-fuzzy γ-contraction for a suitable increasing function γ. In this interpretation, the fixed point of Ψ corresponds to a stable timing configuration toward which the iterative adjustment process converges.

This example models a regular correction mechanism in which the current signal plan is gradually moved toward a target plan. The role of the k-fuzzy metric is to measure nearness across multiple uncertainty scales, such as short-term and long-term traffic fluctuations.

Example 5.2 (UAV trajectory tracking as a k-fuzzy γ-weak contraction)**.** *Let*

$$X = [-1, 1] \times [-1, 1] \times [-\pi/4, \pi/4],$$

where a point

$$x = (\tilde{x}, \tilde{y}, \tilde{\phi}) \in X$$

represents the tracking error in lateral position, altitude, and roll angle of a reduced unmanned aerial vehicle (UAV) model. These error variables are affected by wind disturbance, sensor noise, actuator imperfections, and parameter mismatch. To reflect such uncertainty, define

$$D(x, z) = w_1|\tilde{x} - \hat{x}| + w_2|\tilde{y} - \hat{y}| + w_3|\tilde{\phi} - \hat{\phi}|,$$

where

$$z = (\hat{x}, \hat{y}, \hat{\phi}) \in X$$

and $w_1, w_2, w_3 > 0$ *are reliability weights.*

For

$$p^k = (p_1, \dots, p_k) \in (0, \infty)^k,$$

define

$$F(x, z, p^k) = \exp\left(-\sum_{i=1}^{k} \frac{D(x, z)}{p_i}\right).$$

Then $(X, F, *)$*, with the product t-norm, forms a k-fuzzy metric environment for the reduced tracking-error dynamics.*

Let

$$h = (0, 0, 0)$$

denote the desired zero-error equilibrium, and consider the perturbed update rule

$$\Psi_n(x) = (1 - \alpha_n)x + \alpha_n h + \varepsilon_n, \qquad 0 < \alpha_n < 1,$$

where ε_n *is a bounded perturbation term representing disturbance and temporary uncertainty. Assume that*

$$\alpha_n \to \alpha \in (0, 1) \qquad \text{and} \qquad \varepsilon_n \to 0 \quad \text{as } n \to \infty.$$

Because of the perturbation term, the decrease in error is not purely governed by a single direct contraction inequality. Instead, the closeness between successive iterates is more naturally described through a weaker contractive mechanism involving the current state, its image, and the next correction step. For this reason, the UAV update process is more suitably interpreted as a k-fuzzy γ*-weak contraction rather than a strict k-fuzzy* γ*-contraction.*

The fixed point of the limiting map

$$\Psi(x) = (1 - \alpha)x + \alpha h$$

is

$$x^* = h = (0, 0, 0),$$

which corresponds to a stable tracking configuration. Thus, the k-fuzzy γ*-weak contraction framework captures the practical fact that convergence toward the desired flight state may still occur even when the control loop is affected by uncertainty and non-ideal transient perturbations.*

Remark 5.1. *The two examples above are intentionally formulated at different levels of contractive strength. The traffic-signal timing model represents a more regular iterative correction process and is therefore naturally associated with a k-fuzzy γ-contraction. In contrast, the UAV tracking model includes bounded perturbations and transient uncertainty, which makes the weaker notion of k-fuzzy γ-weak contraction more appropriate. In this way, the examples illustrate not only the applicability of the proposed framework, but also the structural relation between the two classes of mappings.*

6. Conclusions

In this paper, we developed a fixed point theory for γ-type mappings in k-FMSs. We proved the existence and uniqueness of fixed points for k-fuzzy γ-contractions on G-complete spaces and extended the result to the broader class of k-fuzzy γ-weak contractions. This framework generalizes several known results on fuzzy contractive mappings and enlarges the range of admissible operators in k-fuzzy metric spaces. The examples illustrate the applicability of the theory, while future research may address common fixed points, stochastic extensions, and applications to nonlinear problems.

Author contributions

Buthinah Bin Dehaish, Sara Alshehri, Jawaher Aali: Conceptualization, Formal analysis, Writing–original draft, Writing–review and editing. Each author has read and approved the final version of the paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no competing interests.

References

1. I. Kramosil, J. Michálek, Fuzzy metrics and statistical metric spaces, *Kybernetika*, **11** (1975), 336–344.
2. A. George, P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets Syst.*, **64** (1994), 395–399. https://doi.org/10.1016/0165-0114(94)90162-7
3. M. Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy Sets Syst.*, **27** (1988), 385–389. https://doi.org/10.1016/0165-0114(88)90064-4

4. D. Miheţ, Fuzzy Ψ-contractive mappings in non-Archimedean fuzzy metric spaces, *Fuzzy Sets Syst.*, **159** (2008), 739–744. https://doi.org/10.1016/j.fss.2007.07.006
5. D. Miheţ, A class of contractions in fuzzy metric spaces, *Fuzzy Sets Syst.*, **161** (2010), 1131–1137. https://doi.org/10.1016/j.fss.2009.09.018
6. P. Salimi, C. Vetro, P. Vetro, Some new fixed point results in non-Archimedean fuzzy metric spaces, *Nonlinear Anal. Model. Control*, **18** (2013), 344–358. https://doi.org/10.15388/NA.18.3.14014
7. S. Chauhan, S. Bhatnagar, S. Radenović, Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces, *Le Matematiche*, **68** (2013), 87–98. https://doi.org/10.4418/2013.68.1.8
8. C. Vetro, Fixed points in weak non-Archimedean fuzzy metric spaces, *Fuzzy Sets Syst.*, **162** (2011), 84–90. https://doi.org/10.1016/j.fss.2010.09.018
9. D. Turkoğlu, M. Sangurlu, Fixed point theorems for fuzzy ψ-contractive mappings in fuzzy metric spaces, *J. Intell. Fuzzy Syst.*, **26** (2014), 137–142. https://doi.org/10.3233/IFS-120721
10. M. Sangurlu, D. Turkoğlu, Fixed point theorems for $(\psi \circ \varphi)$-contractions in fuzzy metric spaces, *J. Nonlinear Sci. Appl.*, **8** (2015), 687–694. https://doi.org/10.22436/jnsa.008.05.21
11. S. Sedghi, N. Shobe, T. Dosenović, S. Radenović, Suzuki-type of common fixed point theorems in fuzzy metric spaces, *Math. Slovaca*, **68** (2018), 451–462. https://doi.org/10.1515/ms-2017-0115
12. M. Sangurlu Sezan, Fixed point theorem for new type contractive mappings, *J. Funct. Spaces*, **2019** (2019), 2153563. https://doi.org/10.1155/2019/2153563
13. F. M. Azmi, Fixed-point results for $(\alpha - \psi)$-fuzzy contractive mappings on fuzzy double-controlled metric spaces, *Symmetry*, **15** (2023), 716. https://doi.org/10.3390/sym15030716
14. S. Furqan, N. Saleem, S. Sessa, Fuzzy n-controlled metric space, *Int. J. Anal. Appl.*, **21** (2023), 101. https://doi.org/10.28924/2291-8639-21-2023-101
15. D. Gopal , S. Wutiphol, S. Abhay , S. Satish, The investigation of k-fuzzy metric spaces with the contraction principle in such spaces, *Soft Comput.*, **27** (2023), 11081–11089. https://doi.org/10.1007/s00500-023-07946-y
16. M. Nazam, S. Attique, A. Hussain, H. H. Alsulami, Exploring fixed-point theorems in k-fuzzy metric spaces: A comprehensive study, *Axioms*, **13** (2024), 558. https://doi.org/10.3390/axioms13080558
17. M. Iqbal, A. Batool, A. Hussain, H. H. Alsulami, Fuzzy fixed point theorems in S-metric spaces and applications to navigation and control systems, *Axioms*, **13** (2024), 650. https://doi.org/10.3390/axioms13090650

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