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Research article

Integrable dynamics and compatibility conditions in geometric curve flows

Sameh Shenawy[1], **Bang-Yen Chen**[2], **Elsayed I. Mahmoud**[3,*], **Nasser Bin Turki**[4] **and Abdallah A. Syied**[3,5]

[1] Basic Science Department, Modern Academy for Engineering and Technology, Maadi 4411602, Egypt

[2] Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

[3] Department of Mathematics, Faculty of Science, Zagazig University, P.O. Box 44519, Zagazig, Egypt

[4] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

[5] Faculty of Computers and Information Systems, Egyptian Chinese University, Cairo 4541312, Egypt

* **Correspondence:** Email: ei_abdelgalil@zu.edu.eg.

Abstract: This paper studies inextensible flows of planar and spatial curves by means of the Frenet frame. The velocity field is decomposed into its tangential, normal, and binormal components, so that the metric variation and the evolution of the intrinsic invariants can be computed explicitly. For planar curves, the tangent, normal, and curvature equations are recorded together with a metric-preserving condition, which relates the tangential velocity to the curvature-weighted normal velocity. For space curves in $\mathbb{R}^3$, the Frenet frame evolution and the coupled curvature-torsion system are derived, and the compatibility of the Frenet and temporal matrices is obtained in a general form retaining the metric-variation term; the zero-curvature equation then appears as the inextensible reduction rather than as an initial assumption. Several special motions are examined, including purely tangential and purely binormal flows, tangent-binormal constant motion, and the curvature-dependent flow $r_t = \frac{1}{2}\kappa^2 T + \kappa_s N + \kappa B$. Explicit solutions of the curvature-torsion system are obtained in closed form, including a travelling wave along which curvature and torsion are linearly related, and a localized travelling-wave curvature profile with constant torsion. The results give a unified geometric formulation for inextensible curve motion and clarify the roles of the velocity components in metric preservation, bending, twisting, and integrability.

Keywords: evolution of curves; inextendable curves
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1. Introduction

The geometric evolution of curves is a classical source of interaction between differential geometry, nonlinear partial differential equations, and mathematical models of motion. For a fixed curve, the local geometry is encoded by intrinsic invariants such as curvature and torsion. For an evolving curve, these invariants become dynamical quantities governed by the chosen velocity field. Curve evolution theory therefore converts the Frenet geometry of curves into a system of differential equations describing shape variation, filament motion, elastic deformation, and controlled trajectories.

A central development in this direction is Hasimoto's work on vortex filament motion. By combining the curvature and torsion of a space curve into a complex function, Hasimoto obtained the nonlinear Schrödinger equation from the localized induction approximation [1]. This transformation revealed that the intrinsic geometry of a moving filament may carry soliton structure. Lamb Jr. subsequently studied solitons on moving space curves, further connecting curve flows with integrable evolution equations [2]. Since then, curve evolution has served as a geometric mechanism for deriving and interpreting nonlinear integrable systems [3].

Inextensible curve flows form a distinguished class of geometric motions. In such flows, the arc length is preserved, so the curve may bend or twist without stretching. This condition is natural in the modeling of elastic filaments, vortex lines, fluid interfaces, biological fibers, and string-like mechanical systems. Daniel and Yazaki studied analytical and computational aspects of inextensible plane curve flows, including the role of curvature-adjusted tangential velocities [4]. Later works used alternative moving frames, such as quasi-frames and modified orthogonal frames [5,6], to treat inextensible flows in Euclidean, Galilean [7], Minkowski [8], and related geometries.

Recent contributions have shown that both the ambient geometry and the chosen moving frame affect the resulting evolution system. Turan et al. [9] and Kizilay and Yakut [10] studied inextensible flows by means of alternative orthogonal frames. Gurses examined integrability properties of moving null curves [11]. Further work by Samah Gaber and collaborators [8, 12] treated inextensible curve flows in Minkowski and de-Sitter spaces. These studies indicate that the evolution of curvature and torsion is highly sensitive to the metric signature, the frame construction, and the prescribed decomposition of the velocity field.

The same framework appears in trajectory planning and control. The path of a vehicle, robot, drone, or autonomous agent may be represented by an evolving curve whose velocity is decomposed along Frenet directions [13].

The tangential component controls progression along the path, while the normal and binormal components govern bending, steering, pitch adjustment, and out-of-plane maneuvers. For a space curve with velocity

$$\frac{\partial r}{\partial t} = WT + UN + VB,$$

the preservation of arc length is equivalent to

$$\frac{\partial W}{\partial s} = \kappa U.$$

Thus, inextensibility is not a restriction on the binormal velocity; it is a coupling between tangential motion and curvature-weighted normal motion. This relation prevents artificial stretching and gives a geometric constraint suitable for smooth path generation. The corresponding curvature and torsion equations describe bending and twisting, which are central in autonomous navigation, flight-path optimization, computer-aided route generation, flexible mechanisms, and string-driven motion.

The purpose of this paper is to give a unified Frenet frame formulation of planar and spatial inextensible curve flows and use it to construct explicit inextensible motions together with their curvature-torsion dynamics. The velocity components are kept arbitrary until the metric-preserving condition is imposed. This approach separates the geometric roles of tangential, normal, and binormal motions in arc-length preservation, curvature variation, and torsion evolution. In the planar case, the flow reduces to a curvature equation controlled by the tangential-normal balance. In the spatial case, the binormal component and the torsion produce a coupled curvature-torsion system.

The planar inextensibility condition $W_s = \kappa U$ is a classical compatibility relation for arc-length preserving curve flows and has appeared, for example, in the work of Inoguchi [14] and in the subsequent lecture notes of Kajiwara [15]. Therefore, in the planar part of the present paper, this relation is recalled and rederived only to fix notation and to make the transition to the spatial case self-contained. The same qualification applies to the general apparatus used in the spatial case: the Frenet-frame description of moving space curves, the skew-symmetric temporal evolution matrix, and the zero-curvature representation of inextensible flows are likewise classical [1, 2, 16].

Within this setting, the passage from planar curves to space curves is not a formal repetition. In the spatial case, the binormal component does not enter the metric variation directly, but it enters the Frenet-frame evolution and couples the curvature and torsion through the quantities

$$\alpha = U_s + \kappa W - \tau V, \qquad \beta = V_s + \tau U, \qquad \gamma = \frac{1}{\kappa}(\beta_s + \tau\alpha),$$

so that torsion, binormal motion, and a coupled curvature-torsion system appear, which have no planar counterpart.

The novelty of the present paper lies not in this framework but in what is obtained from it. First, the compatibility condition for the Frenet and temporal matrices is derived in the general form $M_t - Q_s + [M, Q] = -\frac{g_t}{2g}M$, valid before the metric-preserving constraint is imposed; retaining the metric-variation term exhibits the zero-curvature equation as the inextensible reduction rather than as an initial assumption, and so identifies arc-length preservation as the exact mechanism producing the integrable structure. Second, for the tangent-binormal constant motion $W = a$, $U = 0$, $V = b$, the curvature-torsion system is integrated in closed form, yielding a travelling wave in which curvature and torsion are linearly related; the degenerate case in which this proportionality breaks down is identified and treated separately. Third, the curvature-dependent flow $r_t = \frac{1}{2}\kappa^2 T + \kappa_s N + \kappa B$ is introduced, shown to be inextensible identically, and solved by an explicit localized travelling wave with constant torsion whose curvature is strictly positive for all finite (s, t), so that the Frenet frame remains well defined throughout the evolution. Together, these show that the inextensible system in $\mathbb{R}^3$ contains both transport-type reductions, in which the intrinsic invariants are merely carried along the curve, and nonlinear dispersive reductions admitting solitary-wave profiles.

These results provide a coherent geometric description of inextensible curve motion and its induced curvature-torsion dynamics, with applications to filament models, constrained trajectory design, and integrable geometric flows.

2. Evolution of curves in the plane

The computations in this section are included for completeness and to establish the notation used later in the spatial case. The planar inextensibility condition and the corresponding curvature evolution are classical; see, for example, Inoguchi [14] and Kajiwara [15]. Our purpose here is to use these calculations as the two-dimensional reference model from which the space-curve formulation in the following sections is developed. Planar curve evolution gives the local model for metric-preserving shape variation. In this setting, the curve is not fixed; its Frenet frame (see Figure 1 for the Frenet frame of a general curve at two distinct points) and curvature evolve under a prescribed velocity field. The inextensible case is distinguished by the preservation of the intrinsic metric, so that the curve may bend without stretching. This constraint occurs naturally in elastic filaments, interfaces, flexible rods, and planar trajectory design. The aim of this section is to derive the planar Frenet evolution equations and to identify the precise relation between the tangential and normal velocities that preserves arc length.

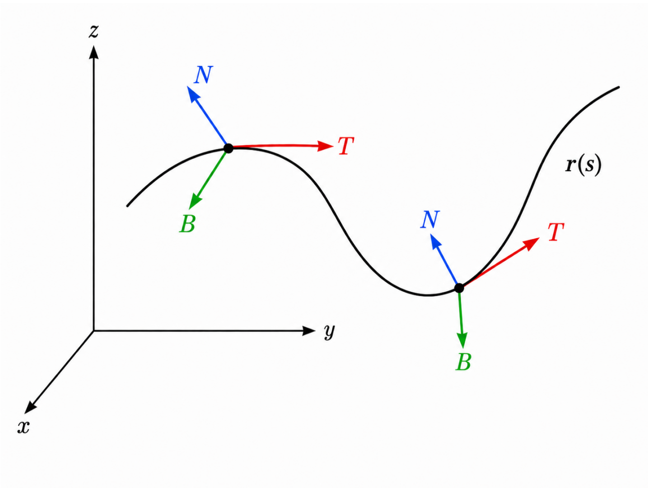


Figure 1. Illustration of a smooth space curve $r(s)$ in three-dimensional Euclidean space together with the Frenet frame at two distinct points. The curve is shown in black, while the tangent vector T, principal normal vector N, and binormal vector B are represented in red, blue, and green, respectively.

The evolution of space curves through the dynamical behavior of the Frenet frame $\{T, N, B\}$ was studied by Mukherjee and Balakrishnan [16]. Their treatment of the sine-Gordon equation produced five new classes of space curves beyond the two families previously obtained by Lamb Jr. [2]. The planar case considered here is the two-dimensional reduction in which curvature is the only intrinsic invariant.

Let $r = r(u, t)$ be a smooth one-parameter family of planar curves, where u is a curve parameter, and t is the evolution parameter. The induced metric is

$$g = \left\langle \frac{\partial r}{\partial u}, \frac{\partial r}{\partial u} \right\rangle. \tag{2.1}$$

The arc-length parameter is

$$s(t,u) = \int_0^u \sqrt{g(t,u')}\,du' + s_0(t), \tag{2.2}$$

where $s_0(t)$ is the value of $s(t,u)$ at $u = 0$. Hence,

$$T = \frac{\partial r}{\partial s} = \frac{1}{\sqrt{g}}\frac{\partial r}{\partial u}. \tag{2.3}$$

The unit normal N and the curvature κ are determined by the planar Frenet equations

$$\frac{\partial T}{\partial s} = \kappa N, \tag{2.4}$$

$$\frac{\partial N}{\partial s} = -\kappa T. \tag{2.5}$$

The orientation of N is fixed by (2.4) and (2.5). Conversely, a prescribed curvature function determines the curve locally up to an orientation-preserving rigid motion of the plane.

Assume that the curve evolves by

$$\frac{\partial r}{\partial t} = WT + UN, \tag{2.6}$$

where W and U are the tangential and normal velocity components, respectively. Throughout this paper, the curve parameter u and the evolution parameter t are assumed to be independent variables whereas s depends on t. Consequently, the mixed partial derivatives commute, i.e.,

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial}{\partial t}\right), \tag{2.7}$$

which forms the basis for deriving the compatibility condition of the Frenet frame evolution.

Using (2.1)–(2.7), we obtain

$$\begin{aligned}\frac{\partial g}{\partial t} &= \frac{\partial}{\partial t}\langle r_u, r_u\rangle = 2\left\langle \frac{\partial}{\partial t}(r_u), r_u\right\rangle = 2\left\langle \sqrt{g}\frac{\partial}{\partial s}\left(\frac{\partial r}{\partial t}\right), \sqrt{g}\frac{\partial r}{\partial s}\right\rangle \\ &= 2g\left\langle \frac{\partial}{\partial s}(WT + UN), T\right\rangle = 2g\left\langle W_s T + \kappa W N + U_s N - \kappa U T, T\right\rangle = 2g(W_s - \kappa U).\end{aligned} \tag{2.8}$$

Thus, the metric variation is governed only by the scalar quantity $W_s - \kappa U$.

The dependence of s on t gives the operator identity

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}\right) &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial u}\left(\frac{\partial}{\partial t}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial u}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}\frac{\partial}{\partial s}\right) \\ &= \frac{\partial}{\partial t}\frac{\partial}{\partial s} + \frac{g_t}{2g}\frac{\partial}{\partial s} = \frac{\partial}{\partial t}\frac{\partial}{\partial s} + (W_s - \kappa U)\frac{\partial}{\partial s}.\end{aligned} \tag{2.9}$$

Equivalently,

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = \frac{\partial}{\partial s}\frac{\partial}{\partial t} - (W_s - \kappa U)\frac{\partial}{\partial s}.$$

The tangent vector evolves as

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\partial r}{\partial s}\right)=\frac{\partial}{\partial s}\left(\frac{\partial r}{\partial t}\right)-(W_s-\kappa U)\frac{\partial r}{\partial s}=\frac{\partial}{\partial s}(WT+UN)-(W_s-\kappa U)T=(U_s+\kappa W)N. \tag{2.10}$$

Since

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial u}\right)=\frac{\partial}{\partial u}\left(\frac{\partial T}{\partial t}\right), \tag{2.11}$$

we compute both sides. From (2.4),

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial u}\right)=\frac{\partial}{\partial t}\left(\sqrt{g}\frac{\partial T}{\partial s}\right)=\frac{\partial}{\partial t}(\sqrt{g}\kappa N)=\frac{g_t}{2\sqrt{g}}\kappa N+\sqrt{g}\kappa_t N+\sqrt{g}\kappa N_t=\sqrt{g}\left[\left(\kappa_t+\kappa W_s-\kappa^2 U\right)N+\kappa N_t\right]. \tag{2.12}$$

On the other hand, by (2.10),

$$\frac{\partial}{\partial u}\left(\frac{\partial T}{\partial t}\right)=\sqrt{g}\frac{\partial}{\partial s}\left[(U_s+\kappa W)N\right]=\sqrt{g}\left[(U_{ss}+\kappa_s W+\kappa W_s)N-\kappa(U_s+\kappa W)T\right]. \tag{2.13}$$

Substitution of (2.12) and (2.13) into (2.11) gives

$$(\kappa_t+\kappa W_s-\kappa^2 U)N+\kappa N_t=(U_{ss}+\kappa_s W+\kappa W_s)N-\kappa(U_s+\kappa W)T. \tag{2.14}$$

Comparing the T- and N-components yields

$$\frac{\partial N}{\partial t}=-(U_s+\kappa W)T, \tag{2.15}$$

$$\frac{\partial \kappa}{\partial t}=U_{ss}+\kappa^2 U+\kappa_s W. \tag{2.16}$$

Equation (2.15) follows directly from differentiating $\langle T,N\rangle=0$ and $|N|=1$, while (2.16) gives the curvature evolution under the general planar flow (2.6).

Definition 2.1. *A plane curve $r(u,t)$ and its associated flow $\partial r/\partial t$ are said to be inextensible if the metric is preserved under the flow, that is,*

$$\frac{\partial g}{\partial t}=0.$$

By Definition 2.1 and (2.8), the flow is inextensible if and only if

$$W_s=\kappa U. \tag{2.17}$$

Under this condition, (2.16) becomes

$$\kappa_t=U_{ss}+\kappa W_s+\kappa_s W=\frac{\partial}{\partial s}(U_s+\kappa W). \tag{2.18}$$

Proposition 2.1. *Let $r(t,u)$ be a planar curve evolving according to*

$$\frac{\partial r}{\partial t}=WT+UN.$$

Then, the flow is inextensible if and only if

$$W_s=\kappa U.$$

In this case, the curvature satisfies

$$\kappa_t=\frac{\partial}{\partial s}(U_s+\kappa W).$$

Proof. The first assertion follows immediately from (2.8). Substituting $W_s = \kappa U$ into the general curvature equation (2.16) gives

$$\kappa_t = U_{ss} + \kappa W_s + \kappa_s W = (U_s + \kappa W)_s.$$

□

This condition is well known in the theory of arc-length preserving planar curve flows; see [14, 15].

Corollary 2.1. *A non-straight planar curve admits no non-trivial purely normal inextensible motion.*

Proof. If $W = 0$, then (2.17) gives $\kappa U = 0$. On every interval where $\kappa \neq 0$, one has $U = 0$. If $\kappa = 0$ on an interval, the curve is locally a straight line. □

On any interval where $\kappa \neq 0$, the inextensibility condition gives

$$U = \frac{W_s}{\kappa}. \tag{2.19}$$

Hence, (2.18) becomes

$$\kappa_t = \frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left(\frac{W_s}{\kappa}\right) + \kappa W\right]. \tag{2.20}$$

Thus, after prescribing W, the inextensibility constraint determines the compatible normal velocity and the induced curvature evolution.

Corollary 2.2. *If $U = 0$, then the planar flow is inextensible if and only if $W_s = 0$. In this case,*

$$\kappa_t = W\kappa_s.$$

In particular, if W is independent of s, and κ is constant along the curve, then the curvature is time-independent.

Proof. For $U = 0$, (2.17) gives $W_s = 0$. Equation (2.16) then gives $\kappa_t = W\kappa_s$. □

For an inextensible curve, the length

$$L(t) = \int ds$$

is preserved. Since

$$\frac{\partial}{\partial t}(ds) = (W_s - \kappa U)ds,$$

we have

$$\frac{dL}{dt} = \int (W_s - \kappa U)\, ds = 0. \tag{2.21}$$

If the curve is closed and positively oriented, and if $A(t)$ denotes the enclosed signed area, then only the normal component contributes to the area variation:

$$\frac{dA}{dt} = \int U\, ds = \int \frac{W_s}{\kappa}\, ds, \qquad \kappa \neq 0. \tag{2.22}$$

Thus, inextensible planar flows preserve the intrinsic metric, while the enclosed signed area may vary through the normal velocity forced by the constraint (2.17).

3. Evolution of a curve in three-dimensional space

Space curve evolution extends planar curve flow by incorporating torsion and binormal motion. This setting is natural in the geometry of elastic filaments, vortex lines, DNA strands, and three-dimensional trajectory design, where bending and twisting occur simultaneously. Inextensible flows form the metric-preserving class: the curve changes its shape, while its arc-length element remains fixed. The resulting equations describe how the Frenet frame, curvature, and torsion respond to tangential, normal, and binormal velocity components.

Let $r = r(u,t)$ be a smooth evolving space curve, and let s denote its arc-length parameter. The Frenet frame $\{T, N, B\}$ satisfies

$$\frac{\partial T}{\partial s} = \kappa N, \tag{3.1}$$

$$\frac{\partial N}{\partial s} = -\kappa T + \tau B, \tag{3.2}$$

$$\frac{\partial B}{\partial s} = -\tau N, \tag{3.3}$$

where $\kappa(s,t)$ and $\tau(s,t)$ are the curvature and torsion. In matrix form,

$$\frac{\partial}{\partial s}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{3.4}$$

Assume that the curve evolves according to

$$\frac{\partial r}{\partial t} = WT + UN + VB, \tag{3.5}$$

where W, U, and V are the tangential, normal, and binormal velocity components, respectively. The metric $g = \langle r_u, r_u \rangle$ and the arc-length parameter are defined as in (2.1) and (2.2).

Using the compatibility of the independent parameters u and t, together with the Frenet equations, we obtain

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{\partial}{\partial t}\langle r_u, r_u \rangle = 2\left\langle \frac{\partial r_u}{\partial t}, r_u \right\rangle = 2\left\langle \sqrt{g}\frac{\partial}{\partial s}\left(\frac{\partial r}{\partial t}\right), \sqrt{g}\frac{\partial r}{\partial s} \right\rangle = 2g\left\langle \frac{\partial}{\partial s}(WT + UN + VB), T \right\rangle \\ &= 2g\,\langle (W_s - \kappa U)T + (U_s + \kappa W - \tau V)N + (V_s + \tau U)B, T \rangle = 2g(W_s - \kappa U). \end{aligned} \tag{3.6}$$

Thus, the binormal component does not enter the metric variation directly; arc-length preservation is controlled by the tangential-normal balance $W_s - \kappa U$.

Using the commutation identity (2.9), the tangent vector satisfies

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial r}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial r}{\partial t}\right) - (W_s - \kappa U)\frac{\partial r}{\partial s} = (U_s + \kappa W - \tau V)N + (V_s + \tau U)B. \tag{3.7}$$

Set

$$\alpha = U_s + \kappa W - \tau V, \qquad \beta = V_s + \tau U. \tag{3.8}$$

Then,

$$T_t = \alpha N + \beta B.$$

Differentiating with respect to u gives

$$\frac{\partial}{\partial u}\left(\frac{\partial T}{\partial t}\right) = \sqrt{g}\frac{\partial}{\partial s}(\alpha N + \beta B) = \sqrt{g}\left[-\kappa\alpha T + (\alpha_s - \tau\beta)N + (\beta_s + \tau\alpha)B\right]. \tag{3.9}$$

On the other hand,

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial u}\right) = \frac{\partial}{\partial t}\left(\sqrt{g}\,\kappa N\right) = \sqrt{g}\left[\left(\kappa_t + \kappa(W_s - \kappa U)\right)N + \kappa N_t\right]. \tag{3.10}$$

The compatibility condition

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial T}{\partial t}\right),$$

therefore, yields, on intervals where $\kappa \neq 0$,

$$N_t = -\alpha T + \gamma B, \tag{3.11}$$

$$\kappa_t = \alpha_s - \tau\beta - \kappa(W_s - \kappa U), \tag{3.12}$$

where

$$\gamma = \frac{1}{\kappa}(\beta_s + \tau\alpha). \tag{3.13}$$

Equivalently,

$$\kappa_t = \frac{\partial}{\partial s}(U_s + \kappa W - \tau V) - \tau(V_s + \tau U) - \kappa(W_s - \kappa U).$$

Since $B = T \times N$, the binormal vector satisfies

$$B_t = T_t \times N + T \times N_t = (\alpha N + \beta B) \times N + T \times (-\alpha T + \gamma B) = -\beta T - \gamma N. \tag{3.14}$$

To derive the torsion equation, start from $\partial_u B = \sqrt{g}\,\partial_s B = -\sqrt{g}\,\tau N$. Then,

$$\frac{\partial}{\partial t}(\partial_u B) = \frac{\partial}{\partial t}(-\sqrt{g}\,\tau N) = -\sqrt{g}\left[\tau N_t + \left(\tau_t + \tau(W_s - \kappa U)\right)N\right], \tag{3.15}$$

while

$$\frac{\partial}{\partial u}(B_t) = \sqrt{g}\frac{\partial}{\partial s}(-\beta T - \gamma N) = -\sqrt{g}\left[(\beta_s - \kappa\gamma)T + (\kappa\beta + \gamma_s)N + \tau\gamma B\right]. \tag{3.16}$$

Using (3.11) in (3.15) and comparing the N-components gives

$$\tau_t = \kappa\beta + \gamma_s - \tau(W_s - \kappa U). \tag{3.17}$$

In terms of W, U, V, this becomes

$$\tau_t = \kappa(V_s + \tau U) - \tau(W_s - \kappa U) + \frac{\partial}{\partial s}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}(V_s + \tau U) + \frac{\tau}{\kappa}(U_s + \kappa W - \tau V)\right]. \tag{3.18}$$

The time evolution of the Frenet frame is,

$$\frac{\partial}{\partial t}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{3.19}$$

where α, β, and γ are given by (3.8) and (3.13). Hence, the frame evolution is skew-symmetric and preserves orthonormality.

Consequently, the intrinsic quantities satisfy

$$\frac{\partial}{\partial t}\begin{pmatrix} \kappa \\ \tau \end{pmatrix} = \begin{pmatrix} -\frac{g_t}{2g} & -\beta \\ \beta & -\frac{g_t}{2g} \end{pmatrix}\begin{pmatrix} \kappa \\ \tau \end{pmatrix} + \begin{pmatrix} \alpha_s \\ \gamma_s \end{pmatrix}. \tag{3.20}$$

The Frenet system and its temporal evolution may be written compactly as

$$F_s = MF, \qquad F_t = QF, \tag{3.21}$$

where

$$F = \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}. \tag{3.22}$$

Theorem 3.1. *Let F satisfy the matrix system* (3.21)*. Then, the compatibility of mixed derivatives with respect to u and t implies*

$$M_t - Q_s + [M, Q] = -\frac{g_t}{2g}M, \tag{3.23}$$

where $[M, Q] = MQ - QM$.

Proof. The compatibility condition is

$$\frac{\partial}{\partial t}\frac{\partial F}{\partial u} = \frac{\partial}{\partial u}\frac{\partial F}{\partial t}. \tag{3.24}$$

Since $\partial_u F = \sqrt{g}F_s = \sqrt{g}MF$, we have

$$\frac{\partial}{\partial t}\frac{\partial F}{\partial u} = \frac{\partial}{\partial t}(\sqrt{g}MF) = \frac{g_t}{2\sqrt{g}}MF + \sqrt{g}M_tF + \sqrt{g}MF_t = \sqrt{g}\left(\frac{g_t}{2g}M + M_t + MQ\right)F. \tag{3.25}$$

Also,

$$\frac{\partial}{\partial u}\frac{\partial F}{\partial t} = \sqrt{g}\frac{\partial}{\partial s}(QF) = \sqrt{g}(Q_s + QM)F. \tag{3.26}$$

Equating (3.25) and (3.26) gives

$$\frac{g_t}{2g}M + M_t + MQ = Q_s + QM, \tag{3.27}$$

which is equivalent to (3.23). □

Remark 3.1. *The explicit matrices in* (3.22) *are*

$$M_t = \begin{pmatrix} 0 & \kappa_t & 0 \\ -\kappa_t & 0 & \tau_t \\ 0 & -\tau_t & 0 \end{pmatrix}, \qquad Q_s = \begin{pmatrix} 0 & \alpha_s & \beta_s \\ -\alpha_s & 0 & \gamma_s \\ -\beta_s & -\gamma_s & 0 \end{pmatrix}, \tag{3.28}$$

and

$$[M, Q] = \begin{pmatrix} 0 & -\tau\beta & \kappa\gamma - \tau\alpha \\ \tau\beta & 0 & -\kappa\beta \\ -\kappa\gamma + \tau\alpha & \kappa\beta & 0 \end{pmatrix}. \tag{3.29}$$

Thus, (3.23) *gives*

$$\begin{pmatrix} 0 & \kappa_t - \alpha_s - \tau\beta & -\beta_s + \kappa\gamma - \tau\alpha \\ -\kappa_t + \alpha_s + \tau\beta & 0 & \tau_t - \gamma_s - \kappa\beta \\ \beta_s - \kappa\gamma + \tau\alpha & -\tau_t + \gamma_s + \kappa\beta & 0 \end{pmatrix} = -\frac{g_t}{2g} \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}. \tag{3.30}$$

The (3.29) *and* (3.30) *entries reproduce* (3.23).

Remark 3.2. *Equations* (3.12) *and* (3.17) *describe the general evolution of curvature and torsion under the velocity field* (3.5)*. Three basic reductions are worth recording:*

(1) If $U = V = 0$, the motion is tangential. The metric is preserved only when $W_s = 0$.
(2) If $W = V = 0$, the motion is normal. The metric is preserved only when $\kappa U = 0$.
(3) If $W = U = 0$, the motion is binormal. The inextensibility condition is automatically satisfied.

The vector field $\partial r/\partial t$ and the tangent $T = \partial r/\partial s$ span a ruled surface swept by the evolving curve. Its infinitesimal swept area is

$$dA_{\text{swept}} = \left| \frac{\partial r}{\partial t} \times \frac{\partial r}{\partial s} \right| ds\, dt = \sqrt{U^2 + V^2}\, ds\, dt. \tag{3.31}$$

Thus, tangential motion does not contribute to the swept area, whereas the normal and binormal components determine the transverse displacement. This quantity is distinct from the enclosed planar area, which is not defined for a general non-planar closed curve.

4. Motion of inextensible curves in three-dimensional space

Definition 4.1. *A space curve $r(u, t)$ and its flow $\partial r/\partial t$ in $\mathbb{R}^3$ are said to be* inextensible *if the induced metric is preserved under the evolution, namely*

$$\frac{\partial g}{\partial t} = 0.$$

By (3.6), Definition 4.1 is equivalent to

$$\kappa U = \frac{\partial W}{\partial s}, \tag{4.1}$$

where κ is the curvature, U is the normal velocity, and W is the tangential velocity. Hence, the tangential and normal components are coupled by the metric constraint. The binormal velocity does not affect the metric variation directly, but it enters the curvature-torsion dynamics through the frame coefficients.

For an inextensible curve, the commutation identity (2.9) reduces to

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}.$$

Moreover, the integrability condition (3.23) becomes the zero-curvature equation

$$M_t - Q_s + [M, Q] = 0, \tag{4.2}$$

where M and Q are the Serret-Frenet and temporal evolution matrices defined in (3.22). This condition is the compatibility equation for the Frenet frame and is the geometric source of several integrable systems associated with space curve flows, including the nonlinear Schrödinger equation in the binormal-flow case.

Consequently, the intrinsic quantities satisfy

$$\frac{\partial}{\partial t}\begin{pmatrix}\kappa\\ \tau\end{pmatrix} = \begin{pmatrix}0 & -\beta\\ \beta & 0\end{pmatrix}\begin{pmatrix}\kappa\\ \tau\end{pmatrix} + \begin{pmatrix}\alpha_s\\ \gamma_s\end{pmatrix}, \tag{4.3}$$

where

$$\alpha = U_s + \kappa W - \tau V, \qquad \beta = V_s + \tau U, \qquad \gamma = \frac{1}{\kappa}\left(\beta_s + \tau\alpha\right).$$

The formulas involving γ are understood on intervals where $\kappa \neq 0$.

Remark 4.1. *Equations* (4.1)–(4.3) *characterize inextensible space-curve flows. They preserve arc length while allowing nontrivial bending and twisting. The system is suited to constrained filament dynamics, vortex filament models, and geometric reductions leading to integrable equations.*

The governing system for an inextensible curve in $\mathbb{R}^3$ follows from (4.3):

$$\begin{aligned} \kappa_t &= \frac{\partial}{\partial s}(U_s + \kappa W - \tau V) - \tau(V_s + \tau U), \\ \tau_t &= \kappa(V_s + \tau U) + \frac{\partial}{\partial s}\left[\frac{1}{\kappa}\frac{\partial}{\partial s}(V_s + \tau U) + \frac{\tau}{\kappa}(U_s + \kappa W - \tau V)\right]. \end{aligned} \tag{4.4}$$

Proposition 4.1. *Let $r_t = WT + UN + VB$ be an inextensible space-curve flow with $\kappa \neq 0$. Then, the normal velocity is determined by the tangential velocity through*

$$U = \frac{W_s}{\kappa}.$$

In particular, the binormal component V may be prescribed freely at the level of metric preservation, while W and U must satisfy (4.1).

Proof. The assertion follows immediately from (4.1). Since V does not appear in (3.6), it is not constrained by metric preservation, although it enters (4.4). □

We now examine two classes of inextensible motions in $\mathbb{R}^3$.

Corollary 4.1. *For an inextensible purely tangential flow, $U = V = 0$, one has $W_s = 0$. Consequently,*

$$\kappa_t = W\kappa_s, \qquad \tau_t = W\tau_s.$$

In particular, κ and τ are time-independent only when they are independent of s or when $W = 0$.

Proof. If $U = V = 0$, the inextensibility condition gives $W_s = 0$. Then,

$$\alpha = \kappa W, \qquad \beta = 0, \qquad \gamma = \tau W.$$

Using (4.4), one obtains

$$\kappa_t = (\kappa W)_s = W\kappa_s, \qquad \tau_t = (\tau W)_s = W\tau_s.$$

□

Corollary 4.2. *Every purely binormal flow $r_t = VB$ is inextensible.*

Proof. For $W = U = 0$, equation (3.6) gives $g_t = 0$ for arbitrary V. □

Example 4.1 (Tangential-only flow)**.** *Consider*

$$\frac{\partial r}{\partial t} = W(s,t)T, \qquad U = V = 0.$$

The inextensibility condition gives $W_s = 0$, so W is independent of s. Moreover,

$$\alpha = \kappa W, \qquad \beta = 0, \qquad \gamma = \tau W.$$

Hence,

$$\kappa_t = W\kappa_s, \qquad \tau_t = W\tau_s.$$

Thus, the invariants are transported along the curve with tangential speed W. They are unchanged only for spatially constant curvature and torsion, or for the trivial case $W = 0$.

Example 4.2 (Binormal-only flow)**.** *Consider*

$$\frac{\partial r}{\partial t} = V(s,t)B, \qquad W = U = 0.$$

The inextensibility condition is automatically satisfied. In this case,

$$\alpha = -\tau V, \qquad \beta = V_s, \qquad \gamma = \frac{1}{\kappa}(V_{ss} - \tau^2 V).$$

Therefore,

$$\kappa_t = -(\tau V)_s - \tau V_s,$$

and

$$\tau_t = \kappa V_s + \frac{\partial}{\partial s}\left[\frac{1}{\kappa}\left(V_{ss} - \tau^2 V\right)\right].$$

The Frenet frame evolves as

$$\begin{aligned} T_t &= -\tau V N + V_s B, \\ N_t &= \tau V T + \frac{1}{\kappa}(V_{ss} - \tau^2 V)B, \\ B_t &= -V_s T - \frac{1}{\kappa}(V_{ss} - \tau^2 V)N. \end{aligned}$$

For the localized induction choice $V = \kappa$, this gives the classical binormal flow of vortex filaments [17].

Remark 4.2. *Tangential inextensible motion transports the intrinsic invariants, whereas binormal motion produces a coupled curvature-torsion system. The latter is the geometric mechanism behind the relation between filament motion and soliton equations.*

Example 4.3. *Consider the tangent-binormal constant motion*

$$W = a, \qquad U = 0, \qquad V = b, \tag{4.5}$$

where a and b are constants. Since $W_s = 0 = \kappa U$, the flow is inextensible. In this case,

$$\alpha = a\kappa - b\tau, \qquad \beta = 0, \qquad \gamma = a\tau - b\frac{\tau^2}{\kappa}.$$

Therefore, (4.4) *reduces to*

$$\begin{aligned} \kappa_t &= a\kappa_s - b\tau_s, \\ \tau_t &= \frac{\partial}{\partial s}\left(a\tau - b\frac{\tau^2}{\kappa}\right). \end{aligned} \tag{4.6}$$

A solution of (4.6) *is*

$$\begin{aligned} \kappa(s,t) &= \frac{bc_1(c_4 + c_5\tanh(c_1 s + c_2 t + c_3))}{ac_1 - c_2}, \qquad ac_1 \neq c_2, \\ \tau(s,t) &= c_4 + c_5\tanh(c_1 s + c_2 t + c_3), \end{aligned} \tag{4.7}$$

where c_1, c_2, c_3, c_4, and c_5 are constants.

Indeed, set

$$z = c_1 s + c_2 t + c_3, \qquad \tau = c_4 + c_5 \tanh z, \qquad \kappa = \frac{bc_1}{ac_1 - c_2}\tau.$$

Then,

$$\kappa_s = \frac{bc_1}{ac_1 - c_2}\tau_s, \qquad \kappa_t = \frac{bc_1}{ac_1 - c_2}\tau_t,$$

and

$$\tau_t = \frac{c_2}{c_1}\tau_s.$$

Consequently,

$$a\kappa_s - b\tau_s = \left(\frac{abc_1}{ac_1 - c_2} - b\right)\tau_s = \frac{bc_2}{ac_1 - c_2}\tau_s = \kappa_t.$$

Moreover, since

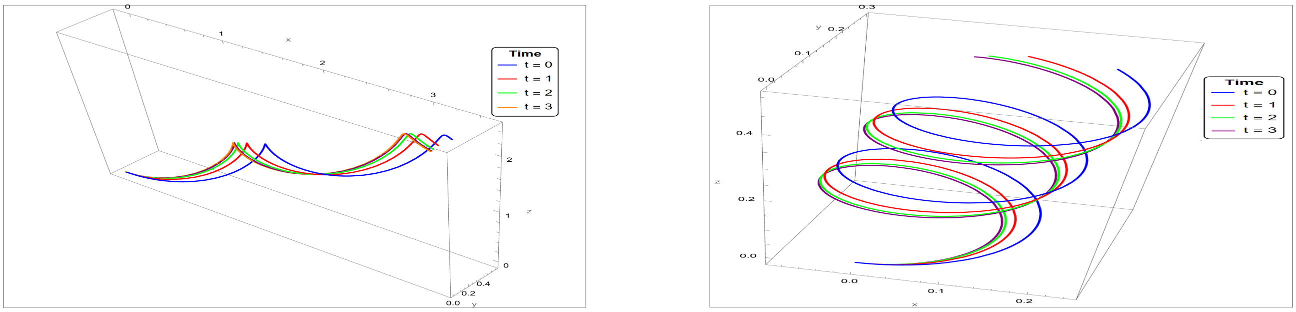
$$a\tau - b\frac{\tau^2}{\kappa} = a\tau - \frac{ac_1 - c_2}{c_1}\tau = \frac{c_2}{c_1}\tau,$$

we get

$$\frac{\partial}{\partial s}\left(a\tau - b\frac{\tau^2}{\kappa}\right) = \frac{c_2}{c_1}\tau_s = \tau_t.$$

Hence, (4.7) *satisfies* (4.6).

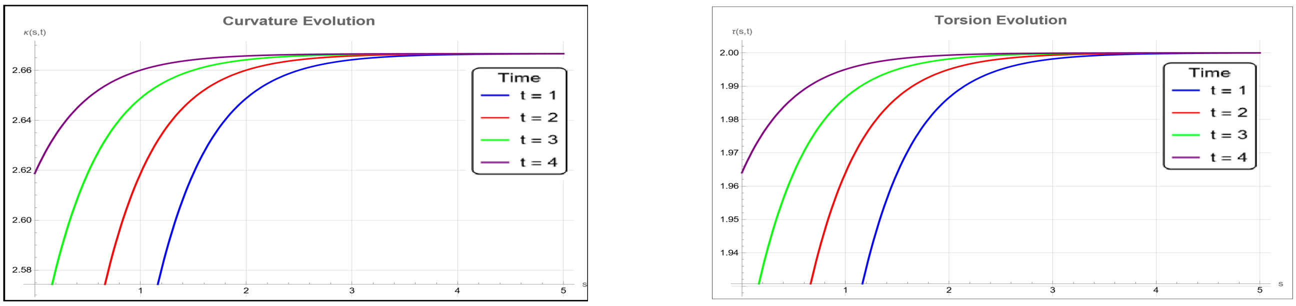
Remark 4.3. *Example 4.3 gives a tangent-binormal flow in which the curvature and torsion are linearly related along the explicit solution; see Figures 2 and 3. Indeed, the derivation of the travelling-wave solution assumes that* $ac_1 \neq c_2$*, which allows the proportionality relation* $\kappa = \frac{bc_1}{ac_1 - c_2}\tau$ *to be obtained. The special case* $ac_1 = c_2$ *is therefore singular, and the above solution family is no longer applicable. In this case, the reduction leading to the proportionality between the curvature and torsion breaks down, and Eq* (4.6) *must be analyzed separately.*



(a) $a = 2, b = 1$

(b) $a = 1, b = 2$

Figure 2. Evolution of an inextensible curve under the velocity $r_t = aT + bB$, for $c_1 = c_4 = c_5 = 1$, $c_2 = 0.5$, and $c_3 = 0$, at different times.



(a) Curvature

(b) Torsion

Figure 3. Evolution of curvature and torsion for the inextensible motion $r_t = aT + bB$, with $a = 2$, $b = 1$, $c_1 = c_4 = c_5 = 1$, $c_2 = 0.5$, and $c_3 = 0$.

Example 4.4. *Consider the curvature-dependent inextensible motion*

$$W = \frac{1}{2}\kappa^2, \qquad U = \kappa_s, \qquad V = \kappa, \qquad \kappa \neq 0. \tag{4.8}$$

Since

$$W_s = \kappa\kappa_s = \kappa U,$$

the flow is inextensible. Furthermore,

$$\alpha = \kappa_{ss} + \frac{1}{2}\kappa^3 - \tau\kappa, \qquad \beta = (1+\tau)\kappa_s,$$

and

$$\gamma = \frac{1+2\tau}{\kappa}\kappa_{ss} + \frac{\tau_s\kappa_s}{\kappa} + \frac{1}{2}\tau\kappa^2 - \tau^2.$$

Thus, (4.4) *gives*

$$\begin{aligned} \kappa_t &= \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s - \kappa\tau_s - (2\tau+\tau^2)\kappa_s, \\ \tau_t &= \kappa(1+\tau)\kappa_s + \frac{\partial}{\partial s}\left(\frac{1+2\tau}{\kappa}\kappa_{ss} + \frac{\tau_s\kappa_s}{\kappa} + \frac{1}{2}\tau\kappa^2 - \tau^2\right). \end{aligned} \tag{4.9}$$

A travelling-wave solution of (4.9) *is*

$$\begin{aligned} \kappa(s,t) &= 2c_1\,Sech(c_1(s-c_2t)), \\ \tau(s,t) &= c_3 \neq 0, \qquad c_2 = c_3(2+c_3) - c_1^2, \end{aligned} \tag{4.10}$$

where c_1, c_2, and c_3 are constants.

To verify (4.10)*, set*

$$z = c_1(s-c_2t), \qquad \kappa = 2c_1 Sech\, z, \qquad \tau = c_3.$$

Then,

$$\kappa_{ss} = c_1^2\kappa - \frac{1}{2}\kappa^3, \qquad \kappa_{sss} = \left(c_1^2 - \frac{3}{2}\kappa^2\right)\kappa_s, \qquad \kappa_t = -c_2\kappa_s.$$

Since $\tau_s = 0$, the first equation of (4.9) *becomes*

$$\kappa_t = \left(c_1^2 - c_3(2+c_3)\right)\kappa_s,$$

which is equivalent to

$$c_2 = c_3(2+c_3) - c_1^2.$$

For the second equation, the bracketed term in (4.9) *reduces to*

$$\frac{1+2c_3}{\kappa}\kappa_{ss} + \frac{1}{2}c_3\kappa^2 - c_3^2 = (1+2c_3)c_1^2 - c_3^2 - \frac{1+c_3}{2}\kappa^2.$$

Hence,

$$\frac{\partial}{\partial s}\left[\frac{1+2c_3}{\kappa}\kappa_{ss} + \frac{1}{2}c_3\kappa^2 - c_3^2\right] = -(1+c_3)\kappa\kappa_s.$$

The second equation of (4.9) *therefore gives*

$$\tau_t = \kappa(1+c_3)\kappa_s - (1+c_3)\kappa\kappa_s = 0,$$

which agrees with $\tau = c_3$. Thus, (4.10) *solves the system* (4.9).

The solution (4.10) *of system* (4.9) *satisfies $\kappa > 0$ for every (s,t) finite, and the Frenet frame remains well defined throughout the evolution.*

Remark 4.4. *Example 4.4 gives a curvature-dependent flow whose curvature has a positive definite localized travelling-wave profile and whose torsion is constant; see Figures 4 and 5. Together, these cases show that the inextensible system* (4.4) *contains both transport-type and nonlinear dispersive reductions.*

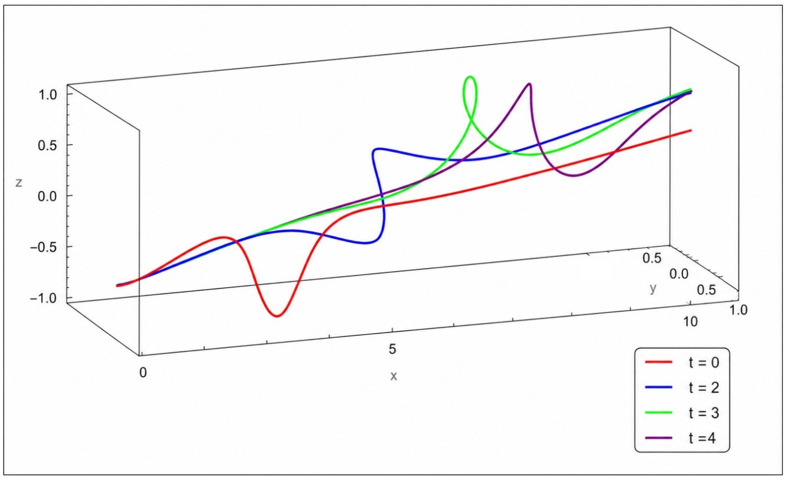


Figure 4. Evolution of an inextensible curve under the velocity $r_t = \frac{1}{2}\kappa^2 T + \kappa_s N + \kappa B$, for the travelling-wave solution (4.10).

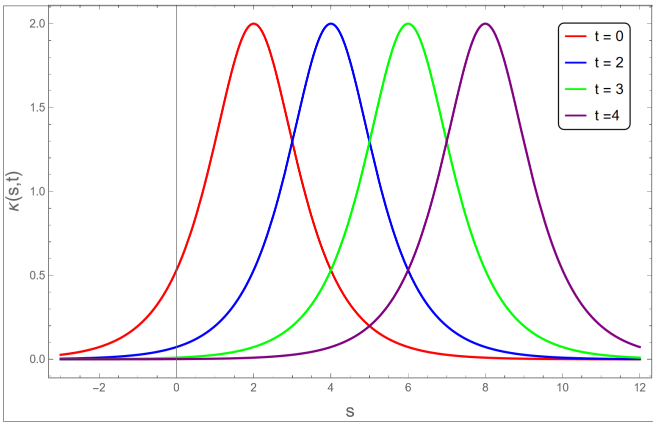


Figure 5. Evolution of curvature for the inextensible motion $r_t = \frac{1}{2}\kappa^2 T + \kappa_s N + \kappa B$, corresponding to (4.10).

5. Conclusions

This paper developed a Frenet frame formulation for the evolution of planar and spatial curves under general geometric velocity fields. The velocity was decomposed into tangential, normal, and binormal components, which made it possible to identify the precise contribution of each component to the metric, the moving frame, and the intrinsic invariants.

For planar curves, the metric variation was computed directly. The inextensibility condition was obtained. Thus, arc-length preservation is governed by a balance between tangential motion and curvature-weighted normal motion. The evolution equations for the tangent vector, normal vector, and curvature were derived. Under the inextensibility constraint, the curvature equation reduces to

$$\kappa_t = \frac{\partial}{\partial s}\left(U_s + \kappa W\right).$$

This formula gives the intrinsic curvature dynamics of the planar flow. It also shows that a non-straight planar curve admits no non-trivial purely normal inextensible motion, while purely tangential inextensible motion transports the curvature according to $\kappa_t = W\kappa_s$.

For space curves in $\mathbb{R}^3$, the inclusion of torsion and binormal velocity yields a skew-symmetric matrix representation of the Frenet frame evolution, preserving its orthonormal structure and leading to a coupled curvature-torsion system that describes bending, twisting, and tangential transport. Moreover, the Frenet and temporal matrices satisfy a compatibility condition that reduces to the zero-curvature equation for inextensible flows, revealing the integrable geometric structure underlying filament dynamics and soliton-type curve evolutions.

The special motions considered in the final section illustrate the scope of the general system. Tangential inextensible flows transport curvature and torsion along the curve. Pure binormal flows are automatically inextensible and generate nontrivial curvature-torsion dynamics. The constant tangent-binormal motion yields an explicit solution in which curvature and torsion are linearly related. The curvature-dependent flow

$$r_t = \frac{1}{2}\kappa^2 T + \kappa_s N + \kappa B$$

produces a nonlinear inextensible system admitting a localized travelling wave for the curvature with constant torsion. These examples show that the derived equations contain both transport-type reductions and nonlinear geometric deformations.

The results give a unified differential-geometric framework for inextensible curve evolution in the plane and in three-dimensional space. They clarify the role of tangential, normal, and binormal velocities in metric preservation, curvature variation, and torsion evolution. The formulas obtained here may serve as a basis for further work on elastic filaments, vortex filament models, constrained trajectory design, and integrable reductions of space curve flows.

Author contributions

Conceptualization: Sameh Shenawy, Bang-Yen Chen, Elsayed I. Mahmoud, and Abdallah A. Syied; Methodology: Sameh Shenawy, Bang-Yen Chen, Elsayed I. Mahmoud, and Abdallah A. Syied; Software: Sameh Shenawy, Bang-Yen Chen, Elsayed I. Mahmoud, Nasser Bin Turki, and Abdallah A. Syied; Validation: Sameh Shenawy, Bang-Yen Chen, Elsayed I. Mahmoud, Nasser Bin Turki, and Abdallah A. Syied; Formal Analysis: Sameh Shenawy, Bang-Yen Chen, Elsayed I. Mahmoud, Nasser Bin Turki, and Abdallah A. Syied; Investigation: Sameh Shenawy, Bang-Yen Chen, Elsayed I. Mahmoud, and Abdallah A. Syied; Resources: Sameh Shenawy, Bang-Yen Chen, Elsayed I. Mahmoud, Nasser Bin Turki, and Abdallah A. Syied; Data Curation: Sameh Shenawy, Bang-Yen Chen, Elsayed I. Mahmoud, Nasser Bin Turki, and Abdallah A. Syied; Writing—Original Draft: Sameh Shenawy, Bang-Yen Chen, Elsayed I. Mahmoud, Nasser Bin Turki, and Abdallah A. Syied; Writing—Review & Editing: Sameh Shenawy, Bang-Yen Chen, Elsayed I. Mahmoud, Nasser Bin Turki, and Abdallah A. Syied; Supervision: Sameh Shenawy, Bang-Yen Chen, and Abdallah A.

Syied; Project administration: Sameh Shenawy, Bang-Yen Chen, and Abdallah A. Syied. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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