Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(8): 25192–25228.
DOI: 10.3934/math.20261013
Received: 08 June 2026
Revised: 20 July 2026
Accepted: 28 July 2026
Published: 14 August 2026

Research article

Fixed point analysis of expectation convergence in financial learning in perturbed metric spaces

Şeyma Bilazeroğlu*

Department of Computer Science, Çankaya University, 06815 Etimesgut, Ankara, Türkiye

* **Correspondence:** Email: sbilazeroglu@cankaya.edu.tr, seymabilazeroglu@gmail.com.

Abstract: Financial markets rarely operate under perfect rationality. Price expectations are formed and revised in environments shaped by limited information, behavioral biases, and institutional frictions. In such settings, investors' perceptions of differences between forecasts may deviate from purely rational benchmarks. In this paper, we develop a mathematical framework for expectation updating that captures these perceptual distortions in a perturbed-metric setting. We introduce a new contractive-type learning condition that simultaneously incorporates individual inertia and institutional adjustment frictions. Unlike classical contraction structures, the proposed condition reflects directional interactions and layered perception effects in the updating process. Under a natural stability requirement linking individual and institutional adjustment intensities, we establish the existence and uniqueness of an equilibrium price expectation and prove the convergence of the learning dynamics. Linear and nonlinear specifications illustrate how behavioral and institutional mechanisms jointly determine not only stability but also persistence and the speed of convergence.

Keywords: fixed point theory; perturbed metric space; generalized contraction; financial learning dynamics; expectation equilibrium
Mathematics Subject Classification: 47H10, 54H25, 91B42, 91G80

1. Introduction

Investing in financial markets, whether in stocks, debt instruments, commodities, exchange rates, or crypto assets, often requires decision-making under uncertainty, limited information, and rapidly changing conditions. One of the most fundamental questions for an investor is often the same: "What is the true, fair, or long-term price of this asset, and how can I predict it?". This question is relevant to novice investors and experienced professionals. However, it is particularly critical for investors who are new to the market and have limited knowledge, since several factors may affect price changes, such as information flow, the way people think and act (behavioral biases), the way institutional investors make decisions (institutional frictions), what analysts expect (forecast revisions), and the collective psychology of the market.

Under such conditions, investors revise their expectations in the presence of incomplete and imperfect information. The literature shows that economic agents do not make decisions under conditions of full information. Slow information flow (Mankiw and Reis [1]), individuals' limited cognitive capacity (Sims [2]), the fact that economic variables are often assessed under partial observation (Evans and Honkapohja [3]), and the dominance of model uncertainty in the decision-making process (Hansen and Sargent [4]) support this view. Furthermore, behavioral studies show that individuals process complex information using simple mental rules (Tversky and Kahneman [5]) and often respond slowly and incompletely to new information (Barberis et al. [6]). Therefore, investors inevitably form their price predictions under conditions of incomplete information, limited attention, heuristic reasoning, and behavioral biases. This raises the following questions: "Can a learning process that begins with incorrect or incomplete information ultimately bring the investor closer to the correct price? Can the investor reach an equilibrium over time, even if they are not an expert?"

Behavioral finance is the study of how investors make decisions not only based on perfect rationality, but also under the influence of cognitive limits, psychological biases, and heuristics (Tversky and Kahneman [5]; Simon [7]; Shleifer [8]; Barberis and Thaler [9], Hirshleifer [10]). Several studies in the literature have revealed that individual investors exhibit behavioral biases, including excessive reliance on previous forecasts, delayed adaptation, and bounded rationality (Tversky and Kahneman [5]; Barberis et al. [6]). On the other hand, institutional investors may take longer to respond to information due to the structure of their committees, risk management practices, and adherence to regulations, which is a well-known mechanism described by the sticky-information model of Mankiw and Reis [1]. It is thought to be a structural factor that slows the process of reaching market consensus. Consequently, individual and institutional updating behaviors differ structurally and shape the convergence dynamics of price expectations in distinct ways. In this study, we build a model that integrates these two update techniques into an overall mathematical framework.

2. Mathematical and economic structure of the model

The behavior exhibited by investors in financial markets when updating their price expectations is often not merely a simple function of the numerical difference between two forecasts. As discussed in the Introduction, market participants do not evaluate information in a fully rational and proportional way, and individual and institutional actors may adjust expectations gradually. These considerations indicate that the classical distance formulation commonly adopted in financial modeling to represent how the difference between two price forecasts is perceived is insufficient in most cases.

Therefore, to represent the deviation of investor perception from the rational component, we adopt a perturbed metric structure introduced by Jleli and Samet [11], in which perceived distance is decomposed into a rational core and a perturbation term. Such perturbed metric structures have attracted increasing attention in recent years, leading to contractivity conditions and convergence results under generalized distance frameworks (Alsahli et al. [12], Alsahli et al. [13], Karapınar et al. [14], Moussaoui and Taş [15], Yalçın [16]). The theoretical foundations of these developments are also presented in monographs on generalized metric spaces and modern fixed point theory (Georgiev and Zennir [17], Karapınar and Agarwal [18], Mebarki et al. [19]), which provide a comprehensive theoretical foundation for this area. Overall, perturbed metric structures provide a flexible analytical framework for modeling deviation-based dynamics beyond classical metric formulations. In this study, this framework enables the systematic incorporation of behavioral and perceptual deviations into the expectation-updating process.

We represent how investors adjust their price expectations in response to new information using the update operator $\mathcal{T}$ in our model. This adjustment process is guided by two key parameters:

- λ represents the update inertia of individual investors (learning speed parameter). Larger λ values indicate that investors are strongly attached to past beliefs and process new information slowly; this is related to behavioral phenomena in the literature such as anchoring, conservatism bias, and underreaction (Tversky and Kahneman [5], Barberis et al. [6]). Small λ values, on the other hand, indicate a faster and more aggressive update process.

- θ reflects the updating friction of institutional investors (institutional lag parameter). In institutional structures, as θ increases, information updates occur more slowly due to the committee processes, risk management protocols, and regulatory frameworks. This is consistent with the sticky expectations mechanism defined by Mankiw and Reis [1]. Faster institutional adaptation is indicated by a smaller θ.

Notation and roles. Throughout the paper, $p \in \mathcal{P}$ denotes an individual investor's price expectation, while $q \in \mathcal{P}$ represents an institutional/analyst benchmark expectation. The update operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ models the individual's expectation revision process, so the learning dynamics are studied in the directed form $p_{n+1} = \mathcal{T}(p_n)$, where q serves as a benchmark reference level.

The combination of these two mechanisms is expressed in our fundamental update inequality:

$$\mathcal{D}(\mathcal{T}p, \mathcal{T}q) \leq \lambda \max\left\{\mathcal{D}(p,q),\ \mathcal{D}(\mathcal{T}p,p),\ \frac{d_0(p,\mathcal{T}q) + d_0(q,\mathcal{T}p)}{2}\right\} + \theta\,\mathcal{D}(\mathcal{T}q,q).$$

This inequality combines individual update inertia (λ), the mutual interaction among market participants (the average term), and the institutional updating friction (θ) into a single mathematical framework. The distance and interaction terms model investors' expectation updates as a layered perception-and-interaction process rather than a one-dimensional difference measure. In this way, cognitive constraints, perceptual biases, and institutional frictions are represented simultaneously within a unified learning framework. Unlike classical contraction conditions defined solely through a single metric term, this structure incorporates perceived distance and cross-interaction components, thereby extending the standard contraction framework.

The model defines the distinction

$$\mathcal{D}(p,q) = d_0(p,q) + \phi(p,q).$$

This formulation reflects the assumption that investors operate at two levels when evaluating price forecasts. Here, $d_0(p,q)$ represents the rational core difference between two forecasts, based on elements such as fundamental economic indicators, macroeconomic expectations, risk premia, and valuation assumptions. Conversely, $\phi(p,q)$ includes behavioral and cognitive biases arising from the investor's perception of this fundamental difference, including limited attention, intuitive decision-making, and misinterpretation of signals (Sims [2]; Tversky and Kahneman [5]; Barberis et al. [6]).

This distinction explains why terms used in contraction inequality are defined through different distance components. While the individual investor's process of updating their forecast is largely influenced by perceptual and behavioral factors, the interaction between the individual investor and institutional expectations (or analyst consensus) often occurs within a more fundamental, rational reference framework. Therefore, defining the term $d_0(p, \mathcal{T}q) + d_0(q, \mathcal{T}p)$, which represents cross-interaction, directly through d_0 rather than perceptual distance, is financially meaningful. This structure reflects the idea that individual investors often rely on the fundamental economic narrative rather than on price levels when approaching institutional expectations.

The fact that the perceived distance $\mathcal{D}$ does not have to be symmetrical is also consistent with financial reality. In practice, an individual investor's perceptual adjustment from their own estimate to an institutional benchmark does not necessarily produce the same evaluation when the difference is considered in the opposite direction. Institutional expectations typically serve as reference benchmarks in market evaluation, whereas individual expectations do not occupy a symmetric position within institutional decision processes. This direction-dependent structure makes it natural and economically meaningful that $\mathcal{D}(p, q) \neq \mathcal{D}(q, p)$.

Finally, establishing contraction through $\mathcal{D}(p, q)$ explicitly defines the direction of the learning process. The model examines how the individual investor's current expectation evolves toward corporate benchmarks and market discipline. Therefore, the reverse distance $\mathcal{D}(q, p)$ is not analyzed; learning dynamics are intentionally defined from the individual investor toward the market benchmark. Accordingly, the asymmetry of the perceived distance does not affect the subsequent fixed-point analysis, since symmetry of $\mathcal{D}$ is not required in the proposed convergence framework.

To clarify the economic meaning of the components appearing in the above inequality, we briefly interpret each term of the contractive structure. In this context, $\mathcal{D}(p, q)$ measures the perceived difference between the individual investor's current prediction and the institutional forecast, reflecting the degree of deviation from the benchmark and the level of initial information heterogeneity. The term $\mathcal{D}(p, \mathcal{T}p)$ represents the investor's response to their own update and indicates how strongly new information is internalized. The interaction term

$$\frac{d_0(p, \mathcal{T}q) + d_0(q, \mathcal{T}p)}{2}$$

captures the cross-adjustment between individual and institutional expectations through the exact metric. Finally, $\mathcal{D}(q, \mathcal{T}q)$ reflects the institutional updating intensity and the degree of adjustment friction. Together, these components ensure that behavioral adaptations, interaction effects, and institutional frictions are incorporated into the same structural inequality. Each of these terms corresponds directly to the components in the contractive inequality above: The maximum structure captures individual adjustment and interaction effects, and the additive term reflects institutional updating friction.

3. Preliminaries

In this section, we present the fundamental concepts that will be used to obtain the main results of the study. The learning dynamics examined are addressed under a generalized distance framework, where systematic deviations and perceptual distortions can affect distance measurement, rather than classical metric structures.

Definition 3.1 (Perturbed Metric Space [11]). Let $\mathcal{P}$ be a non-empty set. Let $\mathcal{D}$ and ϕ be mappings defined on $\mathcal{P}\times\mathcal{P}$ with non-negative values. We define $\mathcal{D}$ as a perturbed metric on the set $\mathcal{P}$ concerning the mapping ϕ if the difference mapping

$$(\mathcal{D}-\phi)(p,q) := \mathcal{D}(p,q)-\phi(p,q)$$

defines a metric on $\mathcal{P}$. That is, for all $p,q,r\in\mathcal{P}$, the conditions:

(1) $(\mathcal{D}-\phi)(p,q)\geq 0$ (non-negativity),

(2) $(\mathcal{D}-\phi)(p,q)=0$ if and only if $p=q$ (separation axiom),

(3) $(\mathcal{D}-\phi)(p,q)=(\mathcal{D}-\phi)(q,p)$ (symmetry),

(4) $(\mathcal{D}-\phi)(p,q)\leq(\mathcal{D}-\phi)(p,r)+(\mathcal{D}-\phi)(r,q)$ (triangle inequality),

are satisfied. Within this framework, the function ϕ is called the perturbation mapping, the function $d_0=\mathcal{D}-\phi$ is referred to as the exact metric corresponding to $\mathcal{D}$, and the ordered triple $(\mathcal{P},\mathcal{D},\phi)$ is defined as a perturbed metric space.

Definition 3.2 (Convergence, Completeness, and Continuity in Perturbed Metric Spaces [11]). Let the ordered triple $(\mathcal{P},\mathcal{D},\phi)$ be a perturbed metric space, and let $\{p_n\}\subset\mathcal{P}$ be a sequence. Additionally, let $\mathcal{T}:\mathcal{P}\to\mathcal{P}$ be a self-mapping. The following concepts describe convergence, completeness, and continuity within the perturbed framework:

(1) A sequence $\{p_n\}$ is said to be perturbed convergent in $(\mathcal{P},\mathcal{D},\phi)$ if it converges in the exact metric space $(\mathcal{P},d_0)$, where $d_0=\mathcal{D}-\phi$.

(2) A sequence $\{p_n\}$ is called a perturbed Cauchy sequence in $(\mathcal{P},\mathcal{D},\phi)$ if it is a Cauchy sequence with respect to the exact metric d_0 in the exact metric space $(\mathcal{P},d_0)$.

(3) The space $(\mathcal{P},\mathcal{D},\phi)$ is said to be complete if every perturbed Cauchy sequence converges (in the exact metric sense); that is, $(\mathcal{P},d_0)$ is a complete metric space, or equivalently, all perturbed Cauchy sequences in $(\mathcal{P},\mathcal{D},\phi)$ converge in $(\mathcal{P},\mathcal{D},\phi)$.

(4) A mapping $\mathcal{T}$ is called perturbed continuous if it is continuous with respect to the exact metric d_0 in the exact metric space $(\mathcal{P},d_0)$.

These definitions enable classical fixed-point arguments to remain valid even when the perturbation term affects distance measurements. The contraction conditions defined within this framework enable the equilibrium behavior of learning and updating processes to be examined.

Within perturbed metric spaces, the Banach contraction principle has been extended by Jleli and Samet [11]. For completeness, we recall their main result.

Theorem 3.3 (Jleli–Samet [11], Theorem 3.1). *Let $(\mathcal{P},\mathcal{D},\phi)$ be a complete perturbed metric space, where the perceived distance $\mathcal{D}$ is defined by*

$$\mathcal{D}(p,q)=d_0(p,q)+\phi(p,q),$$

and let $\mathcal{T}:\mathcal{P}\to\mathcal{P}$ be a self-mapping. Assume that

(1) $\mathcal{T}$ *is perturbed continuous,*

(2) There exists a constant $\kappa \in (0,1)$ *such that*

$$\mathcal{D}(\mathcal{T}p, \mathcal{T}q) \leq \kappa\, \mathcal{D}(p,q) \quad \textit{for all } p, q \in \mathcal{P}.$$

Then $\mathcal{T}$ *admits a unique fixed point in* $\mathcal{P}$.

The contractive condition in Theorem 4.1 does not reduce to the Banach-type contraction considered above. In particular, it is not expressed solely through $\mathcal{D}(p,q)$, but involves additional interaction terms and an additive institutional component.

4. Equilibrium and convergence of the learning dynamics

In this section, we establish the main convergence result of the model and show that the proposed updating mechanism admits a unique equilibrium under suitable conditions.

Theorem 4.1 (Equilibrium of the learning dynamics)**.** *Let* $(\mathcal{P}, \mathcal{D}, \phi)$ *be a complete perturbed metric space, where*

$$\mathcal{D}(p,q) = d_0(p,q) + \phi(p,q), \qquad p, q \in \mathcal{P},$$

d_0 *is the associated exact metric on* $\mathcal{P}$ *and* $\phi \geq 0$*. Let* $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ *be a perturbed continuous updating operator. Assume that there exist constants* $\lambda \in [0,1)$ *and* $\theta \in [0,1)$ *with*

$$\lambda + \theta < 1$$

such that for all $p, q \in \mathcal{P}$,

$$\mathcal{D}(\mathcal{T}p, \mathcal{T}q) \leq \lambda \max\left\{\mathcal{D}(p,q),\ \mathcal{D}(\mathcal{T}p, p),\ \frac{d_0(p, \mathcal{T}q) + d_0(q, \mathcal{T}p)}{2}\right\} + \theta\, \mathcal{D}(\mathcal{T}q, q). \tag{4.1}$$

Then, the operator $\mathcal{T}$ *admits a unique fixed point* $p^* \in \mathcal{P}$*, that is,* $\mathcal{T}p^* = p^*$*. Moreover, for any initial price expectation* $p_0 \in \mathcal{P}$*, the iterative sequence* $\{p_n\}$ *defined by*

$$p_{n+1} = \mathcal{T}p_n, \quad n \in \mathbb{N},$$

converges to p^* *with respect to the exact metric* d_0.

Remark 4.2 (Mathematical interpretation). In Theorem 4.1, condition $\lambda+\theta < 1$ ensures that the effective contraction constant obtained in the proof satisfies $\eta = \lambda + \theta \in (0,1)$. This guarantees that the sequence $\{\mathcal{D}(p_{n+1}, p_n)\}$ decreases geometrically, and that the iterative process is contractive with respect to the exact metric d_0.

Remark 4.3 (Economic interpretation). In this theorem, parameters λ and θ are assumed to belong to $[0,1)$, together with the condition $\lambda + \theta < 1$. These assumptions are not merely technical requirements for contraction but also admit a natural financial interpretation.

The learning parameter λ measures how strongly an individual investor adjusts expectations in response to new information. The restriction $\lambda < 1$ reflects gradual belief updating, which is consistent with behavioral phenomena such as conservatism, anchoring, and delayed adjustment. Likewise, parameter θ represents the responsiveness of institutional expectations, and condition $\theta < 1$ reflects that institutional decisions are updated progressively rather than instantaneously because of regulatory procedures, committee decisions, and risk management practices.

The condition $\lambda + \theta < 1$ therefore represents a stability requirement that ensures neither individual behavior nor institutional signals dominates the learning process. Under this balanced adjustment mechanism, expectations evolve in a controlled manner and converge toward a unique equilibrium despite incomplete information and behavioral frictions.

Remark 4.4 (Relation to classical contractive conditions). The contractive condition in Theorem 4.1 is formulated in a perturbed metric space rather than in a classical metric space. Therefore, its relationship with the classical Banach, Kannan, and Chatterjea contractions should be interpreted within the perturbed metric framework. Below, we demonstrate how the condition in (4.1) may be reduced to certain classical contraction forms with certain parameter choices and suitable dominance requirements of the maximum term.

First, if $\theta = 0$ and

$$\max\left\{\mathcal{D}(p,q),\ \mathcal{D}(\mathcal{T}p,p),\ \frac{d_0(p,\mathcal{T}q)+d_0(q,\mathcal{T}p)}{2}\right\} = \mathcal{D}(p,q)$$

then the contraction condition (4.1) becomes

$$\mathcal{D}(\mathcal{T}p,\mathcal{T}q) \le \lambda\mathcal{D}(p,q).$$

Consequently, the proposed contraction reduces to a Banach-type contraction within the perturbed metric structure.

Second, if $\theta = 0$ and

$$\max\left\{\mathcal{D}(p,q),\ \mathcal{D}(\mathcal{T}p,p),\ \frac{d_0(p,\mathcal{T}q)+d_0(q,\mathcal{T}p)}{2}\right\} = \frac{d_0(p,\mathcal{T}q)+d_0(q,\mathcal{T}p)}{2}$$

then we have

$$\mathcal{D}(\mathcal{T}p,\mathcal{T}q) \le \frac{\lambda}{2}\left(d_0(p,\mathcal{T}q)+d_0(q,\mathcal{T}p)\right).$$

Therefore, in this case (4.1), the Chatterjea-type contraction condition is recovered in the perturbed metric space.

Finally, assume that the maximum term in (4.1) is attained at $\mathcal{D}(\mathcal{T}p,p)$, that is,

$$\max\left\{\mathcal{D}(p,q),\ \mathcal{D}(\mathcal{T}p,p),\ \frac{d_0(p,\mathcal{T}q)+d_0(q,\mathcal{T}p)}{2}\right\} = \mathcal{D}(\mathcal{T}p,p).$$

Then, the contractive condition (4.1) reduces to

$$\mathcal{D}(\mathcal{T}p,\mathcal{T}q) \le \lambda\mathcal{D}(\mathcal{T}p,p) + \theta\mathcal{D}(\mathcal{T}q,q).$$

If, in addition, $\lambda = \theta$, we obtain

$$\mathcal{D}(\mathcal{T}p,\mathcal{T}q) \le \lambda\mathcal{D}(\mathcal{T}p,p) + \lambda\mathcal{D}(\mathcal{T}q,q),$$

or equivalently,

$$\mathcal{D}(\mathcal{T}p,\mathcal{T}q) \leq \lambda\left[\mathcal{D}(\mathcal{T}p,p)+\mathcal{D}(\mathcal{T}q,q)\right].$$

This yields the Kannan-type contractive form in the perturbed metric framework. Thus, condition (4.1) recovers the structural form of the Kannan-type contraction in the perturbed metric setting.

As a result, Theorem 4.1 unifies several classical contractive structures as special cases within the perturbed metric framework, while its novelty lies in combining the perturbed metric structure with the generalized contractive inequality (4.1).

Proof. Let $p_0 \in \mathcal{P}$ be an arbitrary initial price expectation and define the iterative sequence using an updating operator $\mathcal{T}$ as follows:

$$p_{n+1} = \mathcal{T}p_n, \qquad n \in \mathbb{N}.$$

Fixing $n \geq 1$ and applying (4.1) with $(p,q) = (p_n, p_{n-1})$ yields

$$\begin{aligned}\mathcal{D}(p_{n+1},p_n) &= \mathcal{D}(\mathcal{T}p_n,\mathcal{T}p_{n-1})\\ &\leq \lambda\max\left\{\mathcal{D}(p_n,p_{n-1}),\ \mathcal{D}(\mathcal{T}p_n,p_n),\ \frac{d_0(p_n,\mathcal{T}p_{n-1})+d_0(p_{n-1},\mathcal{T}p_n)}{2}\right\}\\ &\quad+\theta\,\mathcal{D}(\mathcal{T}p_{n-1},p_{n-1}).\end{aligned}$$

The third quantity in the maximum corresponds to the interaction term appearing in (4.1). After substituting $\mathcal{T}p_{n-1} = p_n$ and $\mathcal{T}p_n = p_{n+1}$, it reduces to

$$\frac{d_0(p_n,p_n)+d_0(p_{n-1},p_{n+1})}{2} = \frac{1}{2}d_0(p_{n-1},p_{n+1}).$$

Hence, we have,

$$\mathcal{D}(p_{n+1},p_n) \leq \lambda\max\left\{\mathcal{D}(p_n,p_{n-1}),\ \mathcal{D}(p_{n+1},p_n),\ \frac{1}{2}d_0(p_{n-1},p_{n+1})\right\}+\theta\,\mathcal{D}(p_n,p_{n-1}).$$

Set

$$A_n := \mathcal{D}(p_n,p_{n-1}), \qquad B_n := \mathcal{D}(p_{n+1},p_n), \qquad C_n := \frac{1}{2}d_0(p_{n-1},p_{n+1}).$$

Thus,

$$B_n \leq \lambda\max\{A_n,B_n,C_n\}+\theta A_n. \tag{4.2}$$

Case I: If $\max\{A_n,B_n,C_n\} = A_n$, then from (4.2),

$$B_n \leq (\lambda+\theta)A_n.$$

Case II: If $\max\{A_n,B_n,C_n\} = B_n$, then from (4.2),

$$B_n \leq \lambda B_n + \theta A_n,$$

so

$$B_n \leq \frac{\theta}{1-\lambda}A_n.$$

Case III: If $\max\{A_n, B_n, C_n\} = C_n$, then from (4.2),

$$B_n \leq \frac{\lambda}{2} d_0(p_{n-1}, p_{n+1}) + \theta A_n.$$

Using the triangle inequality for the metric d_0,

$$d_0(p_{n-1}, p_{n+1}) \leq d_0(p_{n-1}, p_n) + d_0(p_n, p_{n+1}).$$

Since $\phi \geq 0$, we have $d_0(u, v) \leq \mathcal{D}(u, v)$ for all u, v, then

$$d_0(p_{n-1}, p_n) \leq A_n, \qquad d_0(p_n, p_{n+1}) \leq B_n.$$

Therefore,

$$d_0(p_{n-1}, p_{n+1}) \leq A_n + B_n,$$

and consequently

$$B_n \leq \frac{\lambda}{2}(A_n + B_n) + \theta A_n = \frac{\lambda}{2} B_n + \left(\frac{\lambda}{2} + \theta\right) A_n.$$

This yields

$$B_n \leq \frac{\lambda + 2\theta}{2 - \lambda} A_n.$$

Combining the three cases, define

$$\eta_1 := \lambda + \theta, \qquad \eta_2 := \frac{\theta}{1 - \lambda}, \qquad \eta_3 := \frac{\lambda + 2\theta}{2 - \lambda}, \qquad \eta := \max\{\eta_1, \eta_2, \eta_3\}.$$

We now show that $\eta = \eta_1$ and $\eta \in (0, 1)$ under $\lambda \in [0, 1)$ and $\lambda + \theta < 1$. Indeed, since

$$\lambda + \theta < 1,$$

equivalently

$$1 - \lambda - \theta > 0,$$

we have,

$$\eta_1 - \eta_3 = \frac{(\lambda + \theta)(2 - \lambda) - (\lambda + 2\theta)}{2 - \lambda} = \frac{\lambda(1 - \lambda - \theta)}{2 - \lambda} > 0,$$
$$\eta_3 - \eta_2 = \frac{(\lambda + 2\theta)(1 - \lambda) - \theta(2 - \lambda)}{(2 - \lambda)(1 - \lambda)} = \frac{\lambda(1 - \lambda - \theta)}{(2 - \lambda)(1 - \lambda)} > 0.$$

Hence, $\eta_1 > \eta_3 > \eta_2$, so $\eta = \eta_1 = \lambda + \theta$. In particular, $\eta \in (0, 1)$ because $\lambda + \theta < 1$. Thus, for all $n \geq 1$,

$$\mathcal{D}(p_{n+1}, p_n) \leq \eta\, \mathcal{D}(p_n, p_{n-1}). \tag{4.3}$$

Iterating (4.3) gives

$$\mathcal{D}(p_{n+1}, p_n) \leq \eta^n\, \mathcal{D}(p_1, p_0), \qquad n \geq 0.$$

Since

$$\mathcal{D}(p,q) = d_0(p,q) + \phi(p,q)$$

with $\phi(p,q) \geq 0$ for all $p, q \in \mathcal{P}$, it follows that

$$d_0(p,q) \leq \mathcal{D}(p,q).$$

Therefore,

$$d_0(p_{n+1}, p_n) \leq \eta^n \, \mathcal{D}(p_1, p_0), \qquad n \geq 0.$$

For $m > n$, by the triangle inequality for d_0,

$$d_0(p_m, p_n) \leq \sum_{k=n}^{m-1} d_0(p_{k+1}, p_k) \leq \sum_{k=n}^{m-1} \eta^k \, \mathcal{D}(p_1, p_0) \leq \frac{\eta^n}{1-\eta} \, \mathcal{D}(p_1, p_0).$$

Letting $n \to \infty$ shows that $\{p_n\}$ is a Cauchy sequence in the complete metric space $(\mathcal{P}, d_0)$. Hence, there exists $p^* \in \mathcal{P}$ such that

$$\lim_{n\to\infty} d_0(p_n, p^*) = 0.$$

To prove that p^* is a fixed point, we invoke the perturbed continuity of $\mathcal{T}$, which, by Definition 3.2, means continuity with respect to the exact metric d_0. Since

$$d_0(p_n, p^*) \longrightarrow 0,$$

perturbed continuity yields

$$d_0(\mathcal{T} p_n, \mathcal{T} p^*) \longrightarrow 0.$$

Since

$$\mathcal{T} p_n = p_{n+1},$$

and the sequence $\{p_{n+1}\}$ has the same limit p^* because $d_0(p_{n+1}, p_n) \to 0$, we obtain

$$d_0(\mathcal{T} p^*, p^*) = 0$$

which implies

$$\mathcal{T} p^* = p^*.$$

Finally, for uniqueness, let $p^*, q^* \in \mathcal{P}$ be two fixed points of $\mathcal{T}$. Applying (4.1) and using $\mathcal{T} p^* = p^*$, $\mathcal{T} q^* = q^*$, we get

$$\begin{aligned}
\mathcal{D}(p^*, q^*) &= \mathcal{D}(\mathcal{T} p^*, \mathcal{T} q^*) \\
&\leq \lambda \max \left\{ \mathcal{D}(p^*, q^*),\ \mathcal{D}(p^*, p^*),\ \frac{d_0(p^*, q^*) + d_0(q^*, p^*)}{2} \right\} + \theta \, \mathcal{D}(q^*, q^*) \\
&= \lambda \max \{ \mathcal{D}(p^*, q^*),\ d_0(p^*, q^*) \} \leq \lambda \, \mathcal{D}(p^*, q^*).
\end{aligned}$$

Thus,

$$(1-\lambda)\mathcal{D}(p^*, q^*) \leq 0.$$

Since $\lambda < 1$ and $\mathcal{D}(p^*, q^*) \geq 0$, it follows that

$$\mathcal{D}(p^*, q^*) = 0.$$

By the definition of the perturbed metric,

$$\mathcal{D}(p^*, q^*) = d_0(p^*, q^*) + \phi(p^*, q^*).$$

Since both d_0 and ϕ are nonnegative, the equality

$$d_0(p^*, q^*) + \phi(p^*, q^*) = 0$$

implies

$$d_0(p^*, q^*) = 0 \quad \text{and} \quad \phi(p^*, q^*) = 0.$$

Hence, by the separation property of the exact metric d_0,

$$p^* = q^*.$$

Therefore, $\mathcal{T}$ admits a unique fixed point in $\mathcal{P}$, and the iterates converge to p^* with respect to the exact metric d_0. □

5. Illustrative examples of the learning dynamics

In this section, we examine concrete examples that illustrate how investor expectations are updated to reveal the economic significance of the theoretical results presented in previous sections. Our objective is to demonstrate, in an intuitive and calculable manner, that price expectations can converge to a single equilibrium value in an environment where behavioral and institutional frictions coexist, and to identify the parameters that determine the speed of this convergence.

The section proceeds in two steps. First, we analyze a benchmark linear update mechanism, which preserves the classical structure studied in earlier contributions and enables direct comparison with existing results. This example clarifies how the perturbed distance framework operates under a transparent linear adjustment rule.

Second, we introduce a nonlinear expectation dynamic incorporating bounded institutional amplification. This extension enables us to examine how nonlinear corrections affect the admissible parameter region required for contractive stability, and how convergence speed becomes more sensitive to the interaction between individual learning and institutional smoothing.

5.1. Example 1: Expectation dynamics under a linear update

In this example, the convergence behavior of price expectations toward a common equilibrium value is examined under a linear update mechanism. The framework enables individual and institutional forecasts to be interpreted within the same mathematical structure, while preserving the single-update dynamic considered in prior research. The objective is to maintain the classical linear adjustment rule and to demonstrate how the behavioral parameters introduced in the main theorem affect the convergence speed without altering the underlying benchmark structure.

The model considers two representative actors in the market whose adjustment speeds are characterized by the parameters that appear in the main theorem of this paper. The first actor, P, represents the individual investor. Individual daily price expectations are denoted by $p \in \mathcal{P}$, where $\mathcal{P}$ is the set of possible price forecasts. The updating behavior reflects attachment to past beliefs and gradual incorporation of new information, consistent with anchoring and conservatism tendencies documented in the behavioral finance literature. While new information is not ignored, previous forecasts continue to serve as a strong reference point, generating inertia in the learning process.

The second actor, Q, represents analyst consensus or the institutional benchmark. Its price expectation is denoted by $q \in \mathcal{P}$. Although both actors form expectations within the same set $\mathcal{P}$, the institutional side processes information through more structured procedures and is subject to update friction arising from coordination mechanisms, reporting delays, and regulatory considerations. Consequently, even when P and Q observe the same market information, their expectations p and q may evolve at different effective speeds.

In this context, the fundamental question is: Do the expectations of these actors, who initially have different price forecasts, converge to a single equilibrium price in the long run, even when perceptual distortions exist, and through which parameters can the speed of this convergence be interpreted?

Let the set $\mathcal{P}$ be defined as the set of daily price forecasts formed by market participants, and let the transformation $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ represent a generic forecast update process. For any $p \in \mathcal{P}$, the next step's updated price forecast is denoted by $\mathcal{T}(p)$. To examine this question mathematically, the expectation update mechanism is modeled by the following linear operator:

$$\mathcal{T}(p) = \mu \cdot p + (1 - \mu) \cdot p_{\text{market}} + \epsilon,$$

where p_{market} represents the institutional benchmark or the market benchmark (e.g., the average price of the previous period or analyst consensus). Parameter $\mu \in (0, 1)$ captures the effective adjustment intensity. Larger values of μ indicate stronger attachment to previous beliefs and slower adaptation to new information. Within the general framework of the main theorem, this coefficient is interpreted either as the individual learning parameter λ (investor-focused setting) or as the institutional friction parameter θ (institution-focused setting). The term ϵ denotes a bounded noise component. To isolate the pure learning dynamics in this example, we set $\epsilon = 0$.

To model how investors perceive differences among price forecasts, a perturbed-metric approach is adopted rather than the classical metric structure. In this context, the basic metric

$$d_0(p, q) = |p - q|$$

is selected, and the perceptual deviation is defined as a symmetric and linear multiplier

$$\phi(p, q) = \kappa|p - q|, \qquad \kappa \in [0, 1).$$

Accordingly, the total distance perceived by the investor takes the form

$$\mathcal{D}(p, q) = d_0(p, q) + \phi(p, q) = (1 + \kappa)|p - q|.$$

This specification preserves the linear structure of the update rule while embedding the model into the perturbed metric framework. In particular, the update operator remains linear, whereas the distance concept reflects perceptual amplification.

Under this formulation, the same rational difference $|p-q|$ is uniformly scaled by the factor $(1+\kappa)$. Thus, perceptual distortions affect the magnitude of distances without altering the structural properties of the linear update rule. The symmetric choice of ϕ is made deliberately to ensure direct comparability with classical linear models and previous studies.

Under this linear structure, the equilibrium (fixed point) can be explicitly calculated. Indeed, if p^* is a fixed point, it must satisfy the condition $p^* = \mathcal{T}(p^*)$. When the linear update operator

$$\mathcal{T}(p) = \mu p + (1-\mu)p_{\text{market}}$$

is used, this equation directly yields

$$p^* = p_{\text{market}}.$$

Therefore, the equilibrium price in the model coincides exactly with the market anchor. The convergence mechanism is also observed in a similar manner. For any two $p, q \in \mathcal{P}$,

$$\mathcal{D}(\mathcal{T}p, \mathcal{T}q) = (1+\kappa)|\mu(p-q)| = \mu\,\mathcal{D}(p,q).$$

Thus, $\mathcal{T}$ is a Banach-type contraction on the perturbed metric space $(\mathcal{P}, \mathcal{D}, \phi)$ with contraction constant μ. Whenever $\mu < 1$, the iterative sequence converges to the unique equilibrium independently of the initial value by Theorem 3.3.

It is important to observe that although $\kappa \neq 0$, the perturbation $\phi(p,q) = \kappa|p-q|$ acts merely as a uniform scaling of the exact metric. Consequently, perceptual amplification modifies the magnitude of distances but does not affect the contraction property of the operator. This linear benchmark therefore, represents a special case of the general contraction framework established in Theorem 4.1. In this present setting, the convergence speed is governed solely by the effective adjustment coefficient μ, whereas in the general model, the contraction bound depends on the combined parameters $\lambda+\theta$. Finally, to connect this benchmark with Theorem 4.1, we use λ and θ as upper-bound parameters in (4.1) and perform two complementary parameter settings.

Although the update operator $\mathcal{T}$ is defined on a single forecast space $\mathcal{P}$, parameter μ is interpreted differently depending on whether the updating agent is the individual investor ($\mu = \lambda$) or the institutional benchmark ($\mu = \theta$).

In the numerical analysis, we therefore consider two complementary settings. In the investor-focused setting, the adjustment speed is identified with the individual learning parameter ($\mu = \lambda$), while θ is kept fixed. In the institution-focused setting, we set $\mu = \theta$ and keep λ fixed. This design preserves the linear update structure while enabling the separate influence of individual inertia and institutional friction on the contraction inequality to be examined.

From this point onward, to directly compare the present results with the linear expectation-updating benchmark considered in the previous study [20], the same market anchor, the same perceptual distortion structure, and similar initial values are maintained throughout the numerical analysis. Thus, the effects of parameters λ and θ introduced in this study on the convergence rate can be examined solely by varying these parameters.

Specifically, setting the adjustment coefficient to $\mu = 0.25$ reproduces the update rule

$$\mathcal{T}(p) = 0.25p + 75$$

used in the previous study [20]. In this framework, this value is obtained either by choosing $\lambda = 0.25$ in the investor-focused setting or by choosing $\theta = 0.25$ in the institution-focused setting. This preserves the linear benchmark structure while enabling the separate roles of individual inertia and institutional friction to be examined.

In the numerical analysis, the following common structural settings are maintained to ensure direct comparability with the previous study:

- The market anchor $p_{\text{market}} = 100$ is fixed. This value represents a stable reference level, such as an analyst consensus or a short-term market average. Hence, the linear updating operator becomes

$$\mathcal{T}(p) = \mu\, p + (1 - \mu)\, 100. \tag{5.1}$$

- The perturbation coefficient representing perceptual deviation is chosen as $\kappa = 0.1$. Accordingly, the perceived distance between two price forecasts is defined as

$$\mathcal{D}(p, q) = |p - q| + 0.1|p - q| = 1.1|p - q|. \tag{5.2}$$

- The set of initial price estimates is taken as $\mathcal{P} = \{90, 95, 100, 105, 110\}$, representing heterogeneous initial expectations around the market anchor.
- In the investor-focused setting, the internal adjustment coefficient is identified with the individual learning parameter ($\mu = \lambda$), while the institutional coefficient θ is kept fixed at a moderate level. In this case, the updating rule becomes

$$p_{n+1} = \lambda p_n + (1 - \lambda)\, 100.$$

  Thus, variations in λ directly determine the speed of convergence, enabling the isolated effect of increasing individual inertia to be examined.
- In the institution-focused setting, the adjustment coefficient is identified with the institutional friction parameter ($\mu = \theta$), while the individual learning parameter λ is fixed at a moderate level. The corresponding updating rule is

$$p_{n+1} = \theta p_n + (1 - \theta)\, 100.$$

  Here, changes in θ determine the convergence speed, enabling the isolated examination of institutional update friction without altering the underlying linear structure.

Under the parameter configurations defined in Tables 1 and 2, the convergence speeds of iterations starting from different initial values toward the equilibrium price are compared. The numerical analysis is designed to separately examine the effects of individual learning inertia and institutional update friction while preserving the same linear benchmark structure.

Table 1. Parameter cases in the investor-focused setting ($\mu = \lambda$, θ fixed).

Case	λ	θ	Description
A1	0.10	0.10	Fast adjustment (low individual inertia)
A2	0.25	0.10	Benchmark individual inertia
A3	0.40	0.10	High individual inertia

Table 2. Parameter cases in the institution-focused setting ($\mu = \theta$, λ fixed).

Case	$\boldsymbol{\lambda}$	$\boldsymbol{\theta}$	Description
B1	0.40	0.10	Fast institutional adjustment
B2	0.40	0.25	Benchmark institutional friction
B3	0.40	0.40	High institutional friction

The choice of fixed parameter values in each setting is motivated by economic interpretation and methodological consistency. In the investor-focused setting, the institutional coefficient θ is kept at a relatively small positive value to represent background institutional friction without allowing it to dominate the adjustment dynamics. This enables the isolated examination of how increasing individual inertia affects convergence speed. Conversely, in the institution-focused setting, the individual learning parameter λ is set to a moderate value. This reflects the empirical observation that investor inertia is rarely negligible in practice while ensuring that the sufficient contraction condition in Theorem 4.1 remains structurally verifiable across all tested cases. In this way, the two settings provide complementary perspectives on the separate roles of individual and institutional adjustment mechanisms.

We now provide a simple sufficient condition ensuring that the contractive inequality (4.1) holds under the present linear structure. For arbitrary $p, q \in \mathcal{P}$, by Eqs (5.1) and (5.2), we have

$$\mathcal{D}(\mathcal{T}p, \mathcal{T}q) = 1.1|\mathcal{T}(p) - \mathcal{T}(q)| = 1.1\,\mu\,|p - q|.$$

Moreover, we can always conclude that

$$\max\left\{\mathcal{D}(p, q), \mathcal{D}(\mathcal{T}p, p), \frac{d_0(p, \mathcal{T}q) + d_0(q, \mathcal{T}p)}{2}\right\} \geq \mathcal{D}(p, q).$$

Since the term $\theta\,\mathcal{D}(\mathcal{T}q, q)$ is nonnegative, the right-hand side of the contraction condition (4.1) satisfies

$$\text{RHS of inequality (4.1)} \;\geq \lambda\,\mathcal{D}(p, q) + \theta\,\mathcal{D}(\mathcal{T}q, q) \geq \lambda\,\mathcal{D}(p, q) = \lambda \cdot 1.1|p - q|.$$

Hence, whenever $\mu \leq \lambda$, we guarantee the desired result:

$$\text{LHS} = \mathcal{D}(\mathcal{T}p, \mathcal{T}q) = 1.1\,\mu\,|p - q| \leq \lambda \cdot 1.1|p - q| \leq \text{RHS}.$$

Note that the sufficient condition reduces to requiring that the effective adjustment coefficient satisfies

$$\mu \leq \lambda.$$

In the investor-focused setting, this holds trivially since $\mu = \lambda$. In the institution-focused setting, this requirement becomes $\theta \leq \lambda$. All parameter configurations reported in Tables 1 and 2 are chosen so that this condition is satisfied.

In all parameter configurations considered in this example (Cases A1–A3 and B1–B3) given by Tables 1 and 2, condition $\mu \leq \lambda$ is satisfied. Consequently, the contraction inequality (4.1) holds for all $(p, q) \in \mathcal{P} \times \mathcal{P}$. Together with the equilibrium requirement $\lambda + \theta < 1$, which ensures the validity of the fixed point framework of Theorem 4.1 and is fulfilled in each case, the sufficient conditions of Theorem 4.1 are verified for all six parameter configurations. Therefore, the update operator $\mathcal{T}$ admits a unique fixed point under each parameter combination. For completeness and transparency, the explicit numerical evaluations of the left-hand and right-hand sides of inequality (4.1) are reported in the appendix.

While the existence and uniqueness of the equilibrium are now guaranteed analytically, the numerical simulations below illustrate how different parameter levels affect the speed of convergence toward this equilibrium.

Figures 1 and 2 illustrate the iteration dynamics of the linear update rule under the investor-focused and institution-focused parameter configurations, respectively. In all cases, trajectories starting from heterogeneous initial forecasts converge to the common fixed point $p^* = 100$, in agreement with the analytical contraction result established above.

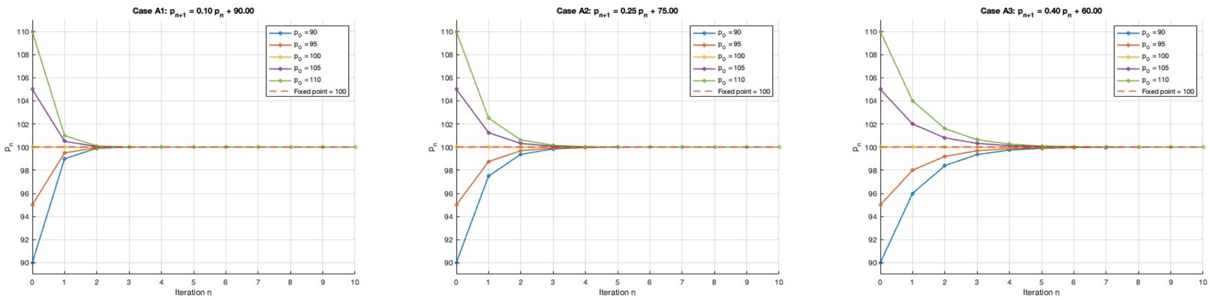

Figure 1. Iteration paths of the linear expectation update rule for Cases A1–A3 with heterogeneous initial values $p_0 \in \{90, 95, 100, 105, 110\}$. All trajectories converge to the common fixed point $p^* = 100$.

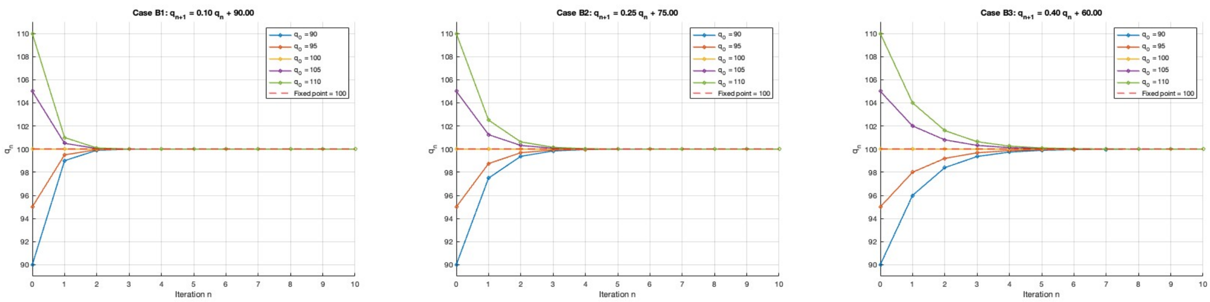

Figure 2. Iteration paths of the linear expectation update rule for Cases B1–B3 with heterogeneous initial values $p_0 \in \{90, 95, 100, 105, 110\}$. All trajectories converge to the common fixed point $p^* = 100$.

The speed of convergence varies systematically across parameter levels. Since the distance to the equilibrium evolves according to

$$|p_n - p^*| = \mu^n |p_0 - p^*|,$$

the adjustment coefficient μ directly determines the contraction rate. Smaller values of μ imply that the deviation from equilibrium shrinks more rapidly at each iteration step, leading to faster stabilization of expectations. Conversely, when μ is closer to one, past beliefs receive a stronger weight in the updating rule, so deviations decay more slowly and the adjustment path becomes more gradual. From a financial perspective, this indicates that lower learning inertia or weaker institutional friction enables market participants to incorporate benchmark information more efficiently, resulting in rapid coordination around the equilibrium price. In contrast, stronger inertia reflects a greater persistence of prior beliefs, which delays the alignment of expectations even though convergence remains guaranteed by the contraction property.

As can be seen more clearly in Figure 3, when the same initial value is considered, the contraction speed is directly governed by the effective adjustment coefficient μ. In particular, the benchmark case $\mu = 0.25$, which corresponds to the update rule $\mathcal{T}(p) = 0.25p + 75$ used in [20], lies between the fast-adjustment case ($\mu = 0.10$) and the high-inertia case ($\mu = 0.40$). This confirms that this framework preserves the classical linear benchmark structure while extending it by explicitly separating individual inertia and institutional friction.

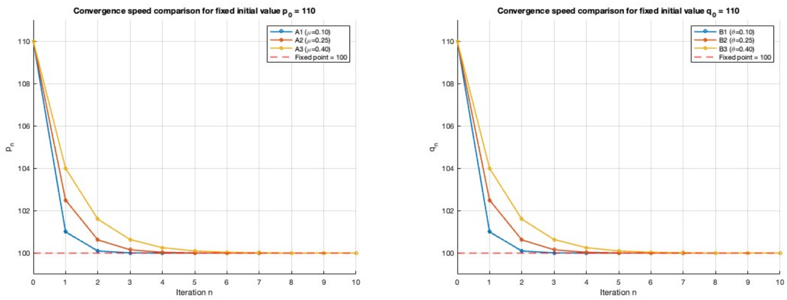

Figure 3. Direct comparison of convergence speeds for a fixed initial value $p_0 = 110$. Left panel: investor-focused cases A1–A3 ($\mu = \lambda$). Right panel: institution-focused cases B1–B3 ($\mu = \theta$). Smaller adjustment coefficients produce faster geometric convergence toward the fixed point $p^* = 100$.

Thus, the numerical illustrations support the theoretical contraction result and its behavioral interpretation within the expectation formation framework.

5.2. Example 2: Nonlinear expectation dynamics with slow convergence

While Example 5.1 illustrates convergence under a linear benchmark structure, real financial adjustment processes often display a substantially slower path toward equilibrium. In many markets, price expectations do not immediately align with benchmark values. Instead, revisions occur gradually due to behavioral inertia, anchoring, delayed information processing, and institutional coordination constraints.

In [20], it was observed that the nonlinear operator

$$\mathcal{T}(p) = \frac{p}{2} + 5\ \arctan(0.1(p - 100)) + 50 \tag{5.3}$$

converges to its equilibrium considerably more slowly than its linear counterpart. Although a fixed point exists, the adjustment trajectory is persistent and gradual. This behavior reflects a financial environment in which learning dynamics are affected by bounded rationality and institutional frictions.

Motivated by this observation, we reconsider this nonlinear structure within the perturbed metric framework developed in Theorem 4.1. The objective is not to accelerate convergence, but to determine under which structural parameter conditions equilibrium is guaranteed to exist even when the adjustment process is intrinsically slow.

We fix the market anchor at

$$p_{\text{market}} = 100,$$

and define the nonlinear updating operator $\mathcal{T}_{\lambda,\theta} : \mathcal{P} \to \mathcal{P}$ by

$$\mathcal{T}_{\lambda,\theta}(p) = (1 - \lambda - \theta)p + (\lambda + \theta)\, p_{\text{market}} + \theta\, c\, \arctan(\nu(p - p_{\text{market}})), \tag{5.4}$$

where $\mathcal{P}$ denotes the set of admissible price forecasts.

The first two terms reproduce the linear mean-reversion mechanism introduced in Example 5.1. Indeed, ignoring the nonlinear component,

$$\mathcal{T}(p) = (1 - \lambda - \theta)p + (\lambda + \theta)100$$

can be rewritten as

$$p_{n+1} = p_n - (\lambda + \theta)(p_n - 100),$$

which shows that the deviation from the benchmark shrinks proportionally at rate $(\lambda + \theta)$. Hence, in the absence of nonlinear effects, the model reduces to a classical linear adjustment rule.

The third term,

$$\theta c\, \arctan(\nu(p - 100)),$$

introduces a bounded nonlinear institutional correction. Parameter $c > 0$ determines the maximal amplitude of this adjustment, since

$$|\arctan(x)| \le \frac{\pi}{2},$$

so the nonlinear contribution is globally bounded by $\theta c \frac{\pi}{2}$. Parameter $\nu > 0$ controls the sensitivity of the response: For small deviations from the benchmark, $\arctan(\nu(p - 100)) \approx \nu(p - 100)$, and the operator behaves locally like a linear rule, whereas for larger deviations, the correction saturates and prevents disproportionately large institutional reactions. This choice captures a realistic financial mechanism in which expectations are continuously pulled toward a benchmark; yet, institutional revisions remain moderate and gradually implemented rather than strictly proportional to forecast errors.

Our objective of this example is to determine explicit restrictions on (λ, θ) under which the nonlinear operator (5.4) satisfies the contractive condition of Theorem 4.1, thereby guaranteeing the existence and uniqueness of the equilibrium expectation.

Throughout this example, we keep the same perturbed distance structure as in Example 5.1, namely

$$d_0(p, q) = |p - q|, \qquad \phi(p, q) = \kappa|p - q|, \qquad \mathcal{D}(p, q) = (1 + \kappa)|p - q|, \quad \kappa \in [0, 1).$$

Let $p, q \in \mathcal{P}$ be arbitrary. Using (5.4) (with $p_{\text{market}} = 100$), we write

$$\mathcal{T}_{\lambda,\theta}(p) - \mathcal{T}_{\lambda,\theta}(q) = (1 - \lambda - \theta)(p - q) + \theta c\Big(\arctan\left(\nu(p - 100)\right) - \arctan\left(\nu(q - 100)\right)\Big).$$

Since $|\arctan'(x)| \le 1$ for all $x \in \mathbb{R}$, the Mean Value Theorem implies that

$$|\arctan(u) - \arctan(v)| \le |u - v| \quad \text{for all } u, v \in \mathbb{R},$$

and hence we obtain the bound

$$\left|\arctan(\nu(p-100)) - \arctan(\nu(q-100))\right| \le \nu|p-q|.$$

Using $\lambda + \theta < 1$ (hence $1-\lambda-\theta > 0$), it follows that

$$|\mathcal{T}_{\lambda,\theta}(p) - \mathcal{T}_{\lambda,\theta}(q)| \le (1-\lambda-\theta)|p-q| + \theta c\nu|p-q| = (1-\lambda-\theta+\theta c\nu)|p-q|. \tag{5.5}$$

Therefore,

$$\mathcal{D}(\mathcal{T}_{\lambda,\theta}p, \mathcal{T}_{\lambda,\theta}q) = (1+\kappa)|\mathcal{T}_{\lambda,\theta}(p) - \mathcal{T}_{\lambda,\theta}(q)| \le (1+\kappa)(1-\lambda-\theta+\theta c\nu)|p-q|. \tag{5.6}$$

Next, note that for the right-hand side of (4.1), we always have

$$\max\left\{\mathcal{D}(p,q),\ \mathcal{D}(\mathcal{T}p,p),\ \frac{d_0(p,\mathcal{T}q) + d_0(q,\mathcal{T}p)}{2}\right\} \ge \mathcal{D}(p,q) = (1+\kappa)|p-q|,$$

and $\theta\,\mathcal{D}(\mathcal{T}q,q) \ge 0$. Since the maximum term dominates $\mathcal{D}(p,q)$ and the last term is nonnegative, the right-hand side of (4.1) is bounded below by $\lambda\mathcal{D}(p,q)$. Hence,

$$\text{RHS of (4.1)} \ge \lambda\,\mathcal{D}(p,q) = \lambda(1+\kappa)|p-q|. \tag{5.7}$$

Combining (5.6) and (5.7), a sufficient condition for (4.1) to hold for all $p, q \in \mathcal{P}$ is

$$(1+\kappa)(1-\lambda-\theta+\theta c\nu)|p-q| \le \lambda(1+\kappa)|p-q|.$$

Cancelling the common factor $(1+\kappa)|p-q|$ yields the explicit restriction

$$1-\lambda-\theta+\theta c\nu \le \lambda \quad \Longleftrightarrow \quad \lambda \ge \frac{1-\theta+\theta c\nu}{2}. \tag{5.8}$$

In our numerical design we fix $c = 20$ and $\nu = 0.1$, hence $c\nu = 2$, and (5.8) reduces to

$$\lambda \ge \frac{1+\theta}{2}.$$

Together with the equilibrium requirement $\lambda + \theta < 1$ from Theorem 4.1, for the interval

$$\frac{1+\theta}{2} \le \lambda < 1-\theta$$

to be nonempty, its lower bound must be strictly smaller than its upper bound. Hence, we require

$$\frac{1+\theta}{2} < 1-\theta,$$

which is equivalent to

$$\theta < \frac{1}{3}.$$

Therefore, we obtain the admissible parameter region

$$0 \le \theta < \frac{1}{3}, \quad \text{and} \quad \frac{1+\theta}{2} \le \lambda < 1-\theta. \tag{5.9}$$

Figure 4 illustrates the admissible parameter region determined by the theoretical restrictions in (5.9). The admissible region represents combinations of individual learning intensity and institutional adjustment strength that guarantee stable convergence of price expectations. The parameter configurations C1–C3 used in the numerical simulations are displayed within this region.

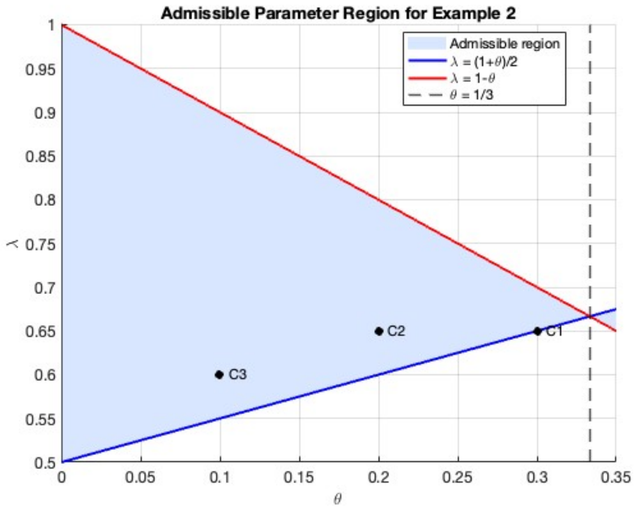

Figure 4. Admissible parameter region defined by the theoretical constraints $0 \leq \theta < 1/3$ and $(1 + \theta)/2 \leq \lambda < 1 - \theta$, $\lambda \geq 0$. The shaded area represents the set of parameter pairs satisfying the contraction conditions of Theorem 4.1. Points C1–C3 correspond to the parameter configurations examined in the numerical simulations.

This admissible region reflects the specific sufficient contractive inequality (4.1) employed in Theorem 4.1. The narrowing of the parameter range does not arise from the nonlinear operator, but from the particular structure of the contraction condition used to verify stability within the perturbed metric framework.

In other words, the presence of the nonlinear institutional correction term modifies the analytical upper bound appearing in (4.1), and hence reduces the set of parameters values for which contractivity can be certified through this sufficient condition. This should be interpreted as a feature of the sufficient verification approach rather than as evidence of structural instability of the model.

From an economic perspective, however, the contraction framework carries a meaningful interpretation. The contractive inequality enforces a disciplined form of expectation adjustment: It requires that deviations between forecasts shrink in a controlled and systematic manner despite perceptual distortions and institutional nonlinear corrections. Thus, although the admissible region becomes narrower, the resulting stability guarantee corresponds to a stricter notion of convergence. In financial terms, this means that even in the presence of institutional smoothing and bounded nonlinear reactions, the aggregate adjustment mechanism must retain a net stabilizing force in order for equilibrium to be guaranteed.

For comparison, when $\lambda + \theta = 0.5$, the linear core of the operator reduces to

$$(1 - \lambda - \theta)p + (\lambda + \theta)100 = \frac{p}{2} + 50.$$

Moreover, fixing $c = 20$ and $\nu = 0.1$, the nonlinear term coincides with (5.3) precisely when $\theta c = 5$, i.e., $\theta = 0.25$, and hence $\lambda = 0.25$. However, the symmetric configuration $(\lambda, \theta) = (0.25, 0.25)$, which reproduces the nonlinear specification studied in [20], does not belong to the admissible region (5.9). Therefore, within this perturbed metric framework, Theorem 4.1 does not provide an analytical guarantee of existence and uniqueness for that parameter choice. This does not imply that a fixed point fails to exist. Rather, the condition (5.9) is sufficient but not necessary. Convergence may still occur outside this region, yet it is not certified by Theorem 4.1.

The parameter configurations reported in Table 3 are selected from the admissible region (5.9). Hence, for each case, the sufficient contractive condition of Theorem 4.1 is analytically verified, and the existence of a unique fixed point is guaranteed.

Table 3. Verification of admissibility conditions for Example 5.2.

Case	λ	$\theta < 1/3$?	$\lambda + \theta < 1$?	$(1+\theta)/2$	$1-\theta$	Admissible?
C1	0.65	0.30 ✓	0.95 ✓	0.65	0.70	✓
C2	0.65	0.20 ✓	0.85 ✓	0.60	0.80	✓
C3	0.60	0.10 ✓	0.70 ✓	0.55	0.90	✓

The three configurations are chosen to represent different positions within the admissible region and, therefore, different regimes of adjustment dynamics. Case C1 lies close to the boundary of the region, where λ attains its minimal admissible value relative to θ. In this configuration, the effective adjustment intensity $\lambda + \theta$ is close to one, and the corresponding Lipschitz bound $L(\lambda, \theta)$ approaches unity. Hence, the contraction effect is weak. Consequently, convergence toward equilibrium is guaranteed but expected to be very slow. Case C2 represents an interior configuration, where the parameters are sufficiently separated from the boundary. Here, the contraction mechanism is stronger, leading to a moderately slow but more stable convergence path. Finally, Case C3 corresponds to a configuration with a smaller institutional weight θ. In this regime, the nonlinear amplification is weaker and the contraction effect becomes more pronounced, so convergence toward equilibrium is expected to occur faster.

Although the admissible region (5.9) guarantees the contractive property analytically, we additionally report the full numerical verification of inequality (4.1) for all ordered pairs $(p, q) \in \mathcal{P} \times \mathcal{P}$ and for Cases C1–C3 in the appendix.

In the next step, we illustrate these theoretical findings through numerical iterations over the finite forecast set $\mathcal{P} = \{90, 95, 100, 105, 110\}$, enabling a direct comparison with the linear benchmark presented in Example 5.1.

In general, the nonlinear specification (5.4) leads to a genuinely nonlinear fixed point equation,

$$p = \mathcal{T}_{\lambda,\theta}(p),$$

for which a closed-form analytical solution is typically not available. Accordingly, the equilibrium level must, in general, be approximated numerically.

The graphical simulations reported in Figure 5 indicate that the equilibrium lies close to the market anchor $p_{\text{market}} = 100$ for all admissible parameter configurations. The intersection of $\mathcal{T}(x)$ with the identity line $y = x$ provides a numerical confirmation of the fixed point.

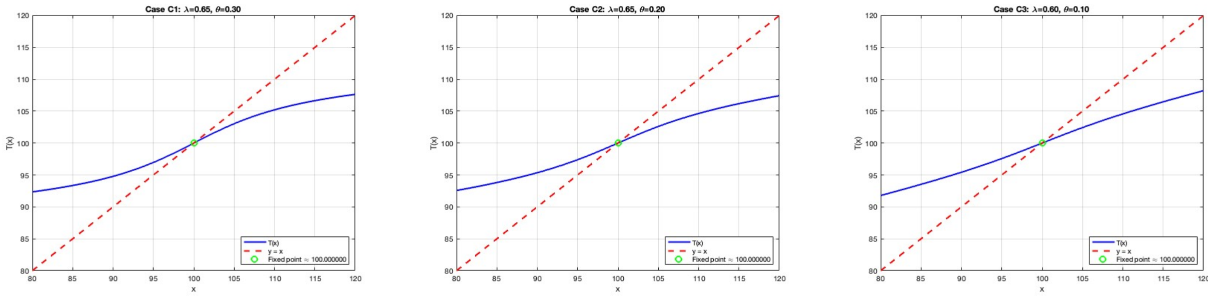

Figure 5. Intersection of the nonlinear update function $\mathcal{T}(x)$ and the identity line $y = x$ for Cases C1–C3. The intersection point (marked in green) gives the numerically computed fixed-point, which is guaranteed to exist and be unique by Theorem 4.1.

However, this operator is constructed in a benchmark-centered form. Since the nonlinear institutional correction is written as $\arctan(\nu(p - p_{\text{market}}))$, it vanishes when $p = p_{\text{market}}$. Consequently,

$$\mathcal{T}_{\lambda,\theta}(p_{\text{market}}) = p_{\text{market}},$$

so the market anchor is an explicit fixed point of the system. The fact that p_{market} is a fixed point is not accidental but reflects the benchmark-centered design of the operator. The nonlinear institutional correction disappears when the forecast coincides with the market anchor, which is economically consistent with the idea that no institutional adjustment is triggered in the absence of deviation.

Having established the admissible parameter region and the analytical contraction bound, we now examine how these theoretical results are reflected in the numerical dynamics. In particular, we investigate whether the differences in parameter configurations within the admissible region lead to visibly distinct convergence patterns.

Figure 6 illustrates the iteration paths of the nonlinear expectation update rule for Cases C1–C3. In all configurations, expectations converge to the common equilibrium level $p^* = 100$. However, the adjustment dynamics differ substantially across parameter regimes.

Figures 6 and 7 jointly illustrate the adjustment dynamics of the nonlinear expectation update rule.

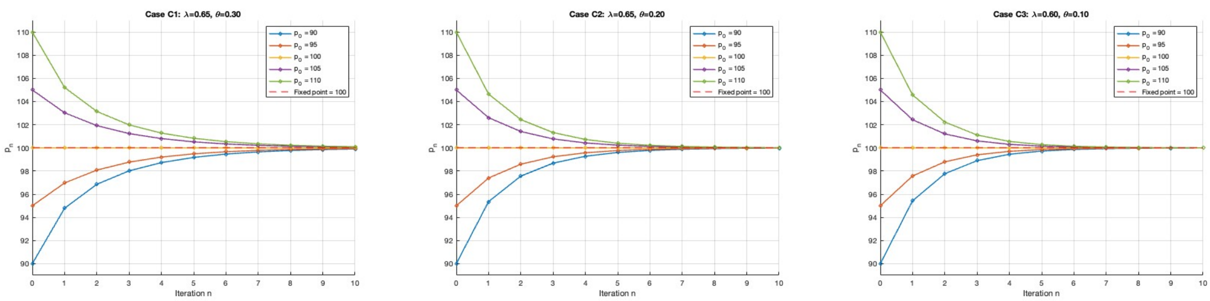

Figure 6. Iteration paths of the nonlinear expectation update rule for Cases C1–C3 with heterogeneous initial values $p_0 \in \{90, 95, 100, 105, 110\}$. All trajectories converge to the common fixed point $p^* = 100$.

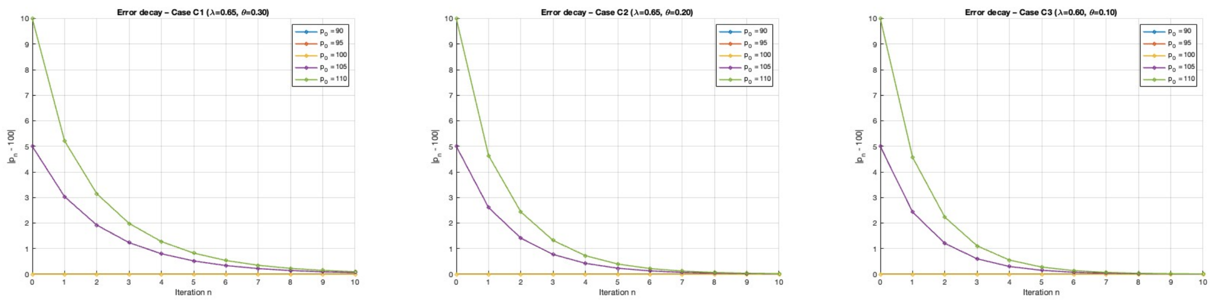

Figure 7. Decay of the absolute deviation $|p_n - 100|$ under the nonlinear expectation update rule for Cases C1–C3 and heterogeneous initial values $p_0 \in \{90, 95, 100, 105, 110\}$. All trajectories converge toward the equilibrium level $p^* = 100$.

The differences in convergence speed can be directly related to the analytical upper bound obtained in (5.5). Indeed, the nonlinear operator admits the Lipschitz-type bound

$$|\mathcal{T}_{\lambda,\theta}(p) - \mathcal{T}_{\lambda,\theta}(q)| \leq (1 - \lambda - \theta + \theta cv)|p - q|.$$

Hence, the effective contraction level is governed by the coefficient

$$L(\lambda, \theta) = 1 - \lambda - \theta + \theta cv.$$

When $L(\lambda, \theta)$ is closer to 1, the contraction effect becomes weaker and deviations shrink more slowly. Conversely, smaller values of $L(\lambda, \theta)$ produce faster error decay.

From an economic perspective, the term $1-\lambda-\theta$ represents the residual persistence of past forecasts, while θcv captures the amplification effect induced by nonlinear institutional smoothing. An increase in θ therefore has two opposing effects: It strengthens the linear pull toward the benchmark through $\lambda + \theta$ and simultaneously increases the nonlinear amplification term θcv. When the latter dominates, the effective contraction weakens, leading to slower convergence.

Although convergence toward the common equilibrium $p^* = 100$ is guaranteed in all configurations, the speed of error reduction differs substantially across parameter regimes. In Case C1, where the institutional nonlinear component carries a relatively strong weight, the absolute deviation from equilibrium decays more gradually. Expectations respond cautiously to forecast errors, and the adjustment process remains persistent. This reflects a market environment in which institutional coordination and bounded rationality moderate immediate reactions to new information, thereby slowing the revision of forecasts. Case C2 represents a more balanced configuration, where individual learning and institutional smoothing coexist without either channel dominating excessively. The convergence path remains stable, while persistence is less pronounced than in Case C1. In Case C3, where the institutional influence is comparatively weaker, the deviation declines more rapidly. Expectations react more directly to discrepancies from the benchmark, and convergence occurs faster. This regime resembles a market in which individual learning dominates institutional inertia, leading to quicker absorption of market information.

Overall, the nonlinear institutional mechanism does not alter the equilibrium level, but significantly shapes the persistence and speed of expectation adjustment. Higher institutional intensity generates smoother, more cautious revisions of expectations, yet at the cost of slower informational efficiency. In short, a higher θ generates slower error decay (institutional moderation dominates), whereas a smaller θ yields faster adjustment (individual learning dominates).

5.3. Example 3: Nonlinear perturbation of perceived forecast distances

The preceding example considered a nonlinear expectation updating rule under a linear perturbation of the exact metric. To show that the proposed framework is not restricted to perturbations that merely rescale the Euclidean distance, we now keep the same nonlinear updating operator as in Example 2 but replace the linear perturbation by a genuinely nonlinear one.

Let

$$\mathcal{P} = \{90, 95, 100, 105, 110\}, \qquad d_0(p, q) = |p - q|,$$

and define

$$\phi(p, q) = 0.5|\sin(0.1(p - q))|.$$

Then the perceived distance becomes

$$\mathcal{D}(p, q) = d_0(p, q) + \phi(p, q) = |p - q| + 0.5|\sin(0.1(p - q))|.$$

We use the same nonlinear updating operator introduced in Example 2:

$$\mathcal{T}_{\lambda,\theta}(p) = (1 - \lambda - \theta)p + (\lambda + \theta)100 + 20\theta \arctan(0.1(p - 100)).$$

This perturbation is nonlinear and bounded. Unlike the linear choice $\phi(p, q) = \kappa|p - q|$, it does not simply multiply the exact metric by a constant. Therefore, it represents a non-uniform perceptual distortion of forecast differences.

To obtain an analytical admissible region for this nonlinear perturbation, we use the estimate established in (5.5):

$$\left|\mathcal{T}_{\lambda,\theta}(p) - \mathcal{T}_{\lambda,\theta}(q)\right| \le (1 - \lambda + \theta)|p - q|.$$

Moreover, since $|\sin x| \le |x|$, the perturbed distance satisfies

$$\begin{aligned}\mathcal{D}(u, v) &= |u - v| + 0.5\left|\sin(0.1(u - v))\right| \\ &\le |u - v| + 0.05|u - v| \\ &= 1.05|u - v|.\end{aligned}$$

Consequently,

$$\mathcal{D}\left(\mathcal{T}_{\lambda,\theta}(p), \mathcal{T}_{\lambda,\theta}(q)\right) \le 1.05(1 - \lambda + \theta)|p - q|.$$

On the other hand, the maximum term on the right-hand side of (4.1) is at least $\mathcal{D}(p, q)$, and $\mathcal{D}(p, q) \ge |p - q|$. Hence,

$$\text{RHS} \ge \lambda\mathcal{D}(p, q) \ge \lambda|p - q|.$$

Therefore, a sufficient condition for (4.1) is

$$1.05(1 - \lambda + \theta) \le \lambda,$$

or equivalently,

$$\lambda \ge \frac{21(1 + \theta)}{41}.$$

Combining this inequality with $\lambda + \theta < 1$ gives the analytical sufficient admissible region

$$0 \le \theta < \frac{10}{31}, \quad \text{and} \quad \frac{21(1 + \theta)}{41} \le \lambda < 1 - \theta.$$

The analytical admissible region provides a guaranteed parameter set under which the assumptions of Theorem 4.1 are satisfied directly from the derived estimates. Outside this region, the analytical bounds are no longer sufficient to guarantee the contraction condition. However, the contractive inequality may still hold and can be verified by direct evaluation for specific parameter configurations.

To examine the effect of the nonlinear perturbation on the analytical admissible region, we retain the same parameter configurations considered in Example 5.2:

$$C_1 : (\lambda, \theta) = (0.65, 0.30), \qquad C_2 : (\lambda, \theta) = (0.65, 0.20), \qquad C_3 : (\lambda, \theta) = (0.60, 0.10).$$

Unlike Example 5.2, where all three parameter configurations satisfied the analytical admissible conditions, the stronger sufficient condition derived for the nonlinear perturbation excludes Case C_1, while Cases C_2 and C_3 remain inside the admissible region (see Figure 8). This comparison enables us to investigate whether the theoretical admissible region is conservative by testing a parameter configuration that lies outside the analytically derived region.

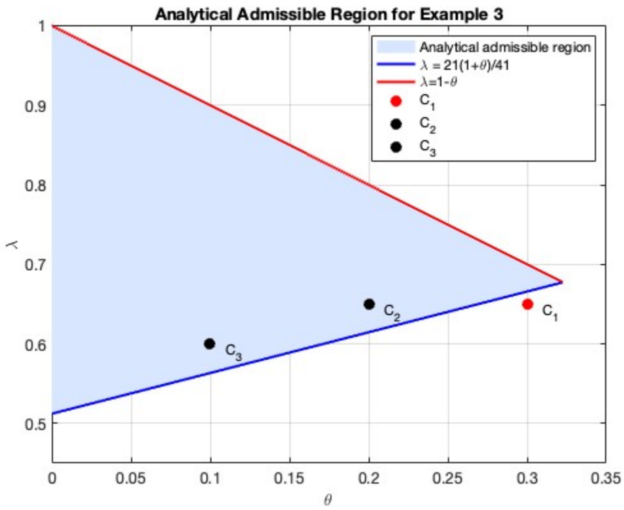

Figure 8. Analytical admissible parameter region defined by the sufficient conditions $\lambda \geq \frac{21(1+\theta)}{41}$ and $\lambda + \theta < 1$. The shaded area represents the set of parameter pairs satisfying the analytical contraction conditions derived for the nonlinear perturbation. Points C_1–C_3 denote the parameter configurations examined in the numerical verification.

For each case, the contractive condition

$$\mathcal{D}(\mathcal{T}p, \mathcal{T}q) \leq \lambda \max \left\{ \mathcal{D}(p, q), \mathcal{D}(\mathcal{T}p, p), \frac{d_0(p, \mathcal{T}q) + d_0(q, \mathcal{T}p)}{2} \right\} + \theta \mathcal{D}(\mathcal{T}q, q)$$

is checked for all ordered pairs $(p, q) \in \mathcal{P} \times \mathcal{P}$.

The computations confirm that the inequality holds for all 25 ordered pairs in each of the cases C_1, C_2, and C_3. A complete pairwise verification of the left- and right-hand sides of (4.1) is provided in appendix. Hence, Theorem 4.1 guarantees the existence and uniqueness of the equilibrium expectation under this nonlinear perturbation structure.

Since the updating operator is identical to that of Example 5.2, the corresponding iteration sequences remain unchanged. Therefore, our objective of this example is not to obtain a different convergence pattern, but to demonstrate that the proposed fixed point framework also applies when the perturbation is genuinely nonlinear rather than a simple linear rescaling of the exact metric. Furthermore, the successful verification of Case C_1 shows that the analytical admissible region is sufficient but conservative, and that additional admissible parameter configurations may be identified through direct verification of the contractive inequality.

5.4. Example 4: Nonlinear expectation adjustment on a discrete forecast grid

To further illustrate the relevance of the proposed contraction framework, we consider a simple expectation updating rule defined on the discrete forecast set

$$\mathcal{P} = \{90, 95, 100, 105, 110\}.$$

These values represent price expectations commonly used by market participants, with forecasts tending to cluster around round-number benchmarks.

We define the expectation update operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ by

$$\mathcal{T}(90) = 100, \quad \mathcal{T}(95) = 100, \quad \mathcal{T}(100) = 100, \quad \mathcal{T}(105) = 95, \quad \mathcal{T}(110) = 90.$$

Economically, this rule captures a stylized correction mechanism toward the benchmark price level 100. When expectations fall significantly below the benchmark, investors revise their forecasts upward toward the consensus level. Expectations already close to the benchmark quickly align with it, whereas excessively optimistic expectations are corrected downward, reflecting the adjustment of overly optimistic beliefs.

It is immediate that

$$\mathcal{T}(100) = 100,$$

so that $p^* = 100$ is a fixed point of the operator. Moreover, the iterative process converges to this equilibrium from any initial forecast in $\mathcal{P}$:

$$90 \mapsto 100, \quad 95 \mapsto 100, \quad 100 \mapsto 100, \quad 105 \mapsto 95 \mapsto 100, \quad 110 \mapsto 90 \mapsto 100.$$

Hence, the equilibrium expectation 100 is globally attracting on the forecast grid $\mathcal{P}$.

We next observe that the classical Banach contraction principle is not applicable with respect to the exact metric

$$d_0(p, q) = |p - q|.$$

Indeed, taking $p = 105$ and $q = 110$ yields

$$\mathcal{T}(105) = 95, \qquad \mathcal{T}(110) = 90,$$

and therefore

$$d_0(\mathcal{T}p, \mathcal{T}q) = |95 - 90| = 5 = d_0(p, q).$$

Consequently, there exists no constant $L < 1$, satisfying

$$d_0(\mathcal{T}p, \mathcal{T}q) \leq Ld_0(p, q)$$

for all $p, q \in P$, and the operator is not a Banach contraction.

We now derive a parameter region for which the contractive inequality of Theorem 4.1 holds. Although a complete verification requires checking all pairs $(p, q) \in \mathcal{P} \times \mathcal{P}$, the admissible parameter region is determined by the most restrictive cases. In particular, the pairs $(110, 100)$ and $(100, 105)$ generate the binding inequalities, while the remaining pairs satisfy the contractive inequality automatically once these conditions are met. This classification of the pairs is summarized in Table 4.

Table 4. Pairwise classification for inequality (4.1) in Example 4.

Group	Pairs (p, q)	Typical LHS	Implication
Group 1	$(90, 90)$, $(90, 95)$ $(90, 100)$, $(95, 90)$ $(95, 95)$, $(95, 100)$ $(100, 90)$, $(100, 95)$ $(100, 100)$	0	Automatically satisfied
Group 2	$(110, 100)$, $(100, 110)$	11	$\lambda \geq \frac{1}{2}$
Group 3	$(100, 105)$, $(105, 100)$	5.5	$\lambda + 2\theta \geq 1$
Group 4	$(90, 105)$, $(105, 90)$ $(95, 110)$, $(110, 95)$	5.5 or 11	Satisfied if Group 2–3 hold

Consider first the pair $(p, q) = (110, 100)$. Since

$$\mathcal{T}(110) = 90, \qquad \mathcal{T}(100) = 100,$$

we obtain

$$\mathcal{D}(\mathcal{T}p, \mathcal{T}q) = 1.1|90 - 100| = 11.$$

Furthermore,

$$\mathcal{D}(p, q) = 11, \qquad \mathcal{D}(\mathcal{T}p, p) = 1.1|90 - 110| = 22,$$

and

$$\frac{d_0(p, \mathcal{T}q) + d_0(q, \mathcal{T}p)}{2} = \frac{|110 - 100| + |100 - 90|}{2} = 10.$$

Hence, the maximal term equals 22, and the contractive inequality yields

$$11 \leq 22\lambda,$$

which implies

$$\lambda \geq \frac{1}{2}.$$

Next, consider the pair $(p, q) = (100, 105)$. In this case

$$\mathcal{T}(100) = 100, \qquad \mathcal{T}(105) = 95,$$

and therefore

$$\mathcal{D}(\mathcal{T}p, \mathcal{T}q) = 1.1|100 - 95| = 5.5.$$

Moreover,

$$\mathcal{D}(p,q) = 5.5, \qquad \mathcal{D}(\mathcal{T}p,p) = 0,$$

and

$$\frac{d_0(p,\mathcal{T}q) + d_0(q,\mathcal{T}p)}{2} = \frac{|100-95| + |105-100|}{2} = 5.$$

Thus, the maximal term equals 5.5 and

$$\mathcal{D}(\mathcal{T}q,q) = 1.1|95-105| = 11.$$

The inequality becomes

$$5.5 \leq 5.5\lambda + 11\theta,$$

which implies

$$\lambda + 2\theta \geq 1.$$

Combining these conditions with the requirement $\lambda + \theta < 1$ from Theorem 4.1, we obtain the admissible parameter region as follows:

$$\lambda \geq \frac{1}{2}, \qquad \lambda + 2\theta \geq 1, \qquad \lambda + \theta < 1. \tag{5.10}$$

Figure 9 illustrates the admissible parameter region determined by the theoretical restrictions in (5.10). We choose the following parameter configurations from this admissible region

$$\text{D1} : (\lambda,\theta) = (0.50, 0.25), \quad \text{D2} : (\lambda,\theta) = (0.55, 0.30), \quad \text{D3} : (\lambda,\theta) = (0.60, 0.20). \tag{5.11}$$

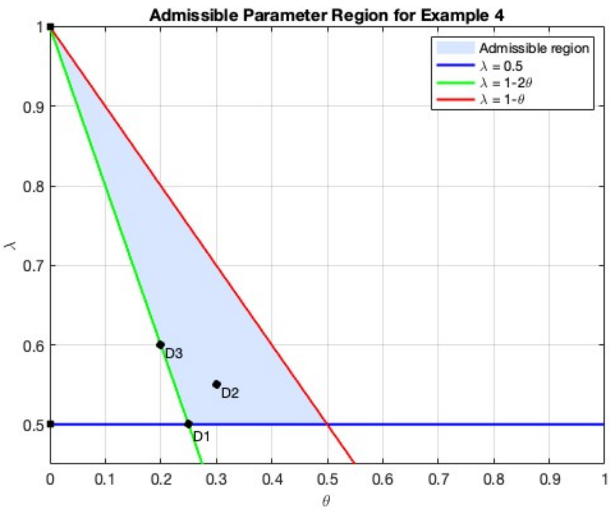

Figure 9. Admissible parameter region defined by the theoretical constraints $\lambda \geq \frac{1}{2}$, $\lambda + 2\theta \geq 1$, and $\lambda + \theta < 1$. The shaded area represents the set of parameter pairs satisfying the contraction conditions of Theorem 4.1. Points D1–D3 correspond to the parameter configurations examined in the numerical simulations.

Table 5 reports the numerical values of the left-hand and right-hand sides of (4.1) for all 25 ordered pairs and for each parameter configuration given in (5.11). As can be seen from this table, inequality (4.1) holds for all 25 ordered pairs in each parameter configuration. This confirms numerically that the sufficient contractive condition of Theorem 4.1 is satisfied for the discrete nonlinear operator considered in this example. Therefore, although the classical Banach contraction principle is not applicable to this operator, the perturbed contraction condition established in Theorem 4.1 guarantees the existence and uniqueness of the equilibrium forecast $p^* = 100$.

Table 5. The LHS and RHS values in the inequality (4.1) for Cases D1–D3.

		Case D1		Case D2		Case D3	
p	q	LHS	RHS	LHS	RHS	LHS	RHS
90	90	0.0000	8.2500	0.0000	9.3500	0.0000	8.8000
90	95	0.0000	6.8750	0.0000	7.7000	0.0000	7.7000
90	100	0.0000	5.5000	0.0000	6.0500	0.0000	6.6000
90	105	5.5000	11.0000	5.5000	12.3750	5.5000	12.1000
90	110	11.0000	16.5000	11.0000	18.7000	11.0000	17.6000
95	90	0.0000	6.5000	0.0000	7.4250	0.0000	6.7000
95	95	0.0000	4.1250	0.0000	4.6750	0.0000	4.4000
95	100	0.0000	2.7500	0.0000	3.0250	0.0000	3.3000
95	105	5.5000	8.2500	5.5000	9.3500	5.5000	8.8000
95	110	11.0000	13.7500	11.0000	15.6750	11.0000	14.3000
100	90	0.0000	8.2500	0.0000	9.3500	0.0000	8.8000
100	95	0.0000	4.1250	0.0000	4.6750	0.0000	4.4000
100	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100	105	5.5000	5.5000	5.5000	6.3250	5.5000	5.5000
100	110	11.0000	11.0000	11.0000	12.6500	11.0000	11.0000
105	90	5.5000	11.0000	5.5000	12.3750	5.5000	12.1000
105	95	5.5000	6.8750	5.5000	7.7000	5.5000	7.7000
105	100	5.5000	5.5000	5.5000	6.0500	5.5000	6.6000
105	105	0.0000	8.2500	0.0000	9.3500	0.0000	8.8000
105	110	5.5000	13.0000	5.5000	14.8500	5.5000	13.4000
110	90	11.0000	13.7500	11.0000	15.4000	11.0000	15.4000
110	95	11.0000	12.3750	11.0000	13.7500	11.0000	14.3000
110	100	11.0000	11.0000	11.0000	12.1000	11.0000	13.2000
110	105	5.5000	13.7500	5.5000	15.4000	5.5000	15.4000
110	110	0.0000	16.5000	0.0000	18.7000	0.0000	17.6000
Result		The contraction inequality (4.1) holds for all 25 pairs					

6. Discussion

The framework proposed in this study extends the classical fixed-point approach by directly incorporating perceptual deviations into the distance structure while preserving the mathematical soundness of the convergence analysis. Unlike classical contraction models defined in standard metric spaces, the proposed approach distinguishes the exact metric, which represents objective differences between forecasts, and the perturbation component, which represents how investors perceive these differences. Thus, behavioral and institutional effects can be addressed not merely as assumptions embedded within the update operator, but directly as part of the mathematical model's distance structure.

The four examples presented in this study demonstrate that the proposed contraction framework is applicable to a wide range of expectation updating dynamics. The framework can accommodate not only linear update mechanisms but also nonlinear update rules, nonlinear perturbation functions, and discrete forecast spaces. In particular, Examples 2 and 3 show that the theoretically derived admissible parameter regions are based on sufficient conditions and are therefore conservative in nature. The fact that certain parameter combinations outside the analytical region also satisfy the contraction inequality highlights that deriving sharper sufficient conditions, or even necessary and sufficient conditions, remains an important direction for future research.

From a financial perspective, the model can interpret individual learning inertia and institutional update friction jointly within a single convergence framework. These parameters not only guarantee contraction mathematically but also explain the persistence of expectation errors and the rate at which market expectations converge toward equilibrium. Thus, the proposed framework provides a mathematically consistent connection between generalized fixed-point theory and financial expectation dynamics inspired by behavioral finance.

However, the study has some inherent limitations. The model is deterministic and addresses learning processes using a single equilibrium point. In the future, stochastic learning models, multi-agent interaction structures, time-dependent perturbation functions, and empirical calibration studies using real financial data could further expand the scope of application of the proposed framework.

7. Conclusions

In this study, a generalized fixed-point framework for financial expectation dynamics on perturbed metric spaces is proposed. Behavioral learning inertia, institutional update friction, and perceptual biases are unified under a single contraction condition; under this framework, the existence and uniqueness of the equilibrium expectation, as well as the convergence of the learning process, are proven under explicit parameter conditions.

The theoretical results are supported by examples covering linear, nonlinear, and discrete expectation-updating mechanisms. These examples demonstrate that the proposed approach extends classical contraction models to a more general framework and enables analysis of behavioral and institutional mechanisms within a single mathematical structure. In conclusion, this study demonstrates that perturbed metric spaces are not merely an abstract generalization; they also provide a natural and effective mathematical framework for analyzing financial learning processes that can model behavioral and institutional mechanisms.

Use of Generative-AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that they have no conflict of interest.

References

1. N. G. Mankiw, R. Reis, Sticky information versus sticky prices: A proposal to replace the new Keynesian Phillips curve, *Q. J. Econ.*, **117** (2002), 1295–1328. https://doi.org/10.1162/003355302320935034
2. C. A. Sims, Implications of rational inattention, *J. Monetary Econ.*, **50** (2003), 665–690. https://doi.org/10.1016/S0304-3932(03)00029-1
3. G. W. Evans, S. Honkapohja, *Learning and expectations in macroeconomics*, Princeton: Princeton University Press, 2001.
4. L. P. Hansen, T. J. Sargent, *Robustness*, Princeton: Princeton University Press, 2007.
5. A. Tversky, D. Kahneman, Judgment under uncertainty: Heuristics and biases, *Science*, **185** (1974), 1124–1131. https://doi.org/10.1126/science.185.4157.1124
6. N. Barberis, A. Shleifer, R. Vishny, A model of investor sentiment, *J. Financ. Econ.*, **49** (1998), 307–343. https://doi.org/10.1016/S0304-405X(98)00027-0
7. H. A. Simon, A behavioral model of rational choice, *Q. J. Econ.*, **69** (1955), 99–118. https://doi.org/10.2307/1884852
8. A. Shleifer, *Inefficient markets: An introduction to behavioral finance*, Oxford: Oxford University Press, 2000.
9. N. Barberis, R. H. Thaler, Behavioral finance, In: *Handbook of the economics of finance*, Amsterdam: Elsevier, 2003, 1053–1128.
10. D. Hirshleifer, Behavioral finance, *Annu. Rev. Financ. Econ.*, **7** (2015), 133–159. https://doi.org/10.1146/annurev-financial-092214-043752
11. M. Jleli, B. Samet, On Banach's fixed point theorem in perturbed metric spaces, *J. Appl. Anal. Comput.*, **15** (2025), 993–1001. https://doi.org/10.11948/20240242
12. G. Alsahli, E. Karapınar, P. Shahi, On rational contractions in perturbed metric spaces, *Res. Math.*, **12** (2025), 2508578. https://doi.org/10.1080/27684830.2025.2508578
13. G. Alsahli, P. Shahi, E. Karapınar, On perturbed-S_τ contractions, *AIMS Mathematics*, **10** (2025), 11976–11985. https://doi.org/10.3934/math.2025541
14. E. Karapınar, C. M. Păcurar, P. Shahi, Interpolative contractions on perturbed metric spaces, *An. Univ. Ovidius Constanta Ser. Mat.*, **34** (2026), 169–181. https://doi.org/10.2478/auom-2026-0009
15. A. Moussaoui, N. Taş, New perturbed parametric metric spaces for distance modeling: Fixed-point analysis with applications to the fixed-circle problem, *Fix. Point Theory Alg. Sci. Eng.*, **2026** (2026), 15. https://doi.org/10.1186/s13663-026-00829-5
16. C. Yalçın, Perturbed statistical convergence, *Results Nonlinear Anal.*, **8** (2025), 305–313. https://doi.org/10.31838/rna/2025.08.02.023

17. S. G. Georgiev, K. Zennir, *Multiple fixed-point theorems and applications in the theory of ODEs, FDEs and PDEs*, New York: Chapman & Hall/CRC Press, 2020. https://doi.org/10.1201/9781003028727

18. E. Karapınar, R. P. Agarwal, *Fixed point theory in generalized metric spaces*, Cham: Springer, 2022. https://doi.org/10.1007/978-3-031-14969-6

19. K. Mebarki, S. G. Georgiev, S. Djebali, K. Zennir, *Fixed point theorems with applications*, New York: Chapman & Hall/CRC Press, 2023. https://doi.org/10.1201/9781003381969

20. Ş. Bilazeroglu, A behavioral perspective on price convergence via perturbed metric spaces with an extended contraction, *Turk. J. Math. Comput. Sci.*, **18** (2026), 32–45. https://doi.org/10.47000/tjmcs.1761322

Appendix

Example 1: Detailed numerical verification

This appendix provides the complete numerical verification of the contractive inequality (4.1) for the linear updating rule introduced in Example 5.1 under all parameter configurations A1–A3 and B1–B3.

Throughout the computations, the perturbed distance is given by

$$\mathcal{D}(p,q) = (1+\kappa)|p-q|, \qquad \kappa = 0.1,$$

and the linear operator takes the form

$$\mathcal{T}(p) = \mu p + (1-\mu)100,$$

where the effective adjustment coefficient μ is identified either with λ (investor-focused setting) or with θ (institution-focused setting).

For each ordered pair $(p,q) \in \mathcal{P} \times \mathcal{P}$, with $\mathcal{P} = \{90, 95, 100, 105, 110\}$, we compute

$$\text{LHS} = \mathcal{D}(\mathcal{T}p, \mathcal{T}q),$$

and

$$\text{RHS} = \lambda \max\left\{\mathcal{D}(p,q),\ \mathcal{D}(\mathcal{T}p,p),\ \frac{d_0(p,\mathcal{T}q) + d_0(q,\mathcal{T}p)}{2}\right\} + \theta\, \mathcal{D}(\mathcal{T}q,q).$$

The following tables report the numerical values of the left-hand and right-hand sides of (4.1) for all 25 ordered pairs and for each parameter configuration considered in Example 5.1.

Table 6. The LHS and RHS values in the inequality (4.1) for Cases A1–A3.

		Case A1		Case A2		Case A3	
p	q	LHS	RHS	LHS	RHS	LHS	RHS
90	90	0.0000	1.9800	0.0000	2.8875	0.0000	3.3000
90	95	0.5500	1.4850	1.3750	2.4750	2.2000	2.9700
90	100	1.1000	1.1000	2.7500	2.7500	4.4000	4.4000
90	105	1.6500	2.1450	4.1250	4.5375	6.6000	6.9300
90	110	2.2000	3.1900	5.5000	6.3250	8.8000	9.4600
95	90	0.5500	1.6650	1.3750	2.2313	2.2000	2.8600
95	95	0.0000	0.9900	0.0000	1.4438	0.0000	1.6500
95	100	0.5500	0.5500	1.3750	1.3750	2.2000	2.2000
95	105	1.1000	1.5950	2.7500	3.1625	4.4000	4.6200
95	110	1.6500	2.6400	4.1250	4.9500	6.6000	7.1500
100	90	1.1000	1.8700	2.7500	2.8875	4.4000	4.8000
100	95	0.5500	0.8250	1.3750	1.4438	2.2000	2.3100
100	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100	105	0.5500	0.8250	1.3750	1.4438	2.2000	2.3100
100	110	1.1000	1.8700	2.7500	2.8875	4.4000	4.8000
105	90	1.6500	2.5300	4.1250	4.6750	6.6000	7.1500
105	95	1.1000	1.4850	2.7500	3.1625	4.4000	4.6200
105	100	0.5500	0.5500	1.3750	1.3750	2.2000	2.2000
105	105	0.0000	0.9900	0.0000	1.4438	0.0000	1.6500
105	110	0.5500	1.6650	1.3750	2.2313	2.2000	2.8600
110	90	2.2000	3.1900	5.5000	6.3250	8.8000	9.4600
110	95	1.6500	2.1450	4.1250	4.5375	6.6000	6.9300
110	100	1.1000	1.1000	2.7500	2.7500	4.4000	4.4000
110	105	0.5500	1.4850	1.3750	2.4750	2.2000	2.9700
110	110	0.0000	1.9800	0.0000	2.8875	0.0000	3.3000
Result		The contraction inequality (4.1) holds for all 25 pairs					

Table 7. The LHS and RHS values in the inequality (4.1) for Cases B1–B3.

		Case B1		Case B2		Case B3	
p	q	LHS	RHS	LHS	RHS	LHS	RHS
90	90	0.0000	7.9200	0.0000	7.2188	0.0000	6.6000
90	95	0.5500	5.9400	1.3750	6.1875	2.2000	5.9400
90	100	1.1000	4.4000	2.7500	5.5000	4.4000	6.6000
90	105	1.6500	8.5800	4.1250	9.0750	6.6000	9.9000
90	110	2.2000	12.7600	5.5000	12.6500	8.8000	13.2000
95	90	0.5500	6.6600	1.3750	5.9719	2.2000	5.7200
95	95	0.0000	3.9600	0.0000	3.6094	0.0000	3.3000
95	100	0.5500	2.2000	1.3750	2.7500	2.2000	3.3000
95	105	1.1000	6.3800	2.7500	6.6000	4.4000	6.6000
95	110	1.6500	10.5600	4.1250	10.1750	6.6000	9.9000
100	90	1.1000	7.4800	2.7500	7.2188	4.4000	7.2000
100	95	0.5500	3.3000	1.3750	3.6094	2.2000	3.3000
100	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100	105	0.5500	3.3000	1.3750	3.6094	2.2000	3.3000
100	110	1.1000	7.4800	2.7500	7.2188	4.4000	7.2000
105	90	1.6500	10.1200	4.1250	9.5156	6.6000	9.9000
105	95	1.1000	5.9400	2.7500	6.1875	4.4000	6.6000
105	100	0.5500	2.2000	1.3750	2.7500	2.2000	3.3000
105	105	0.0000	3.9600	0.0000	3.6094	0.0000	3.3000
105	110	0.5500	6.6600	1.3750	5.9719	2.2000	5.7200
110	90	2.2000	12.7600	5.5000	12.6500	8.8000	13.2000
110	95	1.6500	8.5800	4.1250	9.0750	6.6000	9.9000
110	100	1.1000	4.4000	2.7500	5.5000	4.4000	6.6000
110	105	0.5500	5.9400	1.3750	6.1875	2.2000	5.9400
110	110	0.0000	7.9200	0.0000	7.2188	0.0000	6.6000
Result		The contraction inequality (4.1) holds for all 25 pairs					

In configurations A1–A3 and B1–B3, inequality (4.1) holds for all 25 ordered pairs in $\mathcal{P} \times \mathcal{P}$. This confirms numerically that the sufficient contractive condition of Theorem 4.1 is satisfied for the linear benchmark model.

Example 2: Detailed numerical verification

This appendix provides the complete numerical verification of the contractive inequality (4.1) for the nonlinear operator defined in (5.4) under the parameter configurations C1–C3 reported in Table 3.

Throughout the computations, we use the perturbed distance

$$\mathcal{D}(p, q) = (1 + \kappa)|p - q|, \qquad \kappa = 0.1,$$

and the nonlinear operator

$$\mathcal{T}_{\lambda,\theta}(p) = (1 - \lambda - \theta)p + (\lambda + \theta)100 + 20\theta \arctan(0.1(p - 100)).$$

For each pair $(p, q) \in \mathcal{P} \times \mathcal{P}$, with $\mathcal{P} = \{90, 95, 100, 105, 110\}$, we compute

$$\text{LHS} = \mathcal{D}(\mathcal{T}_{\lambda,\theta}(p), \mathcal{T}_{\lambda,\theta}(q)),$$

and

$$\text{RHS} = \lambda \max\left\{\mathcal{D}(p, q),\ \mathcal{D}(\mathcal{T} p, p),\ \frac{d_0(p, \mathcal{T} q) + d_0(q, \mathcal{T} p)}{2}\right\} + \theta\, \mathcal{D}(\mathcal{T} q, q).$$

The following table reports the numerical values of the left-hand and right-hand sides for all 25 ordered pairs and for all admissible parameter configurations.

Table 8. The LHS and RHS values in the inequality (4.1) for Cases C1–C3.

		Case C1		Case C2		Case C3	
p	q	LHS	RHS	LHS	RHS	LHS	RHS
90	90	0.0000	5.0031	0.0000	5.0101	0.0000	4.1805
90	95	2.3986	4.2245	2.2407	4.3583	2.3579	3.8663
90	100	5.7336	7.1500	5.1058	7.1500	5.0279	6.6000
90	105	9.0687	11.3745	7.9708	11.2520	7.6979	10.1830
90	110	11.4673	15.8799	10.2115	15.4788	10.0558	13.7972
95	90	2.3986	5.1549	2.2407	5.0872	2.3579	4.1357
95	95	0.0000	2.0567	0.0000	2.0539	0.0000	1.4180
95	100	3.3351	3.5750	2.8650	3.5750	2.6700	3.3000
95	105	6.6701	7.7995	5.7301	7.8467	5.3400	6.6820
95	110	9.0687	12.3049	7.9708	12.0820	7.6979	10.9357
100	90	5.7336	9.3031	5.1058	9.1847	5.0279	8.1195
100	95	3.3351	4.3750	2.8650	4.3750	2.6700	3.8500
100	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100	105	3.3351	4.3750	2.8650	4.3750	2.6700	3.8500
100	110	5.7336	9.3031	5.1058	9.1847	5.0279	8.1195
105	90	9.0687	12.3049	7.9708	12.0820	7.6979	10.9357
105	95	6.6701	7.7995	5.7301	7.8467	5.3400	6.6820
105	100	3.3351	3.5750	2.8650	3.5750	2.6700	3.3000
105	105	0.0000	2.0567	0.0000	2.0539	0.0000	1.4180
105	110	2.3986	5.1549	2.2407	5.0872	2.3579	4.1357
110	90	11.4673	15.8799	10.2115	15.4788	10.0558	13.7972
110	95	9.0687	11.3745	7.9708	11.2520	7.6979	10.1830
110	100	5.7336	7.1500	5.1058	7.1500	5.0279	6.6000
110	105	2.3986	4.2245	2.2407	4.3583	2.3579	3.8663
110	110	0.0000	5.0031	0.0000	5.0101	0.0000	4.1805
Result		The contraction inequality (4.1) holds for all 25 pairs					

As can be seen from Table 8, the inequality (4.1) holds for all 25 pairs in each parameter configuration. This confirms numerically that the sufficient contractive condition derived in Section 5 is satisfied for the selected admissible cases.

Example 3: Detailed numerical verification

This appendix provides the complete numerical verification of the contractive inequality (4.1) for the nonlinear operator used in Example 5.3 under the parameter configurations C1–C3 reported in Table 3.

In contrast to Example 5.2, we now use the nonlinear perturbation

$$\phi(p,q) = 0.5\left|\sin(0.1(p-q))\right|,$$

together with the exact metric

$$d_0(p,q) = |p-q|.$$

Accordingly, the perturbed distance is defined by

$$\mathcal{D}(p,q) = d_0(p,q) + \phi(p,q) = |p-q| + 0.5\left|\sin(0.1(p-q))\right|.$$

The nonlinear operator is the same as in Example 5.2:

$$\mathcal{T}_{\lambda,\theta}(p) = (1-\lambda-\theta)p + (\lambda+\theta)100 + 20\theta\,\arctan(0.1(p-100)).$$

For each pair $(p,q) \in \mathcal{P}\times\mathcal{P}$, where

$$\mathcal{P} = \{90, 95, 100, 105, 110\},$$

we compute

$$\text{LHS} = \mathcal{D}\left(\mathcal{T}_{\lambda,\theta}(p), \mathcal{T}_{\lambda,\theta}(q)\right),$$

and

$$\text{RHS} = \lambda \max\left\{\mathcal{D}(p,q),\ \mathcal{D}(\mathcal{T}p,p),\ \frac{d_0(p,\mathcal{T}q) + d_0(q,\mathcal{T}p)}{2}\right\} + \theta\,\mathcal{D}(\mathcal{T}q,q).$$

The following table reports the numerical values of the left-hand and right-hand sides for all 25 ordered pairs and for all three parameter configurations.

Table 9. The LHS and RHS values in the inequality (4.1) for Cases C1–C3 under the nonlinear perturbation.

		Case C1		Case C2		Case C3	
p	q	LHS	RHS	LHS	RHS	LHS	RHS
90	90	0.0000	4.7671	0.0000	4.7716	0.0000	3.9813
90	95	2.2887	4.0256	2.1381	4.1517	2.2499	3.6825
90	100	5.4614	6.7735	4.8654	6.7735	4.7915	6.2524
90	105	8.6114	10.6940	7.5776	10.5770	7.3201	9.5692
90	110	10.8566	14.8009	9.6835	14.4183	9.5376	12.8415
95	90	2.2887	4.9112	2.1381	4.5286	2.2499	3.7126
95	95	0.0000	1.9626	0.0000	2.1369	0.0000	1.8899
95	100	3.1812	3.4058	2.7334	3.4058	2.5475	3.1438
95	105	6.3487	7.3932	5.4580	7.2763	5.0879	6.5224
95	110	8.6114	11.5796	7.5776	11.1969	7.3201	9.8680
100	90	5.4614	8.2789	4.8654	7.8962	4.7915	6.8212
100	95	3.1812	4.0256	2.7334	3.9086	2.5475	3.4138
100	100	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100	105	3.1812	4.0256	2.7334	3.9086	2.5475	3.4138
100	110	5.4614	8.2789	4.8654	7.8962	4.7915	6.8212
105	90	8.6114	11.5796	7.5776	11.1969	7.3201	9.8680
105	95	6.3487	7.3932	5.4580	7.2763	5.0879	6.5224
105	100	3.1812	3.4058	2.7334	3.4058	2.5475	3.1438
105	105	0.0000	1.9626	0.0000	2.1369	0.0000	1.8899
105	110	2.2887	4.9112	2.1381	4.5286	2.2499	3.7126
110	90	10.8566	14.8009	9.6835	14.4183	9.5376	12.8415
110	95	8.6114	10.6940	7.5776	10.5770	7.3201	9.5692
110	100	5.4614	6.7735	4.8654	6.7735	4.7915	6.2524
110	105	2.2887	4.0256	2.1381	4.1517	2.2499	3.6825
110	110	0.0000	4.7671	0.0000	4.7716	0.0000	3.9813
Result		The contraction inequality (4.1) holds for all 25 pairs					

As shown in Table 9, inequality (4.1) holds for all 25 ordered pairs under each parameter configuration. Therefore, the sufficient contractive condition of Theorem 4.1 remains valid when the linear perturbation is replaced by the nonlinear and non-uniform perturbation

$$\phi(p,q) = 0.5\left|\sin(0.1(p-q))\right|.$$

This numerical verification confirms that the applicability of the proposed framework is not restricted to perturbations that merely rescale the exact metric.