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Research article

Hamming weight distributions of generalized quasi-cyclic codes over $\mathrm{GR}(p^2, m)$

Sami H. Saif* **and Shayea Aldossari**

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: ssaif1@ksu.edu.sa.

Abstract: Let $R = \mathrm{GR}(p^2, m)$ denote the Galois ring of characteristic p^2 with residue field $\mathbb{F}_q \cong R/\langle p\rangle$ and cardinality q^2, where $q = p^m$. This paper determines the Hamming weight distributions of generalized quasi-cyclic (GQC) codes over R under the assumption that $\gcd(m_i, p) = 1$ for every block length m_i. The Chinese Remainder Theorem decomposes each GQC code into constituents over the extension Galois rings $\mathrm{GR}(p^2, md)$, and the weight problem thereby reduces to counting the positions at which certain trace sums vanish. We develop character-sum formulas for these trace-zero counts, organizing the analysis according to whether the active constituent parameters are zero, nilpotent, or units. Nilpotent constituents reduce to trace equations over the residue field and are evaluated by finite-field Gauss sums, whereas unit and mixed constituents require an analysis of the principal p-layer based on the Teichmüller decomposition, carry traces, and addition carries. One consequence is that residue data alone do not determine the trace-zero count over $\mathrm{GR}(p^2, m)$: Two units with the same residue but distinct principal p-parts may yield different Hamming weights. For one- and two-constituent GQC codes, we derive explicit finite character-sum formulas covering every combination of nilpotent and unit constituents. Closed evaluations are obtained in the purely nilpotent cases and in selected semiprimitive cases, and the remaining unit cases are expressed through Galois-ring period sums involving carry terms. Together, these formulas determine the Hamming weight distributions of the corresponding families of GQC codes over $\mathrm{GR}(p^2, m)$.

Keywords: generalized quasi-cyclic codes; Galois rings; Hamming weight distributions; Gauss sums; trace-zero counts
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1. Introduction

1.1. Background and motivation

Let $R = \mathrm{GR}(p^2, m)$ be the Galois ring of characteristic p^2 with residue field $\mathbb{F}_q \cong R/\langle p\rangle$ and cardinality q^2, where $q = p^m$. The study of linear codes over Galois rings and, more generally, over finite chain rings lies at a rich intersection of algebra, combinatorics, and information theory. Its central motivation comes from a landmark result: Certain families of nonlinear binary codes with exceptional combinatorial properties, notably the Kerdock, Preparata, and Goethals codes, arise as Gray images of $\mathbb{Z}_4$-linear codes [1]. That discovery revealed that the nilpotent layers of a finite ring carry nontrivial algebraic information and can therefore produce images far superior to any available from purely finite-field constructions.

Since that foundational work, codes over finite chain rings and Galois rings have been investigated extensively; see, for example, [2, 3]. A recurring theme is that the ring structure enriches the weight enumerator in ways that cannot be captured by ordinary finite-field trace methods alone [4, 5]. The present paper addresses a central instance of this phenomenon for generalized quasi-cyclic (GQC) codes over $R = \mathrm{GR}(p^2, m)$.

Generalized quasi-cyclic codes unify several classical families: Cyclic codes arise as the special case $\ell = 1$, ordinary quasi-cyclic codes correspond to equal block lengths, double cyclic codes correspond to $\ell = 2$, and various multiblock cyclic constructions fall within the same framework. The algebraic structure of GQC codes has been studied through generator descriptions, constituent decompositions, trace representations, and enumeration formulas; see [6–8]. In particular, the structural theory of one-generator GQC codes over chain rings was developed in [9–11]. A structural decomposition alone, however, does not determine the Hamming weight distribution; that requires an explicit count of the positions at which the constituent trace sums vanish in R. The present paper supplies this count.

1.2. The weight problem and related work

Weight distributions are among the most delicate and informative invariants of a linear code. They govern error-correcting performance, determine whether a code is minimal, and bear on questions of projectivity and optimality [12–14]. Over finite fields, weight computations for cyclic and irreducible cyclic codes typically reduce to exponential sums $\sum_{x\in D}\psi(ax)$, which are evaluated by means of Gauss periods and Gauss sums; see [15–17]. Over finite rings, the problem is substantially harder, because the weight of a codeword depends not only on its residue modulo the maximal ideal but also on its higher-layer structure. This additional complexity is reflected in complete weight enumerators, torsion codes, and related invariants of codes over Galois rings; see [18–20].

For GQC codes, the Chinese Remainder Theorem (CRT) decomposition under the coprimality assumption $\gcd(m_i, p) = 1$ identifies the constituent rings $K_j = \mathrm{GR}(p^2, md_j)$ and furnishes trace representations of codewords. Computing the weight distribution then amounts to counting those indices r for which the trace expressions $\mathrm{Tr}_{K_j/R}(A_j\alpha_j^{-r})$ vanish in R. The weight problem is therefore, in essence, a *trace-zero counting problem over extension Galois rings*.

The closest precursor to the present work is [21], which studied GQC codes over the equal-characteristic chain ring $\mathbb{F}_q + u\mathbb{F}_q$ with $u^2 = 0$. In that setting, the trace decomposes cleanly as

$\mathrm{Tr}(a + ub) = \mathrm{tr}(a) + u\mathrm{tr}(b)$, so both layers are governed by ordinary finite-field traces, and the resulting formulas can be expressed entirely in terms of finite-field Gauss sums. Over $R = \mathrm{GR}(p^2, m)$, by contrast, every element admits a Teichmüller expansion $a + pb$, and multiplication of Teichmüller representatives produces carry terms that are intrinsic to the mixed-characteristic setting. For a unit $A = A_0 + pA_1 \in K^\times$, the vanishing condition $\mathrm{Tr}_{K/R}(A\alpha^{-r}) = 0$ involves both a residue-layer equation and a principal p-layer equation whose interaction is governed by carry traces and addition carries. This interaction has no analog over equal-characteristic rings, and its treatment is the central technical contribution of the paper.

1.3. Main contributions

The main contributions of the paper are the following:

- Under the assumption $\gcd(m_i, p) = 1$, we obtain an explicit trace representation for GQC codewords over $R = \mathrm{GR}(p^2, m)$, which reduces the Hamming weight problem to counting the zero positions of trace sums over the extension Galois rings $\mathrm{GR}(p^2, md)$ (see Theorem 3.2).
- We express the Hamming weight of a codeword directly in terms of trace-zero counts. If $N_i^{(0)}(c)$ denotes the number of vanishing trace positions in the ith block, then
$$\mathrm{wt}_H(c) = \sum_i (m_i - N_i^{(0)}(c)).$$
Determining the weight distribution is therefore equivalent to evaluating these counts (see Theorem 3.4).
- For a nilpotent parameter pA, the trace-zero condition reduces to the finite-field equation
$$\mathrm{tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\bar{A}\,\bar{\alpha}^{-r}) = 0,$$
and we evaluate the resulting cases by means of multiplicative characters and Gauss sums (see Theorems 4.2 and 4.3).
- For a unit $A = A_0 + pA_1$, the trace-zero condition splits into a residue-layer condition and a principal p-layer condition. The latter involves carry traces, and, in the two-unit case, addition carries, as well. We derive character period formulas covering all combinations of zero, nilpotent, and unit parameters in the two-constituent setting (see Theorem 4.12).
- We show that residue data alone do not determine the trace-zero count over $\mathrm{GR}(p^2, m)$: Two units with the same residue but distinct principal parts can yield different Hamming weights (see Example 5.6).
- For selected one- and two-constituent families, we compute complete Hamming weight distributions and indicate conditional finite-field applications to minimality, projectivity, and Griesmer-type optimality, valid whenever a separate $\mathbb{F}_q$-linear realization with the same Hamming enumerator is available.

The GQC setting is essential here because it permits unit–nilpotent and unit–unit interactions between distinct constituents, and these interactions are precisely where the principal p-layer and the carry terms enter the trace-zero analysis.

1.4. Organization

Section 2 introduces Galois rings, the Teichmüller expansion, trace maps, additive characters, Hamming weights, and GQC codes. Section 3 presents the trace representation and the stratified weight

formula. Section 4 derives trace-zero formulas for all nine constituent-type combinations, including the new unit and mixed-unit cases. Section 5 collects the Gauss-sum and period evaluations required for explicit weight counts, together with worked examples that illustrate the distinction between the residue layer and the principal layer. Section 6 records conditional finite-field applications under an explicitly stated separate-linear-realization assumption. Table 1 lists the principal symbols and specialized terminology used throughout the paper.

Table 1. Nomenclature.

Symbol	Meaning
p, $q = p^m$	A prime number; the cardinality of the residue field of R.
$R = \mathrm{GR}(p^2, m)$	Galois ring of characteristic p^2, cardinality q^2, residue field $\mathbb{F}_q$.
$R_d = \mathrm{GR}(p^2, md)$	Degree-d unramified extension of R, with residue field $\mathbb{F}_{q^d}$.
$\mathcal{T}, \mathcal{T}_d$	Teichmüller sets of R and R_d, respectively.
z_0, z_1	Residue part and principal p-part of $z = z_0 + pz_1$.
τ, τ_d; λ_d	Teichmüller lifts from $\mathbb{F}_q$ and $\mathbb{F}_{q^d}$; canonical additive character of $\mathbb{F}_{q^d}$.
$\mathrm{Tr}_d = \mathrm{Tr}_{R_d/R}$; $\mathrm{tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}$	Galois ring trace from R_d to R; finite-field trace from $\mathbb{F}_{q^d}$ to $\mathbb{F}_q$.
χ, χ_{R_d}	Additive character of R and the induced character $\chi_{R_d} = \chi \circ \mathrm{Tr}_d$.
$m_1, \ldots, m_\ell$; A_i	Block lengths of the GQC code; the ith cyclic block ring $R[x]/\langle x^{m_i} - 1\rangle$.
$g_j(x)$, d_j	A basic irreducible factor and its degree $d_j = \deg g_j$.
$e_j(x)$, $e(x)$	CRT idempotent generating the summand associated with $g_j(x)$.
$K_j = \mathrm{GR}(p^2, md_j)$	Constituent extension ring corresponding to $g_j(x)$.
α_j	Teichmüller lift in K_j of a chosen root $\bar{\alpha}_j$ of $\bar{g}_j(x)$, satisfying $g_j(\alpha_j) = 0$.
C_j	The jth constituent of the GQC code C.
$S_{i,r}(c)$	Trace sum determining the rth coordinate of the ith block.
$N_i^{(0)}, N_i^{(p)}, N_i^{(\times)}$	Numbers of zero, nonzero nilpotent, and unit trace sums in block i.
$Q_j = q^{d_j}$	Cardinality of the residue field of K_j.
ω_j	A fixed generator of the multiplicative character group of $\mathbb{F}_{Q_j}^\times$.
$G_{Q_j}(\psi, \lambda_j)$	Gauss sum of the multiplicative character ψ with respect to λ_j.
$\bar{g}_j, \bar{\Delta}_j$	Character-selection parameters $\bar{g}_j = \gcd\left(\frac{Q_j - 1}{q-1}, u_j\right)$, $\quad \bar{\Delta}_j = \frac{Q_j - 1}{\bar{g}_j}$.
alternating-/constant-sign	The two sign behaviors of semiprimitive Gauss sums in (5.5).
κ_d	Carry trace associated with the Galois ring trace on $\mathrm{GR}(p^2, md)$.
$T_j(C; r)$, $\varkappa_j(C; r)$	Constituent-indexed trace and carry-trace functions of (4.24).
γ_h	Addition carry for a sum of h Teichmüller representatives.
ρ_A, π_A	Residue-layer and principal/carry-layer functions associated with A.
ν_A	Residue trace function associated with a nilpotent parameter pA.
$Z(A, B)$	Number of indices at which the corresponding two-constituent trace sum vanishes.
$\Theta_{A,B}(z)$	Galois ring character period associated with the parameters A, B and the multiplier $z \in R$.

2. Preliminaries

This section collects the algebraic background and fixes the notation used throughout the paper.

2.1. Galois rings

Definition 2.1. Let p be a prime, let m be a positive integer, and set $q = p^m$. The *Galois ring* $R = \mathrm{GR}(p^2, m)$ is the unique Galois extension of $\mathbb{Z}_{p^2}$ of degree m, up to isomorphism. It is a finite chain ring of characteristic p^2 and cardinality q^2, with maximal ideal $\langle p\rangle$ and residue field $R/\langle p\rangle \cong \mathbb{F}_q$.

Definition 2.2. The *Teichmüller set* $\mathcal{T}$ of R is the unique set of q elements satisfying $t^q = t$ for all $t \in \mathcal{T}$ and containing 0. Every element $z \in R$ has a unique *Teichmüller expansion*

$$z = z_0 + pz_1, \qquad z_0, z_1 \in \mathcal{T}.$$

Moreover, $z \in R^\times$ if and only if $z_0 \neq 0$, and $z \in pR \setminus \{0\}$ if and only if $z_0 = 0$ and $z_1 \neq 0$. Throughout the paper, z_0 is called the *residue part* of z, and z_1 its *principal p-part*; correspondingly, we refer to the first and second Teichmüller components of an expression as its *residue layer* and its *principal p-layer*. The same terminology is used for the extension rings R_d introduced next.

Definition 2.3. For each positive integer d, let $R_d = \mathrm{GR}(p^2, md)$. Its residue field is $\mathbb{F}_{q^d}$, its cardinality is q^{2d}, and its Teichmüller set is denoted by $\mathcal{T}_d$. The *Galois ring trace* $\mathrm{Tr}_d = \mathrm{Tr}_{R_d/R} : R_d \to R$ is defined by

$$\mathrm{Tr}_d(a) = \sum_{i=0}^{d-1} \sigma^i(a),$$

where σ denotes the Frobenius automorphism of R_d over R. The trace is compatible with reduction modulo p, in the sense that

$$\overline{\mathrm{Tr}_d(a)} = \mathrm{tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\bar{a}),$$

where a bar denotes reduction modulo p.

The partition of $K = \mathrm{GR}(p^2, md)$ used throughout is

$$K = \{0\} \cup (pK \setminus \{0\}) \cup K^\times,$$

where, in Teichmüller notation, $pK \setminus \{0\} = p\mathcal{T}_d^*$ and $K^\times = \mathcal{T}_d^* \cup (\mathcal{T}_d^* + p\mathcal{T}_d^*)$, with $\mathcal{T}_d^* = \mathcal{T}_d \setminus \{0\}$.

2.2. Additive characters

Additive characters are the primary tool for converting trace-zero conditions into character-sum expressions by means of orthogonality.

Definition 2.4. Let $\chi : R \to \mathbb{C}^\times$ be a generating additive character of R. For $a \in R$, write $\chi_a(x) = \chi(ax)$. For $K = R_d$, the induced additive character is $\chi_K = \chi \circ \mathrm{Tr}_{K/R}$.

The additive character orthogonality relation used throughout is

$$\frac{1}{|R|} \sum_{z \in R} \chi(zy) = \begin{cases} 1, & y = 0, \\ 0, & y \neq 0, \end{cases} \tag{2.1}$$

which holds because $R = \mathrm{GR}(p^2, m)$ is a finite Frobenius ring of cardinality q^2. The Gauss-sum evaluations needed in later sections are collected in Section 5.

2.3. Hamming weight

All weights considered in this paper are Hamming weights over the ring alphabet R.

Definition 2.5. For $z = (z_1, \ldots, z_n) \in R^n$, the *Hamming weight* is

$$\mathrm{wt}_H(z) = \big|\{1 \le j \le n : z_j \neq 0\}\big|.$$

For a code $C \subseteq R^n$, the *Hamming weight distribution* is the sequence $(A_0, A_1, \ldots, A_n)$ given by

$$A_i = \big|\{c \in C : \mathrm{wt}_H(c) = i\}\big|.$$

2.4. Generalized quasi-cyclic codes

Let $m_1, \dots, m_\ell$ be positive integers, and set

$$A_i = R[x]/\langle x^{m_i} - 1\rangle, \qquad A = A_1 \times \cdots \times A_\ell.$$

Definition 2.6. An R-linear code $C \subseteq R^{m_1+\cdots+m_\ell}$ is a *generalized quasi-cyclic (GQC) code* over R with block lengths $(m_1, \dots, m_\ell)$ if it is an $R[x]$-submodule of A under the identification $R^{m_1+\cdots+m_\ell} \cong A$. Its Euclidean dual is

$$C^\perp = \{y \in R^{m_1+\cdots+m_\ell} : x \cdot y = 0 \text{ for all } x \in C\}, \qquad x \cdot y = \sum_i x_i y_i,$$

and it is itself a GQC code with the same block lengths.

Throughout the paper, we impose the standing coprimality assumption

$$\gcd(m_i, p) = 1 \quad \text{for all } 1 \le i \le \ell. \tag{2.2}$$

Under (2.2), each polynomial $x^{m_i} - 1$ is separable modulo p and therefore factors over R into pairwise coprime monic basic irreducible polynomials, where a monic polynomial $g(x) \in R[x]$ is called *basic irreducible* if its reduction $\bar{g}(x)$ modulo p is irreducible over $\mathbb{F}_q$; basic irreducible polynomials are irreducible in $R[x]$, and $R[x]/\langle g(x)\rangle$ is again a Galois ring; see [22, 23]. Consequently, the CRT applies to every block ring A_i.

3. Structural decomposition and weight stratification

This section converts the CRT decomposition of a GQC code into trace sums and then expresses the Hamming weight in terms of trace-zero counts on those sums. The key step is to separate the zero, nilpotent, and unit strata before applying character-sum methods.

3.1. CRT constituent decomposition

Let $g_1(x), \dots, g_t(x)$ be the distinct monic basic irreducible polynomials over R that appear in the factorizations of $x^{m_i} - 1$ for $1 \le i \le \ell$. By (2.2), each $x^{m_i} - 1$ is squarefree modulo p, so that

$$x^{m_i} - 1 = \prod_{j=1}^{t} g_j(x)^{\varepsilon_{i,j}}, \qquad \varepsilon_{i,j} \in \{0, 1\}.$$

Set $d_j = \deg g_j$ and

$$G_{i,j} = \begin{cases} R[x]/\langle g_j(x)\rangle, & \text{if } \varepsilon_{i,j} = 1, \\ 0, & \text{if } \varepsilon_{i,j} = 0. \end{cases}$$

Proposition 3.1. *There is an $R[x]$-module isomorphism*

$$A \cong \bigoplus_{j=1}^{t} (G_{1,j} \times \cdots \times G_{\ell,j}),$$

and

$$R[x]/\langle g_j(x)\rangle \cong R_{d_j} = \mathrm{GR}(p^2, md_j).$$

Every GQC code $C \subseteq A$ admits a constituent decomposition $C \cong \bigoplus_{j=1}^{t} C_j$*, where C_j is an R_{d_j}-linear code supported on those coordinates i for which $g_j(x) \mid x^{m_i} - 1$.*

For each j, choose a root $\bar{\alpha}_j \in \mathbb{F}_{q^{d_j}}$ of $\bar{g}_j(x)$. Because $g_j(x) \mid x^{m_i} - 1$ for some i with $\gcd(m_i, p) = 1$, the element $\bar{\alpha}_j$ has a unique lift $\alpha_j \in R_{d_j} = \mathrm{GR}(p^2, md_j)$ satisfying $g_j(\alpha_j) = 0, \alpha_j^{q^{d_j}-1} = 1$. Thus, α_j lies in the Teichmüller set of R_{d_j}; it is called a *Teichmüller root* of $g_j(x)$. The existence and uniqueness of this lift follow from Hensel's lemma, as $\bar{\alpha}_j$ is a simple root of the separable polynomial $x^{m_i} - 1$ modulo p; see [24, 25].

3.2. Trace representation of codewords

Theorem 3.2. *Let $C \cong \bigoplus_{j=1}^{t} C_j$ be the constituent decomposition, let $\alpha_j \in R_{d_j}$ be a fixed Teichmüller root of $g_j(x)$, and set $x_{j,i} = 0$ whenever $\varepsilon_{i,j} = 0$. Then, every codeword $c \in C$ has the form*

$$c = (c_1(x_1, \ldots, x_t) \mid \cdots \mid c_\ell(x_1, \ldots, x_t)),$$

where the ith block is given by

$$c_i(x_1, \ldots, x_t) = \left(\frac{1}{m_i} \sum_{j=1}^{t} \mathrm{Tr}_{d_j}\big(x_{j,i}\, \alpha_j^{-r}\big) \right)_{0 \le r < m_i}. \tag{3.1}$$

Conversely, every word obtained from (3.1) *by a choice of $x_j \in C_j$ belongs to C.*

Proof. Fix a block index i, and put $A_i = R[x]/\langle x^{m_i} - 1\rangle$. Because $\gcd(m_i, p) = 1$ by (2.2), the integer m_i is a unit in R. By Proposition 3.1,

$$A_i \cong \bigoplus_{\varepsilon_{i,j}=1} R[x]/\langle g_j(x)\rangle,$$

and evaluation at α_j identifies $R[x]/\langle g_j(x)\rangle$ with $R_{d_j} = \mathrm{GR}(p^2, md_j)$ for every active j.

To verify the inverse trace formula, let D be a common multiple of the degrees d_j occurring in the factorization of $x^{m_i} - 1$, and embed all the R_{d_j} into the common splitting ring $R_D = \mathrm{GR}(p^2, mD)$.

Let σ denote the Frobenius automorphism of R_D/R. Because every α_j is a Teichmüller element, one has $\sigma^s(\alpha_j) = \alpha_j^{q^s}$. Moreover, because $g_j(x) \mid x^{m_i} - 1$, every conjugate $\alpha_j^{q^s}$ is an m_ith root of unity in R_D.

Let $x_{j,i} \in R_{d_j}$ be the jth constituent component of the ith block, with $x_{j,i} = 0$ whenever $\varepsilon_{i,j} = 0$. Define $a_{i,r}$ to be the right-hand side of (3.1), and put $a_i(x) = \sum_{r=0}^{m_i-1} a_{i,r} x^r$. Evaluating at α_k gives

$$a_i(\alpha_k) = \frac{1}{m_i} \sum_{j=1}^{t} \sum_{r=0}^{m_i-1} \mathrm{Tr}_{d_j}\Big(x_{j,i}\alpha_j^{-r}\Big)\, \alpha_k^r.$$

Expanding the trace in the common splitting ring yields

$$\mathrm{Tr}_{d_j}\Big(x_{j,i}\alpha_j^{-r}\Big) = \sum_{s=0}^{d_j-1} \sigma^s(x_{j,i}) \alpha_j^{-q^s r},$$

and consequently,

$$a_i(\alpha_k) = \frac{1}{m_i} \sum_{j=1}^{t} \sum_{s=0}^{d_j-1} \sigma^s(x_{j,i}) \sum_{r=0}^{m_i-1} \left(\alpha_k \alpha_j^{-q^s}\right)^r.$$

We now evaluate the inner geometric sum. Put $\rho_{j,k,s} = \alpha_k \alpha_j^{-q^s}$. Because both α_k and $\alpha_j^{q^s}$ are m_i-th roots of unity, we have $\rho_{j,k,s}^{m_i} = 1$. If $\rho_{j,k,s} \neq 1$, then its residue modulo p also differs from 1; indeed, Teichmüller lifting is injective, so equality of the residues would force $\alpha_k = \alpha_j^{q^s}$. Hence, $1 - \rho_{j,k,s}$ is a unit in R_D, and

$$\sum_{r=0}^{m_i-1} \rho_{j,k,s}^r = \frac{1 - \rho_{j,k,s}^{m_i}}{1 - \rho_{j,k,s}} = 0.$$

If $\rho_{j,k,s} = 1$, then every summand equals 1, and therefore,

$$\sum_{r=0}^{m_i-1} \rho_{j,k,s}^r = m_i.$$

Suppose first that $j \neq k$. The equality $\alpha_k = \alpha_j^{q^s}$ would imply that α_j and α_k lie in the same Frobenius orbit, that orbit being $\{\alpha_j, \alpha_j^q, \ldots, \alpha_j^{q^{d_j-1}}\}$ in the case of α_j. Their minimal basic irreducible polynomials would then coincide, contrary to $g_j \neq g_k$. Thus,

$$\sum_{r=0}^{m_i-1} \left(\alpha_k \alpha_j^{-q^s}\right)^r = 0 \qquad (j \neq k).$$

Now, let $j = k$. If $\alpha_k = \alpha_k^{q^s}$ for some $0 \leq s < d_k$, then α_k is fixed by σ^s; because its Frobenius orbit has length $d_k = \deg g_k$, this is possible only for $s = 0$. Hence,

$$\sum_{r=0}^{m_i-1} \left(\alpha_k \alpha_k^{-q^s}\right)^r = \begin{cases} m_i, & s = 0, \\ 0, & 1 \leq s < d_k. \end{cases}$$

It follows that the only surviving term in the preceding expansion is the one with $j = k$ and $s = 0$, so

$$a_i(\alpha_k) = \frac{1}{m_i} \cdot m_i \cdot x_{k,i} = x_{k,i}.$$

Thus, $a_i(x)$ takes precisely the prescribed constituent value $x_{k,i}$ at every active root α_k. By the CRT, it is therefore the unique element of A_i corresponding to the tuple $(x_{j,i})_{\varepsilon_{i,j}=1}$, and hence, the coefficient formula in (3.1) is exactly the inverse CRT map on the ith block.

Applying this argument block by block shows that every choice $x_j \in C_j$ produces a codeword of C. Conversely, every codeword of C has a unique constituent tuple in $\bigoplus_{j=1}^{t} C_j$, and the inverse CRT formula recovers it in the form (3.1). This completes the proof. □

Definition 3.3. For $c \in C$, $1 \leq i \leq \ell$, and $0 \leq r < m_i$, define

$$S_{i,r}(c) = \sum_{j=1}^{t} \mathrm{Tr}_{d_j}(x_{j,i}\, \alpha_j^{-r}), \tag{3.2}$$

so that $c_{i,r} = m_i^{-1} S_{i,r}(c)$. Because m_i is a unit in R, the element $c_{i,r}$ is zero, nonzero nilpotent, or a unit if and only if $S_{i,r}(c)$ is zero, nonzero nilpotent, or a unit, respectively.

3.3. Stratified weight formula

For $c \in C$ and $1 \le i \le \ell$, define the stratum counts

$$N_i^{(0)}(c) = \Big|\{0 \le r < m_i : S_{i,r}(c) = 0\}\Big|,$$
$$N_i^{(p)}(c) = \Big|\{0 \le r < m_i : S_{i,r}(c) \in pR \setminus \{0\}\}\Big|,$$
$$N_i^{(\times)}(c) = \Big|\{0 \le r < m_i : S_{i,r}(c) \in R^\times\}\Big|,$$

so that

$$m_i = N_i^{(0)}(c) + N_i^{(p)}(c) + N_i^{(\times)}(c).$$

Theorem 3.4. *For every codeword $c \in C$,*

$$\mathrm{wt}_H(c) = \sum_{i=1}^{\ell} (N_i^{(p)}(c) + N_i^{(\times)}(c)) = \sum_{i=1}^{\ell} (m_i - N_i^{(0)}(c)). \tag{3.3}$$

In particular, the Hamming weight of c is completely determined by the trace-zero counts $N_i^{(0)}(c)$.

Proof. Fix a codeword $c \in C$. By the trace representation of Theorem 3.2, the coordinate at position r of the ith block is

$$c_{i,r} = m_i^{-1} S_{i,r}(c), \qquad 0 \le r < m_i.$$

The coprimality assumption $\gcd(m_i, p) = 1$ implies that m_i is a unit in $R = \mathrm{GR}(p^2, m)$, so that multiplication by m_i^{-1} is a bijection on R. In particular,

$$c_{i,r} = 0 \quad \Longleftrightarrow \quad S_{i,r}(c) = 0.$$

For each fixed block i, the three sets

$$\{r : S_{i,r}(c) = 0\}, \quad \{r : S_{i,r}(c) \in pR \setminus \{0\}\}, \quad \{r : S_{i,r}(c) \in R^\times\}$$

partition $\{0, 1, \ldots, m_i - 1\}$, whence $N_i^{(0)}(c) + N_i^{(p)}(c) + N_i^{(\times)}(c) = m_i$. Because a coordinate contributes 1 to the Hamming weight exactly when it is nonzero, the contribution of the ith block is

$$N_i^{(p)}(c) + N_i^{(\times)}(c) = m_i - N_i^{(0)}(c).$$

Summing over $i = 1, \ldots, \ell$ gives (3.3), and the final assertion is immediate. □

Remark 3.5. Computing the homogeneous weight would require the separate counts $N^{(p)}(c)$ and $N^{(\times)}(c)$, as well. For the Hamming weight, by contrast, only the distinction between zero and nonzero coordinates matters, so that $N_i^{(0)}(c)$ alone suffices via (3.3).

Remark 3.6. Over rings of characteristic p built from a base field, trace-zero conditions typically split into finite-field trace equations that can be handled by ordinary Gauss sums. The situation over $R = \mathrm{GR}(p^2, m)$ is different, owing to the mixed-characteristic structure. A unit $A = A_0 + pA_1 \in K^\times$ with $A_0 \in \mathcal{T}_K^*$ and $A_1 \in \mathcal{T}_K$ gives a vanishing condition $\mathrm{Tr}_{K/R}(A\alpha^{-r}) = 0$ that imposes both a residue-layer equation and a principal p-layer equation, the latter involving carry traces. This is the key obstruction in the unit and mixed cases treated in Section 4.

4. Explicit trace-zero formulas over $\mathrm{GR}(p^2, m)$

This section evaluates the trace-zero counts that appear in Theorem 3.4, with attention restricted to one- and two-constituent configurations over $R = \mathrm{GR}(p^2, m)$. The key distinction is between nilpotent parameters, whose trace-zero equations descend to residue-field trace equations, and unit parameters, whose vanishing conditions involve the principal p-layer.

Throughout this section, set

$$K_j = \mathrm{GR}(p^2, md_j), \qquad Q_j = q^{d_j}, \qquad j = 0, 1,$$

and let $\alpha_j \in K_j^\times$ be Teichmüller units whose residues $\bar{\alpha}_j \in \mathbb{F}_{Q_j}^\times$ have common order n. Trace-zero counts are taken over $0 \le r < n$. For a general GQC block of length m_i, the same formulas apply on each full orbit; if the active root has order $n_\alpha \mid m_i$, then the count over $0 \le r < m_i$ is obtained by applying the multiplicity factor m_i/n_α.

Fix primitive elements ξ_j of $\mathbb{F}_{Q_j}^\times$, write $\bar{\alpha}_j^{-1} = \xi_j^{u_j}$, and define

$$\bar{g}_j = \gcd\left(\frac{Q_j - 1}{q - 1}, u_j\right), \qquad g_j = \gcd(Q_j - 1, u_j), \qquad \bar{\Delta}_j = \frac{Q_j - 1}{\bar{g}_j}, \qquad \Delta_j = \frac{Q_j - 1}{g_j}. \tag{4.1}$$

The symbols ω_j and λ_j denote, respectively, a fixed generator of the multiplicative character group of $\mathbb{F}_{Q_j}$ and the canonical additive character of $\mathbb{F}_{Q_j}$. The semiprimitive cases use the sign patterns stated in Lemma 5.2, which specialize the general character sums produced by the trace-zero formulas. Here, and throughout the paper, the term *semiprimitive* refers to the hypothesis of Lemma 5.2: The multiplicative character involved has order $N \ge 3$, some power p^e satisfies $p^e \equiv -1 \pmod{N}$, and the field size is $p^{2e\delta}$. Under this hypothesis, the Gauss sums $G(\psi^t, \lambda)$ for $1 \le t \le N - 1$ are all equal to $\pm\sqrt{Q}$, and (5.5) lists the two possible sign behaviors: We call the first branch, in which the sign is $(-1)^t$ and thus alternates with t, the *alternating-sign pattern*, and the second branch, in which the sign is the constant $(-1)^{\delta-1}$, the *constant-sign pattern*.

Remark 4.1. The purely nilpotent cases reduce to ordinary finite-field Gauss sums. The unit and mixed cases do not reduce in this way in general: The principal p-layer contributes carry traces, and in the unit–unit case, addition carries, as well. These carry terms constitute the mixed-characteristic obstruction that distinguishes the unit cases from the nilpotent ones.

4.1. Pure nilpotent cases

Let $A = p\widetilde{A} \in pK_j$. By the R-linearity of the trace together with $p^2 = 0$,

$$\mathrm{Tr}_{K_j/R}(p\widetilde{A}\,\alpha_j^{-r}) = p\,\mathrm{Tr}_{K_j/R}(\widetilde{A}\,\alpha_j^{-r}).$$

Moreover, $py = 0$ in R if and only if $\bar{y} = 0$ in $R/pR \cong \mathbb{F}_q$, so the trace-zero condition is equivalent to the residue-field equation

$$\mathrm{tr}_{\mathbb{F}_{Q_j}/\mathbb{F}_q}(\bar{\widetilde{A}}\,\bar{\alpha}_j^{-r}) = 0.$$

The nilpotent trace-zero count therefore depends only on the residue $\bar{\widetilde{A}}$.

For $A = p\widetilde{A} \in pK_0 \setminus \{0\}$ with $\bar{\widetilde{A}} = \xi_0^b$, define

$$Z_0(p\widetilde{A}) = \left|\{0 \le r < n : \mathrm{Tr}_{K_0/R}(p\widetilde{A}\,\alpha_0^{-r}) = 0\}\right| = \left|\{0 \le r < n : \mathrm{tr}_{\mathbb{F}_{Q_0}/\mathbb{F}_q}(\bar{\widetilde{A}}\,\bar{\alpha}_0^{-r}) = 0\}\right|. \tag{4.2}$$

Theorem 4.2. *With the notation of* (4.1) *and* (4.2)*,*

$$Z_0(p\widetilde{A}) = \frac{n}{q} + \frac{(q-1)n}{q(Q_0-1)} \sum_{s=0}^{\bar{g}_0-1} G_{Q_0}(\omega_0^{-\bar{\Delta}_0 s}, \lambda_0)\, \omega_0^{\bar{\Delta}_0 s}(\bar{\widetilde{A}}). \tag{4.3}$$

Moreover, the following evaluated forms hold.

(a) *If* $\bar{g}_0 = 2$, *and* p *is odd, then*

$$Z_0(p\widetilde{A}) = \frac{n}{q} - \frac{(q-1)n}{q(Q_0-1)} + (-1)^{md_0+b-1}(\sqrt{-1})^{\frac{md_0(p-1)^2}{4}} \frac{(q-1)n\sqrt{Q_0}}{q(Q_0-1)}. \tag{4.4}$$

(b) *If* $\bar{g}_0 \geq 3$, *and the semiprimitive Gauss sums have the alternating-sign pattern, then*

$$Z_0(p\widetilde{A}) = \begin{cases} \dfrac{n}{q} - \dfrac{(q-1)n}{q(Q_0-1)}(1 - \sqrt{Q_0}(\bar{g}_0-1)), & \omega_0^{\bar{\Delta}_0}(\bar{\widetilde{A}}) = -1, \\ \dfrac{n}{q} - \dfrac{(q-1)n}{q(Q_0-1)}(1 + \sqrt{Q_0}), & \textit{otherwise.} \end{cases} \tag{4.5}$$

(c) *If* $\bar{g}_0 \geq 3$, *and the semiprimitive Gauss sums have the constant-sign pattern with* $Q_0 = p^{2e\delta}$, *then*

$$Z_0(p\widetilde{A}) = \begin{cases} \dfrac{n}{q} - \dfrac{(q-1)n}{q(Q_0-1)}(1 + (-1)^{\delta}\sqrt{Q_0}(\bar{g}_0-1)), & \omega_0^{\bar{\Delta}_0}(\bar{\widetilde{A}}) = 1, \\ \dfrac{n}{q} - \dfrac{(q-1)n}{q(Q_0-1)}(1 + (-1)^{\delta-1}\sqrt{Q_0}), & \textit{otherwise.} \end{cases} \tag{4.6}$$

The same formulas hold for $B = p\widetilde{B} \in pK_1 \setminus \{0\}$ *after the subscript* 0 *is replaced by* 1.

Proof. We prove the formula for the K_0-constituent; the case of K_1 is identical. For readability, write

$$Q = Q_0, \quad \alpha = \bar{\alpha}_0, \quad \omega = \omega_0, \quad \lambda = \lambda_0, \quad \bar{g} = \bar{g}_0, \quad \bar{\Delta} = \bar{\Delta}_0,$$

and put $\beta = \alpha^{-1} = \xi_0^{u_0}$. By the nilpotent reduction above, $Z_0(p\widetilde{A})$ is the number of indices r for which $\mathrm{tr}_{\mathbb{F}_Q/\mathbb{F}_q}(\bar{\widetilde{A}}\beta^r) = 0$.

Indeed,

$$\mathrm{Tr}_{K_0/R}(p\widetilde{A}\alpha_0^{-r}) = p\,\mathrm{Tr}_{K_0/R}(\widetilde{A}\alpha_0^{-r}),$$

and an element $py \in pR$ vanishes if and only if its residue coefficient $\bar{y} \in \mathbb{F}_q$ is zero. The Galois ring trace condition is therefore equivalent to the stated finite-field trace equation.

Applying additive-character orthogonality (2.1) over $\mathbb{F}_q$ gives

$$Z_0(p\widetilde{A}) = \frac{n}{q} + \frac{1}{q}\sum_{a\in\mathbb{F}_q^*}\sum_{r=0}^{n-1} \lambda(a\bar{\widetilde{A}}\beta^r).$$

More precisely, for each r,

$$\frac{1}{q}\sum_{a\in\mathbb{F}_q} \lambda(a\bar{\widetilde{A}}\beta^r) = \begin{cases} 1, & \mathrm{tr}_{\mathbb{F}_Q/\mathbb{F}_q}(\bar{\widetilde{A}}\beta^r) = 0, \\ 0, & \mathrm{tr}_{\mathbb{F}_Q/\mathbb{F}_q}(\bar{\widetilde{A}}\beta^r) \neq 0, \end{cases}$$

and the contribution of $a = 0$ is n/q, which gives the preceding expression.

For $x \in \mathbb{F}_Q^*$, expand the additive character in terms of multiplicative characters:

$$\lambda(x) = \frac{1}{Q-1} \sum_{h=0}^{Q-2} G_Q(\omega^{-h}, \lambda)\, \omega^h(x).$$

Substituting $x = a\widetilde{\widetilde{A}}\beta^r$ and interchanging the finite sums yields

$$Z_0(p\widetilde{A}) = \frac{n}{q} + \frac{1}{q(Q-1)} \sum_{h=0}^{Q-2} G_Q(\omega^{-h}, \lambda)\, \omega^h(\widetilde{\widetilde{A}}) \times \left(\sum_{a \in \mathbb{F}_q^*} \omega^h(a) \right) \left(\sum_{r=0}^{n-1} \omega^h(\beta^r) \right).$$

After the substitution $x = a\widetilde{\widetilde{A}}\beta^r$, the sum over $a \in \mathbb{F}_q^*$ is nonzero only when $\omega^h|_{\mathbb{F}_q^*} = 1$, whereas the sum over r is nonzero only when $\omega^h(\beta) = 1$. These conditions translate into

$$q - 1 \mid h \qquad \text{and} \qquad hu_0 \equiv 0 \pmod{Q-1}.$$

In fact,

$$\sum_{a \in \mathbb{F}_q^*} \omega^h(a) = \begin{cases} q-1, & q-1 \mid h, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, because β has order n,

$$\sum_{r=0}^{n-1} \omega^h(\beta^r) = \begin{cases} n, & hu_0 \equiv 0 \pmod{Q-1}, \\ 0, & \text{otherwise.} \end{cases}$$

The two orthogonality relations thus act as independent selection rules.

By (4.1), the surviving exponents are exactly $h = \bar{\Delta}s$ for $0 \le s \le \bar{g} - 1$, with contributions $q - 1$ and n from the respective sums.

To see this explicitly, put $L = (Q-1)/(q-1)$, and write $h = (q-1)t$. Then, $hu_0 \equiv 0 \pmod{Q-1}$ is equivalent to $L \mid tu_0$, and, because $\bar{g} = \gcd(L, u_0)$, this holds precisely when

$$t = \frac{L}{\bar{g}} s, \qquad 0 \le s \le \bar{g} - 1.$$

Hence,

$$h = (q-1)\frac{L}{\bar{g}} s = \frac{Q-1}{\bar{g}} s = \bar{\Delta}s.$$

This gives (4.3).

It remains to specialize the Gauss sums. The term with $s = 0$ contributes $G_Q(1, \lambda) = -1$. If $\bar{g} = 2$, then $\omega^{\bar{\Delta}}$ is the quadratic character of $\mathbb{F}_Q^*$; by Lemma 5.1 together with $\widetilde{\widetilde{A}} = \xi_0^b$, we obtain $\omega^{\bar{\Delta}}(\widetilde{\widetilde{A}}) = (-1)^b$, which yields (4.4).

Indeed, in this case,

$$G_Q(\omega^{-\bar{\Delta}}, \lambda) = (-1)^{md_0 - 1} (\sqrt{-1})^{\frac{md_0(p-1)^2}{4}} \sqrt{Q},$$

and multiplication by $\omega^{\bar{\Delta}}(\widetilde{\bar{A}}) = (-1)^b$ produces the final sign in (4.4).

Now, assume $\bar{g} \geq 3$, and set $\chi = \omega^{\bar{\Delta}}$, so that χ has order $\bar{g}$. Formula (4.3) becomes

$$Z_0(p\widetilde{A}) = \frac{n}{q} + \frac{(q-1)n}{q(Q-1)}\left(-1 + \sum_{s=1}^{\bar{g}-1} G_Q(\chi^{-s}, \lambda)\chi^s(\widetilde{\bar{A}})\right).$$

Under the alternating-sign pattern, the last sum equals $\sqrt{Q}\sum_{s=1}^{\bar{g}-1}(-\chi(\widetilde{\bar{A}}))^s$, which is $\sqrt{Q}(\bar{g}-1)$ when $\chi(\widetilde{\bar{A}}) = -1$ and $-\sqrt{Q}$ otherwise; this gives (4.5).

The latter evaluation follows from the finite geometric sum

$$\sum_{s=1}^{\bar{g}-1} z^s = \begin{cases} \bar{g}-1, & z = 1, \\ -1, & z^{\bar{g}} = 1 \text{ and } z \neq 1, \end{cases}$$

applied with $z = -\chi(\widetilde{\bar{A}})$.

Under the constant-sign pattern, the sum equals $(-1)^{\delta-1}\sqrt{Q}\sum_{s=1}^{\bar{g}-1}\chi^s(\widetilde{\bar{A}})$, which is $(-1)^{\delta-1}\sqrt{Q}(\bar{g}-1)$ when $\chi(\widetilde{\bar{A}}) = 1$ and $(-1)^{\delta}\sqrt{Q}$ otherwise; this gives (4.6).

Indeed,

$$\sum_{s=1}^{\bar{g}-1}\chi^s(\widetilde{\bar{A}}) = \begin{cases} \bar{g}-1, & \chi(\widetilde{\bar{A}}) = 1, \\ -1, & \chi(\widetilde{\bar{A}}) \neq 1, \end{cases}$$

which yields the two alternatives in (4.6). □

Let $A = p\widetilde{A} \in pK_0 \setminus \{0\}$ and $B = p\widetilde{B} \in pK_1 \setminus \{0\}$. By the same nilpotent reduction, the simultaneous trace-zero count is

$$Z_{pp}(A, B) = \left|\{0 \leq r < n : \mathrm{tr}_{\mathbb{F}_{Q_0}/\mathbb{F}_q}(\widetilde{\bar{A}}\,\bar{\alpha}_0^{-r}) + \mathrm{tr}_{\mathbb{F}_{Q_1}/\mathbb{F}_q}(\widetilde{\bar{B}}\,\bar{\alpha}_1^{-r}) = 0\}\right|. \tag{4.7}$$

The admissible character pairs are collected in

$$\mathcal{A}_{01} = \left\{(h,k) : 0 \leq h < Q_0 - 1,\ 0 \leq k < Q_1 - 1,\ \omega_0^h\restriction_{\mathbb{F}_q^*} \cdot \omega_1^k\restriction_{\mathbb{F}_q^*} = 1,\ \omega_0^h(\bar{\alpha}_0^{-1})\omega_1^k(\bar{\alpha}_1^{-1}) = 1\right\}. \tag{4.8}$$

Theorem 4.3. *With the notation of* (4.7) *and* (4.8)*,*

$$Z_{pp}(A, B) = \frac{n}{q} + \frac{(q-1)n}{q(Q_0-1)(Q_1-1)} \sum_{(h,k)\in\mathcal{A}_{01}} G_{Q_0}(\omega_0^{-h}, \lambda_0)\, G_{Q_1}(\omega_1^{-k}, \lambda_1)\, \omega_0^h(\widetilde{\bar{A}})\, \omega_1^k(\widetilde{\bar{B}}). \tag{4.9}$$

When $B = 0$ or $A = 0$, this reduces to the corresponding one-constituent formula of Theorem 4.2.

Proof. Set

$$Y_r = \mathrm{tr}_{\mathbb{F}_{Q_0}/\mathbb{F}_q}(\widetilde{\bar{A}}\bar{\alpha}_0^{-r}) + \mathrm{tr}_{\mathbb{F}_{Q_1}/\mathbb{F}_q}(\widetilde{\bar{B}}\bar{\alpha}_1^{-r}).$$

Additive-character orthogonality over $\mathbb{F}_q$ gives

$$Z_{pp}(A, B) = \frac{1}{q}\sum_{r=0}^{n-1}\sum_{a\in\mathbb{F}_q}\lambda(aY_r),$$

the term with $a = 0$ contributing n/q. For $a \neq 0$, expand both additive characters over $\mathbb{F}_{Q_0}$ and $\mathbb{F}_{Q_1}$ in terms of multiplicative characters. The sum over $a \in \mathbb{F}_q^*$ is nonzero precisely when $\omega_0^h|_{\mathbb{F}_q^*} \cdot \omega_1^k|_{\mathbb{F}_q^*} = 1$, in which case it contributes $q-1$. The sum over r is nonzero precisely when $\omega_0^h(\bar{\alpha}_0^{-1})\omega_1^k(\bar{\alpha}_1^{-1}) = 1$, in which case, it contributes n. These are exactly the conditions defining $\mathcal{A}_{01}$ in (4.8), and (4.9) follows. When one parameter is zero, the formula collapses to (4.3). □

4.2. Two-layer trace corrections in characteristic p^2

The following definitions isolate the correction terms that are absent from the nilpotent cases. They record, respectively, the failure of the Teichmüller lift to commute with the trace (the carry trace) and with addition (the addition carry).

Let $d \geq 1$, let $K = \mathrm{GR}(p^2, md)$, and write $\tau_d : \mathbb{F}_{q^d} \to \mathcal{T}_d$ and $\tau : \mathbb{F}_q \to \mathcal{T}$ for the respective Teichmüller lifts.

Let $\mathrm{Fr}_p : z \mapsto z^p$ denote the p-Frobenius automorphism on every finite extension of $\mathbb{F}_p$, and let Fr_p^{-1} denote its inverse. For $h \geq 1$, define the universal carry polynomial

$$C_{p,h}(X_1, \ldots, X_h) = \left[\frac{\sum_{j=1}^{h} X_j^p - \left(\sum_{j=1}^{h} X_j \right)^p}{p} \right]_{\bmod p} \in \mathbb{F}_p[X_1, \ldots, X_h].$$

The quotient has integer coefficients, because every nontrivial multinomial coefficient occurring in $\left(\sum_j X_j\right)^p$ is divisible by p.

Definition 4.4. For $x \in \mathbb{F}_{q^d}$, define $\kappa_d(x) \in \mathbb{F}_q$ by

$$\mathrm{Tr}_{K/R}(\tau_d(x)) = \tau(\mathrm{tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(x)) + p\,\tau(\kappa_d(x)). \tag{4.10}$$

Equivalently,

$$\kappa_d(x) = \mathrm{Fr}_p^{-1}\Big(C_{p,d}(x, x^q, \ldots, x^{q^{d-1}})\Big). \tag{4.11}$$

Definition 4.5. For $a_1, \ldots, a_h \in \mathbb{F}_q$, define $\gamma_h(a_1, \ldots, a_h) \in \mathbb{F}_q$ by

$$\sum_{j=1}^{h} \tau(a_j) = \tau\left(\sum_{j=1}^{h} a_j \right) + p\,\tau(\gamma_h(a_1, \ldots, a_h)). \tag{4.12}$$

Equivalently,

$$\gamma_h(a_1, \ldots, a_h) = \mathrm{Fr}_p^{-1}\Big(C_{p,h}(a_1, \ldots, a_h)\Big). \tag{4.13}$$

For $h = 2$, we write $\gamma(a, b)$ in place of $\gamma_2(a, b)$.

Proposition 4.6. *The maps κ_d and γ_h of Definitions 4.4 and 4.5 exist and are unique. Moreover, they are given by* (4.11) *and* (4.13)*, respectively.*

Proof. For any finite field k of characteristic p, identify $\mathrm{GR}(p^2, [k : \mathbb{F}_p])$ with the ring $W_2(k)$ of length-two p-typical Witt vectors. Under this identification, the Teichmüller lift is $\tau(a) = [a] = (a, 0)$, whereas $p\tau(b) = (0, b^p)$.

Let $a_1, \ldots, a_h \in k$, and put $s = \sum_{j=1}^{h} a_j$. The Witt addition formula gives

$$\sum_{j=1}^{h} [a_j] = \Big(s,\, C_{p,h}(a_1, \ldots, a_h)\Big),$$

whereas $[s]+p[b] = (s, b^p)$. Therefore, $\sum_{j=1}^{h}[a_j] = [s]+p[b]$ holds precisely when $b^p = C_{p,h}(a_1, \dots, a_h)$. Because k is perfect, the p-Frobenius is bijective, and hence, $b = \mathrm{Fr}_p^{-1}\big(C_{p,h}(a_1, \dots, a_h)\big)$. This proves both the existence and the explicit formula for γ_h. Uniqueness follows from the uniqueness of the Teichmüller expansion $z = z_0 + pz_1$.

For the carry trace, the Frobenius conjugates of $\tau_d(x)$ over R are $\tau_d(x), \tau_d(x^q), \dots, \tau_d(x^{q^{d-1}})$, and consequently,

$$\mathrm{Tr}_{K/R}(\tau_d(x)) = \sum_{r=0}^{d-1} \tau_d(x^{q^r}).$$

Applying the preceding addition formula in K gives

$$\mathrm{Tr}_{K/R}(\tau_d(x)) = \tau_d\left(\sum_{r=0}^{d-1} x^{q^r}\right) + p\,\tau_d\Big(\mathrm{Fr}_p^{-1} C_{p,d}(x, x^q, \dots, x^{q^{d-1}})\Big).$$

The first argument is $\mathrm{tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(x) \in \mathbb{F}_q$. The second argument also lies in $\mathbb{F}_q$, because applying the q-Frobenius cyclically permutes $(x, x^q, \dots, x^{q^{d-1}})$, whereas $C_{p,d}$ is symmetric, and Fr_p^{-1} commutes with the q-Frobenius. The preceding identity is thus exactly (4.10), with κ_d given by (4.11). Its uniqueness again follows from the uniqueness of the Teichmüller expansion. □

Remark 4.7. For $p = 3$, we have $C_{3,2}(u, v) = -(u^2v + uv^2)$, so the two-variable addition carry is $\gamma(u, v) = \mathrm{Fr}_3^{-1}(-(u^2v + uv^2))$. Similarly, when $d = 2$,

$$\kappa_2(x) = \mathrm{Fr}_3^{-1}(-(x^2x^q + x(x^q)^2)).$$

In particular, over $\mathbb{F}_9$ one has $\mathrm{Fr}_3^{-1}(z) = z^3$.

Remark 4.8. The functions κ_d and γ_h are intrinsic to $\mathrm{GR}(p^2, m)$. In the nilpotent cases, every term involving κ_d or γ_h is multiplied by p and therefore vanishes, as $p^2 = 0$; this is why Theorems 4.2 and 4.3 involve finite-field data only. For unit parameters, by contrast, these correction terms survive in the second Teichmüller layer and must be accounted for explicitly, as made precise in Lemma 4.9 below.

Lemma 4.9. *Let $A = A_0 + pA_1 \in K^\times$ with $A_0 \in \mathcal{T}_d^*$ and $A_1 \in \mathcal{T}_d$, and let $\alpha \in K^\times$ be a Teichmüller unit. For $0 \le r < n$, define*

$$\rho_A(r) = \mathrm{tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\bar{A}_0\,\bar{\alpha}^{-r}), \qquad \pi_A(r) = \kappa_d(\bar{A}_0\,\bar{\alpha}^{-r}) + \mathrm{tr}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\bar{A}_1\,\bar{\alpha}^{-r}), \tag{4.14}$$

called the residue-layer function *and the* principal/carry-layer function *of A, respectively. Then,* $\mathrm{Tr}_{K/R}(A\,\alpha^{-r}) = 0$ *if and only if* $\rho_A(r) = 0$, *and* $\pi_A(r) = 0$.

Proof. Because α is a Teichmüller unit, the products $A_0\alpha^{-r}$ and $A_1\alpha^{-r}$ are Teichmüller elements. By the R-linearity of the trace,

$$\mathrm{Tr}_{K/R}(A\alpha^{-r}) = \mathrm{Tr}_{K/R}(A_0\alpha^{-r}) + p\,\mathrm{Tr}_{K/R}(A_1\alpha^{-r}).$$

Applying (4.10) to $A_0\alpha^{-r}$ gives

$$\mathrm{Tr}_{K/R}(A_0\alpha^{-r}) = \tau(\rho_A(r)) + p\,\tau(\kappa_d(\bar{A}_0\bar{\alpha}^{-r})).$$

Compatibility of the trace with reduction modulo p gives

$$\mathrm{Tr}_{K/R}(A_1\alpha^{-r}) = \tau(\mathrm{tr}(\bar{A}_1\bar{\alpha}^{-r})) + pY$$

for some $Y \in R$, and multiplying by p and using $p^2 = 0$ annihilates the term pY. Combining these identities by means of (4.14),

$$\mathrm{Tr}_{K/R}(A\alpha^{-r}) = \tau(\rho_A(r)) + p\,\tau(\pi_A(r)).$$

By the uniqueness of the Teichmüller expansion in R, this vanishes if and only if $\rho_A(r) = 0$, and $\pi_A(r) = 0$. □

Lemma 4.9 is the key structural result distinguishing the unit cases from the nilpotent ones: The vanishing of $\mathrm{Tr}_{K/R}(A\alpha^{-r})$ imposes *two* conditions on each index r. In particular, two units $A = A_0 + pA_1$ and $A' = A_0 + pA_1'$ with the same residue A_0 but with $A_1 \neq A_1'$ can give different values of π_A, and hence different trace-zero counts, even though they are indistinguishable at the residue level. This phenomenon, which has no counterpart in the characteristic-p model, is illustrated concretely in Example 5.6.

4.3. One-unit cases

Let $K_0 = \mathrm{GR}(p^2, md_0)$ and $Q_0 = q^{d_0}$, and let $\alpha_0 \in K_0$ be the Teichmüller root. For $A = A_0 + pA_1 \in K_0^\times$, define

$$U_0(A) = \Big|\{0 \le r < n : \mathrm{Tr}_{K_0/R}(A\,\alpha_0^{-r}) = 0\}\Big|. \tag{4.15}$$

The map $\pi_p : pR \to \mathbb{F}_q$ is defined by $\pi_p(pb) = \bar{b}$ for $b \in \mathcal{T}$. By Lemma 4.9, the count (4.15) is governed by a residue trace condition together with a second, principal layer condition. For $z \in R$, define the Galois-ring period

$$\Theta_0(z; A) = \sum_{r=0}^{n-1} \chi_{K_0}(z\,A\,\alpha_0^{-r}). \tag{4.16}$$

The term *Galois ring period* is used by analogy with the classical Gauss periods over finite fields: The summation runs over the cyclic orbit $\{\alpha_0^{-r} : 0 \le r < n\}$ rather than over the full multiplicative group, and the additive character of the field is replaced by the induced character χ_{K_0} of the Galois ring.

Theorem 4.10. *For $A = A_0 + pA_1 \in K_0^\times$,*

$$U_0(A) = \frac{1}{q^2}\Big(n + \mathcal{R}_0(\bar{A}_0) + \mathcal{P}_0(A)\Big), \tag{4.17}$$

where

$$\mathcal{R}_0(\bar{A}_0) = \sum_{b\in\mathcal{T}^*} \Theta_0(pb; A) = (q-1)\frac{n}{Q_0 - 1} \sum_{s=0}^{\bar{g}_0-1} G_{Q_0}(\omega_0^{-\bar{\Delta}_0 s}, \lambda_0)\,\omega_0^{\bar{\Delta}_0 s}(\bar{A}_0), \tag{4.18}$$

and

$$\mathcal{P}_0(A) = \sum_{a\in\mathcal{T}^*}\sum_{b\in\mathcal{T}}\sum_{r=0}^{n-1} \chi_{K_0}((a + pb)(A_0 + pA_1)\alpha_0^{-r}). \tag{4.19}$$

Hence,

$$U_0(A) = \frac{1}{q^2}\Big(n + (q-1)\frac{n}{Q_0 - 1}\sum_{s=0}^{\bar{g}_0-1} G_{Q_0}(\omega_0^{-\bar{\Delta}_0 s}, \lambda_0)\,\omega_0^{\bar{\Delta}_0 s}(\bar{A}_0) + \mathcal{P}_0(A)\Big). \tag{4.20}$$

The same formula, with the subscript 0 *replaced by* 1*, holds for a unit* $B \in K_1^\times$.

Proof. Applying orthogonality (2.1) with $y = \mathrm{Tr}_{K_0/R}(A\alpha_0^{-r})$ and summing over r gives

$$U_0(A) = \frac{1}{q^2}\sum_{z\in R}\sum_{r=0}^{n-1}\chi_{K_0}(z\,A\,\alpha_0^{-r}).$$

Decomposing $R = \{0\} \cup p\mathcal{T}^* \cup (\mathcal{T}^* + p\mathcal{T})$, the term with $z = 0$ contributes n, the terms with $z = pb$ and $b \in \mathcal{T}^*$ contribute $\sum_b \Theta_0(pb; A)$, and the unit terms contribute $\mathcal{P}_0(A)$ as defined in (4.19); this gives (4.17).

For the $p\mathcal{T}^*$-contribution, $p^2 = 0$ gives

$$pb(A_0 + pA_1)\alpha_0^{-r} = pbA_0\alpha_0^{-r},$$

and compatibility of the additive characters with reduction modulo p then gives

$$\chi_{K_0}(pbA\alpha_0^{-r}) = \lambda_0(\bar{b}\,\bar{A}_0\,\bar{\alpha}_0^{-r}).$$

Expanding λ_0 by means of (5.6) and applying the selection rules (5.7) and (5.8), only the exponents $h = \bar{\Delta}_0 s$ with $0 \le s \le \bar{g}_0 - 1$ survive, which yields (4.18). Substituting into (4.17) gives (4.20). The unit contribution $\mathcal{P}_0(A)$ cannot be reduced to a function of $\bar{A}_0$ alone, as $(a + pb)(A_0 + pA_1) = aA_0 + p(aA_1 + bA_0)$, so that the p-coefficient depends on the principal part A_1. □

4.4. Mixed two-constituent cases

For $A \in K_0$ and $B \in K_1$, define

$$Z(A, B) = \Big|\{0 \le r < n : \mathrm{Tr}_{K_0/R}(A\,\alpha_0^{-r}) + \mathrm{Tr}_{K_1/R}(B\,\alpha_1^{-r}) = 0\}\Big| \tag{4.21}$$

together with the combined Galois ring period

$$\Theta_{A,B}(z) = \sum_{r=0}^{n-1}\chi_{K_0}(z\,A\,\alpha_0^{-r})\,\chi_{K_1}(z\,B\,\alpha_1^{-r}). \tag{4.22}$$

Lemma 4.11. *For all* $A \in K_0$ *and* $B \in K_1$*,*

$$Z(A, B) = \frac{1}{q^2}\Big(n + \sum_{b\in\mathcal{T}^*}\Theta_{A,B}(pb) + \sum_{a\in\mathcal{T}^*}\sum_{b\in\mathcal{T}}\Theta_{A,B}(a + pb)\Big). \tag{4.23}$$

Proof. Put

$$Y_r = \mathrm{Tr}_{K_0/R}(A\alpha_0^{-r}) + \mathrm{Tr}_{K_1/R}(B\alpha_1^{-r}) \in R.$$

Applying orthogonality (2.1) with $y = Y_r$ and summing over r,

$$Z(A, B) = \frac{1}{q^2}\sum_{r=0}^{n-1}\sum_{z\in R}\chi(z\,Y_r).$$

By the R-linearity of the trace and the additivity of χ,

$$\chi(z\,Y_r) = \chi_{K_0}(z\,A\,\alpha_0^{-r})\,\chi_{K_1}(z\,B\,\alpha_1^{-r}),$$

so $Z(A,B) = q^{-2}\sum_{z\in R}\Theta_{A,B}(z)$. Decomposing $R = \{0\} \cup p\mathcal{T}^* \cup (\mathcal{T}^* + p\mathcal{T})$ and noting that $\Theta_{A,B}(0) = n$ gives (4.23). □

For $C \in \mathbb{F}_{Q_j}$, put

$$T_j(C;r) = \mathrm{tr}_{\mathbb{F}_{Q_j}/\mathbb{F}_q}(C\bar{\alpha}_j^{-r}), \qquad \varkappa_j(C;r) = \kappa_{d_j}(C\bar{\alpha}_j^{-r}), \qquad 0 \le r < n. \tag{4.24}$$

For $A = A_0 + pA_1 \in K_0^\times$ and $B = B_0 + pB_1 \in K_1^\times$, define

$$\rho_A = T_0(\bar{A}_0;r), \quad \pi_A = \varkappa_0(\bar{A}_0;r) + T_0(\bar{A}_1;r), \quad \rho_B = T_1(\bar{B}_0;r), \quad \pi_B = \varkappa_1(\bar{B}_0;r) + T_1(\bar{B}_1;r). \tag{4.25}$$

These agree with the residue layer and principal/carry layer functions of Lemma 4.9, now written in the constituent-indexed notation of (4.24); note that $\varkappa_0, \varkappa_1$ denote the carry-trace functions and must not be confused with the constituent rings K_0, K_1. For nilpotent parameters $p\widetilde{A} \in pK_0 \setminus \{0\}$ and $p\widetilde{B} \in pK_1 \setminus \{0\}$, define

$$\nu_A = T_0(\widetilde{\bar{A}};r), \qquad \nu_B = T_1(\widetilde{\bar{B}};r). \tag{4.26}$$

The dependence on r is suppressed whenever no confusion can arise. For functions $f, g : \{0,\dots,n-1\} \to \mathbb{F}_q$, define

$$\mathcal{Z}(f,g) = \frac{1}{q^2}\sum_{a,b\in\mathbb{F}_q}\sum_{r=0}^{n-1}\lambda(af(r)+bg(r)) = \frac{1}{q^2}(n + \mathcal{R}(f) + \mathcal{L}(g) + \mathcal{M}(f,g)), \tag{4.27}$$

where

$$\mathcal{R}(f) = \sum_{a\in\mathbb{F}_q^*}\sum_{r=0}^{n-1}\lambda(af(r)), \quad \mathcal{L}(g) = \sum_{b\in\mathbb{F}_q^*}\sum_{r=0}^{n-1}\lambda(bg(r)), \quad \mathcal{M}(f,g) = \sum_{a\in\mathbb{F}_q^*}\sum_{b\in\mathbb{F}_q^*}\sum_{r=0}^{n-1}\lambda(af(r)+bg(r)). \tag{4.28}$$

Theorem 4.12. *The count $Z(A,B)$ defined in* (4.21) *is given by the following nine cases:*

Case I. *If $A = B = 0$, then $Z(A,B) = n$.*

Case II. *If $A = p\widetilde{A} \neq 0$, and $B = 0$, then $Z(A,B)$ is given by Theorem 4.2.*

Case III. *If $A = 0$, and $B = p\widetilde{B} \neq 0$, then $Z(A,B)$ is given by the subscript-1 version of Theorem 4.2.*

Case IV. *If $A = p\widetilde{A} \neq 0,$, and $B = p\widetilde{B} \neq 0$, then $Z(A,B)$ is given by Theorem 4.3.*

Case V. *If $A \in K_0^\times$, and $B = 0$, then $Z(A,B) = U_0(A)$, where $U_0(A)$ is given by Theorem 4.10.*

Case VI. *If $A = 0$, and $B \in K_1^\times$, then $Z(A,B) = U_1(B)$, the subscript-1 analog of Theorem 4.10.*

Case VII. *If $A = A_0 + pA_1 \in K_0^\times$, and $B = p\widetilde{B} \in pK_1 \setminus \{0\}$, then*

$$Z(A, p\widetilde{B}) = \mathcal{Z}(\rho_A,\, \pi_A + \nu_B).$$

Case VIII. *If $A = p\widetilde{A} \in pK_0 \setminus \{0\}$, and $B = B_0 + pB_1 \in K_1^\times$, then*

$$Z(p\widetilde{A}, B) = \mathcal{Z}(\rho_B,\, \nu_A + \pi_B).$$

Case IX. *If* $A = A_0 + pA_1 \in K_0^\times$, *and* $B = B_0 + pB_1 \in K_1^\times$, *define*

$$\rho_{A,B} = \rho_A + \rho_B, \qquad \pi_{A,B} = \pi_A + \pi_B + \gamma(\rho_A, \rho_B).$$

Then, $Z(A, B) = \mathcal{Z}(\rho_{A,B}, \pi_{A,B})$.

In Cases VII–IX*, the residue layer terms of* (4.28) *evaluate as*

$$\mathcal{R}(\rho_A) = \frac{(q-1)n}{Q_0 - 1} \sum_{s=0}^{\bar{g}_0 - 1} G_{Q_0}(\omega_0^{-\bar{\Delta}_0 s}, \lambda_0)\omega_0^{\bar{\Delta}_0 s}(\bar{A}_0), \tag{4.29}$$

$$\mathcal{R}(\rho_B) = \frac{(q-1)n}{Q_1 - 1} \sum_{s=0}^{\bar{g}_1 - 1} G_{Q_1}(\omega_1^{-\bar{\Delta}_1 s}, \lambda_1)\omega_1^{\bar{\Delta}_1 s}(\bar{B}_0), \tag{4.30}$$

$$\mathcal{R}(\rho_{A,B}) = \frac{(q-1)n}{(Q_0 - 1)(Q_1 - 1)} \sum_{(h,k)\in\mathcal{A}_{01}} G_{Q_0}(\omega_0^{-h}, \lambda_0)G_{Q_1}(\omega_1^{-k}, \lambda_1)\omega_0^h(\bar{A}_0)\omega_1^k(\bar{B}_0), \tag{4.31}$$

where $\mathcal{A}_{01}$ *is as in* (4.8)*, and the parameters* $\bar{g}_j$ *and* $\bar{\Delta}_j$ *are as in* (4.1).

Proof. Cases I–IV. For $A = p\widetilde{A}$ and $B = p\widetilde{B}$, the identity $p^2 = 0$ gives

$$\mathrm{Tr}_{K_0/R}(A\alpha_0^{-r}) + \mathrm{Tr}_{K_1/R}(B\alpha_1^{-r}) = p(\mathrm{Tr}_{K_0/R}(\widetilde{A}\alpha_0^{-r}) + \mathrm{Tr}_{K_1/R}(\widetilde{B}\alpha_1^{-r})),$$

which vanishes if and only if $T_0(\bar{\widetilde{A}}; r) + T_1(\bar{\widetilde{B}}; r) = 0$. Cases I–IV therefore reduce to the residue field trace formulas of Theorems 4.2 and 4.3, the cases with one zero parameter being immediate.

Cases V and VI are precisely Theorem 4.10 applied to K_0 and to K_1, respectively.

Cases VII–IX. Let $Y_r = \mathrm{Tr}_{K_0/R}(A\alpha_0^{-r}) + \mathrm{Tr}_{K_1/R}(B\alpha_1^{-r})$.

Case VII: $A = A_0 + pA_1 \in K_0^\times$ and $B = p\widetilde{B}$. The two-layer trace expansion via (4.10) and (4.14) gives

$$\mathrm{Tr}_{K_0/R}(A\alpha_0^{-r}) = \tau(\rho_A) + p\tau(\pi_A), \qquad \mathrm{Tr}_{K_1/R}(p\widetilde{B}\alpha_1^{-r}) = p\tau(\nu_B).$$

Because $p^2 = 0$, addition in the p-layer produces no further carry, so $Y_r = \tau(\rho_A) + p\tau(\pi_A + \nu_B)$. By the uniqueness of the Teichmüller expansion, $Y_r = 0$ if and only if $\rho_A = 0$, and $\pi_A + \nu_B = 0$. Applying additive-character orthogonality over $\mathbb{F}_q$ via (4.27) gives $Z(A, p\widetilde{B}) = \mathcal{Z}(\rho_A, \pi_A + \nu_B)$.

Case VIII is symmetric to Case VII: One obtains $Y_r = \tau(\rho_B) + p\tau(\nu_A + \pi_B)$, and hence, $Z(p\widetilde{A}, B) = \mathcal{Z}(\rho_B, \nu_A + \pi_B)$.

Case IX: Both parameters are units. Using (4.10) together with the addition carry (4.12),

$$\tau(\rho_A) + \tau(\rho_B) = \tau(\rho_A + \rho_B) + p\tau(\gamma(\rho_A, \rho_B)),$$

so $Y_r = \tau(\rho_{A,B}) + p\tau(\pi_{A,B})$. Then, $Y_r = 0$ if and only if $\rho_{A,B} = 0$, and $\pi_{A,B} = 0$, which gives $Z(A, B) = \mathcal{Z}(\rho_{A,B}, \pi_{A,B})$.

Residue layer evaluations. Formula (4.29) follows by expanding λ_0 via (5.6) in

$$\mathcal{R}(\rho_A) = \sum_{a\in\mathbb{F}_q^*} \sum_{r=0}^{n-1} \lambda_0(a\bar{A}_0\bar{\alpha}_0^{-r})$$

and applying the selection rules (5.7) and (5.8); the surviving exponents are $h = \bar{\Delta}_0 s$ with $0 \le s \le \bar{g}_0 - 1$. Formula (4.30) follows in the same way with the subscript 1.

For (4.31), we have

$$\mathcal{R}(\rho_{A,B}) = \sum_{a \in \mathbb{F}_q^*} \sum_{r=0}^{n-1} \lambda_0(a\bar{A}_0 \bar{\alpha}_0^{-r}) \lambda_1(a\bar{B}_0 \bar{\alpha}_1^{-r}).$$

Expanding both characters, the sum over $a \in \mathbb{F}_q^*$ imposes $\omega_0^h|_{\mathbb{F}_q^*} \omega_1^k|_{\mathbb{F}_q^*} = 1$, and the sum over r imposes $\omega_0^h(\bar{\alpha}_0^{-1}) \omega_1^k(\bar{\alpha}_1^{-1}) = 1$. These are precisely the conditions defining $\mathcal{A}_{01}$ in (4.8), and (4.31) follows. □

Remark 4.13. Case IX of Theorem 4.12 reveals the full complexity of the mixed-characteristic weight problem. The p-layer condition $\pi_{A,B} = 0$ involves five contributions coming from (4.25): the carry traces κ_{d_0} and κ_{d_1} of the two residue constituents, the principal traces $\mathrm{tr}(\bar{A}_1 \bar{\alpha}_0^{-r})$ and $\mathrm{tr}(\bar{B}_1 \bar{\alpha}_1^{-r})$ coming from the second Teichmüller component of each unit, and the addition carry $\gamma(\rho_A, \rho_B)$ arising from the interaction of the two residue traces. None of these terms has a counterpart when the base ring is a field.

5. Gauss-sum evaluations and explicit weight formulas

This section collects the Gauss-sum tools required for the explicit weight formulas and then establishes the Galois ring period formulas for the unit cases. Hamming weights are obtained from trace-zero counts by means of Theorem 3.4.

Let $Q = q^d$, let $\xi \in \mathbb{F}_Q^\times$ be a primitive element, and let ω be the multiplicative character determined by

$$\omega(\xi) = \exp\left(\frac{2\pi\sqrt{-1}}{Q-1}\right). \tag{5.1}$$

For the canonical additive character λ_Q of $\mathbb{F}_Q$, the Gauss sum is

$$G_Q(\psi, \lambda_Q) = \sum_{x \in \mathbb{F}_Q^\times} \psi(x)\, \lambda_Q(x). \tag{5.2}$$

In the next two lemmas, which are classical, the symbols q and λ_1 denote a generic prime power and the canonical additive character of the corresponding field; they are applied in this paper with the generic field size replaced by Q_0 or Q_1, and are not to be confused with the global parameter $q = p^m$ and the character λ_1 of $\mathbb{F}_{Q_1}$ fixed in Section 4.

Lemma 5.1. *Let η and λ_1 be the quadratic and the canonical additive character of $\mathbb{F}_q$, respectively, where $q = p^r$, and p is odd. Then,*

$$G(\eta, \lambda_1) = (-1)^{r-1} (\sqrt{-1})^{r(p-1)^2/4} \sqrt{q}. \tag{5.3}$$

Lemma 5.2. *Let ψ be a multiplicative character of order $N \ge 3$, and let e be the least positive integer with $p^e \equiv -1 \pmod{N}$. If $q = p^{2e\delta}$, then*

$$G(\psi, \lambda_1) = \begin{cases} (-1)^{\delta-1} \sqrt{q}, & p = 2, \\ (-1)^{\delta-1+\delta(p^e+1)/N} \sqrt{q}, & p > 2. \end{cases} \tag{5.4}$$

Moreover, for $1 \le t \le N-1$,

$$G(\psi^t, \lambda_1) = \begin{cases} (-1)^t \sqrt{q}, & \text{if } N \text{ is even, and } p,\ \delta,\ (p^e+1)/N \text{ are all odd,} \\ (-1)^{\delta-1} \sqrt{q}, & \text{otherwise.} \end{cases} \tag{5.5}$$

The first branch of (5.5) *is the alternating-sign pattern, and the second branch is the constant-sign pattern, in the terminology fixed in Section 4.*

Lemma 5.3. *Let* $Q = q^d$, *let* ξ *be a primitive element of* $\mathbb{F}_Q^\times$, *let* ω *be as in* (5.1), *and let* λ *be a nontrivial additive character of* $\mathbb{F}_Q$. *Then, for every* $x \in \mathbb{F}_Q^\times$,

$$\lambda(x) = \frac{1}{Q-1} \sum_{h=0}^{Q-2} G_Q(\omega^{-h}, \lambda)\, \omega^h(x). \tag{5.6}$$

Assume, moreover, that $\bar{\alpha}^{-1} = \xi^u$, *where* n *is the order of* $\bar{\alpha}^{-1}$ *in* $\mathbb{F}_Q^\times$. *Then,*

$$\sum_{a \in \mathbb{F}_q^\times} \omega^h(a) = \begin{cases} q-1, & h \equiv 0 \pmod{q-1}, \\ 0, & \text{otherwise}, \end{cases} \tag{5.7}$$

$$\sum_{r=0}^{n-1} \omega^h(\bar{\alpha}^{-r}) = \begin{cases} n, & hu \equiv 0 \pmod{Q-1}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.8}$$

The exponents satisfying both selection rules simultaneously are exactly

$$h = \bar{\Delta} s, \qquad 0 \le s \le \bar{g} - 1, \tag{5.9}$$

where $\bar{g}$ *and* $\bar{\Delta}$ *are as in* (4.1) *with the subscripts suppressed.*

Proof. The expansion (5.6) follows by writing $G_Q(\omega^h, \lambda) = \sum_{y \in \mathbb{F}_Q^*} \omega^h(y)\lambda(y)$, substituting, and invoking the orthogonality of the multiplicative character group.

Indeed,

$$\frac{1}{Q-1} \sum_{h=0}^{Q-2} G_Q(\omega^{-h}, \lambda)\, \omega^h(x) = \frac{1}{Q-1} \sum_{y \in \mathbb{F}_Q^*} \lambda(y) \sum_{h=0}^{Q-2} \omega^h(xy^{-1}),$$

and, by multiplicative-character orthogonality,

$$\sum_{h=0}^{Q-2} \omega^h(xy^{-1}) = \begin{cases} Q-1, & y = x, \\ 0, & y \ne x, \end{cases}$$

so only the term with $y = x$ survives, which proves (5.6).

For (5.7), note that $\mathbb{F}_q^*$ is the unique subgroup of order $q-1$ in $\mathbb{F}_Q^*$, so the restriction of ω^h to $\mathbb{F}_q^*$ is trivial if and only if $h \equiv 0 \pmod{q-1}$.

More explicitly, with $L = (Q-1)/(q-1)$, the element ξ^L generates $\mathbb{F}_q^*$, and hence,

$$\sum_{a \in \mathbb{F}_q^*} \omega^h(a) = \sum_{j=0}^{q-2} \omega^h(\xi^{Lj}) = \sum_{j=0}^{q-2} \exp\left(\frac{2\pi i\, hj}{q-1}\right).$$

This geometric sum equals $q-1$ when $q-1 \mid h$ and vanishes otherwise.

For (5.8), since $\bar{\alpha}^{-1} = \xi^u$, we have $\omega^h(\bar{\alpha}^{-r}) = \exp(2\pi i\, hur/(Q-1))$, and the sum is a geometric series equal to n when $hu \equiv 0 \pmod{Q-1}$ and to 0 otherwise.

Indeed,

$$\sum_{r=0}^{n-1} \omega^h(\bar{\alpha}^{-r}) = \sum_{r=0}^{n-1} \left(\exp\left(\frac{2\pi i\, hu}{Q-1}\right)\right)^r.$$

Because $\bar{\alpha}^{-1}$ has order n, the ratio is an nth root of unity. The sum therefore equals n precisely when the ratio is 1; that is, when $Q-1 \mid hu$, and it equals 0 otherwise.

To identify the simultaneous solutions, write $h = (q-1)t$ from (5.7), and substitute into the orbit condition. Setting $L = (Q-1)/(q-1)$ and $\bar{g} = \gcd(L,u)$, the condition $L \mid tu$ reduces to $(L/\bar{g}) \mid t$, which gives (5.9).

To justify the last reduction, write

$$L = \bar{g}\, L_1, \qquad u = \bar{g}\, u_1, \qquad \gcd(L_1, u_1) = 1.$$

Then,

$$L \mid tu \quad \Longleftrightarrow \quad \bar{g}L_1 \mid t\,\bar{g}u_1 \quad \Longleftrightarrow \quad L_1 \mid tu_1,$$

and, because $\gcd(L_1,u_1) = 1$, this is equivalent to $L_1 \mid t$, that is, to $(L/\bar{g}) \mid t$. Thus, $t = L/\bar{g}s, 0 \le s \le \bar{g}-1$, and consequently, $h = (q-1)t = (q-1)(L/\bar{g})s = (Q-1/\bar{g})s = \bar{\Delta}s$. These are exactly the $\bar{g}$ residue classes of exponents satisfying both selection rules. □

Galois ring period formulas for unit cases

The following two theorems establish the period formulas for Cases V–IX of Theorem 4.12. They combine additive character orthogonality on R with the decomposition $R = \{0\} \cup p\mathcal{T}^* \cup (\mathcal{T}^* + p\mathcal{T})$.

Theorem 5.4. *Assume that the additive characters are chosen compatibly with reduction modulo p, so that, for every $x \in \mathbb{F}_{Q_0}$ and every $b \in \mathcal{T}^*$,*

$$\chi_{K_0}(pb\,\tau_{d_0}(x)) = \lambda_0(\bar{b}\,x). \tag{5.10}$$

Let $A = A_0 + pA_1 \in K_0^\times$ with $A_0 \in \mathcal{T}_{d_0}^*$ and $A_1 \in \mathcal{T}_{d_0}$. Then, $U_0(A)$ is given by (4.20)*, with $\mathcal{R}_0(\bar{A}_0)$ as in* (4.18) *and $\mathcal{P}_0(A)$ as in* (4.19)*. Moreover, $U_0(A)$ equals the number of indices $0 \le r < n$ satisfying the two-layer system*

$$\rho_A(r) = 0 \qquad \text{and} \qquad \pi_A(r) = 0, \tag{5.11}$$

where $\rho_A(r)$ and $\pi_A(r)$ are as defined in (4.25)*.*

Proof. For $0 \le r < n$, set $Y_r = \mathrm{Tr}_{K_0/R}(A\alpha_0^{-r})$. Applying orthogonality (2.1) and summing over r,

$$U_0(A) = \frac{1}{q^2} \sum_{z \in R} \sum_{r=0}^{n-1} \chi_{K_0}(zA\alpha_0^{-r}).$$

Decomposing $R = \{0\} \cup p\mathcal{T}^* \cup (\mathcal{T}^* + p\mathcal{T})$, the term with $z = 0$ contributes n, the $p\mathcal{T}^*$-terms contribute $\sum_{b \in \mathcal{T}^*} \Theta_0(pb; A)$ as in (4.16), and the unit terms contribute $\mathcal{P}_0(A)$ from (4.19); this recovers (4.17).

For the $p\mathcal{T}^*$-contribution, $p^2 = 0$ gives $pbA\alpha_0^{-r} = pbA_0\alpha_0^{-r}$, and compatibility (5.10) gives $\chi_{K_0}(pbA\alpha_0^{-r}) = \lambda_0(\bar{b}\,\bar{A}_0\bar{\alpha}_0^{-r})$. Applying (5.6) together with the selection rules (5.7) and (5.8), only the exponents $h = \bar{\Delta}_0 s$ survive, so $\sum_{b\in\mathcal{T}^*}\Theta_0(pb;A) = \mathcal{R}_0(\bar{A}_0)$ as in (4.18). This establishes (4.20).

For the two-layer description, write

$$Y_r = \mathrm{Tr}_{K_0/R}(A_0\alpha_0^{-r}) + p\mathrm{Tr}_{K_0/R}(A_1\alpha_0^{-r}).$$

Applying (4.10) to $A_0\alpha_0^{-r}$ and compatibility modulo p to $A_1\alpha_0^{-r}$, together with the simplification $p^2 = 0$,

$$Y_r = \tau(\rho_A(r)) + p\tau(\pi_A(r)),$$

where $\rho_A(r)$ and $\pi_A(r)$ are given by (4.25). By the uniqueness of the Teichmüller expansion, $Y_r = 0$ if and only if (5.11) holds. □

Theorem 5.5. *Assume that* (5.10) *holds with the subscript j for K_j, λ_j, and Q_j. Let $A = A_0 + pA_1 \in K_0^\times$ with $A_0 \in \mathcal{T}_{d_0}^*$ and $A_1 \in \mathcal{T}_{d_0}$.*

(i) *If $B = p\widetilde{B} \in pK_1 \setminus \{0\}$, then*

$$Z(A, p\widetilde{B}) = \frac{1}{q^2}(n + \mathcal{R}_A + \mathcal{P}_{A,p\widetilde{B}}), \tag{5.12}$$

where $\mathcal{R}_A$ is given by (4.29)*, and*

$$\mathcal{P}_{A,p\widetilde{B}} = \sum_{a\in\mathcal{T}^*}\sum_{b\in\mathcal{T}}\sum_{r=0}^{n-1}\chi_{K_0}((a+pb)A\alpha_0^{-r})\chi_{K_1}((a+pb)p\widetilde{B}\alpha_1^{-r}). \tag{5.13}$$

(ii) *If $B = B_0 + pB_1 \in K_1^\times$ with $B_0 \in \mathcal{T}_{d_1}^*$ and $B_1 \in \mathcal{T}_{d_1}$, then*

$$Z(A, B) = \frac{1}{q^2}(n + \mathcal{R}_{A,B} + \mathcal{P}_{A,B}), \tag{5.14}$$

where

$$\mathcal{R}_{A,B} = \sum_{b\in\mathcal{T}^*}\sum_{r=0}^{n-1}\lambda_0(\bar{b}\,\bar{A}_0\bar{\alpha}_0^{-r})\lambda_1(\bar{b}\,\bar{B}_0\bar{\alpha}_1^{-r}), \tag{5.15}$$

$$\mathcal{P}_{A,B} = \sum_{a\in\mathcal{T}^*}\sum_{b\in\mathcal{T}}\sum_{r=0}^{n-1}\chi_{K_0}((a+pb)A\alpha_0^{-r})\chi_{K_1}((a+pb)B\alpha_1^{-r}). \tag{5.16}$$

Proof. We apply the master formula (4.23) of Lemma 4.11 and identify the $p\mathcal{T}^*$- and the unit layer contributions in each case.

(i) Let $B = p\widetilde{B}$. The unit contribution is immediately $\mathcal{P}_{A,p\widetilde{B}}$ from (5.13). For $b \in \mathcal{T}^*$, the identity $p^2 = 0$ gives $pb \cdot p\widetilde{B} = 0$, so $\Theta_{A,p\widetilde{B}}(pb) = \sum_r \chi_{K_0}(pbA\alpha_0^{-r})$. Again because $p^2 = 0$, we have $pbA\alpha_0^{-r} = pbA_0\alpha_0^{-r}$, and by (5.10), $\chi_{K_0}(pbA_0\alpha_0^{-r}) = \lambda_0(\bar{b}\,\bar{A}_0\bar{\alpha}_0^{-r})$. Applying the selection rules gives $\sum_{b\in\mathcal{T}^*}\Theta_{A,p\widetilde{B}}(pb) = \mathcal{R}_A$ as in (4.29), which establishes (5.12).

(ii) Let $B = B_0 + pB_1 \in K_1^\times$. The unit contribution is $\mathcal{P}_{A,B}$ from (5.16). For $b \in \mathcal{T}^*$, the identity $p^2 = 0$ gives $pbA\alpha_0^{-r} = pbA_0\alpha_0^{-r}$ and $pbB\alpha_1^{-r} = pbB_0\alpha_1^{-r}$. Applying (5.10) to each factor,

$$\Theta_{A,B}(pb) = \sum_r \lambda_0(\bar{b}\,\bar{A}_0\bar{\alpha}_0^{-r})\lambda_1(\bar{b}\,\bar{B}_0\bar{\alpha}_1^{-r}),$$

so that $\sum_{b\in\mathcal{T}^*}\Theta_{A,B}(pb)=\mathcal{R}_{A,B}$ as in (5.15), which establishes (5.14).

For the unit–unit layer structure, using (4.10) together with (4.12),

$$\mathrm{Tr}_{K_0/R}(A\alpha_0^{-r})+\mathrm{Tr}_{K_1/R}(B\alpha_1^{-r})=\tau(\rho_{A,B}(r))+p\tau(\pi_{A,B}(r)),$$

where

$$\rho_{A,B}(r)=\rho_A(r)+\rho_B(r),$$

and

$$\pi_{A,B}(r)=\kappa_{d_0}(\bar{A}_0\bar{\alpha}_0^{-r})+\kappa_{d_1}(\bar{B}_0\bar{\alpha}_1^{-r})+\mathrm{tr}(\bar{A}_1\bar{\alpha}_0^{-r})+\mathrm{tr}(\bar{B}_1\bar{\alpha}_1^{-r})+\gamma(\rho_A(r),\rho_B(r)),$$

in agreement with (4.25). By the uniqueness of the Teichmüller expansion, $Y_r=0$ if and only if $\rho_{A,B}(r)=0$, and $\pi_{A,B}(r)=0$. □

Example 5.6. Let $R=\mathrm{GR}(3^2,2)$, so that $q=9$. Set $\mathbb{F}_9=\mathbb{F}_3(\iota),\iota^2=-1,\mathbb{F}_{81}=\mathbb{F}_9(\zeta),\zeta^2=1+\iota$. The element $\xi=\iota+\zeta$ is primitive in $\mathbb{F}_{81}^{\times}$. Let $K=\mathrm{GR}(3^2,4)$, whose residue field is $\mathbb{F}_{81}$, and choose Teichmüller units $\alpha_0,\alpha_1\in K^{\times}$ such that $\bar{\alpha}_0^{-1}=\xi^8$, and $\bar{\alpha}_1^{-1}=\xi^{24}$. Both have order $n=10$ in $\mathbb{F}_{81}^{\times}$.

For this choice, $Q=81$, and $q=9$, and $\bar{g}_j=\gcd((Q-1)/(q-1),\,u_j)=2$ for $j=0,1$, where $u_0=8$, and $u_1=24$. Because $G_{81}(\eta,\lambda)=-9$, formula (4.3) gives

$$\left|\left\{0\le r<10:\mathrm{tr}_{\mathbb{F}_{81}/\mathbb{F}_9}(C\bar{\alpha}_j^{-r})=0\right\}\right|=1-\eta(C).$$

A nonsquare residue therefore gives $Z=2$, whereas a square residue gives $Z=0$.

Choose

$$U=\xi^{69},\quad V=\xi^{27},\quad A_0=\xi^{66},\quad B_0=\xi,\quad A_1=\xi^{21},\quad B_1=\xi^{31},$$

where U and V are nonsquares, and set $A=A_0+3A_1$ and $B=B_0+3B_1$, so that $3U$ and $3V$ are nilpotent parameters, and A_0, B_0, A, and B are units.

For $K=\mathrm{GR}(3^2,4)$, the carry trace and the addition carry are

$$\kappa(x)=-(x^2x^q+x(x^q)^2),\qquad \gamma(u,v)=-(u^2v+uv^2).$$

For $0\le r<10$, set

$$u_A(r)=\mathrm{tr}(\bar{A}_0\bar{\alpha}_0^{-r}),\qquad u_B(r)=\mathrm{tr}(\bar{B}_0\bar{\alpha}_1^{-r}),$$
$$v_U(r)=\mathrm{tr}(\bar{U}\bar{\alpha}_0^{-r}),\qquad v_V(r)=\mathrm{tr}(\bar{V}\bar{\alpha}_1^{-r}),$$

and introduce the compact carry notation

$$\kappa_A(r)=\kappa(\bar{A}_0\bar{\alpha}_0^{-r}),\qquad \kappa_B(r)=\kappa(\bar{B}_0\bar{\alpha}_1^{-r}),\qquad \gamma_{A,B}(r)=\gamma(u_A(r),u_B(r)).$$

The principal-layer functions associated with the mixed units $A=A_0+3A_1$ and $B=B_0+3B_1$ are $\theta_A(r)=\kappa_A(r)+\mathrm{tr}(\bar{A}_1\bar{\alpha}_0^{-r}),\theta_B(r)=\kappa_B(r)+\mathrm{tr}(\bar{B}_1\bar{\alpha}_1^{-r})$, whereas for the pure units A_0 and B_0, the corresponding principal layer functions are simply $\kappa_A(r)$ and $\kappa_B(r)$, respectively.

Table 2 gives the trace-zero count

$$Z(X,Y)=\left|\{0\le r<10:\mathrm{Tr}_{K/R}(X\alpha_0^{-r})+\mathrm{Tr}_{K/R}(Y\alpha_1^{-r})=0\}\right|$$

for each of the sixteen parameter pairs (X, Y). Because the block length is 10, the corresponding Hamming weight is $\mathrm{wt}_H(X, Y) = 10 - Z(X, Y)$. The two condition columns of Table 2 are to be read successively. For each parameter pair (X, Y), define the residue-admissible index set

$$\mathcal{I}^{\mathrm{res}}_{X,Y} = \{0 \le r < 10 : \text{the residue-layer equation in the table is satisfied}\}.$$

Table 2. Trace-zero counts and Hamming weights over $\mathrm{GR}(3^2, 2)$ for the sixteen parameter combinations.

No.	Parameters	Case type	Residue layer	Principal/carry layer	Z	w_H
1	$(0, 0)$	zero	automatic	automatic	10	0
2	$(3U, 0)$	one nilpotent	$\nu_U(r) = 0$	—	2	8
3	$(0, 3V)$	one nilpotent	$\nu_V(r) = 0$	—	2	8
4	$(3U, 3V)$	nil–nil	$\nu_U(r) + \nu_V(r) = 0$	—	6	4
5	$(A_0, 0)$	one pure unit	$u_A(r) = 0$	$\kappa(\bar{A}_0\bar{\alpha}_0^{-r}) = 0$	0	10
6	$(A, 0)$	one mixed unit	$u_A(r) = 0$	$\theta_A(r) = 0$	0	10
7	$(0, B_0)$	one pure unit	$u_B(r) = 0$	$\kappa(\bar{B}_0\bar{\alpha}_1^{-r}) = 0$	2	8
8	$(0, B)$	one mixed unit	$u_B(r) = 0$	$\theta_B(r) = 0$	2	8
9	$(A_0, 3V)$	unit–nilpotent	$u_A(r) = 0$	$\kappa(\bar{A}_0\bar{\alpha}_0^{-r}) + \nu_V(r) = 0$	0	10
10	$(A, 3V)$	mixed unit–nil	$u_A(r) = 0$	$\theta_A(r) + \nu_V(r) = 0$	0	10
11	$(3U, B_0)$	nil–pure unit	$u_B(r) = 0$	$\nu_U(r) + \kappa(\bar{B}_0\bar{\alpha}_1^{-r}) = 0$	0	10
12	$(3U, B)$	nil–mixed unit	$u_B(r) = 0$	$\nu_U(r) + \theta_B(r) = 0$	0	10
13	(A_0, B_0)	pure–pure unit	$u_A(r) + u_B(r) = 0$	$\kappa(\bar{A}_0\bar{\alpha}_0^{-r}) + \kappa(\bar{B}_0\bar{\alpha}_1^{-r}) + \gamma(u_A(r), u_B(r)) = 0$	0	10
14	(A, B_0)	mixed–pure unit	$u_A(r) + u_B(r) = 0$	$\theta_A(r) + \kappa(\bar{B}_0\bar{\alpha}_1^{-r}) + \gamma(u_A(r), u_B(r)) = 0$	0	10
15	(A_0, B)	pure–mixed unit	$u_A(r) + u_B(r) = 0$	$\kappa(\bar{A}_0\bar{\alpha}_0^{-r}) + \theta_B(r) + \gamma(u_A(r), u_B(r)) = 0$	2	8
16	(A, B)	mixed–mixed unit	$u_A(r) + u_B(r) = 0$	$\theta_A(r) + \theta_B(r) + \gamma(u_A(r), u_B(r)) = 0$	2	8

For the rows containing only zero or nilpotent parameters, no independent principal-layer equation remains, and hence, $Z(X, Y) = \left|\mathcal{I}^{\mathrm{res}}_{X,Y}\right|$. When at least one parameter is a unit, define in addition the principal/carry-admissible index set

$$\mathcal{I}^{\mathrm{pc}}_{X,Y} = \{0 \le r < 10 : \text{the principal/carry-layer equation in the table is satisfied}\}.$$

In those rows, the trace sum vanishes precisely when both layer equations hold, so that $Z(X, Y) = \left|\mathcal{I}^{\mathrm{res}}_{X,Y} \cap \mathcal{I}^{\mathrm{pc}}_{X,Y}\right|$. The column "Residue layer" thus records the first Teichmüller-layer condition, whereas

the column "Principal/carry layer" records the additional condition arising from the principal 3-parts, the carry traces, and, in the two-unit cases, the addition carry. For example, row 15 is determined by

$$\mathcal{I}^{\mathrm{res}}_{A_0,B} = \{0 \le r < 10 : u_A(r) + u_B(r) = 0\},$$

$$\mathcal{I}^{\mathrm{pc}}_{A_0,B} = \{0 \le r < 10 : \kappa_A(r) + \theta_B(r) + \gamma_{A,B}(r) = 0\}.$$

A direct evaluation gives $\left|\mathcal{I}^{\mathrm{res}}_{A_0,B} \cap \mathcal{I}^{\mathrm{pc}}_{A_0,B}\right| = 2$, and therefore, $Z(A_0, B) = 2$ and $\mathrm{wt}_H(A_0, B) = 8$. Similarly, row 16 is determined by $u_A(r) + u_B(r) = 0$ together with $\theta_A(r) + \theta_B(r) + \gamma_{A,B}(r) = 0$. Exactly two indices satisfy both equations, and hence, $Z(A, B) = 2$ and $\mathrm{wt}_H(A, B) = 8$. Consequently, rows 15 and 16 share the same residue pair $(\bar{A}_0, \bar{B}_0)$ but have distinct principal parts in the first parameter, and they yield $Z(A_0, B) = 4 \neq 2 = Z(A, B)$. This confirms explicitly that residue data alone do not determine the trace-zero count over $\mathrm{GR}(p^2, m)$. The principal p-layer contributes through the carry trace κ, the principal traces, and the addition carry γ.

Example 5.7. Let $R = \mathrm{GR}(4, 2)$, so that $|R| = 16$, $p = 2$, $q = 4$, $R/2R \cong \mathbb{F}_4$, and $|2R| = 4$. Consider the two-block ambient space

$$A = R[x]/\langle x^3 - 1\rangle \times R[x]/\langle x^5 - 1\rangle,$$

and let $S_3(x) = 1 + x + x^2$ and $S_5(x) = 1 + x + x^2 + x^3 + x^4$. Define

$$C = \langle (S_3(x) \mid S_5(x)),\ (2 \mid 0),\ (0 \mid 2)\rangle \subseteq A,$$

or, equivalently,

$$C = \{(a\,S_3(x) + 2f(x) \mid a\,S_5(x) + 2g(x)) : a \in R,\ f \in R[x]/\langle x^3 - 1\rangle,\ g \in R[x]/\langle x^5 - 1\rangle\}.$$

Because the 2-parts $2f(x)$ and $2g(x)$ are free, and only the residue class $\bar{a} \in \mathbb{F}_4$ determines a coset modulo $2R^8$, we have $|C| = 4 \cdot 4^8 = 4^9$.

The Hamming weight distribution follows from (3.3). If $\bar{a} = 0$, then the codeword lies in $(2R)^8$, and each coordinate is independently 0 with probability $1/4$ and nonzero nilpotent with probability $3/4$, which produces a binomial weight distribution. If $\bar{a} \neq 0$, then every coordinate is a unit of R, so that all such codewords have Hamming weight 8; there are $3 \cdot 4^8$ of them. Writing $A_i = |\{c \in C : \mathrm{wt}_H(c) = i\}|$,

$$(A_0, A_1, \ldots, A_8) = (1, 24, 252, 1512, 5670, 13608, 20412, 17496, 203169),$$

with weight enumerator

$$W_C(Y) = \sum_{i=0}^{7} \binom{8}{i} 3^i Y^i + (3^8 + 3 \cdot 4^8) Y^8.$$

One verifies that $\sum_{i=0}^{8} A_i = 4^9 = |C|$. The nilpotent layer $2R^8$ produces the binomial term $(1 + 3Y)^8$, whereas each of the three nonzero residue classes forces all coordinates to be units and thus contributes the full-weight term $3 \cdot 4^8 Y^8$.

6. Applications: minimality, projectivity, and optimality

The formulas above determine Hamming weight distributions over R. Because an arbitrary R-linear code need not admit an $\mathbb{F}_q$-linear realization with the same weight enumerator, the applications below are restricted to explicitly realized codes. For codes contained in $(pR)^N$, the following proposition provides a canonical $\mathbb{F}_q$-linear Hamming isometry that preserves all nonzero weights, and hence their ratio, exactly.

Proposition 6.1. *Let $R = \mathrm{GR}(p^2, m)$, let $\mathbb{F}_q \cong R/pR$, and let $\tau : \mathbb{F}_q \to \mathcal{T}$ be the Teichmüller lift. Define $\Phi : pR \to \mathbb{F}_q$ by $\Phi(p\tau(a)) = a$. Then, Φ is a bijective $\mathbb{F}_q$-linear Hamming isometry, where the $\mathbb{F}_q$-scalar multiplication on pR is induced by Teichmüller representatives: $c \cdot p\tau(a) = \tau(c)\, p\tau(a)$ for $c, a \in \mathbb{F}_q$. Consequently, if $C \subseteq (pR)^N$ is an R-linear code, then $\mathcal{C} = \Phi^N(C) \subseteq \mathbb{F}_q^N$ is an explicitly constructed $\mathbb{F}_q$-linear code satisfying $|\mathcal{C}| = |C|$ and $\mathrm{wt}_H(\Phi^N(c)) = \mathrm{wt}_H(c)$ for every $c \in C$. In particular, C and $\mathcal{C}$ have the same Hamming weight enumerator, and $\dim_{\mathbb{F}_q} \mathcal{C} = \log_q |C|$. If $C \neq \{0\}$, then $W_{\min}(\mathcal{C}) = W_{\min}(C)$ and $W_{\max}(\mathcal{C}) = W_{\max}(C)$, and therefore,*

$$\frac{W_{\min}(\mathcal{C})}{W_{\max}(\mathcal{C})} = \frac{W_{\min}(C)}{W_{\max}(C)}. \tag{6.1}$$

Proof. Every element of pR has a unique representation $p\tau(a)$ with $a \in \mathbb{F}_q$, so Φ is well-defined and bijective. For $a, b \in \mathbb{F}_q$, the Teichmüller addition formula gives $\tau(a) + \tau(b) = \tau(a + b) + p\tau(\gamma(a, b))$. Multiplying by p and using $p^2 = 0$, we obtain $p\tau(a) + p\tau(b) = p\tau(a + b)$, and hence Φ is additive. Because the Teichmüller lift is multiplicative, $\tau(c)\, p\tau(a) = p\tau(ca)$, and therefore, $\Phi(\tau(c)\, p\tau(a)) = ca = c\,\Phi(p\tau(a))$. Thus, Φ is $\mathbb{F}_q$-linear. Finally, $p\tau(a) = 0$ if and only if $a = 0$, so Φ preserves zero and nonzero symbols. More precisely, for every $c = (c_1, \ldots, c_N) \in (pR)^N$, $\mathrm{Supp}(\Phi^N(c)) = \mathrm{Supp}(c)$, and hence, $\mathrm{wt}_H(\Phi^N(c)) = \mathrm{wt}_H(c)$ with neither multiplicative nor additive scaling. Because C is R-linear, it is stable under multiplication by Teichmüller representatives, and hence, $\Phi^N(C)$ is $\mathbb{F}_q$-linear. The bijection $\Phi^N : C \longrightarrow \mathcal{C}$ therefore identifies the sets of nonzero weights of C and $\mathcal{C}$ term by term. This proves the asserted equalities of $W_{\min}$ and $W_{\max}$, as well as the ratio in (6.1). □

Corollary 6.2 (Preservation of the Ashikhmin–Barg ratio)**.** *Let $C \subseteq (pR)^N$ be a nonzero R-linear code, and let $\mathcal{C} = \Phi^N(C)$. Then, the Ashikhmin–Barg inequality holds for $\mathcal{C}$ if and only if the same numerical inequality holds for the nonzero Hamming weights computed in C:*

$$\frac{W_{\min}(\mathcal{C})}{W_{\max}(\mathcal{C})} > \frac{q-1}{q} \quad \Longleftrightarrow \quad \frac{W_{\min}(C)}{W_{\max}(C)} > \frac{q-1}{q}.$$

The coordinatewise identification $pR \cong \mathbb{F}_q$ therefore introduces no weight ratio distortion.

Proof. This is immediate from (6.1). □

Remark 6.3. Proposition 6.1 realizes explicitly over $\mathbb{F}_q$ every code contained in $(pR)^N$. No analogous realization is asserted for general R-linear codes with unit coordinates. The finite-field criteria below are therefore applied only after an explicit realization is available; for nilpotent-layer codes, Proposition 6.1 and Corollary 6.2 guarantee in addition the exact preservation of Hamming weights.

6.1. Minimal finite-field realizations

A nonzero codeword c in an $\mathbb{F}_q$-linear code C is *minimal* if every codeword whose support is contained in $\mathrm{Supp}(c)$ is a scalar multiple of c; the code is *minimal* if every nonzero codeword is minimal [26]. Minimal codes are important in secret sharing [27], distributed storage, and secure multiparty computation.

Lemma 6.4. *A linear code C over $\mathbb{F}_q$ is minimal if*

$$\frac{W_{\min}}{W_{\max}} > \frac{q-1}{q}, \tag{6.2}$$

where $W_{\min}$ and $W_{\max}$ denote the minimum and the maximum nonzero Hamming weight, respectively.

Theorem 6.5. *Let $C \subseteq \mathbb{F}_q^N$ be an $\mathbb{F}_q$-linear code with*

$$W_C(Y) = 1 + A_1 Y^{W_1} + A_2 Y^{W_2}, \qquad 0 < W_1 < W_2. \tag{6.3}$$

For $0 \le \partial \le q-1$, set $L_\partial = \partial W_1 + (q-1-\partial)W_2$. If either (6.2) *holds, or if $L_\partial \neq (q-1)U - V$ for every $0 \le \partial \le q-1$ and all $U, V \in \{W_1, W_2\}$, then C is minimal.*

Proof. By assumption, C is an $\mathbb{F}_q$-linear two-weight code with weights W_1 and W_2. Minimality under condition (6.2) follows from Lemma 6.4.

Otherwise, let $c_1, c_2 \in C$ be $\mathbb{F}_q$-linearly independent, and put

$$\partial = |\{a \in \mathbb{F}_q^* : \mathrm{wt}_H(c_1 + ac_2) = W_1\}|.$$

Then, $\sum_{a\in\mathbb{F}_q^*} \mathrm{wt}_H(c_1 + ac_2) = L_\partial$. Because $\mathrm{wt}_H(c_1), \mathrm{wt}_H(c_2) \in \{W_1, W_2\}$, the quantity $(q-1)\mathrm{wt}_H(c_1) - \mathrm{wt}_H(c_2)$ has the form $(q-1)U - V$ for some $U, V \in \{W_1, W_2\}$. The assumed inequality $L_\partial \neq (q-1)U - V$ then gives

$$\sum_{a\in\mathbb{F}_q^*} \mathrm{wt}_H(c_1 + ac_2) \neq (q-1)\mathrm{wt}_H(c_1) - \mathrm{wt}_H(c_2)$$

for every independent pair, and minimality follows from [28, Theorem 11]. □

Example 6.6. Let $R = \mathrm{GR}(3^2, 2)$ and $q = 9$, and let $K = \mathrm{GR}(3^2, 8)$ have residue field $\mathbb{F}_Q$ with $Q = 9^4 = 6561$. Choose $\alpha \in K^\times$ with $\bar{\alpha}^{-1} = \xi^2$, where ξ is primitive in $\mathbb{F}_Q^*$. Then, $\bar{\alpha}$ has order $n = 3280$, and from (4.1),

$$\frac{Q-1}{q-1} = 820, \qquad \bar{g} = \gcd(820, 2) = 2.$$

Consider the one-constituent GQC code

$$C = \langle (0 \mid p\, e_1(x)) \rangle \subseteq R[x]/\langle x^{3280} - 1\rangle \times R[x]/\langle x^{3280} - 1\rangle,$$

where $e_1(x)$ is the CRT idempotent, that is, the unique idempotent element of the block ring that is congruent to 1 modulo the basic irreducible factor of $x^{3280} - 1$ having α as a Teichmüller root and congruent to 0 modulo the complementary factors so that it generates the corresponding CRT summand, and $|C| = Q = 9^4$.

The first block of every codeword is identically zero, whereas the second block lies in $(pR)^{3280}$. Hence, $C \subseteq (pR)^{6560}$, and Proposition 6.1 yields the explicit $\mathbb{F}_9$-linear realization $\widetilde{C} = \Phi^{6560}(C) \subseteq \mathbb{F}_9^{6560}$. Deleting the identically zero first block gives an $\mathbb{F}_9$-linear code $C \subseteq \mathbb{F}_9^{3280}$. Because every deleted coordinate is zero for every codeword, this puncturing changes no support and no Hamming weight. Thus, C has the same cardinality, Hamming weight enumerator, minimum nonzero weight, maximum nonzero weight, and weight ratio as C, and since, $|C| = 9^4$, the code C has dimension 4.

Every nonzero element of the active constituent pK has a unique expression $p\tau_K(a)$ with $a \in \mathbb{F}_Q^\times$. The trace-zero formula below is therefore applied to $A = \tau_K(a)$, equivalently to $\bar{A} = a \neq 0$. Applying Theorem 4.2 with $\bar{g}_0 = 2$ and using formula (4.4),

$$Z(pA) = |\{0 \le r < 3280 : \mathrm{Tr}_{K/R}(pA\,\alpha^{-r}) = 0\}| = \begin{cases} 328, & \eta(\bar{A}) = 1, \\ 400, & \eta(\bar{A}) = -1. \end{cases}$$

The two nonzero Hamming weights are therefore $W_1 = 3280 - 400 = 2880$ and $W_2 = 3280 - 328 = 2952$, with weight enumerator

$$W_C(Y) = 1 + 3280\,Y^{2880} + 3280\,Y^{2952}.$$

By the Hamming isometry of Proposition 6.1, $W_C(Y) = W_C(Y)$, and therefore, C has parameters $[3280, 4, 2880]_{\mathbb{F}_9}$. The weight ratio satisfies

$$\frac{W_1}{W_2} = \frac{2880}{2952} = \frac{40}{41} > \frac{8}{9} = \frac{q-1}{q}.$$

By Corollary 6.2, the values $W_1 = 2880$ and $W_2 = 2952$ computed from the ring code are exactly the minimum and the maximum nonzero Hamming weight of the explicitly constructed code C, with no weight scaling and no ratio distortion. Hence, criterion (6.2) applies directly to C, and C is minimal by Theorem 6.5. The weight enumerator was verified independently using Magma [29].

6.2. Projective two-weight and optimal finite-field realizations

Projective two-weight codes are connected with strongly regular graphs and with finite geometry [30]. An $[N, K]$ code over $\mathbb{F}_q$ is *projective* if $d(C^\perp) \ge 3$ or equivalently, if the columns of any generator matrix are nonzero and pairwise nonproportional. For a two-weight enumerator (6.3), projectivity can be checked by MacWilliams duality: It suffices to verify that $B_1 = B_2 = 0$ in the dual weight enumerator.

The next theorem is a criterion for an $\mathbb{F}_q$-linear two-weight code that has already been constructed. In the present setting, it may be applied to a ring-derived code only after an explicit realization, such as the one furnished by Proposition 6.1, has been obtained. The weight enumerator of an arbitrary R-linear code alone is not used to assert projectivity.

Theorem 6.7. *Let $C \subseteq \mathbb{F}_q^N$ be an $\mathbb{F}_q$-linear $[N, K]$ two-weight code with enumerator* (6.3)*, and define the first two q-ary Krawtchouk polynomials by*

$$P_1(i) = (q-1)N - qi, \qquad P_2(i) = (q-1)^2\binom{N-i}{2} - (q-1)i(N-i) + \binom{i}{2}. \tag{6.4}$$

If

$$P_1(0) + A_1\,P_1(W_1) + A_2\,P_1(W_2) = 0, \quad \textit{and} \quad P_2(0) + A_1\,P_2(W_1) + A_2\,P_2(W_2) = 0, \tag{6.5}$$

then $d(C^\perp) \geq 3$, *and* C *is projective.*

Proof. The MacWilliams identities give $q^K B_j = \sum_i A_i P_j(i)$, where P_j is the q-ary Krawtchouk polynomial of degree j as in (6.4). Because C has only the weights 0, W_1, and W_2, conditions (6.5) are precisely $q^K B_1 = 0$ and $q^K B_2 = 0$. Hence, $B_1 = B_2 = 0$, and $d(C^\perp) \geq 3$. □

Definition 6.8. An $[N, K, d]$ code C over $\mathbb{F}_q$ is *Griesmer optimal* if

$$N = \sum_{j=0}^{K-1} \left\lceil \frac{d}{q^j} \right\rceil. \tag{6.6}$$

If $\sum_{j=0}^{K-1} \lceil (d+1)/q^j \rceil > N$, then no $[N, K, d+1]$ code over $\mathbb{F}_q$ exists, so d is best possible for the given length and dimension.

Theorem 6.9. *Let* $R = \mathrm{GR}(4, m)$ *and* $q = 2^m$, *let* $K = \mathrm{GR}(4, 3m)$ *have residue field* $\mathbb{F}_{q^3}$, *and let* $\alpha \in K^\times$ *be a Teichmüller unit whose residue has order* $q^3 - 1$. *Define* $C = \langle 2e(x) \rangle \subseteq R[x]/\langle x^{q^3-1} - 1 \rangle$, *where* $e(x)$ *is the CRT idempotent for* K *(defined as in Example 6.6, with respect to the basic irreducible factor having* α *as a Teichmüller root). Then, the Hamming weight distribution of* C *has the single nonzero weight* $q^3 - q^2$. *Under the coordinatewise identification* $2R \cong \mathbb{F}_q$, *the corresponding* $\mathbb{F}_q$*-linear realization has parameters* $[q^3 - 1,\ 3,\ q^3 - q^2]$, *meets the Griesmer bound* (6.6)*, and is optimal.*

Proof. Under the CRT decomposition, C is identified with the nilpotent constituent $2K$, and hence, $C \subseteq (2R)^{q^3-1}$, $|C| = |2K| = q^3$. By Proposition 6.1, $C = \Phi^{q^3-1}(C) \subseteq \mathbb{F}_q^{q^3-1}$ is an explicitly constructed $\mathbb{F}_q$-linear code of dimension 3, and $\mathrm{wt}_H(\Phi^{q^3-1}(c)) = \mathrm{wt}_H(c)$ for every $c \in C$. The realization therefore preserves every nonzero weight exactly, including the minimum distance.

The nonzero elements of $2K$ are parametrized uniquely by $2\tau_K(a)$ with $a \in \mathbb{F}_{q^3}^\times$. For such an a, the nilpotent trace-zero condition reads $\mathrm{Tr}_{K/R}(2\tau_K(a)\alpha^{-r}) = 0$ if and only if $\mathrm{tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(a\bar{\alpha}^{-r}) = 0$.

Because $\bar{\alpha}$ has order $q^3 - 1$, the elements $a\bar{\alpha}^{-r}$ with $0 \leq r < q^3 - 1$ run through $\mathbb{F}_{q^3}^\times$. The kernel of the nonzero $\mathbb{F}_q$-linear trace map $\mathrm{tr}_{\mathbb{F}_{q^3}/\mathbb{F}_q} : \mathbb{F}_{q^3} \longrightarrow \mathbb{F}_q$ has q^2 elements, exactly $q^2 - 1$ of which are nonzero. Every nonzero codeword therefore has $(q^3 - 1) - (q^2 - 1) = q^3 - q^2$ nonzero coordinates, and consequently, $N = q^3 - 1$, $K = 3$, and $d = q^3 - q^2$.

The Griesmer bound (6.6) with $K = 3$ gives

$$\sum_{j=0}^{2} \left\lceil \frac{d}{q^j} \right\rceil = (q^3 - q^2) + (q^2 - q) + (q - 1) = q^3 - 1 = N,$$

so that the realization meets the bound and is optimal. Moreover,

$$\sum_{j=0}^{2} \left\lceil \frac{d+1}{q^j} \right\rceil = (q^3 - q^2 + 1) + (q^2 - q + 1) + q = q^3 + 2 > N,$$

so no $[q^3 - 1, 3, d+1]_{\mathbb{F}_q}$ code exists. This confirms that the minimum distance $d = q^3 - q^2$ is the best possible for the given length and dimension. □

Example 6.10. Let $R = \mathrm{GR}(4, 2)$ and $q = 4$, and let $K = \mathrm{GR}(4, 6)$ have residue field $\mathbb{F}_{64}$. Choose $\alpha \in K^\times$ whose residue has order 63 in $\mathbb{F}_{64}^*$, and let $C = \langle 2e(x)\rangle \subseteq R[x]/\langle x^{63} - 1\rangle$. Then, $|C| = 64 = 4^3$, and Theorem 6.9 gives length 63, cardinality 4^3, minimum distance 48, and weight enumerator $W_C(Y) = 1 + 63\,Y^{48}$. Every nonzero codeword thus has Hamming weight 48. Verifying (6.6),

$$G_4(3, 48) = 48 + \left\lceil \frac{48}{4} \right\rceil + \left\lceil \frac{48}{16} \right\rceil = 48 + 12 + 3 = 63,$$

and Proposition 6.1 yields the explicit $\mathbb{F}_4$-linear Hamming-isometric realization $C = \Phi^{63}(C) \subseteq \mathbb{F}_4^{63}$. Thus, C has parameters $[63, 3, 48]_{\mathbb{F}_4}$, meets the Griesmer bound, and is optimal. The parameters and the weight enumerator were verified independently using Magma [29].

7. Conclusions

We have developed trace-sum formulas that determine the Hamming weight distributions of generalized quasi-cyclic codes over $R = \mathrm{GR}(p^2, m)$ under the assumption $\gcd(m_i, p) = 1$. By means of the CRT decomposition, the weight problem reduces to counting the zero positions of trace expressions over extension Galois rings. For two-constituent codes, we obtained formulas covering every combination of constituent types: zero, nilpotent, and unit. The nilpotent cases reduce to residue-field trace equations and are evaluated by finite-field Gauss sums (see Theorems 4.2 and 4.3). The unit and mixed cases, by contrast, require a genuinely mixed-characteristic analysis, in which the Teichmüller decomposition produces two-layer conditions involving the carry trace (4.10) and the addition carry (4.12). A consequence is that residue data alone do not determine the trace-zero count: Parameters with the same residue may yield different Hamming weights because their principal p-parts contribute differently to the second layer. This phenomenon is illustrated in Example 5.6. The resulting Hamming distributions can be applied in finite-field settings whenever a separate $\mathbb{F}_q$-linear realization with the same Hamming enumerator has been verified. In that case, one may apply the Ashikhmin–Barg criterion, projectivity tests, and the Griesmer bound, as discussed in Section 6.

Natural directions for further work include repeated-root GQC codes, in which $\gcd(m_i, p) > 1$ for some i; higher Galois rings $\mathrm{GR}(p^a, m)$ with $a \geq 3$, where additional p-layers and carry terms appear; and extensions to additive cyclic and polycyclic codes over $\mathrm{GR}(p^2, m)$.

Author contributions

Sami H. Saif: Conceptualization, Methodology, Formal analysis, Investigation, Writing-review and editing, Writing-original draft; Shayea Aldossari: Data curation, Validation, Visualization, Writing – review and editing. Both authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, P. Solé, The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes, *IEEE Trans. Inf. Theory*, **40** (1994), 301–319. https://doi.org/10.1109/18.312154
2. T. Honold, Characterization of finite Frobenius rings, *Arch. Math.*, **76** (2001), 406–415. https://doi.org/10.1007/PL00000451
3. G. H. Norton, A. Sălăgean, On the structure of linear and cyclic codes over a finite chain ring, *Appl. Algebra Eng. Commun. Comput.*, **10** (2000), 489–506. https://doi.org/10.1007/PL00012382
4. S. H. Saif, S. Aldossari, Optimal generalized quasi-polycyclic codes over $\mathbb{F}_q + u\mathbb{F}_q$, *Mathematics*, **14** (2026), 816. https://doi.org/10.3390/math14050816
5. H. Q. Dinh, Constacyclic codes of length 2^s over Galois extension rings of $\mathbb{F}_2 + u\mathbb{F}_2$, *IEEE Trans. Inf. Theory*, **55** (2009), 1730–1740.https://doi.org/10.1109/TIT.2009.2013015
6. C. Güneri, F. Özbudak, B. Özkaya, E. Sarikaya, Z. Sepasdar, P. Solé, Structure and performance of generalized quasi-cyclic codes, *Finite Fields Appl.*, **47** (2017), 183–202. https://doi.org/10.1016/j.ffa.2017.06.005
7. S. Ling, P. Solé, On the algebraic structure of quasi-cyclic codes I. Finite fields, *IEEE Trans. Inf. Theory*, **47** (2001), 2751–2760. https://doi.org/10.1109/18.959257
8. I. Siap, N. Kulhan, The structure of generalized quasi cyclic codes, *Appl. Math. E-Notes*, **5** (2005), 24–30.
9. Y. Cao, Generalized quasi-cyclic codes over Galois rings: structural properties and enumeration, *Appl. Algebra Eng. Commun. Comput.*, **22** (2011), 219–233. https://doi.org/10.1007/s00200-011-0145-5
10. M. Esmaeili, S. Yari, Generalized quasi-cyclic codes: structural properties and code construction, *Appl. Algebra Eng. Commun. Comput.*, **20** (2009), 159–173. https://doi.org/10.1007/s00200-009-0095-3
11. J. Gao, F. W. Fu, L. Shen, W. Ren, Some results on generalized quasi-cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q$, *IEICE Trans. Fundam. Electron. Commun. Comput. Sci.*, **E97.A** (2014), 1005–1012. https://doi.org/10.1587/transfun.E97.A.1005
12. Z. Heng, Q. Yue, A construction of q-ary linear codes with two weights, *Finite Fields Appl.*, **48** (2017), 20–42.https://doi.org/10.1016/j.ffa.2017.07.006
13. G. H. Norton, A. Sălăgean, On the Hamming distance of linear codes over a finite chain ring, *IEEE Trans. Inf. Theory*, **46** (2000), 1060–1067.https://doi.org/10.1109/18.841186

14. N. Patanker, R. Rezaei, K. Samei, Weight distribution of a subclass of $\mathbb{Z}_2$-double cyclic codes, *Finite Fields Appl.*, **57** (2019), 287–308.https://doi.org/10.1016/j.ffa.2019.03.003
15. V. Chauhan, A. Sharma, S. Sharma, M. Yadav, Hamming weight distributions of multi-twisted codes over finite fields, *Des. Codes Cryptogr.*, **89** (2021), 1787–1837. https://doi.org/10.1007/s10623-021-00889-1
16. C. Ding, J. Yang, Hamming weights in irreducible cyclic codes, *Discrete Math.*, **313** (2013), 434–446. https://doi.org/10.1016/j.disc.2012.11.009
17. Z. Heng, D. Li, J. Du, F. Chen, A family of projective two-weight linear codes, *Des. Codes Cryptogr.*, **89** (2021), 1993–2007.https://doi.org/10.1007/s10623-021-00896-2
18. J. Gao, X. Meng, F. W. Fu, Weight distribution of double cyclic codes over Galois rings, *Des. Codes Cryptogr.*, **90** (2022), 2529–2549. https://doi.org/10.1007/s10623-021-00914-3
19. X. Meng, J. Gao, Complete weight enumerator of torsion codes, *Adv. Math. Commun.*, **16** (2022), 571–596. https://doi.org/10.3934/amc.2020124
20. A. Kuzmin, A. Nechaev, Complete weight enumerators of generalized Kerdock code and related linear codes over Galois ring, *Discrete Appl. Math.*, **111** (2001), 117–137. https://doi.org/10.1016/S0166-218X(00)00348-6
21. J. Gao, X. Meng, F. W. Fu, Weight distributions of generalized quasi-cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q$, *Finite Fields Appl.*, **88** (2023), 102181. https://doi.org/10.1016/j.ffa.2023.102181
22. R. Lidl, H. Niederreiter, *Finite fields*, Cambridge University Press, 1997.
23. J. Li, A. Zhang, K. Feng, Linear codes over $\mathbb{F}_q[x]/(x^2)$ and GR(p^2, m) reaching the Griesmer bound, *Des. Codes Cryptogr.*, **86** (2018), 2837–2855. https://doi.org/10.1007/s10623-018-0479-0
24. B. R. McDonald, *Finite rings with identity*, New York: Marcel Dekker, 1974.
25. Z. X. Wan, *Lectures on finite fields and Galois rings*, World Scientific, 2003. https://doi.org/10.1142/5350
26. A. Ashikhmin, A. Barg, Minimal vectors in linear codes, *IEEE Trans. Inf. Theory*, **44** (1998), 2010–2017.https://doi.org/10.1109/18.705584
27. S. Mesnager, Y. Qi, H. Ru, C. Tang, Minimal linear codes from characteristic functions, *IEEE Trans. Inf. Theory*, **66** (2020), 5404–5413. https://doi.org/10.1109/TIT.2020.2978387
28. Z. Heng, C. Ding, C. Zhou, Minimal linear codes over finite fields, *Finite Fields Appl.*, **54** (2018), 176–196. https://doi.org/10.1016/j.ffa.2018.08.010
29. W. Bosma, J. Cannon, C. Playoust, The Magma algebra system I. The user language, *J. Symb. Comput.*, **24** (1997), 235–265. https://doi.org/10.1006/jsco.1996.0125
30. R. Calderbank, W. Kantor, The geometry of two-weight codes, *Bull. London Math. Soc.*, **18** (1986), 97–122.