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Research article

Optimistic multigranulation S-approximation spaces and three-way decisions

Jing Huang*

College of Mathematics and Information Engineering, Longyan University, Longyan, 364012, China

* **Correspondence:** Email: 2044631554@qq.com; Tel: +8618760192515.

Abstract: This paper studies a multigranulation extension of S-approximation spaces based on an optimistic fusion principle. Instead of using a single knowledge mapping, we consider a family of knowledge mappings defined on the same universe and evaluated by a common decision mapping. Given this setting, we introduce optimistic lower and upper approximation operators and derive the corresponding acceptance, rejection, and deferment regions from the viewpoint of three-way decisions. Several basic properties of the proposed operators are established, including duality and monotonicity-type relations. We also examine how the new model is connected with the underlying single-granularity S-approximation spaces. In addition, when the involved granularities satisfy a non-contradictory condition, we prove that optimistic combination preserves decisions' consistency and yields extremal regional behavior, namely, the largest acceptance region and the smallest deferment region among admissible combination strategies. The obtained results extend the theory of S-approximation spaces to a multigranulation setting and provide a formal basis for three-way decision analysis with multi-source knowledge. A semi-realistic multi-source assessment case study and a quantitative comparison with classical optimistic multigranulation rough set, majority voting, and averaging-based fusion strategies are also presented to illustrate the applicability and decisiveness of the proposed model.

Keywords: S-approximation space; multigranulation rough set; optimistic fusion; three-way decision; uncertainty reasoning
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1. Introduction and basic definitions

In many information processing tasks, the same object may be observed, measured, or described by more than one source. Each source may provide a different granularity of knowledge, and these granularities may not always lead to identical approximation results. A central question is therefore how such heterogeneous descriptions can be incorporated into one approximation framework without losing their individual decision information. This question is particularly relevant for rough-set-based models, where uncertainty is usually represented through lower and upper approximations.

Pawlak's rough set theory provides a basic mathematical tool for studying uncertainty through indiscernibility and approximation [1]. In its classical form, the approximation process is usually generated by a single granularity. However, this setting is often too restrictive when objects are described by multiple information sources, different attribute groups, or several scales of observation. To overcome this limitation, multigranulation rough sets have been investigated in various forms. Typical developments include neighborhood-based multigranulation rough sets [2], matrix-based updating methods for approximations [3], fuzzy covering models [4], generalized multi-scale decision tables [5], multi-scale fuzzy granulation [6], dynamic target approximation [7], and granular-ball rough sets [8]. These studies show that multigranulation structures are useful for representing knowledge obtained from different viewpoints.

Although multigranulation rough sets have been widely studied, their direct use does not immediately answer a related question in S-approximation spaces. The theory of S-approximation spaces, introduced by Hooshmandasl et al. [9], separates the description of an object from the evaluation of a decision. In this framework, a knowledge mapping assigns descriptions to objects, and a decision mapping determines whether a description supports a target concept. This separation makes S-approximation spaces different from approximation models generated only by equivalence relations. It also makes them suitable for further extensions in which the source of knowledge and the decision criterion are treated independently. Existing studies have connected S-approximation spaces with three-way decision semantics [10], neighborhood systems [11], and intuitionistic fuzzy information [12].

The decision meaning of an approximation model becomes clearer when it is interpreted through three-way decisions. Instead of forcing each object into a binary classification, three-way decision theory divides objects into acceptance, rejection, and deferment regions [13]. This viewpoint has been discussed in terms of triadic thinking and decision interpretation [14]. In decision-theoretic rough sets and their interval-valued extensions, the deferment region describes those objects for which a definite decision is not yet justified by the available information [15]. Related studies on granular structures and uncertainty measures also indicate that the deferment region carries meaningful undecided information rather than being a negligible residual part [16]. Sequential three-way decision models also show that a delayed decision may be useful when additional information can reduce the decision risk [17].

These observations suggest that multigranulation structures, S-approximation spaces, and three-way decisions are closely related. Nevertheless, their integration is not straightforward. On the one hand, multigranulation rough sets focus on combining several granularities. On the other hand, S-approximation spaces emphasize the interaction between a knowledge mapping and a decision mapping. If several knowledge mappings are available, then it is necessary to determine how these mappings should jointly generate lower and upper approximations, and how the corresponding acceptance, rejection, and deferment regions should be obtained.

Some recent studies have considered multigranulation and three-way decision models from different perspectives. For example, multigranulation sequential three-way decisions have been studied by using multiple thresholds [18], and multi-granularity variable precision fuzzy rough sets have been combined with TOPSIS (technique for order preference by similarity to ideal solution) for decision-making problems [19]. In dynamic multi-scale decision information systems, sequential three-way decision models have been applied to optimal scale selection [20] and local optimal scale selection [21].

Recent developments also include cross-entropy and overlap-function-based sequential three-way decision methods [22] and multi-granularity variable precision neutrosophic rough sets with optimistic, pessimistic, and compromise strategies for group decision-making [23]. These works confirm the importance of granularity combinations in three-way decision analysis. However, most existing S-approximation models are still formulated with a single knowledge mapping, and a systematic optimistic multigranulation version of S-approximation spaces has not been fully developed.

To make the theoretical contribution more explicit, Table 1 compares the proposed framework with closely related models. The comparison shows that the novelty of the present study is not only the use of several granularities, but also the way in which these granularities are embedded in an S-approximation setting through a common decision mapping and then interpreted as three-way decision regions.

Table 1. Comparison between the proposed model and related approximation/decision frameworks.

Framework	Knowledge representation	Role of decision mapping	Fusion/approximation rule	Three-way decision regions
Classical rough sets	One equivalence relation or one granulation	Implicitly determined by inclusion or indiscernibility	One lower–upper approximation pair	Positive, negative, and boundary regions from one granulation
Multigranulation rough sets	Several relations, coverings, neighborhoods, or scales	Usually embedded in the granulation structure	Optimistic combination of rough approximations	Regions generated from combined rough approximations
Classical S-approximation spaces	One knowledge mapping $K : U \to 2^W$	Explicit $S : 2^W \times 2^W \to \{0, 1\}$ separates description and evaluation	$\mathcal{L}_G(P)$ and $\mathcal{U}_G(P)$ from one K and one S	Acc_G, Rej_G, and Def_G for one source
Proposed optimistic multigranulation S-approximation spaces	Indexed family $(K_i)_{i=1}^m$ on the same universe	A common S evaluates all granular descriptions	$\mathcal{L}^o_{\mathcal{K}}(P) = \bigcup_{i=1}^m \mathcal{L}_{G_i}(P)$ and $\mathcal{U}^o_{\mathcal{K}}(P) = \bigcap_{i=1}^m \mathcal{U}_{G_i}(P)$	Multi-source acceptance, rejection, and deferment from support and non-exclusion

The present paper is motivated by the following problem. Suppose that a finite family of knowledge mappings is defined on the same object universe and evaluated by a common decision mapping. If at least one granularity provides reliable support for a target concept, should this local support be allowed to influence the global lower approximation? Conversely, if no granularity supports the complement of the target concept, can the object be retained in the global upper approximation? These questions lead naturally to an optimistic fusion rule. Compared with conservative combination rules, which usually require simultaneous support from several granularities [2,4,24], an optimistic rule allows useful evidence from individual granularities to participate in the final approximation result.

Based on this motivation, this paper develops an optimistic multigranulation S-approximation model. A family of knowledge mappings is introduced into the framework of S-approximation spaces, while the decision mapping is kept common to all granularities. The resulting model provides a way to construct optimistic lower and upper approximations and to interpret the induced approximation regions through three-way decisions.

The main contributions of this paper are summarized as follows.

- A multigranulation version of S-approximation spaces is formulated by using a finite family of knowledge mappings together with a common decision mapping.
- Optimistic lower and upper approximation operators are introduced, and their fundamental

properties, including duality, monotonicity, and set-theoretic inclusion relations, are studied.
- The relationship between the optimistic multigranulation model and the associated single-granularity S-approximation spaces is established, which clarifies how individual granularities contribute to the final approximation regions.
- A three-way decision interpretation of the proposed model is constructed. Under a non-contradictory condition, optimistic fusion is shown to preserve decision's consistency, enlarge the admissible acceptance region, and reduce the deferment region.
- The proposed rule is compared with classical optimistic multigranulation rough set, majority-voting, and averaging-based fusion strategies, and a semi-realistic multi-source assessment case study is provided to illustrate the computation of the approximation regions.

Before introducing the proposed model, we recall the basic notions used throughout the paper. Let U and W be finite nonempty sets. The set U denotes the universe of objects, and W denotes the feature domain. For any $P \subseteq W$, the complement of P in W is denoted by P^c.

Definition 1.1. [9] An S-approximation space is a quadruple

$$G = (U, W, K, S),$$

where U and W are finite nonempty sets, $K : U \to 2^W$ is a knowledge mapping, and

$$S : 2^W \times 2^W \to \{0, 1\}$$

is a decision mapping. For each $P \subseteq W$, the lower and upper approximations of P induced by G are defined respectively by

$$\mathcal{L}_G(P) = \{x \in U : S(K(x), P) = 1\}, \qquad \mathcal{U}_G(P) = \{x \in U : S(K(x), P^c) = 0\}.$$

In this definition, the mapping K and the mapping S play different roles. The former produces a knowledge description of an object, while the latter evaluates this description with respect to a target set. Thus, the pair $(\mathcal{L}_G(P), \mathcal{U}_G(P))$ records two decision statuses generated by S: Support for P and non-support for P^c. This distinction will be used below to introduce several knowledge mappings without changing the decision criterion.

Definition 1.2. *[10] Let $G = (U, W, K, S)$ be an* S*-approximation space and let $P \subseteq W$. The three-way decision regions associated with P are defined by*

$$\begin{aligned}
\mathrm{Acc}_G(P) &= \{x \in U : S(K(x), P) = 1 \text{ and } S(K(x), P^c) = 0\},\\
\mathrm{Rej}_G(P) &= \{x \in U : S(K(x), P) = 0 \text{ and } S(K(x), P^c) = 1\},\\
\mathrm{Def}_G(P) &= \{x \in U : S(K(x), P) = S(K(x), P^c)\}.
\end{aligned}$$

The regions $\mathrm{Acc}_G(P)$, $\mathrm{Rej}_G(P)$, and $\mathrm{Def}_G(P)$ correspond, respectively, to acceptance, rejection, and deferment. They provide the decision-theoretic interpretation of an S-approximation space and will serve as the reference model for the multigranulation construction.

In what follows, we consider a family of single-granularity S-approximation spaces

$$G_i = (U, W, K_i, S), \quad i = 1, 2, \ldots, m,$$

where all spaces share the same object universe U, the feature domain W, and the decision mapping S. The mappings $K_i : U \to 2^W$ represent different knowledge granularities. The next section constructs an optimistic multigranulation S-approximation space from this family and analyzes the approximation operators induced by it.

The remainder of this paper is arranged as follows. Section 2 develops the optimistic multigranulation S-approximation framework and records its basic properties. Section 3 examines the optimistic knowledge-fusion mechanism, compares it with other fusion strategies, and constructs the resulting three-way decision regions. Section 4 investigates the structural behavior of the proposed construction under non-contradictory conditions. Section 5 presents a semi-realistic multi-source assessment case study. Section 6 summarizes the conclusions and points out possible directions for future work.

2. Optimistic multigranulation S-approximation space model

2.1. Basic definitions of optimistic multigranulation S-approximation spaces

In a multigranulation setting, the same object may be described in several different ways, where each description corresponds to a particular knowledge source, viewpoint, or granularity level. Some granularities may focus on local characteristics, while others may reflect more global structural features. Hence, the descriptions generated by different granularities are not necessarily identical, but they may still provide complementary information for approximation and decision analysis. To incorporate this situation into the framework of S-approximation spaces, we introduce an optimistic multigranulation model.

We begin with an abstract formulation based on a multi-knowledge mapping, which is used to describe the general form of multigranulation approximation. After that, we consider the special case induced by a finite family of single-granularity S-approximation spaces, since this setting will be used in the subsequent theoretical analysis.

In the revised notation, target subsets of the feature domain are denoted as $P, Q, R \subseteq W$. Knowledge mappings are denoted as K_i, and the induced family of granular descriptions is denoted as $\mathcal{K}$. For repeated decision responses, we use the compact notation

$$r_i(x; P) = S(K_i(x), P), \qquad \bar{r}_i(x; P) = S(K_i(x), P^c).$$

The optimistic lower and upper operators are written as $\mathcal{L}^o_{\mathcal{K}}$ and $\mathcal{U}^o_{\mathcal{K}}$, and the three-way regions are written as Acc, Rej, and Def.

Definition 2.1. *A quadruple $M^o = (U, W, \mathcal{K}, S)$ is called a multigranulation S-approximation space, where U and W are finite nonempty sets,*

$$\mathcal{K} : U \to 2^{2^W}$$

is a multi-knowledge mapping such that $\mathcal{K}(x) \neq \emptyset$ for each $x \in U$, and

$$S : 2^W \times 2^W \to \{0, 1\}$$

is a decision mapping. For any $P \subseteq W$, define the optimistic lower and upper approximations of P as

$$\mathcal{L}^o_{\mathcal{K}}(P) = \{x \in U \mid \exists Q \in \mathcal{K}(x) \text{ such that } S(Q, P) = 1\},$$
$$\mathcal{U}^o_{\mathcal{K}}(P) = \{x \in U \mid \forall Q \in \mathcal{K}(x),\ S(Q, P^c) = 0\}.$$

In Definition 2.1, $\mathcal{K}(x)$ is written as a set of knowledge descriptions. This notation is adopted for the abstract multi-knowledge mapping because the optimistic lower and upper approximation operators depend only on existential and universal conditions over the descriptions in $\mathcal{K}(x)$. Therefore, repeated identical descriptions do not change the resulting approximation sets. When it is necessary to indicate the source of each granularity, we use the indexed family $\{K_i\}_{i=1}^m$ and the corresponding single-granularity S-approximation spaces $G_i = (U, W, K_i, S)$ introduced below.

Example 2.2. *Let*

$$U = \{x\}, \quad W = \{a, b, c\},$$

and suppose that three granularities generate the following descriptions for the object x:

$$K_1(x) = \{a, b\}, \quad K_2(x) = \{a, b\}, \quad K_3(x) = \{c\}.$$

Then the indexed family of descriptions is

$$(K_i(x))_{i=1}^3 = (\{a, b\}, \{a, b\}, \{c\}),$$

where the first and second components come from different granularities. By contrast, the set-valued notation used in Definition 2.1 gives

$$\mathcal{K}(x) = \{\{a, b\}, \{c\}\},$$

where the repeated description $\{a, b\}$ is merged.

Now let $P = \{a, b\}$ and define the decision mapping S as

$$S(Q, P) = 1 \iff Q \subseteq P.$$

Then

$$S(K_1(x), P) = 1, \qquad S(K_2(x), P) = 1, \qquad S(K_3(x), P) = 0.$$

Hence, under the indexed-family interpretation,

$$\exists i \in \{1, 2, 3\} \text{ such that } S(K_i(x), P) = 1.$$

Under the set-valued interpretation, we also have

$$\exists Q \in \mathcal{K}(x) \text{ such that } S(Q, P) = 1,$$

because $Q = \{a, b\} \in \mathcal{K}(x)$. Therefore, the membership of x in the optimistic lower approximation is unchanged.

Similarly, for the optimistic upper approximation, the universal condition over repeated identical descriptions is also unchanged:

$$\forall i \in \{1, 2, 3\},\ S(K_i(x), P^c) = 0 \iff \forall Q \in \mathcal{K}(x),\ S(Q, P^c) = 0.$$

Thus, although the set-valued notation may merge identical descriptions generated by different granularities, this merging does not affect the optimistic lower and upper approximations. The source information can still be retained in later componentwise analysis through the indexed mappings K_i, $i = 1, 2, \ldots, m$.

The word "optimistic" refers to the existential nature of the lower approximation. An object enters $\mathcal{L}^o_{\mathcal{K}}(P)$ once at least one description in $\mathcal{K}(x)$ supports P. The upper approximation is more restrictive: x is kept in $\mathcal{U}^o_{\mathcal{K}}(P)$ only when no description in $\mathcal{K}(x)$ supports P^c. Thus, the model separates positive support for P from possible evidence against P.

The complete information flow of the proposed model can be summarized as

$$\begin{aligned} \{K_i\}_{i=1}^m \text{ on } U \quad &\Longrightarrow \quad \mathcal{K}(x) = \{K_1(x), K_2(x), \ldots, K_m(x)\} \quad \Longrightarrow \quad \{r_i(x;P), \bar{r}_i(x;P)\}_{i=1}^m \\ &\Longrightarrow \quad \mathcal{L}^o_{\mathcal{K}}(P),\ \mathcal{U}^o_{\mathcal{K}}(P) \quad \Longrightarrow \quad \mathrm{Acc}_{M^o}(P),\ \mathrm{Rej}_{M^o}(P),\ \mathrm{Def}_{M^o}(P). \end{aligned}$$

Definition 2.3. *Let*

$$G_i = (U, W, K_i, S), \quad i = 1, 2, \ldots, m,$$

be a finite family of single-granularity S*-approximation spaces with the same object domain U, feature domain W, and decision mapping S, where each*

$$K_i : U \to 2^W$$

is a knowledge mapping. The optimistic multigranulation S*-approximation space induced by $\{G_i\}_{i=1}^m$ is defined as*

$$M^o = (U, W, \mathcal{K}, S), \qquad \mathcal{K}(x) = (K_i(x))_{i=1}^m, \quad x \in U.$$

For $P \subseteq W$, using the response values $r_i(x;P)$ and $\bar{r}_i(x;P)$, the optimistic lower and upper approximations of P are written as follows:

$$\begin{aligned} \mathcal{L}^o_{\mathcal{K}}(P) &= \{x \in U \mid r_i(x;P) = 1 \textit{ for some } i \in \{1, 2, \ldots, m\}\}, \\ \mathcal{U}^o_{\mathcal{K}}(P) &= \{x \in U \mid \bar{r}_i(x;P) = 0 \textit{ for all } i \in \{1, 2, \ldots, m\}\}. \end{aligned}$$

With the response values $r_i(x;P)$ and $\bar{r}_i(x;P)$, the optimistic rule has a compact interpretation. Membership in $\mathcal{L}^o_{\mathcal{K}}(P)$ requires one positive response among the values $r_i(x;P)$, whereas membership in $\mathcal{U}^o_{\mathcal{K}}(P)$ requires all values $\bar{r}_i(x;P)$ to be zero. Thus, a single supporting granularity can contribute to acceptance, while exclusion is avoided only when no granularity supports P^c.

The corresponding pessimistic multigranulation model, in which acceptance usually depends on simultaneous support from all granularities, is not considered in this paper and will be investigated in future work.

2.2. *Basic properties of optimistic multigranulation* S*-approximation spaces*

We now study several fundamental properties of the operators $\mathcal{L}^o_{\mathcal{K}}$ and $\mathcal{U}^o_{\mathcal{K}}$. These results show that the proposed optimistic model satisfies the natural structural requirements expected for approximation operators.

Unless otherwise stated, the results in this subsection concern the induced optimistic multigranulation space introduced in Definition 2.2.

Proposition 2.4 (Duality)**.** *For every $P \subseteq W$, we have*

$$\mathcal{L}^o_{\mathcal{K}}(P^c) = (\mathcal{U}^o_{\mathcal{K}}(P))^c, \quad \mathcal{U}^o_{\mathcal{K}}(P^c) = (\mathcal{L}^o_{\mathcal{K}}(P))^c.$$

Proof. For $x \in U$, we have

$$\begin{aligned} x \in \mathcal{L}^o_{\mathcal{K}}(P^c) &\iff \exists i \in \{1, 2, \dots, m\},\ \bar{r}_i(x; P) = 1 \\ &\iff \neg(\forall i \in \{1, 2, \dots, m\},\ \bar{r}_i(x; P) = 0) \\ &\iff x \notin \mathcal{U}^o_{\mathcal{K}}(P). \end{aligned}$$

Thus

$$\mathcal{L}^o_{\mathcal{K}}(P^c) = (\mathcal{U}^o_{\mathcal{K}}(P))^c.$$

Substituting P^c for P gives

$$\mathcal{U}^o_{\mathcal{K}}(P^c) = (\mathcal{L}^o_{\mathcal{K}}(P))^c.$$

□

To analyze the behavior of these operators under set inclusion, we use the following monotonicity assumption on the decision mapping.

Definition 2.5 (Monotonicity)**.** *The decision mapping S is called monotonic if, for all $K, P, Q \subseteq W$, we have*

$$P \subseteq Q \text{ and } S(K, P) = 1 \Longrightarrow S(K, Q) = 1.$$

This property states that if a knowledge description supports a target set, then it also supports any larger target set containing it. The assumption is natural for many decision mappings used in rough set and granular computing models. For example, the rules $S(K, R) = 1 \iff K \cap R \neq \emptyset$, $S(K, R) = 1 \iff |K \cap R| \geq t$, and threshold rules based on a monotone overlap or containment score all satisfy monotonicity. Since W is finite in the present setting, monotonicity can be checked directly by verifying all pairs $P \subseteq Q \subseteq W$ and all possible descriptions generated by the mappings K_i.

Proposition 2.6. *If S is monotonic and $P \subseteq Q \subseteq W$, then*

$$\mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{L}^o_{\mathcal{K}}(Q), \quad \mathcal{U}^o_{\mathcal{K}}(P) \subseteq \mathcal{U}^o_{\mathcal{K}}(Q).$$

Proof. Let $P \subseteq Q$. Monotonicity and $Q^c \subseteq P^c$ give

$$\begin{aligned} x \in \mathcal{L}^o_{\mathcal{K}}(P) &\iff \exists i \in \{1, 2, \dots, m\},\ r_i(x; P) = 1 \\ &\Longrightarrow \exists i \in \{1, 2, \dots, m\},\ r_i(x; Q) = 1 \\ &\iff x \in \mathcal{L}^o_{\mathcal{K}}(Q), \end{aligned}$$

$$\begin{aligned} x \notin \mathcal{U}^o_{\mathcal{K}}(Q) &\iff \exists j \in \{1, 2, \dots, m\},\ \bar{r}_j(x; Q) = 1 \\ &\Longrightarrow \exists j \in \{1, 2, \dots, m\},\ \bar{r}_j(x; P) = 1 \\ &\iff x \notin \mathcal{U}^o_{\mathcal{K}}(P). \end{aligned}$$

Hence

$$\mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{L}^o_{\mathcal{K}}(Q), \qquad \mathcal{U}^o_{\mathcal{K}}(P) \subseteq \mathcal{U}^o_{\mathcal{K}}(Q).$$

□

The following proposition describes the behavior of optimistic lower approximations with respect to union and intersection.

Proposition 2.7. *If S is monotonic, then for any $P, Q \subseteq W$, we have*

$$\mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q) \subseteq \mathcal{L}^o_{\mathcal{K}}(P \cup Q),$$

and

$$\mathcal{L}^o_{\mathcal{K}}(P \cap Q) \subseteq \mathcal{L}^o_{\mathcal{K}}(P) \cap \mathcal{L}^o_{\mathcal{K}}(Q).$$

Proof. For $R \in \{P, Q\}$, $R \subseteq P \cup Q$. Hence

$$\begin{aligned} x \in \mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q) &\Longleftrightarrow \exists R \in \{P, Q\},\ \exists i,\ r_i(x; R) = 1 \\ &\Longrightarrow \exists i,\ S(K_i(x), P \cup Q) = 1 \\ &\Longleftrightarrow x \in \mathcal{L}^o_{\mathcal{K}}(P \cup Q). \end{aligned}$$

Since $P \cap Q \subseteq P$ and $P \cap Q \subseteq Q$, we have

$$\begin{aligned} x \in \mathcal{L}^o_{\mathcal{K}}(P \cap Q) &\Longleftrightarrow \exists i,\ S(K_i(x), P \cap Q) = 1 \\ &\Longrightarrow \exists i,\ (r_i(x; P) = 1 \text{ and } r_i(x; Q) = 1) \\ &\Longrightarrow x \in \mathcal{L}^o_{\mathcal{K}}(P) \cap \mathcal{L}^o_{\mathcal{K}}(Q). \end{aligned}$$

The two implications yield the stated inclusions. □

Example 2.8. *Let*

$$U = \{u_1, u_2, u_3\}, \quad W = \{w_1, w_2, w_3, w_4\}, \quad P = \{w_1, w_2\}, \quad Q = \{w_3, w_4\}.$$

Define

$$S(K, R) = 1 \Longleftrightarrow |K \cap R| \geq 2.$$

Consider two knowledge mappings $K_1, K_2 : U \to 2^W$ given by

$$\begin{aligned} &K_1(u_1) = \{w_1, w_2\}, \quad K_1(u_2) = \varnothing, \quad K_1(u_3) = \{w_1, w_3\}, \\ &K_2(u_1) = \varnothing, \quad K_2(u_2) = \{w_3, w_4\}, \quad K_2(u_3) = \varnothing. \end{aligned}$$

Then

$$\mathcal{L}^o_{\mathcal{K}}(P) = \{u_1\}, \qquad \mathcal{L}^o_{\mathcal{K}}(Q) = \{u_2\}.$$

Since

$$P \cup Q = W,$$

we obtain

$$\mathcal{L}^o_{\mathcal{K}}(P \cup Q) = \{u_1, u_2, u_3\}.$$

For u_3, separate support fails, whereas $K_1(u_3) = \{w_1, w_3\}$ contributes through $P \cup Q$. Hence

$$\mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q) \subset \mathcal{L}^o_{\mathcal{K}}(P \cup Q).$$

The inclusion in Proposition 2.6 may therefore be strict.

Example 2.9. *Let*

$$U = \{u_1, u_2, u_3\}, \quad W = \{w_1, w_2, w_3\}, \quad P = \{w_1, w_2\}, \quad Q = \{w_2, w_3\}.$$

Define

$$S(K, R) = 1 \iff |K \cap R| \geq 2.$$

Let $K_1, K_2 : U \to 2^W$ *be defined as follows:*

$$K_1(u_1) = \{w_1, w_2\}, \quad K_1(u_2) = \varnothing, \quad K_1(u_3) = \{w_1, w_2\},$$
$$K_2(u_1) = \varnothing, \quad K_2(u_2) = \{w_2, w_3\}, \quad K_2(u_3) = \{w_2, w_3\}.$$

Then

$$\mathcal{L}^o_{\mathcal{K}}(P) = \{u_1, u_3\}, \qquad \mathcal{L}^o_{\mathcal{K}}(Q) = \{u_2, u_3\}.$$

However,

$$P \cap Q = \{w_2\}.$$

No $K_i(x)$ *has two common elements with* $\{w_2\}$*, so*

$$\mathcal{L}^o_{\mathcal{K}}(P \cap Q) = \varnothing.$$

Therefore,

$$\mathcal{L}^o_{\mathcal{K}}(P \cap Q) \subset \mathcal{L}^o_{\mathcal{K}}(P) \cap \mathcal{L}^o_{\mathcal{K}}(Q).$$

Thus u_3 *is supported separately for* P *and* Q*, but not for* $P \cap Q$*.*

We next establish the analogous inclusion properties for optimistic upper approximations.

Proposition 2.10. *If* S *is monotonic, then for any* $P, Q \subseteq W$*, we have*

$$\mathcal{U}^o_{\mathcal{K}}(P) \cup \mathcal{U}^o_{\mathcal{K}}(Q) \subseteq \mathcal{U}^o_{\mathcal{K}}(P \cup Q),$$

and

$$\mathcal{U}^o_{\mathcal{K}}(P \cap Q) \subseteq \mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q).$$

Proof. Since $(P \cup Q)^c \subseteq P^c$ and $(P \cup Q)^c \subseteq Q^c$, we have

$$\begin{aligned} x \notin \mathcal{U}^o_{\mathcal{K}}(P \cup Q) &\iff \exists i,\ S(K_i(x), (P \cup Q)^c) = 1 \\ &\implies x \notin \mathcal{U}^o_{\mathcal{K}}(P) \text{ and } x \notin \mathcal{U}^o_{\mathcal{K}}(Q). \end{aligned}$$

Taking complements gives

$$\mathcal{U}^o_{\mathcal{K}}(P) \cup \mathcal{U}^o_{\mathcal{K}}(Q) \subseteq \mathcal{U}^o_{\mathcal{K}}(P \cup Q).$$

Likewise, $P^c, Q^c \subseteq (P \cap Q)^c$. By the contraposition of monotonicity,

$$\begin{aligned} x \in \mathcal{U}^o_{\mathcal{K}}(P \cap Q) &\iff \forall i,\ S(K_i(x), (P \cap Q)^c) = 0 \\ &\implies \forall i,\ (\bar{r}_i(x; P) = 0 \text{ and } \bar{r}_i(x; Q) = 0) \\ &\iff x \in \mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q). \end{aligned}$$

Thus

$$\mathcal{U}^o_{\mathcal{K}}(P \cap Q) \subseteq \mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q).$$

□

Example 2.11. *Let*

$$U = \{u_1, u_2, u_3\}, \quad W = \{w_1, w_2, w_3, w_4\}, \quad P = \{w_1, w_2\}, \quad Q = \{w_3, w_4\}.$$

Define

$$S(K, R) = 1 \iff |K \cap R| \geq 2.$$

Let $K_1, K_2 : U \to 2^W$ *be given by*

$$K_1(u_1) = \{w_1, w_3\}, \quad K_1(u_2) = \{w_2, w_4\}, \quad K_1(u_3) = \{w_3, w_4\},$$
$$K_2(u_1) = \varnothing, \quad K_2(u_2) = \varnothing, \quad K_2(u_3) = \{w_1, w_2\}.$$

Since

$$P^c = \{w_3, w_4\}, \qquad Q^c = \{w_1, w_2\},$$

we get

$$\mathcal{U}^o_{\mathcal{K}}(P) = \{u_1, u_2\}, \qquad \mathcal{U}^o_{\mathcal{K}}(Q) = \{u_1, u_2\}.$$

However,

$$P \cup Q = W,$$

and hence

$$(P \cup Q)^c = \varnothing.$$

The empty set receives no support under this S*, so*

$$\mathcal{U}^o_{\mathcal{K}}(P \cup Q) = U.$$

Hence

$$\mathcal{U}^o_{\mathcal{K}}(P) \cup \mathcal{U}^o_{\mathcal{K}}(Q) \subset \mathcal{U}^o_{\mathcal{K}}(P \cup Q).$$

Thus the union-related upper inclusion may also be strict.

Example 2.12. *Let*

$$U = \{u_1, u_2\}, \qquad W = \{w_1, w_2, w_3, w_4\}, \qquad P = \{w_1, w_2\}, \qquad Q = \{w_1, w_3\}.$$

Define

$$S(K, R) = 1 \iff |K \cap R| \geq 2.$$

Let $K_1, K_2 : U \to 2^W$ *be given by*

$$K_1(u_1) = \{w_1\}, \quad K_1(u_2) = \{w_2, w_3\},$$
$$K_2(u_1) = \varnothing, \quad K_2(u_2) = \varnothing.$$

Since

$$P^c = \{w_3, w_4\}, \qquad Q^c = \{w_2, w_4\},$$

the set $K_1(u_2) = \{w_2, w_3\}$ *supports neither* P^c *nor* Q^c*. Hence*

$$\mathcal{U}^o_{\mathcal{K}}(P) = U, \qquad \mathcal{U}^o_{\mathcal{K}}(Q) = U.$$

On the other hand, we have

$$P \cap Q = \{w_1\}, \qquad (P \cap Q)^c = \{w_2, w_3, w_4\}.$$

Since

$$|K_1(u_2) \cap (P \cap Q)^c| = |\{w_2, w_3\}| = 2,$$

we have

$$u_2 \notin \mathcal{U}^o_{\mathcal{K}}(P \cap Q).$$

Moreover, $u_1 \in \mathcal{U}^o_{\mathcal{K}}(P \cap Q)$*, so*

$$\mathcal{U}^o_{\mathcal{K}}(P \cap Q) = \{u_1\}.$$

Consequently,

$$\mathcal{U}^o_{\mathcal{K}}(P \cap Q) \subset \mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q).$$

The second inclusion in Proposition 2.9 may therefore be strict.

For the deferment region to have the usual three-way decision interpretation, the lower approximation should be included in the upper approximation. In the present multigranulation model, however, this inclusion does not hold automatically. Therefore, we introduce a non-contradictory condition that guarantees the required inclusion.

Definition 2.13. *Let* $P \subseteq W$*. The family of granularities is said to be non-contradictory with respect to* P *if, for every* $x \in U$*, we have*

$$(\exists i \in \{1, 2, \ldots, m\} \text{ such that } r_i(x; P) = 1) \Rightarrow \Big(\forall j \in \{1, 2, \ldots, m\},\ \bar{r}_j(x; P) = 0\Big).$$

This condition rules out the possibility that an object supported by one granularity is simultaneously rejected by another granularity with respect to the same target concept. It is reasonable in multi-source knowledge systems in which the sources are regarded as admissible descriptions of the same target and direct support for P is not allowed to coexist with direct support for P^c. In applications, the condition can be verified by computing, for every $x \in U$, the response vectors $(r_i(x; P))_{i=1}^m$ and $(\bar{r}_i(x; P))_{i=1}^m$, equivalently, by checking whether $\mathrm{Sup}_i(P) \cap \mathrm{Exc}_j(P) = \varnothing$ for all $i, j \in \{1, 2, \ldots, m\}$. After the response values have been computed, this verification requires only finite pairwise set intersections.

The condition is also necessary for the standard three-way interpretation. For example, let $U = \{x\}$, $W = \{w_1, w_2\}$, and $P = \{w_1\}$, and let $S(K, R) = 1$ if and only if $K \cap R \neq \varnothing$. If $K_1(x) = \{w_1\}$ and $K_2(x) = \{w_2\}$, then $x \in \mathcal{L}^o_{\mathcal{K}}(P)$ because the first granularity supports P, but $x \notin \mathcal{U}^o_{\mathcal{K}}(P)$ because the second granularity supports P^c. Hence, $\mathcal{L}^o_{\mathcal{K}}(P) \nsubseteq \mathcal{U}^o_{\mathcal{K}}(P)$, and the usual acceptance–rejection–deferment partition is not valid.

Remark 2.14. *The non-contradictory condition can also be interpreted via the conflict region. For* $P \subseteq W$*, define*

$$\mathrm{Conf}_{M^o}(P) = \left(\bigcup_{i=1}^m \mathrm{Sup}_i(P)\right) \cap \left(\bigcup_{j=1}^m \mathrm{Exc}_j(P)\right).$$

Then $\mathrm{Conf}_{M^o}(P)$ *consists of the objects that are supported for acceptance by at least one granularity and supported for rejection by at least one possibly different granularity. Hence, we have*

$$\mathrm{Conf}_{M^o}(P) = \varnothing$$

is equivalent to the non-contradictory condition in the preceding definition.

It should be noted that $\mathrm{Conf}_{M^o}(P)$ is different from the deferment region $\mathrm{Def}_{M^o}(P)$ defined below. The former describes inconsistent multi-source evidence, whereas the latter describes insufficient evidence for making either an acceptance or a rejection decision.

Proposition 2.15. *If the family of granularities is non-contradictory with respect to P, then*

$$\mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{U}^o_{\mathcal{K}}(P).$$

Proof. For $x \in U$, the non-contradictory condition gives the implication chain

$$\begin{aligned} x \in \mathcal{L}^o_{\mathcal{K}}(P) &\iff \exists i \in \{1, 2, \dots, m\},\ r_i(x; P) = 1 \\ &\implies \forall j \in \{1, 2, \dots, m\},\ \bar{r}_j(x; P) = 0 \\ &\iff x \in \mathcal{U}^o_{\mathcal{K}}(P). \end{aligned}$$

Therefore

$$\mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{U}^o_{\mathcal{K}}(P).$$

By Proposition 2.13, the inclusion $\mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{U}^o_{\mathcal{K}}(P)$ holds under the non-contradictory condition. We can therefore define the deferment region and approximation accuracy in the usual way. □

Definition 2.16. *Let $P \subseteq W$. Suppose that $\mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{U}^o_{\mathcal{K}}(P)$. The deferment region of P is defined by*

$$\mathrm{Def}_{M^o}(P) = \mathcal{U}^o_{\mathcal{K}}(P) \setminus \mathcal{L}^o_{\mathcal{K}}(P).$$

If, in addition, $\mathcal{U}^o_{\mathcal{K}}(P) \neq \emptyset$, the approximation accuracy is defined by

$$\alpha_{M^o}(P) = \frac{|\mathcal{L}^o_{\mathcal{K}}(P)|}{|\mathcal{U}^o_{\mathcal{K}}(P)|}.$$

The assumption $\mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{U}^o_{\mathcal{K}}(P)$ is needed to maintain the standard three-way decision semantics. Under this inclusion, the deferment region is well defined as the part of the upper approximation that has not yet been accepted by the lower approximation.

3. Optimistic knowledge combination and the construction of three-way decisions

Having introduced the optimistic multigranulation approximation operators, we next analyze their connections with the approximation structures generated by the corresponding single-granularity spaces. This connection is useful because it reveals, in set-theoretic terms, how the information supplied by different granularities is integrated under the optimistic fusion principle.

Proposition 3.1 (Relation to the single-granularity case)**.** *Let $\{G_i = (U, W, K_i, S)\}_{i=1}^m$ be m single-granularity* S*-approximation spaces, and let*

$$M^o = (U, W, \mathcal{K}, S)$$

be the induced optimistic multigranulation S*-approximation space, where*

$$\mathcal{K}(x) = (K_i(x))_{i=1}^m, \qquad x \in U.$$

Then, for every $P \subseteq W$, we have

$$\mathcal{L}^o_{\mathcal{K}}(P) = \bigcup_{i=1}^m \mathcal{L}_{G_i}(P), \qquad \mathcal{U}^o_{\mathcal{K}}(P) = \bigcap_{i=1}^m \mathcal{U}_{G_i}(P).$$

Proof. For $x \in U$,

$$\begin{aligned} x \in \mathcal{L}^o_{\mathcal{K}}(P) &\iff \exists i \in \{1,2,\ldots,m\},\ r_i(x;P) = 1 \\ &\iff \exists i \in \{1,2,\ldots,m\},\ x \in \mathcal{L}_{G_i}(P) \\ &\iff x \in \bigcup_{i=1}^{m} \mathcal{L}_{G_i}(P), \end{aligned}$$

$$\begin{aligned} x \in \mathcal{U}^o_{\mathcal{K}}(P) &\iff \forall i \in \{1,2,\ldots,m\},\ \bar{r}_i(x;P) = 0 \\ &\iff \forall i \in \{1,2,\ldots,m\},\ x \in \mathcal{U}_{G_i}(P) \\ &\iff x \in \bigcap_{i=1}^{m} \mathcal{U}_{G_i}(P). \end{aligned}$$

Hence

$$\mathcal{L}^o_{\mathcal{K}}(P) = \bigcup_{i=1}^{m} \mathcal{L}_{G_i}(P), \qquad \mathcal{U}^o_{\mathcal{K}}(P) = \bigcap_{i=1}^{m} \mathcal{U}_{G_i}(P).$$

The proposition shows that optimistic fusion combines the lower approximations by union and the upper approximations by intersection. Thus, positive support from any granularity is preserved, while membership in the upper approximation requires non-exclusion by all granularities. □

We next define the three-way decision regions generated by the optimistic approximation operators $\mathcal{L}^o_{\mathcal{K}}$ and $\mathcal{U}^o_{\mathcal{K}}$.

Definition 3.2 (Three-way decision regions induced by M^o)**.** *Let $M^o = (U, W, \mathcal{K}, S)$ be an optimistic multigranulation* S*-approximation space and let $P \subseteq W$. Suppose that*

$$\mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{U}^o_{\mathcal{K}}(P).$$

Then the three-way decision regions induced by M^o with respect to P are defined by

$$\begin{aligned} \mathrm{Acc}_{M^o}(P) &= \mathcal{L}^o_{\mathcal{K}}(P), \\ \mathrm{Rej}_{M^o}(P) &= U \setminus \mathcal{U}^o_{\mathcal{K}}(P), \\ \mathrm{Def}_{M^o}(P) &= \mathcal{U}^o_{\mathcal{K}}(P) \setminus \mathcal{L}^o_{\mathcal{K}}(P). \end{aligned}$$

Under this condition, these three regions form a valid three-way partition of U.

This convention matches the acceptance–rejection–deferment interpretation recalled in Definition 1.2. Without the inclusion

$$\mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{U}^o_{\mathcal{K}}(P),$$

the three displayed sets need not form a three-way partition, and the term "three-way decision regions" is not used here.

3.1. Comparison with related optimistic fusion strategies

To clarify the position of the proposed model, this subsection compares it with several related optimistic fusion strategies. We focus on the classical optimistic multigranulation rough set model, the majority voting rule, and the averaging-based fusion rule. These strategies are selected because they are all based on non-pessimistic aggregation principles. In particular, majority voting is a standard and interpretable aggregation mechanism in ensemble learning [24]. Averaging-based fusion is also widely used in classifier combination and multicriteria aggregation. The sum rule and related fixed combination rules have been studied in classifier combination frameworks [25], and ordered weighted averaging aggregation operators provide a classical foundation for average-type aggregation in multicriteria decision-making [26]. Therefore, these rules provide simple but representative baselines for comparing how multi-source responses can be fused.

For a fixed target concept $P \subseteq W$, recall that

$$r_i(x;P) = S(K_i(x), P), \qquad \bar{r}_i(x;P) = S(K_i(x), P^c), \quad i = 1, 2, \ldots, m.$$

Let

$$c_P(x) = \sum_{i=1}^{m} r_i(x;P), \qquad c_{P^c}(x) = \sum_{i=1}^{m} \bar{r}_i(x;P)$$

be the support count and the exclusion count of x, respectively.

Under the non-contradictory condition, $\mathrm{Conf}_{M^o}(P) = \emptyset$. Hence the standard three-way regions induced by the proposed optimistic S-approximation model M^o can be written as

$$\mathrm{Acc}_{M^o}(P) = \{x \in U : \exists i \in \{1, 2, \ldots, m\},\ r_i(x;P) = 1\},$$

$$\mathrm{Rej}_{M^o}(P) = \{x \in U : \exists i \in \{1, 2, \ldots, m\},\ \bar{r}_i(x;P) = 1\},$$

and

$$\mathrm{Def}_{M^o}(P) = U \setminus (\mathrm{Acc}_{M^o}(P) \cup \mathrm{Rej}_{M^o}(P)).$$

Thus, the proposed rule accepts an object once at least one admissible granularity supports P, and rejects it when at least one admissible granularity supports P^c.

For majority-voting fusion, let

$$\tau = \left\lceil \frac{m+1}{2} \right\rceil.$$

Then the majority voting acceptance and rejection regions are given by

$$\mathrm{Acc}_{\mathrm{maj}}(P) = \{x \in U : c_P(x) \geq \tau\},$$

and

$$\mathrm{Rej}_{\mathrm{maj}}(P) = \{x \in U : c_{P^c}(x) \geq \tau\}.$$

Objects satisfying neither condition are placed into the deferment region

$$\mathrm{Def}_{\mathrm{maj}}(P) = U \setminus \left(\mathrm{Acc}_{\mathrm{maj}}(P) \cup \mathrm{Rej}_{\mathrm{maj}}(P)\right).$$

For averaging-based fusion, let

$$\omega_i \geq 0, \qquad \sum_{i=1}^{m} \omega_i = 1,$$

where ω_i denotes the weight or reliability of the i-th granularity. Define the weighted average support degrees as

$$a_P(x) = \sum_{i=1}^{m} \omega_i r_i(x; P), \qquad a_{P^c}(x) = \sum_{i=1}^{m} \omega_i \bar{r}_i(x; P).$$

In applications, the weights may be determined by domain knowledge, experts' reliability assessment, historical validation performance, or data-driven estimation. In the illustrative case study of Section 5, they are specified by normalized source-reliability scores.

Given two thresholds $\theta_a, \theta_b \in (0, 1]$, the averaging-based acceptance and rejection regions are defined as

$$\mathrm{Acc}_{\mathrm{avg}}(P) = \{x \in U : a_P(x) \geq \theta_a\},$$

and

$$\mathrm{Rej}_{\mathrm{avg}}(P) = \{x \in U : a_{P^c}(x) \geq \theta_b\}.$$

The deferment region is

$$\mathrm{Def}_{\mathrm{avg}}(P) = U \setminus \left(\mathrm{Acc}_{\mathrm{avg}}(P) \cup \mathrm{Rej}_{\mathrm{avg}}(P)\right).$$

When $\omega_i = 1/m$ for all i, this rule reduces to the unweighted arithmetic average of the binary responses. In this case, $a_P(x)$ is exactly the proportion of granularities supporting P, while $a_{P^c}(x)$ is the proportion of granularities supporting P^c. The weighted form can further incorporate the reliability of different information sources.

The relationship between the proposed model and the related optimistic fusion strategies is summarized in Table 2. The majority voting and averaging-based rules in this table are benchmark fusion criteria adapted to the present multigranulation S-approximation setting. They are not claimed to be new rough set approximation operators; rather, they are used to show how the proposed optimistic rule differs from common aggregation principles. The voting-type rule is motivated by general ensemble aggregation ideas, whereas the averaging-based rule is motivated by average-type classifier combination and multicriteria aggregation.

Table 2. Comparison with related optimistic fusion strategies.

Strategy	Fusion criterion	Relation to the proposed model
Classical optimistic multigranulation rough set	An object is accepted if at least one granule is contained in the target concept; that is, $K_i(x) \subseteq P$ for some i.	It is an inclusion-based special case of the proposed model when $W = U$ and $S(A, P) = 1$ if and only if $A \subseteq P$.
Proposed optimistic multigranulation S-approximation	An object is accepted if $$S(K_i(x), P) = 1$$ for some $i = 1, 2, \ldots, m$.	It replaces the inclusion relation $A \subseteq P$ with a general decision mapping $S(A, P)$, and therefore allows more flexible support relations.
Majority voting fusion	An object is accepted if $$\sum_{i=1}^{m} r_i(x; P) \geq \tau, \qquad \tau = \left\lceil \frac{m+1}{2} \right\rceil.$$	It filters isolated support by requiring more than half of the granularities to agree. It is stricter than the proposed optimistic rule when $m \geq 2$.
Averaging-based fusion	An object is accepted if $$a_P(x) = \sum_{i=1}^{m} \omega_i r_i(x; P) \geq \theta_a, \qquad \sum_{i=1}^{m} \omega_i = 1.$$	It aggregates multi-source responses by an average-type operator. The unweighted case corresponds to averaging the binary support responses, while the weighted case can incorporate source reliability.

The first row shows the direct connection between the proposed model and the classical optimistic multigranulation rough set model. If $W = U$ and

$$S(A, P) = 1 \quad \Longleftrightarrow \quad A \subseteq P,$$

then the optimistic lower approximation in the proposed model becomes

$$\mathcal{L}^o_{\mathcal{K}}(P) = \{x \in U : \exists i \in \{1, 2, \ldots, m\},\ K_i(x) \subseteq P\},$$

which is exactly the lower approximation in the classical optimistic multigranulation rough set model. Similarly, the upper approximation is obtained from the condition that no granularity supports the complementary concept P^c. Hence, the classical model can be viewed as an inclusion-based special case of the proposed optimistic multigranulation S-approximation model.

The advantage of the proposed model is that the support relation is no longer restricted to set inclusion. The decision mapping S may represent an overlap threshold, a similarity requirement, an expert rule, or another application-dependent admissibility relation. Therefore, the proposed model preserves the optimistic fusion mechanism of multigranulation rough sets while extending the underlying support criterion.

Majority voting and averaging-based fusion provide two useful baselines in the present comparison. The majority voting rule filters isolated support by requiring several granularities to agree. Averaging-based fusion evaluates the average strength of multi-source support and can further reflect different source reliabilities through weights. In contrast, the proposed optimistic rule preserves decisive single-granularity evidence: One admissible granularity is sufficient to trigger acceptance or rejection. Hence, it is especially suitable for situations where different knowledge mappings describe complementary aspects of the same object and where useful information from an individual granularity should not be suppressed by aggregation.

3.2. Decomposition properties of optimistic approximation operators

We now derive exact decomposition properties for the optimistic approximation operators under additional assumptions on the decision mapping.

Theorem 3.3. *Suppose that S is monotonic and that for all $K, P, Q \subseteq W$, we have*

$$S(K, P \cup Q) = 1 \Rightarrow S(K, P) = 1 \text{ or } S(K, Q) = 1.$$

Then, for any $P, Q \subseteq W$,

$$\mathcal{L}^o_{\mathcal{K}}(P \cup Q) = \mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q).$$

Proof. Monotonicity gives support transfer from P or Q to $P \cup Q$, namely

$$\begin{aligned} x \in \mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q) &\Longleftrightarrow \exists i,\ (r_i(x; P) = 1 \text{ or } r_i(x; Q) = 1) \\ &\Longrightarrow \exists i,\ S(K_i(x), P \cup Q) = 1 \\ &\Longleftrightarrow x \in \mathcal{L}^o_{\mathcal{K}}(P \cup Q). \end{aligned}$$

Conversely, the additional assumption yields

$$\begin{aligned} x \in \mathcal{L}^o_{\mathcal{K}}(P \cup Q) &\Longleftrightarrow \exists i,\ S(K_i(x), P \cup Q) = 1 \\ &\Longrightarrow \exists i,\ (r_i(x; P) = 1 \text{ or } r_i(x; Q) = 1) \\ &\Longleftrightarrow x \in \mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q). \end{aligned}$$

Thus

$$\mathcal{L}^o_{\mathcal{K}}(P \cup Q) = \mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q).$$

□

Remark 3.4. *In Theorem 3.2, the monotonicity of S is required to obtain*

$$\mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q) \subseteq \mathcal{L}^o_{\mathcal{K}}(P \cup Q).$$

Indeed, if $x \in \mathcal{L}^o_{\mathcal{K}}(P)$, then for some i, we have

$$r_i(x; P) = 1.$$

Since $P \subseteq P \cup Q$, monotonicity ensures that

$$S(K_i(x), P \cup Q) = 1,$$

which gives $x \in \mathcal{L}^o_{\mathcal{K}}(P \cup Q)$. Without monotonicity, support for P or support for Q does not necessarily extend to support for the larger set $P \cup Q$.

The following example illustrates that the monotonicity assumption cannot be removed, since support may fail to be preserved when the target set is enlarged.

Example 3.5. *Monotonicity in Theorem 3.2 cannot be omitted.*

Let

$$U = \{u_0\}, \qquad W = \{w_1, w_2, w_3\}.$$

Take one knowledge mapping $K_1 : U \to 2^W$; the value of $K_1(u_0)$ is immaterial here. Define

$$S(K, R) = 1 \iff R = \{w_1\},$$

for all $K, R \subseteq W$.

In this case, S is not monotonic. Indeed,

$$\{w_1\} \subseteq \{w_1, w_2\},$$

but

$$S(K, \{w_1\}) = 1, \qquad S(K, \{w_1, w_2\}) = 0.$$

However, S satisfies

$$S(K, R \cup D) = 1 \Rightarrow S(K, R) = 1 \text{ or } S(K, D) = 1.$$

Indeed, $S(K, R \cup D) = 1$ forces $R \cup D = \{w_1\}$, so at least one of R, D equals $\{w_1\}$.

Now take

$$P = \{w_1\}, \qquad Q = \{w_2\}.$$

Then

$$P \cup Q = \{w_1, w_2\}.$$

Since

$$r_1(u_0; P) = S(K_1(u_0), P) = 1,$$

we have

$$u_0 \in \mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q).$$

On the other hand, we have

$$S(K_1(u_0), P \cup Q) = S(K_1(u_0), \{w_1, w_2\}) = 0,$$

and hence

$$u_0 \notin \mathcal{L}^o_{\mathcal{K}}(P \cup Q).$$

Therefore,

$$\mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q) = \{u_0\}, \qquad \mathcal{L}^o_{\mathcal{K}}(P \cup Q) = \varnothing.$$

Thus,

$$\mathcal{L}^o_{\mathcal{K}}(P \cup Q) \neq \mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q).$$

Example 3.6. *Let*

$$U = \{u_1, u_2, u_3\}, \qquad W = \{w_1, w_2, w_3, w_4\}, \qquad P = \{w_1, w_2\}, \qquad Q = \{w_3, w_4\}.$$

Define

$$S(K, R) = 1 \Longleftrightarrow K \cap R \neq \varnothing.$$

In this case, S is monotonic, and

$$K \cap (P \cup Q) \neq \varnothing \Longrightarrow K \cap P \neq \varnothing \text{ or } K \cap Q \neq \varnothing.$$

Hence the assumptions of Theorem 3.2 are satisfied.

Let $K_1, K_2 : U \to 2^W$ be given by

$$\begin{aligned} &K_1(u_1) = \{w_1\}, \quad K_1(u_2) = \varnothing, \quad K_1(u_3) = \{w_2, w_4\}, \\ &K_2(u_1) = \varnothing, \quad K_2(u_2) = \{w_3\}, \quad K_2(u_3) = \varnothing. \end{aligned}$$

Then

$$\mathcal{L}^o_{\mathcal{K}}(P) = \{u_1, u_3\}, \qquad \mathcal{L}^o_{\mathcal{K}}(Q) = \{u_2, u_3\}.$$

Since

$$P \cup Q = W,$$

we also have

$$\mathcal{L}^o_{\mathcal{K}}(P \cup Q) = \{u_1, u_2, u_3\}.$$

Therefore,

$$\mathcal{L}^o_{\mathcal{K}}(P \cup Q) = \mathcal{L}^o_{\mathcal{K}}(P) \cup \mathcal{L}^o_{\mathcal{K}}(Q).$$

This verifies the exact union decomposition in Theorem 3.2.

Theorem 3.7. *Suppose that S is monotonic and that for all $K, P, Q \subseteq W$, we have*

$$S(K, P^c \cup Q^c) = 1 \Rightarrow S(K, P^c) = 1 \text{ or } S(K, Q^c) = 1.$$

Then, for any $P, Q \subseteq W$, we have

$$\mathcal{U}^o_{\mathcal{K}}(P \cap Q) = \mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q).$$

Proof. Because $P^c, Q^c \subseteq (P \cap Q)^c$, monotonicity gives, by contraposition,

$$S(K_i(x), (P \cap Q)^c) = 0 \Longrightarrow \bar{r}_i(x; P) = 0 \text{ and } \bar{r}_i(x; Q) = 0.$$

Hence

$$x \in \mathcal{U}^o_{\mathcal{K}}(P \cap Q) \Longrightarrow x \in \mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q).$$

For the reverse inclusion, use $(P \cap Q)^c = P^c \cup Q^c$. The theorem hypothesis has the contrapositive form

$$S(K, P^c) = 0 \text{ and } S(K, Q^c) = 0 \Longrightarrow S(K, P^c \cup Q^c) = 0.$$

Thus

$$\begin{aligned} x \in \mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q) &\Longleftrightarrow \forall i,\ (\bar{r}_i(x; P) = 0 \text{ and } \bar{r}_i(x; Q) = 0) \\ &\Longrightarrow \forall i,\ S(K_i(x), (P \cap Q)^c) = 0 \\ &\Longleftrightarrow x \in \mathcal{U}^o_{\mathcal{K}}(P \cap Q). \end{aligned}$$

The two inclusions give

$$\mathcal{U}^o_{\mathcal{K}}(P \cap Q) = \mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q).$$

□

Remark 3.8. *In the proof of Theorem 3.6, the monotonicity of S is used to establish*

$$\mathcal{U}^o_{\mathcal{K}}(P \cap Q) \subseteq \mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q).$$

More precisely, because $P^c \subseteq (P \cap Q)^c$ and $Q^c \subseteq (P \cap Q)^c$, monotonicity guarantees that support for P^c or Q^c implies support for $(P \cap Q)^c$. Without this property, the abovementioned inclusion is not guaranteed.

Example 3.9. *Theorem 3.6 may fail without the monotonicity of S.*

Let

$$U = \{u_0\}, \qquad W = \{w_1, w_2, w_3\},$$

and define the decision mapping as

$$S(K, R) = 1 \Longleftrightarrow R = \{w_1\},$$

for all $K, R \subseteq W$. As in Example 3.4, this nonmonotonic mapping still satisfies

$$S(K, R^c \cup D^c) = 1 \Rightarrow S(K, R^c) = 1 \textit{ or } S(K, D^c) = 1,$$

whenever the left-hand side holds, because the union must equal $\{w_1\}$.

Take

$$P = \{w_1, w_3\}, \qquad Q = \{w_2, w_3\}.$$

Then

$$P^c = \{w_2\}, \qquad Q^c = \{w_1\}, \qquad (P \cap Q)^c = \{w_1, w_2\}.$$

Since

$$S(K_1(u_0), (P \cap Q)^c) = S(K_1(u_0), \{w_1, w_2\}) = 0,$$

we get

$$u_0 \in \mathcal{U}^o_{\mathcal{K}}(P \cap Q).$$

Thus, we have

$$\mathcal{U}^o_{\mathcal{K}}(P \cap Q) = \{u_0\}.$$

Moreover,

$$\bar{r}_1(u_0; P) = S(K_1(u_0), \{w_2\}) = 0,$$

so

$$u_0 \in \mathcal{U}^o_{\mathcal{K}}(P).$$

However,

$$\bar{r}_1(u_0; Q) = S(K_1(u_0), \{w_1\}) = 1,$$

which implies

$$u_0 \notin \mathcal{U}^o_{\mathcal{K}}(Q).$$

Therefore,

$$\mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q) = \varnothing.$$

Consequently, we have

$$\mathcal{U}^o_{\mathcal{K}}(P \cap Q) = \{u_0\} \neq \varnothing = \mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q).$$

Example 3.10. *Let*

$$U = \{u_1, u_2, u_3\}, \qquad W = \{w_1, w_2, w_3, w_4\}, \qquad P = \{w_1, w_2\}, \qquad Q = \{w_1, w_3\}.$$

Define

$$S(K, R) = 1 \iff K \cap R \neq \varnothing.$$

The mapping S is monotonic, and

$$K \cap (P^c \cup Q^c) \neq \varnothing \Longrightarrow K \cap P^c \neq \varnothing \text{ or } K \cap Q^c \neq \varnothing,$$

so the hypothesis of Theorem 3.6 is satisfied.

Let $K_1, K_2 : U \to 2^W$ be defined by

$$K_1(u_1) = \{w_1\}, \quad K_1(u_2) = \{w_2\}, \quad K_1(u_3) = \{w_1, w_4\},$$
$$K_2(u_1) = \varnothing, \quad K_2(u_2) = \{w_1\}, \quad K_2(u_3) = \varnothing.$$

Since

$$P^c = \{w_3, w_4\}, \qquad Q^c = \{w_2, w_4\},$$

we obtain

$$\mathcal{U}^o_{\mathcal{K}}(P) = \{u_1, u_2\}, \qquad \mathcal{U}^o_{\mathcal{K}}(Q) = \{u_1\}.$$

Furthermore, we have

$$P \cap Q = \{w_1\}, \qquad (P \cap Q)^c = \{w_2, w_3, w_4\}.$$

A direct calculation gives

$$\mathcal{U}^o_{\mathcal{K}}(P \cap Q) = \{u_1\}.$$

Thus,

$$\mathcal{U}^o_{\mathcal{K}}(P \cap Q) = \mathcal{U}^o_{\mathcal{K}}(P) \cap \mathcal{U}^o_{\mathcal{K}}(Q).$$

This confirms the intersection decomposition in Theorem 3.6.

The properties above provide the approximation-theoretic basis for the three-way decision regions introduced earlier in this section.

In some cases, it is necessary to merge two multigranulation S-approximation spaces into a larger approximation structure. The following definition gives an optimistic rule for such a combination while preserving the identity of each source.

Definition 3.11 (Optimistic combination of multigranulation spaces)**.** *Let*

$$M^{(1)} = (U, W, \mathcal{K}^{(1)}, S) \quad \textit{and} \quad M^{(2)} = (U, W, \mathcal{K}^{(2)}, S)$$

be two multigranulation S*-approximation spaces with the same object domain U, the feature domain W, and the decision mapping S. The symbol*

$$\mathcal{K}^{(1)} \cup \mathcal{K}^{(2)}$$

denotes the pointwise disjoint indexed union; that is, for every $x \in U$, we have

$$(\mathcal{K}^{(1)} \cup \mathcal{K}^{(2)})(x) = \mathcal{K}^{(1)}(x) \cup \mathcal{K}^{(2)}(x).$$

This operation keeps the source identity of every component, even when two component descriptions are equal as subsets of W. The optimistic combination of $M^{(1)}$ and $M^{(2)}$ is defined by

$$M^{(1)} \vee M^{(2)} = (U, W, \mathcal{K}^{(1)} \cup \mathcal{K}^{(2)}, S).$$

That is, the combined multi-knowledge mapping assigns all indexed knowledge descriptions supplied by the two component spaces to each object x. If

$$\mathcal{K}^{(1)}(x) = (Q_s^{(1)}(x))_{s=1}^{m_1}$$

and

$$\mathcal{K}^{(2)}(x) = (Q_t^{(2)}(x))_{t=1}^{m_2},$$

then

$$(\mathcal{K}^{(1)} \cup \mathcal{K}^{(2)})(x) = (Q_1^{(1)}(x), \dots, Q_{m_1}^{(1)}(x), Q_1^{(2)}(x), \dots, Q_{m_2}^{(2)}(x)).$$

Thus, the operation combines the admissible knowledge descriptions associated with each object without deleting duplicate descriptions from different sources.

Consequently, for any $P \subseteq W$, we have

$$\mathcal{L}_{M^{(1)} \vee M^{(2)}}(P) = \{x \in U : \exists Q \in \mathcal{K}^{(1)}(x) \cup \mathcal{K}^{(2)}(x) \text{ such that } S(Q, P) = 1\},$$

and

$$\mathcal{U}_{M^{(1)} \vee M^{(2)}}(P) = \{x \in U : \forall Q \in \mathcal{K}^{(1)}(x) \cup \mathcal{K}^{(2)}(x), S(Q, P^c) = 0\}.$$

Theorem 3.12. *Let*

$$M^{(1)} = (U, W, \mathcal{K}^{(1)}, S) \quad \textit{and} \quad M^{(2)} = (U, W, \mathcal{K}^{(2)}, S)$$

be two optimistic multigranulation S*-approximation spaces with the same object domain U, the feature domain W, and the decision mapping S. Let*

$$M^{(1)} \vee M^{(2)} = (U, W, \mathcal{K}^{(1)} \cup \mathcal{K}^{(2)}, S)$$

be their optimistic combination. For any $P \subseteq W$, suppose that the three-way decision regions induced by $M^{(1)}$, $M^{(2)}$, and $M^{(1)} \vee M^{(2)}$ are well defined, that is,

$$\mathcal{L}_{M^{(1)}}(P) \subseteq \mathcal{U}_{M^{(1)}}(P),$$
$$\mathcal{L}_{M^{(2)}}(P) \subseteq \mathcal{U}_{M^{(2)}}(P),$$
$$\mathcal{L}_{M^{(1)} \vee M^{(2)}}(P) \subseteq \mathcal{U}_{M^{(1)} \vee M^{(2)}}(P).$$

Then

$$\begin{aligned}
\mathrm{Acc}_{M^{(1)} \vee M^{(2)}}(P) &= \mathrm{Acc}_{M^{(1)}}(P) \cup \mathrm{Acc}_{M^{(2)}}(P),\\
\mathrm{Rej}_{M^{(1)} \vee M^{(2)}}(P) &= \mathrm{Rej}_{M^{(1)}}(P) \cup \mathrm{Rej}_{M^{(2)}}(P),\\
\mathrm{Def}_{M^{(1)} \vee M^{(2)}}(P) &= \mathrm{Def}_{M^{(1)}}(P) \cap \mathrm{Def}_{M^{(2)}}(P).
\end{aligned}$$

Proof. For $x \in U$, we have

$$\begin{aligned}
x \in \mathcal{L}_{M^{(1)} \vee M^{(2)}}(P) &\Longleftrightarrow \exists Q \in \mathcal{K}^{(1)}(x) \cup \mathcal{K}^{(2)}(x),\ S(Q, P) = 1\\
&\Longleftrightarrow x \in \mathcal{L}_{M^{(1)}}(P) \cup \mathcal{L}_{M^{(2)}}(P),
\end{aligned}$$

$$\begin{aligned}
x \in \mathcal{U}_{M^{(1)} \vee M^{(2)}}(P) &\Longleftrightarrow \forall Q \in \mathcal{K}^{(1)}(x) \cup \mathcal{K}^{(2)}(x),\ S(Q, P^c) = 0\\
&\Longleftrightarrow x \in \mathcal{U}_{M^{(1)}}(P) \cap \mathcal{U}_{M^{(2)}}(P).
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{Acc}_{M^{(1)} \vee M^{(2)}}(P) &= \mathrm{Acc}_{M^{(1)}}(P) \cup \mathrm{Acc}_{M^{(2)}}(P),\\
\mathrm{Rej}_{M^{(1)} \vee M^{(2)}}(P) &= U \setminus (\mathcal{U}_{M^{(1)}}(P) \cap \mathcal{U}_{M^{(2)}}(P))\\
&= \mathrm{Rej}_{M^{(1)}}(P) \cup \mathrm{Rej}_{M^{(2)}}(P).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{Def}_{M^{(1)} \vee M^{(2)}}(P) &= (\mathcal{U}_{M^{(1)}}(P) \cap \mathcal{U}_{M^{(2)}}(P)) \setminus (\mathcal{L}_{M^{(1)}}(P) \cup \mathcal{L}_{M^{(2)}}(P))\\
&= (\mathcal{U}_{M^{(1)}}(P) \setminus \mathcal{L}_{M^{(1)}}(P)) \cap (\mathcal{U}_{M^{(2)}}(P) \setminus \mathcal{L}_{M^{(2)}}(P))\\
&= \mathrm{Def}_{M^{(1)}}(P) \cap \mathrm{Def}_{M^{(2)}}(P).
\end{aligned}$$

Hence, optimistic combination enlarges the determinate decision parts obtained from the component spaces and leaves only the common undecided part in the deferment region. □

We finally examine the case in which two multigranulation spaces are consistent at the approximation level.

Theorem 3.13. *Assume that for every $x \in U$ and every $P \subseteq W$, we have*

$$x \in \mathcal{L}_{M^{(1)}}(P) \iff x \in \mathcal{L}_{M^{(2)}}(P),$$
$$x \in \mathcal{U}_{M^{(1)}}(P) \iff x \in \mathcal{U}_{M^{(2)}}(P).$$

Then

$$\mathcal{L}_{M^{(1)}}(P) = \mathcal{L}_{M^{(2)}}(P) = \mathcal{L}_{M^{(1)} \vee M^{(2)}}(P),$$
$$\mathcal{U}_{M^{(1)}}(P) = \mathcal{U}_{M^{(2)}}(P) = \mathcal{U}_{M^{(1)} \vee M^{(2)}}(P),$$

and, whenever the corresponding three-way decision regions are well defined, we have

$$\begin{aligned}
\mathrm{Acc}_{M^{(1)} \vee M^{(2)}}(P) &= \mathrm{Acc}_{M^{(1)}}(P) = \mathrm{Acc}_{M^{(2)}}(P),\\
\mathrm{Rej}_{M^{(1)} \vee M^{(2)}}(P) &= \mathrm{Rej}_{M^{(1)}}(P) = \mathrm{Rej}_{M^{(2)}}(P),\\
\mathrm{Def}_{M^{(1)} \vee M^{(2)}}(P) &= \mathrm{Def}_{M^{(1)}}(P) = \mathrm{Def}_{M^{(2)}}(P).
\end{aligned}$$

Proof. The assumptions give

$$\mathcal{L}_{M^{(1)}}(P) = \mathcal{L}_{M^{(2)}}(P), \qquad \mathcal{U}_{M^{(1)}}(P) = \mathcal{U}_{M^{(2)}}(P).$$

Combining these identities with the optimistic union/intersection rule yields

$$\mathcal{L}_{M^{(1)}\vee M^{(2)}}(P) = \mathcal{L}_{M^{(1)}}(P) \cup \mathcal{L}_{M^{(2)}}(P) = \mathcal{L}_{M^{(1)}}(P) = \mathcal{L}_{M^{(2)}}(P),$$
$$\mathcal{U}_{M^{(1)}\vee M^{(2)}}(P) = \mathcal{U}_{M^{(1)}}(P) \cap \mathcal{U}_{M^{(2)}}(P) = \mathcal{U}_{M^{(1)}}(P) = \mathcal{U}_{M^{(2)}}(P).$$

Substitution into

$$\mathrm{Acc}_{M^o}(P) = \mathcal{L}^o_{\mathcal{K}}(P), \qquad \mathrm{Rej}_{M^o}(P) = U \setminus \mathcal{U}^o_{\mathcal{K}}(P), \qquad \mathrm{Def}_{M^o}(P) = \mathcal{U}^o_{\mathcal{K}}(P) \setminus \mathcal{L}^o_{\mathcal{K}}(P)$$

gives the three claimed region equalities.

The result shows that optimistic fusion preserves the approximation structure whenever the component spaces already agree on both lower and upper approximations. Therefore, the optimistic combination rule is stable under approximation-level consistency. □

4. Property analysis under non-contradictory conditions

To give optimistic multigranulation fusion a consistent three-way decision interpretation, possible conflicts among different granularities must be controlled. If an object is accepted by one granularity but rejected by another, then the resulting acceptance and rejection regions may fail to possess a coherent decision-theoretic meaning. In this section, we investigate the structural properties of optimistic combination under the non-contradictory condition introduced in Definition 2.12.

For notational convenience, $\bigvee_{i=1}^{m} G_i$ denotes the optimistic multigranulation space induced by the family $\{G_i\}_{i=1}^{m}$, in agreement with Definition 2.2 and the binary combination rule of Definition 3.11.

For each single-granularity S-approximation space

$$G_i = (U, W, K_i, S), \quad i = 1, 2, \ldots, m,$$

we introduce the following auxiliary sets:

$$\mathrm{Sup}_i(P) = \{x \in U : r_i(x; P) = 1\},$$

$$\mathrm{Exc}_i(P) = \{x \in U : \bar{r}_i(x; P) = 1\},$$

and

$$\mathrm{Def}_i(P) = U \setminus (\mathrm{Sup}_i(P) \cup \mathrm{Exc}_i(P)).$$

Here, $\mathrm{Sup}_i(P)$ consists of the objects supported by the i-th granularity with respect to P, $\mathrm{Exc}_i(P)$ consists of the objects excluded by the i-th granularity through the complement P^c, and $\mathrm{Def}_i(P)$ contains the objects that are neither supported nor excluded by that granularity.

Remark 4.1. *By Definition 2.12, the family $\{G_i\}_{i=1}^{m}$ is non-contradictory with respect to $P \subseteq W$ if and only if*

$$\mathrm{Sup}_i(P) \cap \mathrm{Exc}_j(P) = \varnothing, \qquad \forall i, j \in \{1, 2, \ldots, m\}.$$

This equivalent characterization will be used in the subsequent discussion.

The above condition eliminates direct decision conflicts across granularities. That is, an object cannot be supported by one granularity and, at the same time, excluded by another with respect to the same target concept.

Proposition 4.2. *Assume that the family $\{G_i\}_{i=1}^m$ is non-contradictory with respect to $P \subseteq W$ in the sense of Definition 2.12, and let*

$$M^o = \bigvee_{i=1}^{m} G_i$$

be the optimistic multigranulation space induced by these granularities. Then

$$\mathrm{Acc}_{M^o}(P) \cap \mathrm{Rej}_{M^o}(P) = \varnothing.$$

Proof. Using

$$\mathrm{Acc}_{M^o}(P) = \bigcup_{i=1}^{m} \mathrm{Sup}_i(P), \qquad \mathrm{Rej}_{M^o}(P) = \bigcup_{j=1}^{m} \mathrm{Exc}_j(P),$$

we obtain

$$\begin{aligned}\mathrm{Acc}_{M^o}(P) \cap \mathrm{Rej}_{M^o}(P) &= \left(\bigcup_{i=1}^{m} \mathrm{Sup}_i(P)\right) \cap \left(\bigcup_{j=1}^{m} \mathrm{Exc}_j(P)\right) \\ &= \bigcup_{i=1}^{m} \bigcup_{j=1}^{m} \big(\mathrm{Sup}_i(P) \cap \mathrm{Exc}_j(P)\big) = \varnothing,\end{aligned}$$

because non-contradictoriness gives $\mathrm{Sup}_i(P) \cap \mathrm{Exc}_j(P) = \varnothing$ for all i, j.

Thus, the non-contradictory condition guarantees that optimistic fusion preserves the separation between acceptance and rejection. □

Proposition 4.3. *Assume that $\{G_i\}_{i=1}^m$ is non-contradictory with respect to $P \subseteq W$. Then*

$$U = \mathrm{Acc}_{M^o}(P) \cup \mathrm{Rej}_{M^o}(P) \cup \mathrm{Def}_{M^o}(P),$$

and the three regions are pairwise disjoint.

Proof. Proposition 2.13 gives

$$\mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{U}^o_{\mathcal{K}}(P),$$

with

$$\mathrm{Acc}_{M^o}(P) = \mathcal{L}^o_{\mathcal{K}}(P), \qquad \mathrm{Rej}_{M^o}(P) = U \setminus \mathcal{U}^o_{\mathcal{K}}(P), \qquad \mathrm{Def}_{M^o}(P) = \mathcal{U}^o_{\mathcal{K}}(P) \setminus \mathcal{L}^o_{\mathcal{K}}(P),$$

we have

$$\mathrm{Acc}_{M^o}(P) \cup \mathrm{Def}_{M^o}(P) = \mathcal{U}^o_{\mathcal{K}}(P),$$
$$\mathrm{Acc}_{M^o}(P) \cup \mathrm{Rej}_{M^o}(P) \cup \mathrm{Def}_{M^o}(P) = \mathcal{U}^o_{\mathcal{K}}(P) \cup (U \setminus \mathcal{U}^o_{\mathcal{K}}(P)) = U.$$

Pairwise disjointness follows at once from

$$\begin{aligned}&\mathcal{L}^o_{\mathcal{K}}(P) \cap (\mathcal{U}^o_{\mathcal{K}}(P) \setminus \mathcal{L}^o_{\mathcal{K}}(P)) = \varnothing, \\ &(\mathcal{U}^o_{\mathcal{K}}(P) \setminus \mathcal{L}^o_{\mathcal{K}}(P)) \cap (U \setminus \mathcal{U}^o_{\mathcal{K}}(P)) = \varnothing, \\ &\mathcal{L}^o_{\mathcal{K}}(P) \cap (U \setminus \mathcal{U}^o_{\mathcal{K}}(P)) = \varnothing.\end{aligned}$$

This proposition shows that the optimistic model produces a proper three-way decision partition whenever contradictions among granularities are excluded. □

We next examine how the acceptance and rejection regions change when the target concept is enlarged.

Proposition 4.4. *Assume that the three-way decision regions induced by M^o are well defined for P and Q. If the decision mapping S is monotonic and*

$$P \subseteq Q \subseteq W,$$

then

$$\mathrm{Acc}_{M^o}(P) \subseteq \mathrm{Acc}_{M^o}(Q), \qquad \mathrm{Rej}_{M^o}(P) \supseteq \mathrm{Rej}_{M^o}(Q).$$

Proof. From Proposition 2.5, we have

$$P \subseteq Q \Longrightarrow \mathcal{L}^o_{\mathcal{K}}(P) \subseteq \mathcal{L}^o_{\mathcal{K}}(Q), \qquad \mathcal{U}^o_{\mathcal{K}}(P) \subseteq \mathcal{U}^o_{\mathcal{K}}(Q).$$

Using

$$\mathrm{Acc}_{M^o}(R) = \mathcal{L}^o_{\mathcal{K}}(R), \qquad \mathrm{Rej}_{M^o}(R) = U \setminus \mathcal{U}^o_{\mathcal{K}}(R) \quad (R = P, Q),$$

we obtain

$$\mathrm{Acc}_{M^o}(P) \subseteq \mathrm{Acc}_{M^o}(Q), \qquad \mathrm{Rej}_{M^o}(P) \supseteq \mathrm{Rej}_{M^o}(Q).$$

Consequently, enlarging the target set expands the optimistic acceptance region and reduces the corresponding rejection region.

To compare optimistic combination with other possible fusion mechanisms, we introduce the following admissibility condition.

Definition 4.5. *A combination strategy F is called admissible if, for any $P \subseteq W$, the following conditions are satisfied:*

(1) If $x \in \mathrm{Acc}_F(P)$, then some $i \in \{1, 2, \ldots, m\}$ exists such that $x \in \mathrm{Sup}_i(P)$;
(2) If $x \in \mathrm{Rej}_F(P)$, then some $i \in \{1, 2, \ldots, m\}$ exists such that $x \in \mathrm{Exc}_i(P)$;
(3) The three sets $\mathrm{Acc}_F(P)$, $\mathrm{Rej}_F(P)$, and $\mathrm{Def}_F(P)$ form a partition of U.

This admissibility requirement expresses a minimal rationality principle for fusion rules. An acceptance or rejection decision should not be produced unless it is supported by at least one of the underlying granularities.

Lemma 4.6. *For every admissible combination strategy F, we have*

$$\mathrm{Acc}_F(P) \subseteq \bigcup_{i=1}^{m} \mathrm{Sup}_i(P).$$

Proof. Admissibility gives

$$x \in \mathrm{Acc}_F(P) \Longrightarrow \exists i \in \{1, 2, \ldots, m\},\ x \in \mathrm{Sup}_i(P) \Longleftrightarrow x \in \bigcup_{i=1}^{m} \mathrm{Sup}_i(P).$$

Hence, we have

$$\mathrm{Acc}_F(P) \subseteq \bigcup_{i=1}^{m} \mathrm{Sup}_i(P).$$

This lemma means that an admissible strategy cannot generate acceptance decisions outside the total support supplied by all single-granularity spaces. □

Proposition 4.7. *Assume that the three-way decision regions induced by M^o are well defined for P. Among all admissible combination strategies, we have*

$$\mathrm{Acc}_F(P) \subseteq \bigcup_{i=1}^{m} \mathrm{Sup}_i(P) = \mathrm{Acc}_{M^o}(P).$$

Hence, $\mathrm{Acc}_{M^o}(P)$ *is the maximal acceptance region within the class of admissible strategies.*

Proof. Lemma 4.6 and the optimistic construction give

$$\mathrm{Acc}_F(P) \subseteq \bigcup_{i=1}^{m} \mathrm{Sup}_i(P) = \mathrm{Acc}_{M^o}(P).$$

Thus $\mathrm{Acc}_{M^o}(P)$ is maximal among admissible acceptance regions. □

We now prove the corresponding extremal property for the deferment region.

Proposition 4.8. *Assume that the three-way decision regions induced by M^o are well defined for P. For every admissible combination strategy F, we have*

$$\mathrm{Def}_{M^o}(P) \subseteq \mathrm{Def}_F(P).$$

Consequently, the optimistic combination yields the minimal deferment region in the class of admissible strategies.

Proof. The optimistic rule gives

$$\mathrm{Acc}_{M^o}(P) = \bigcup_{i=1}^{m} \mathrm{Sup}_i(P), \qquad \mathrm{Rej}_{M^o}(P) = \bigcup_{i=1}^{m} \mathrm{Exc}_i(P),$$

and therefore

$$\begin{aligned}\mathrm{Def}_{M^o}(P) &= U \setminus (\mathrm{Acc}_{M^o}(P) \cup \mathrm{Rej}_{M^o}(P)) \\ &= \bigcap_{i=1}^{m} (U \setminus (\mathrm{Sup}_i(P) \cup \mathrm{Exc}_i(P))) = \bigcap_{i=1}^{m} \mathrm{Def}_i(P).\end{aligned}$$

If $x \in \mathrm{Def}_{M^o}(P)$, then $x \notin \mathrm{Sup}_i(P)$ and $x \notin \mathrm{Exc}_i(P)$ for every i. Admissibility excludes x from both $\mathrm{Acc}_F(P)$ and $\mathrm{Rej}_F(P)$; since the three F-regions partition U, $x \in \mathrm{Def}_F(P)$. Hence

$$\mathrm{Def}_{M^o}(P) \subseteq \mathrm{Def}_F(P).$$

Thus, optimistic fusion has two extremal effects among admissible strategies: It maximizes the acceptance region and minimizes the deferment region. □

Proposition 4.9. *Let*

$$M_0 = \bigvee_{i=1}^{m} G_i$$

be the optimistic combination of m single-granularity S*-approximation spaces with respect to $P \subseteq W$, and let G_{m+1} be an additional granularity such that*

$$\{G_1, G_2, \ldots, G_m, G_{m+1}\}$$

is non-contradictory with respect to P. Define

$$M_1 = M_0 \vee G_{m+1} = \bigvee_{i=1}^{m+1} G_i,$$

then

$$\mathrm{Acc}_{M_0}(P) \subseteq \mathrm{Acc}_{M_1}(P),$$
$$\mathrm{Rej}_{M_0}(P) \subseteq \mathrm{Rej}_{M_1}(P),$$

and

$$\mathrm{Def}_{M_1}(P) \subseteq \mathrm{Def}_{M_0}(P).$$

Proof. The added granularity contributes by union to the determinate regions as follows:

$$\mathrm{Acc}_{M_1}(P) = \mathrm{Acc}_{M_0}(P) \cup \mathrm{Sup}_{m+1}(P),$$
$$\mathrm{Rej}_{M_1}(P) = \mathrm{Rej}_{M_0}(P) \cup \mathrm{Exc}_{m+1}(P).$$

Therefore

$$\mathrm{Acc}_{M_0}(P) \subseteq \mathrm{Acc}_{M_1}(P), \qquad \mathrm{Rej}_{M_0}(P) \subseteq \mathrm{Rej}_{M_1}(P).$$

For the deferment region, we have

$$\mathrm{Def}_{M_1}(P) = \mathrm{Def}_{M_0}(P) \cap \mathrm{Def}_{m+1}(P) \subseteq \mathrm{Def}_{M_0}(P),$$

where

$$\mathrm{Def}_{m+1}(P) = U \setminus (\mathrm{Sup}_{m+1}(P) \cup \mathrm{Exc}_{m+1}(P)).$$

The proposition shows that adding a non-contradictory granularity can only increase determinate classifications and cannot enlarge the deferment region. □

Example 4.10. *Let*

$$U = \{u_1, u_2, u_3, u_4, u_5\}, \qquad W = \{w_1, w_2, w_3, w_4\},$$

and let the target set be

$$P = \{w_1, w_2\}, \qquad P^c = \{w_3, w_4\}.$$

Define the decision mapping as

$$S(K, R) = 1 \Longleftrightarrow |K \cap R| \geq 2.$$

Consider the following three single-granularity knowledge mappings $K_1, K_2, K_3 : U \to 2^W$:

$$K_1(u_1) = \{w_1, w_2\}, \quad K_1(u_2) = \{w_1\}, \quad K_1(u_3) = \{w_1, w_3\}, \quad K_1(u_4) = \{w_3, w_4\}, \quad K_1(u_5) = \{w_3\},$$

$$K_2(u_1) = \{w_1\}, \quad K_2(u_2) = \{w_1, w_2\}, \quad K_2(u_3) = \{w_2, w_4\}, \quad K_2(u_4) = \{w_4\}, \quad K_2(u_5) = \{w_4\},$$

and

$$K_3(u_1) = \emptyset, \quad K_3(u_2) = \emptyset, \quad K_3(u_3) = \emptyset, \quad K_3(u_4) = \emptyset, \quad K_3(u_5) = \emptyset.$$

Let

$$G_i = (U, W, K_i, S), \quad i = 1, 2, 3.$$

A direct calculation gives

$$\mathrm{Sup}_1(P) = \{u_1\}, \quad \mathrm{Exc}_1(P) = \{u_4\}, \quad \mathrm{Def}_1(P) = \{u_2, u_3, u_5\},$$

$$\mathrm{Sup}_2(P) = \{u_2\}, \quad \mathrm{Exc}_2(P) = \varnothing, \quad \mathrm{Def}_2(P) = \{u_1, u_3, u_4, u_5\},$$

and

$$\mathrm{Sup}_3(P) = \varnothing, \quad \mathrm{Exc}_3(P) = \varnothing, \quad \mathrm{Def}_3(P) = U.$$

No object lies in both some $\mathrm{Sup}_i(P)$ *and some* $\mathrm{Exc}_j(P)$*; hence, the family*

$$\{G_1, G_2, G_3\}$$

is non-contradictory with respect to P.

Let

$$M_0 = G_1 \vee G_2 \vee G_3,$$

then

$$\begin{aligned} \mathrm{Acc}_{M_0}(P) &= \{u_1, u_2\}, \\ \mathrm{Rej}_{M_0}(P) &= \{u_4\}, \\ \mathrm{Def}_{M_0}(P) &= \{u_3, u_5\}. \end{aligned}$$

Equivalently, we have

$$\mathcal{L}_{M_0}(P) = \{u_1, u_2\}, \qquad \mathcal{U}_{M_0}(P) = \{u_1, u_2, u_3, u_5\}.$$

Thus,

$$\alpha_{M_0}(P) = \frac{|\mathcal{L}_{M_0}(P)|}{|\mathcal{U}_{M_0}(P)|} = \frac{2}{4} = 0.5.$$

Now introduce a fourth granularity $K_4 : U \to 2^W$ *defined by*

$$K_4(u_1) = \{w_1\}, \quad K_4(u_2) = \{w_2\}, \quad K_4(u_3) = \{w_1, w_2\}, \quad K_4(u_4) = \{w_3\}, \quad K_4(u_5) = \{w_3, w_4\}.$$

Let

$$G_4 = (U, W, K_4, S),$$

then

$$\mathrm{Sup}_4(P) = \{u_3\}, \qquad \mathrm{Exc}_4(P) = \{u_5\}, \qquad \mathrm{Def}_4(P) = \{u_1, u_2, u_4\}.$$

The enlarged family

$$\{G_1, G_2, G_3, G_4\}$$

is still non-contradictory with respect to P.

Define

$$M_1 = G_1 \vee G_2 \vee G_3 \vee G_4,$$

then

$$\begin{aligned} \mathrm{Acc}_{M_1}(P) &= \{u_1, u_2, u_3\}, \\ \mathrm{Rej}_{M_1}(P) &= \{u_4, u_5\}, \\ \mathrm{Def}_{M_1}(P) &= \varnothing. \end{aligned}$$

Consequently, we have

$$\mathcal{L}_{M_1}(P) = \{u_1, u_2, u_3\}, \qquad \mathcal{U}_{M_1}(P) = \{u_1, u_2, u_3\},$$

therefore,

$$\alpha_{M_1}(P) = 1.$$

A newly added non-contradictory granularity can turn undecided objects into acceptance or rejection decisions and may reduce the deferment region to the empty set.

In summary, the non-contradictory condition ensures that optimistic multigranulation fusion produces a coherent three-way partition. Under this condition, optimistic fusion enlarges admissible acceptance decisions, reduces the deferment region, and makes the induced decision structure more determinate.

5. A semi-realistic multi-source assessment case study

This section gives a semi-realistic example to demonstrate how the proposed model can be used in a multi-source assessment problem. The example is not intended as a clinical rule; it is used only to show how knowledge mappings, optimistic approximations, and three-way decision regions can be computed.

Suppose that six objects $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ are evaluated by three information sources. The feature domain is

$$W = \{w_1, w_2, w_3, w_4, w_5\},$$

where the symbols may represent five assessment indicators. Let

$$P = \{w_1, w_3, w_4\}, \qquad P^c = \{w_2, w_5\},$$

and use the decision mapping

$$S(K, R) = 1 \Longleftrightarrow |K \cap R| \geq 2.$$

Thus, a source supports a target set when at least two of its reported indicators belong to that target set.

Let K_1, K_2, and K_3 be three knowledge mappings obtained from three sources. Their reported descriptions are listed in Table 3.

Table 3. Descriptions generated by three information sources in the case study.

Object	$K_1(x)$	$K_2(x)$	$K_3(x)$
x_1	$\{w_1, w_3\}$	$\{w_1\}$	$\{w_3, w_4\}$
x_2	$\{w_1\}$	$\{w_3, w_4\}$	$\varnothing$
x_3	$\{w_2, w_5\}$	$\{w_2\}$	$\{w_2, w_5\}$
x_4	$\{w_3\}$	$\{w_2, w_5\}$	$\{w_2\}$
x_5	$\{w_4, w_5\}$	$\{w_1, w_4\}$	$\{w_1\}$
x_6	$\{w_2\}$	$\varnothing$	$\{w_5\}$

For P, the support, exclusion, and deferment sets of the three individual sources are

$$\mathrm{Sup}_1(P) = \{x_1\}, \quad \mathrm{Exc}_1(P) = \{x_3\}, \quad \mathrm{Def}_1(P) = \{x_2, x_4, x_5, x_6\},$$

$$\mathrm{Sup}_2(P) = \{x_2, x_5\}, \quad \mathrm{Exc}_2(P) = \{x_4\}, \quad \mathrm{Def}_2(P) = \{x_1, x_3, x_6\},$$

and

$$\mathrm{Sup}_3(P) = \{x_1\}, \quad \mathrm{Exc}_3(P) = \{x_3\}, \quad \mathrm{Def}_3(P) = \{x_2, x_4, x_5, x_6\}.$$

The pairwise intersections $\mathrm{Sup}_i(P) \cap \mathrm{Exc}_j(P)$ are empty for all $i, j \in \{1, 2, 3\}$. Hence, the three sources are non-contradictory with respect to P.

The optimistic lower approximation is the union of the support sets:

$$\mathcal{L}^o_{\mathcal{K}}(P) = \mathrm{Sup}_1(P) \cup \mathrm{Sup}_2(P) \cup \mathrm{Sup}_3(P) = \{x_1, x_2, x_5\}.$$

The optimistic rejection region is generated by the union of the exclusion sets:

$$\mathrm{Rej}_{M^o}(P) = \mathrm{Exc}_1(P) \cup \mathrm{Exc}_2(P) \cup \mathrm{Exc}_3(P) = \{x_3, x_4\},$$

therefore,

$$\mathcal{U}^o_{\mathcal{K}}(P) = U \setminus \mathrm{Rej}_{M^o}(P) = \{x_1, x_2, x_5, x_6\}.$$

The induced three-way decision regions are

$$\mathrm{Acc}_{M^o}(P) = \{x_1, x_2, x_5\}, \qquad \mathrm{Rej}_{M^o}(P) = \{x_3, x_4\}, \qquad \mathrm{Def}_{M^o}(P) = \{x_6\}.$$

The approximation accuracy is

$$\alpha_{M^o}(P) = \frac{|\mathcal{L}^o_{\mathcal{K}}(P)|}{|\mathcal{U}^o_{\mathcal{K}}(P)|} = \frac{3}{4}.$$

To compare the proposed optimistic rule with the two aggregation baselines on the same data, compute the support and exclusion counts of each object as follows:

	x_1	x_2	x_3	x_4	x_5	x_6
$c_P(x)$	2	1	0	0	1	0
$c_{P^c}(x)$	0	0	2	1	0	0

For majority voting, we take $\tau = 2$. For averaging-based fusion, the weights are assigned according to the assumed reliability of the three information sources in this semi-realistic assessment scenario. Specifically, suppose that the reliability scores of the three sources are

$$\eta_1 = 1, \qquad \eta_2 = 3, \qquad \eta_3 = 1.$$

These scores indicate that the second source is regarded as more reliable than the first and third sources. The normalized weights are then obtained by

$$\omega_i = \frac{\eta_i}{\sum_{j=1}^{3} \eta_j}, \qquad i = 1, 2, 3,$$

which gives

$$\omega = (\omega_1, \omega_2, \omega_3) = (0.2, 0.6, 0.2).$$

Thus, the weighted average support degrees are

$$a_P(x) = 0.2 r_1(x; P) + 0.6 r_2(x; P) + 0.2 r_3(x; P),$$

and

$$a_{P^c}(x) = 0.2\bar{r}_1(x; P) + 0.6\bar{r}_2(x; P) + 0.2\bar{r}_3(x; P).$$

The thresholds are set as

$$\theta_a = \theta_b = 0.5.$$

This averaging-based setting is used as a benchmark fusion rule to illustrate how weighted aggregate support may lead to different three-way decision regions from the proposed optimistic rule. The decision regions and decisive rates obtained by the three fusion rules are summarized in Table 4.

Table 4. Decision regions obtained by different optimistic fusion rules in the case study.

Fusion rule	Acceptance region	Rejection region	Deferment region	Decisive rate
Proposed optimistic rule	$\{x_1, x_2, x_5\}$	$\{x_3, x_4\}$	$\{x_6\}$	5/6
Majority voting ($\tau = 2$)	$\{x_1\}$	$\{x_3\}$	$\{x_2, x_4, x_5, x_6\}$	2/6
Averaging-based fusion ($\omega = (0.2, 0.6, 0.2)$, $\theta_a = \theta_b = 0.5$)	$\{x_2, x_5\}$	$\{x_4\}$	$\{x_1, x_3, x_6\}$	3/6

The comparison shows the different decision attitudes of the three rules. The proposed optimistic rule accepts x_1, x_2, and x_5, because each of them is supported by at least one admissible source, and it rejects x_3 and x_4, because each of them has at least one source supporting P^c. Majority voting accepts only x_1 and rejects only x_3, since these two objects receive support from at least two sources in the same decision direction. Averaging-based fusion, with the second source assigned a larger weight, accepts x_2 and x_5 and rejects x_4, but defers x_1 and x_3 because their support mainly comes from the first and third sources.

Thus, the three fusion rules are not superior or inferior in an absolute sense; rather, they express different attitudes toward multi-source evidence. Majority voting emphasizes numerical agreement, and averaging-based fusion emphasizes weighted aggregate support. The proposed optimistic rule is more decisive because it preserves single-source admissible evidence, which is consistent with the optimistic multigranulation principle.

This example illustrates the practical meaning of the proposed construction. Object x_2 is accepted because the second source provides sufficient support, even though the other two sources do not. Object x_5 is accepted for the same reason. Objects x_3 and x_4 are rejected because at least one source provides sufficient evidence for P^c. Object x_6 remains deferred because none of the sources provides enough support for either P or P^c. Hence, the optimistic model preserves decisive single-source evidence while keeping unsupported and non-excluded objects in the deferment region.

6. Conclusions and future work

This work considered how several knowledge mappings can be incorporated into an S-approximation framework while keeping a common decision mapping. The constructed optimistic multigranulation space M^o treats these mappings as different granular descriptions of the same universe. Its approximation operators $\mathcal{L}^o_{\mathcal{K}}$ and $\mathcal{U}^o_{\mathcal{K}}$ are defined by an optimistic support rule and a non-exclusion rule, respectively. They induce the regions $\mathrm{Acc}_{M^o}(P)$, $\mathrm{Rej}_{M^o}(P)$, and $\mathrm{Def}_{M^o}(P)$, thereby giving the model a three-way decision interpretation.

The obtained results describe the behavior of M^o at both the approximation and decision levels. The duality relations, monotonicity properties, and set-theoretic inclusions show that the new operators retain the basic structure expected from approximation operators. The representation of $\mathcal{L}^o_{\mathcal{K}}(P)$ as a union of single-granularity lower approximations and $\mathcal{U}^o_{\mathcal{K}}(P)$ as an intersection of single-granularity upper approximations explains the role of optimistic fusion. The comparison with classical optimistic multigranulation rough sets, majority-voting fusion, and averaging-based fusion further clarifies that the proposed model preserves the optimistic rough set fusion logic while extending the support criterion through the decision mapping S. Under the non-contradictory condition, acceptance and rejection are separated. Moreover, within the class of admissible strategies, the optimistic rule produces the largest acceptance region and the smallest deferment region. The semi-realistic case study shows how the model can be implemented by constructing several knowledge mappings, computing the response values, and obtaining the three-way regions.

Further work may proceed in several directions. First, weighted multigranulation versions of S-approximation spaces can be developed by incorporating source reliability into the decision mapping or the fusion rule. Second, partially conflicting granularities should be studied, because practical multi-source systems may contain both supporting and opposing evidence. Third, applications to concrete datasets and decision-making problems would help evaluate the usefulness of the proposed model beyond the theoretical setting.

Use of Generative-AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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