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Research article

CR and anti-CR rigidity of conformal mappings on two classes of rank-two Carnot groups

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Abstract: We studied conformal mappings on Carnot groups of type $(2, 1, \ldots, 1)$ and type $(2, 1, \ldots, 1, 2)$. First, for each class, we constructed an explicit CR submanifold M_0 and a diffeomorphism $T : M_0 \to G$ that is a CR diffeomorphism and a Carnot–Carathéodory isometry. Second, for each conformal mapping f on a connected domain in G, we prove that the mapping $T^{-1} \circ f \circ T$ is either CR or anti-CR on M_0.

Keywords: Carnot groups; quasiconformal mappings; conformal mappings; CR mappings
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1. Introduction

It is well-known that a conformal mapping on a domain in the complex plane is either holomorphic or anti-holomorphic. In Euclidean spaces of dimension at least three, Liouville's theorem shows that conformal mappings are much more rigid. They are generated by translations, rotations, dilations, and inversions. A similar rigidity phenomenon appears in Carnot groups. The Heisenberg group is the basic example of a Carnot group with a CR structure. Korányi and Reimann [10, 11] studied conformal and quasiconformal mappings on the Heisenberg group and related them to CR geometry through its realization as a quadratic hypersurface in complex Euclidean space. Later, Wu and Wang [13] obtained an analogous result for the Engel group by using a suitable CR realization. These works show that conformal mappings on certain Carnot groups are closely related to CR and anti-CR mappings.

In this paper, we study conformal mappings on two classes of rank-two Carnot groups: Carnot groups of type $(2, 1, \ldots, 1)$ and Carnot groups of type $(2, 1, \ldots, 1, 2)$. The first main result of the paper is an explicit CR realization of these Carnot groups. For each class, we construct an embedded CR submanifold $M_0 \subset \mathbb{C}^N$ and an explicit diffeomorphism $T : M_0 \to G$. We prove that T is a CR diffeomorphism and a Carnot–Carathéodory isometry. More precisely, we give concrete defining equations for M_0, an explicit formula for T, and a direct verification of the key identity $T_*\widetilde{Z}_1 = Z_1$, where $\widetilde{Z}_1$ generates the CR bundle $T^{1,0}M_0$ and Z_1 is the corresponding complex horizontal vector field on G. This identity shows that the CR structure on M_0 is transported by T to the complex horizontal structure of G. Since T also preserves the orthonormal horizontal frames, it preserves the associated Carnot–Carathéodory distance. This result extends the CR-geometric viewpoint known for the Heisenberg and Engel groups. In the first family, the case $n = 2$ recovers the classical Heisenberg-type quadratic CR model. The case $n = 3$ corresponds to the Engel group and is related to the CR model used by Wu and Wang.

The second main result of the paper is a CR-geometric classification of conformal transformations on these groups. More precisely, after a conformal mapping on G is transferred to the corresponding CR model by T, the induced mapping is either CR or anti-CR. Let $\Omega \subset G$ be a connected domain, and let $f : \Omega \to f(\Omega) \subset G$ be a conformal mapping with respect to the Carnot–Carathéodory distance. Then $\widehat{f} = T^{-1} \circ f \circ T$ is either a CR mapping or an anti-CR mapping on $\widehat{\Omega} = T^{-1}(\Omega) \subset M_0$.

The paper is organized as follows: Section 2 recalls the basic notions concerning Carnot groups, Carnot–Carathéodory distances, CR manifolds, conformal mappings and Pansu differentiability. Section 3 constructs the CR models for the two classes of Carnot groups and proves that the corresponding maps $T : M_0 \to G$ are CR diffeomorphisms and Carnot–Carathéodory isometries. Section 4 proves the CR/anti-CR classification theorem for conformal mappings on the corresponding CR models.

2. Preliminaries

In this section, we recall the basic notions of Carnot groups, Carnot–Carathéodory distances, CR manifolds, CR mappings, conformal mappings, and Pansu differentiability.

Let $(G, *)$ be a connected and simply connected nilpotent Lie group, and let $\mathfrak{g}$ be its Lie algebra. We say that G is a Carnot group of step r if $\mathfrak{g}$ admits a stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r$ such that

$$[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}, \quad 1 \le j \le r-1, \qquad [\mathfrak{g}_1, \mathfrak{g}_r] = \{0\}.$$

The first layer $\mathfrak{g}_1$ is called the horizontal layer.

A Carnot group is also called a stratified nilpotent Lie group, as it plays in sub-Riemannian geometry a role analogous to Euclidean spaces in Riemannian geometry [1]. If $\dim \mathfrak{g}_j = k_j$, then G is said to have type $(k_1, k_2, \ldots, k_r)$. The exponential map $\exp : \mathfrak{g} \to G$ is a global diffeomorphism. Since G is connected, simply connected, and nilpotent, the exponential map is a global diffeomorphism. We use exponential coordinates of the first kind when discussing dilations and Pansu differentiability. For the explicit group laws in Section 3, however, we use adapted coordinates. These coordinates should not be confused with exponential coordinates of the first kind. For $p \in G$, let $\tau_p : G \to G, \tau_p(q) = p * q$, be the left translation. The horizontal distribution of G is the left-invariant distribution $H_pG = (\tau_p)_{*e}\mathfrak{g}_1, p \in G$, where e is the identity element of G. Thus $\mathfrak{g}_1 \subset T_eG, H_pG \subset T_pG$. Since $\mathfrak{g}$ is generated by $\mathfrak{g}_1$, the distribution HG is bracket generating. The Heisenberg group is the step-two example of a Carnot group; the first Heisenberg group $\mathbb{H}^1$ has type $(2, 1)$. The Engel group is the simplest step-three example with two-dimensional horizontal layer, and its type is $(2, 1, 1)$.

In this paper, we consider two concrete classes of Carnot groups with a two-dimensional horizontal layer. These two classes of Carnot groups were also studied in [8].

The first class consists of Carnot groups of type $(2, 1, \dots, 1)$. More precisely, the Lie algebra has a basis $X_1, X_2, \dots, X_{n+1}$ and a stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n$, where

$$\mathfrak{g}_1 = \operatorname{span}\{X_1, X_2\}, \qquad \mathfrak{g}_j = \operatorname{span}\{X_{j+1}\}, \quad 2 \le j \le n.$$

The nonzero Lie brackets are

$$[X_1, X_j] = X_{j+1}, \qquad 2 \le j \le n. \tag{2.1}$$

Thus the whole Lie algebra is generated by the horizontal layer $\mathfrak{g}_1 = \operatorname{span}\{X_1, X_2\}$. Schematically, the generation is

$$\operatorname{span}\{X_1, X_2\} \xrightarrow{[X_1,\cdot]} \operatorname{span}\{X_3\} \xrightarrow{[X_1,\cdot]} \operatorname{span}\{X_4\} \xrightarrow{[X_1,\cdot]} \cdots \xrightarrow{[X_1,\cdot]} \operatorname{span}\{X_{n+1}\}.$$

The second class consists of Carnot groups of type $(2, 1, \dots, 1, 2)$. Let $r \ge 3$. The Lie algebra has a stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r$, where

$$\mathfrak{g}_1 = \operatorname{span}\{X_1, X_2\}, \qquad \mathfrak{g}_j = \operatorname{span}\{X_{j+1}\}, \quad 2 \le j \le r-1, \qquad \mathfrak{g}_r = \operatorname{span}\{X_{r+1}, X_{r+2}\}.$$

The basic generating brackets are

$$[X_1, X_j] = X_{j+1}, \qquad 2 \le j \le r, \qquad [X_2, X_r] = X_{r+2}. \tag{2.2}$$

Schematically, the generation is

$$\operatorname{span}\{X_1, X_2\} \xrightarrow{[X_1,\cdot]} \operatorname{span}\{X_3\} \xrightarrow{[X_1,\cdot]} \cdots \xrightarrow{[X_1,\cdot]} \operatorname{span}\{X_r\} \xrightarrow{[X_1,X_r],\,[X_2,X_r]} \operatorname{span}\{X_{r+1}, X_{r+2}\}.$$

For the type $(2, 1, \dots, 1, 2)$, the Jacobi identity yields the additional bracket relations

$$[X_i, X_j] = (-1)^i X_{r+2}, \qquad i + j = r + 2, \qquad 2 \le i, j \le r. \tag{2.3}$$

These relations are compatible with the skew-symmetry of the Lie bracket only when r is odd. Indeed, skew-symmetry gives

$$[X_i, X_j] = -[X_j, X_i].$$

Using the bracket relations,

$$[X_i, X_j] = (-1)^i X_{r+2}, \qquad [X_j, X_i] = (-1)^j X_{r+2}.$$

Therefore $(-1)^i = -(-1)^j$. Since $i + j = r + 2$, this implies r must be odd.

A locally absolutely continuous curve $\gamma : [a, b] \to G$ is called horizontal if $\gamma'(t) \in H_{\gamma(t)}G$ for almost every $t \in [a, b]$. By the Chow–Rashevskii theorem in [5], we know that if $\mathfrak{g}_1$ is bracket generating and G is connected, then any two points of G can be joined by a horizontal curve.

Choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}_1$. The length of a horizontal curve γ is

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\, dt,$$

where $|\gamma'(t)| = \langle \gamma'(t), \gamma'(t) \rangle^{\frac{1}{2}}$. The associated Carnot–Carathéodory distance is

$$d_{cc}(p, q) = \inf\{\ell(\gamma) : \gamma \text{ is horizontal and joins } p \text{ to } q\}.$$

By the Chow–Rashevskii theorem [5], $d_{cc}(p, q) < \infty$ for all $p, q \in G$. The stratification also gives a family of nonisotropic dilations. If

$$Y = Y_1 + Y_2 + \cdots + Y_r, \qquad Y_j \in \mathfrak{g}_j,$$

then

$$\delta_\lambda(\exp Y) = \exp\left(\lambda Y_1 + \lambda^2 Y_2 + \cdots + \lambda^r Y_r\right), \qquad \lambda > 0.$$

Each δ_λ is a group automorphism of G.

Definition 2.1. *A CR manifold is a smooth manifold M equipped with a complex subbundle $T_{1,0}M \subset \mathbb{C}TM := TM \otimes_{\mathbb{R}} \mathbb{C}$ such that $T_{1,0}M \cap \overline{T_{1,0}M} = \{0\}$, and $[Z_1, Z_2] \in C^\infty(M, T_{1,0}M)$ whenever $Z_1, Z_2 \in C^\infty(M, T_{1,0}M)$.*

The integer $\dim_{\mathbb{C}} T_{1,0}M$ is called the CR dimension of M. A smooth section of $T_{1,0}M$ is called a complex tangent vector field. We also write $T_{0,1}M := \overline{T_{1,0}M}$.

Let $f : (M_1, T_{1,0}M_1) \to (M_2, T_{1,0}M_2)$ be a C^1 mapping between CR manifolds. We say that f is a Cauchy–Riemann mapping, or simply a CR mapping, if $f_*(T_{1,0}M_1) \subset T_{1,0}M_2$. We say that f is an anti-CR mapping if $f_*(T_{1,0}M_1) \subset T_{0,1}M_2$. If f is invertible and both f and f^{-1} are CR mappings, then f is called a CR diffeomorphism.

In the embedded case, if $M \subset \mathbb{C}^N$ is a smooth real submanifold, then its standard CR bundle is

$$T_{1,0}M = \mathbb{C}TM \cap T_{1,0}\mathbb{C}^N.$$

If M is locally defined by smooth real-valued functions $\rho_1 = \cdots = \rho_\ell = 0$ with $d\rho_1, \ldots, d\rho_\ell$ linearly independent on M, then a $(1, 0)$ vector field

$$Z = \sum_{\alpha=1}^{N} a_\alpha \frac{\partial}{\partial \zeta_\alpha}$$

is tangent to M if and only if $Z\rho_j = 0$, $j = 1, \ldots, \ell$.

If $F = (F_1, \ldots, F_{N'})$ is a mapping from a CR manifold M into an embedded CR manifold $M' \subset \mathbb{C}^{N'}$, then F is CR if and only if $\overline{Z}F_\alpha = 0$ for every local complex tangent vector field $Z \in C^\infty(M, T_{1,0}M)$ and for all $\alpha = 1, \ldots, N'$. Similarly, F is anti-CR if and only if $ZF_\alpha = 0$ for every local complex tangent vector field $Z \in C^\infty(M, T_{1,0}M)$ and for all $\alpha = 1, \ldots, N'$.

Let $\Omega, \Omega' \subset G$ be domains, and let $f : \Omega \to \Omega'$ be a homeomorphism. For $p \in \Omega$, define

$$L_f(p, r) = \sup\{d_{cc}(f(p), f(q)) : q \in \Omega,\ d_{cc}(p, q) = r\},$$

and

$$l_f(p, r) = \inf\{d_{cc}(f(p), f(q)) : q \in \Omega,\ d_{cc}(p, q) = r\}.$$

The pointwise distortion of f at p is

$$H_f(p) = \limsup_{r\to 0^+} \frac{L_f(p,r)}{l_f(p,r)}.$$

We say that f is quasiconformal if H_f is uniformly bounded on Ω. More precisely, f is called K-quasiconformal if $H_f(p) \le K$ for almost every $p \in \Omega$. A 1-quasiconformal mapping is called a conformal mapping. Equivalently, f is conformal if $H_f(p) = 1$ for almost every $p \in \Omega$. Heinonen and Koskela studied metric definitions of quasiconformality and quasiconformal mappings in metric spaces with controlled geometry [6, 7]. Modulus methods play an important role in the study of quasiconformal mappings on Carnot groups. Balogh, Fässler and Platis used such methods to study extremal quasiconformal mappings and radial stretch maps in the Heisenberg group [2, 3]. For related results on quasiconformal and quasisymmetric mappings under inner similarity metrics in $\mathbb{R}^n$, see [9].

Definition 2.2. *The mapping f is called Pansu differentiable at $p \in \Omega$ if, as $c \to 0^+$, the mappings*

$$\delta_{1/c} * \tau_{f(p)}^{-1} * f * \tau_p * \delta_c$$

converge locally uniformly to a group homomorphism $D_P f(p) : G \to G$. The homomorphism $D_P f(p)$ is called the Pansu differential, or P-differential, of f at p.

The notion of Pansu differentiability was introduced in [12]. By definition, if $D_P f(p)$ exists, then it commutes with the dilations, $D_P f(p) * \delta_c = \delta_c * D_P f(p)$. Thus $D_P f(p)$ is a homogeneous group homomorphism. We denote by $D_P f_*(p) : \mathfrak{g} \to \mathfrak{g}$ the induced Lie algebra homomorphism. It is characterized by

$$D_P f(p)(\exp X) = \exp(D_P f_*(p)X), \qquad X \in \mathfrak{g}.$$

Since $D_P f(p)$ is homogeneous, the induced homomorphism $D_P f_*(p)$ is grading-preserving [1], namely

$$D_P f_*(p)(\mathfrak{g}_j) \subset \mathfrak{g}_j, \qquad j = 1, \dots, r.$$

If $D_P f(p)$ is an isomorphism, then

$$D_P f_*(p)(\mathfrak{g}_j) = \mathfrak{g}_j, \qquad j = 1, \dots, r.$$

Quasiconformal mappings between domains in a Carnot group are P-differentiable almost everywhere [1]. Every 1-quasiconformal homeomorphism between domains in a Carnot group is smooth. Its inverse is again 1-quasiconformal and hence smooth [4].

Lemma 2.1. *Let $\Omega, \Omega' \subset G$ be domains, and $f : \Omega \to \Omega'$ be a conformal mapping. Then*

$$f_{*p}(H_p G) = H_{f(p)}G, \qquad p \in \Omega.$$

Proof. Since f is conformal, it is smooth and P-differentiable almost everywhere. We first prove

$$f_{*p}(H_p G) \subset H_{f(p)}G$$

at every P-differentiability point p. Let $X \in \mathfrak{g}_1$, and consider the horizontal curve $\gamma(t) = p * \exp(tX)$. Then $\gamma'(0) = (L_p)_{*e}X \in H_p G$. Set

$$h(t) = f(p)^{-1} * f(\gamma(t)).$$

Since f is smooth and $h(0) = e$, there exists $Y \in \mathfrak{g}$ such that $\log h(t) = tY + o(t)$ as $t \to 0$. Moreover,

$$Y = (L_{f(p)^{-1}})_{*f(p)} f_{*p}((L_p)_{*e}X).$$

Write

$$Y = Y_1 + \cdots + Y_r, \qquad Y_j \in \mathfrak{g}_j.$$

Since f is P-differentiable at p, the limit

$$\lim_{t \to 0^+} \delta_{1/t}\left(f(p)^{-1} * f(p * \exp(tX))\right)$$

exists. In exponential coordinates, the dilation $\delta_{1/t}$ multiplies the $\mathfrak{g}_j$-component by t^{-j}. Hence the logarithm of the rescaled curve contains the term $t^{1-j}Y_j + o(t^{1-j})$ in the $\mathfrak{g}_j$-component. If $Y_j \neq 0$ for some $j \geq 2$, this term is unbounded as $t \to 0^+$, contradicting the existence of the Pansu limit. Therefore $Y_j = 0$, $j = 2, \ldots, r$. Thus $Y \in \mathfrak{g}_1$, and consequently

$$f_{*p}((L_p)_{*e}X) \in H_{f(p)}G.$$

This proves $f_{*p}(H_pG) \subset H_{f(p)}G$ at almost every point. Let $E \subset \Omega$ denote the set of Pansu differentiability points of f. Since E has full measure, it is dense in Ω. Fix $p \in \Omega$ and

$$V = a_1X_1(p) + a_2X_2(p) \in H_pG.$$

Choose a sequence $p_k \in E$ such that $p_k \to p$, and set

$$V_k = a_1X_1(p_k) + a_2X_2(p_k) \in H_{p_k}G.$$

Then $V_k \to V$, and for every k, $f_{*p_k}V_k \in H_{f(p_k)}G$. Since f is smooth, $f_{*p_k}V_k \to f_{*p}V$. Since the horizontal bundle HG is a smooth subbundle of TG, the limit belongs to $H_{f(p)}G$. Hence $f_{*p}(H_pG) \subset H_{f(p)}G$ for every $p \in \Omega$. The inverse of a 1-quasiconformal homeomorphism is again 1-quasiconformal. Applying the same regularity theorem to f^{-1}, we conclude that f is a smooth diffeomorphism. Since H_pG and $H_{f(p)}G$ have the same dimension, we have

$$f_{*p}(H_pG) = H_{f(p)}G.$$

□

3. The Carnot groups and their CR realizations

In this section we give CR realizations for the two classes of Carnot groups. For each class, we construct a CR submanifold M_0 in a complex Euclidean space and define a diffeomorphism $T : M_0 \to G$. We prove that this map is both a CR diffeomorphism and a Carnot–Carathéodory isometry.

3.1. The Carnot groups of type $(2,1,\dots,1)$

We first derive the group law from the bracket relations in (2.1).

Let $\mathfrak{g}$ be the nilpotent Lie algebra with basis $X_1,\dots,X_{n+1}$, $n \geq 2$, whose only nonzero brackets between basis elements are

$$[X_1,X_j] = X_{j+1}, \qquad 2 \leq j \leq n.$$

Let G be the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, and set $\mathfrak{n} = \operatorname{span}\{X_2,\dots,X_{n+1}\}$. Define

$$\begin{aligned}\Psi : \mathbb{R}^{n+1} &\longrightarrow G,\\ (x_1,\dots,x_{n+1}) &\longmapsto \exp\left(\sum_{k=2}^{n+1} x_k X_k\right)\exp(x_1X_1).\end{aligned}$$

For $x = (x_1,\dots,x_{n+1})$ and $x' = (x'_1,\dots,x'_{n+1})$, let $v = \sum_{k=2}^{n+1} x_kX_k$, $v' = \sum_{k=2}^{n+1} x'_kX_k$. Using the conjugation identity

$$\exp(x_1X_1)\exp(v')\exp(-x_1X_1) = \exp\left(e^{x_1 \operatorname{ad} X_1}v'\right),$$

we obtain

$$\begin{aligned}\Psi(x)\Psi(x') &= \exp(v)\exp(x_1X_1)\exp(v')\exp(x'_1X_1)\\ &= \exp(v)\exp\left(e^{x_1 \operatorname{ad} X_1}v'\right)\exp((x_1+x'_1)X_1).\end{aligned}$$

Both v and $e^{x_1 \operatorname{ad} X_1}v'$ belong to the abelian algebra $\mathfrak{n}$. Hence the Baker–Campbell–Hausdorff formula reduces to ordinary addition on $\mathfrak{n}$, and therefore

$$\Psi(x)\Psi(x') = \exp\left(v + e^{x_1 \operatorname{ad} X_1}v'\right)\exp((x_1+x'_1)X_1).$$

The bracket relations imply that, for $2 \leq k \leq n+1$,

$$(\operatorname{ad} X_1)^m X_k = \begin{cases} X_{k+m}, & 0 \leq m \leq n+1-k,\\ 0, & m > n+1-k.\end{cases}$$

Consequently,

$$e^{x_1 \operatorname{ad} X_1}X_k = \sum_{m=0}^{n+1-k} \frac{x_1^m}{m!}X_{k+m},$$

and hence

$$e^{x_1 \operatorname{ad} X_1}v' = \sum_{k=2}^{n+1} x'_k \sum_{m=0}^{n+1-k}\frac{x_1^m}{m!}X_{k+m} = \sum_{j=2}^{n+1}\left(\sum_{k=2}^{j}\frac{x_1^{j-k}}{(j-k)!}\,x'_k\right)X_j.$$

It follows that

$$v + e^{x_1 \operatorname{ad} X_1}v' = \sum_{j=2}^{n+1}\left(x_j + \sum_{k=2}^{j}\frac{x_1^{j-k}}{(j-k)!}\,x'_k\right)X_j.$$

By the uniqueness of the decomposition $\exp(w)\exp(tX_1)$, the coordinates of the product satisfy

$$y_1 = x_1 + x_1', \qquad y_j = x_j + \sum_{k=2}^{j} \frac{x_1^{j-k}}{(j-k)!} x_k', \quad 2 \le j \le n+1.$$

The map Ψ is a global diffeomorphism, and the coordinates induced by Ψ are adapted coordinates of the second kind. Under the identification of G with $\mathbb{R}^{n+1}$ through Ψ, the group product $x * x' = y$ is given by

$$\begin{aligned} y_1 &= x_1 + x_1', \qquad\qquad y_2 = x_2 + x_2', \\ y_j &= x_j + x_j' + \sum_{k=2}^{j-1} \frac{x_1^{j-k}}{(j-k)!} x_k', \qquad 3 \le j \le n+1. \end{aligned} \tag{3.1}$$

From the group law (3.1), the left-invariant vector fields are

$$X_1 = \frac{\partial}{\partial x_1}, \qquad X_k = \frac{\partial}{\partial x_k} + \sum_{j=1}^{n+1-k} \frac{x_1^j}{j!} \frac{\partial}{\partial x_{k+j}}, \qquad 2 \le k \le n+1.$$

In particular,

$$X_2 = \frac{\partial}{\partial x_2} + \sum_{j=1}^{n-1} \frac{x_1^j}{j!} \frac{\partial}{\partial x_{j+2}}. \tag{3.2}$$

A direct computation gives that the only nonzero brackets are

$$[X_1, X_k] = X_{k+1}, \qquad 2 \le k \le n.$$

Hence the Lie algebra $\mathfrak{g}$ of G is stratified as $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n$, where

$$\mathfrak{g}_1 = \operatorname{span}\{X_1, X_2\}, \qquad \mathfrak{g}_j = \operatorname{span}\{X_{j+1}\}, \quad 2 \le j \le n.$$

Therefore G is an n-step Carnot group of type $(2, 1, \ldots, 1)$. The horizontal distribution is

$$H_p G = \operatorname{span}\{X_1(p), X_2(p)\}, \qquad p \in G.$$

We choose the left-invariant sub-Riemannian metric for which X_1 and X_2 are orthonormal. The associated Carnot–Carathéodory distance is denoted by d_{cc}^G. Since the horizontal layer is two-dimensional, it carries a natural complex structure after we choose the oriented orthonormal frame X_1, X_2. Define

$$Z_1 = \frac{1}{2}(X_1 - iX_2), \qquad \overline{Z}_1 = \frac{1}{2}(X_1 + iX_2).$$

We shall use the complex line $T_{1,0}G = \operatorname{span}_{\mathbb{C}}\{Z_1\}$ as the horizontal CR direction on G. Its conjugate is $T_{0,1}G = \operatorname{span}_{\mathbb{C}}\{\overline{Z}_1\}$. Let $z = x_1 + ix_2$, and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right).$$

Using (3.2), we obtain

$$Z_1 = \frac{\partial}{\partial z} - \sum_{j=1}^{n-1} \frac{i}{2j!}(\operatorname{Re} z)^j \frac{\partial}{\partial x_{j+2}}. \tag{3.3}$$

We now construct a CR submanifold $M_0 \subset \mathbb{C}^n$. Let $(z, \omega_1, \ldots, \omega_{n-1})$ be the standard coordinates on $\mathbb{C} \times \mathbb{C}^{n-1} = \mathbb{C}^n$, and write

$$z = x_1 + ix_2, \qquad \omega_k = u_k + iv_k, \qquad 1 \le k \le n-1.$$

For $1 \le k \le n-1$, define

$$\rho_k(z, \omega) = \frac{\omega_k - \overline{\omega}_k}{2i} - \sum_{j=1}^{k} \frac{\binom{k}{j}}{k+1-j} z^{k+1-j}\overline{z}^j = v_k - \sum_{j=1}^{k} \frac{\binom{k}{j}}{k+1-j} z^{k+1-j}\overline{z}^j.$$

The polynomial in the second term is real-valued. Indeed, the conjugate of $z^{k+1-j}\overline{z}^j$ is $z^j\overline{z}^{k+1-j}$, and

$$\frac{\binom{k}{j}}{k+1-j} = \frac{\binom{k}{k+1-j}}{j}.$$

Define

$$M_0 = \{(z, \omega_1, \ldots, \omega_{n-1}) \in \mathbb{C}^n : \rho_1 = \cdots = \rho_{n-1} = 0\}.$$

Since $\frac{\partial \rho_k}{\partial v_l} = \delta_{kl}$, the differentials $d\rho_1, \ldots, d\rho_{n-1}$ are linearly independent on M_0. Hence M_0 is a smooth real submanifold of $\mathbb{C}^n$. Its real dimension is $2n - (n-1) = n+1$, which agrees with the real dimension of G. On M_0, the defining equations are

$$v_k = \sum_{j=1}^{k} \frac{\binom{k}{j}}{k+1-j} z^{k+1-j}\overline{z}^j, \qquad 1 \le k \le n-1. \tag{3.4}$$

The standard CR bundle of M_0 is $\widetilde{T}_{1,0}M_0 = \mathbb{C}TM_0 \cap T_{1,0}\mathbb{C}^n$. We claim that it is generated by the vector field

$$\widetilde{Z}_1 = \frac{\partial}{\partial z} + 2i \sum_{j=1}^{n-1} ((z+\overline{z})^j - z^j) \frac{\partial}{\partial \omega_j}. \tag{3.5}$$

Compute

$$\frac{\partial \rho_k}{\partial z} = -\sum_{j=1}^{k} \binom{k}{j} z^{k-j}\overline{z}^j, \qquad \frac{\partial \rho_k}{\partial \omega_j} = \frac{\delta_{kj}}{2i}.$$

Therefore

$$\widetilde{Z}_1 \rho_k = -\sum_{j=1}^{k} \binom{k}{j} z^{k-j}\overline{z}^j + ((z+\overline{z})^k - z^k)$$

$$= -\sum_{j=1}^{k}\binom{k}{j}z^{k-j}\overline{z}^{j} + \sum_{j=1}^{k}\binom{k}{j}z^{k-j}\overline{z}^{j} = 0.$$

Thus $\widetilde{Z}_1$ is tangent to M_0. Moreover, the matrix

$$\left(\frac{\partial\rho_k}{\partial\omega_l}\right)_{1\le k,l\le n-1} = \frac{1}{2i}I_{n-1}$$

has rank $n-1$. Hence the complex tangent space has complex dimension 1. Since $\widetilde{Z}_1$ is a nonzero tangent $(1,0)$ vector field, it generates the CR bundle $\widetilde{T}_{1,0}M_0 = \operatorname{span}_{\mathbb{C}}\{\widetilde{Z}_1\}$. The conjugate vector field is

$$\overline{\widetilde{Z}}_1 = \frac{\partial}{\partial\overline{z}} - 2i\sum_{j=1}^{n-1}((z+\overline{z})^j - \overline{z}^j)\frac{\partial}{\partial\overline{\omega}_j}.$$

Define real horizontal vector fields on M_0 by $\widetilde{X}_1 = \widetilde{Z}_1 + \overline{\widetilde{Z}}_1, \widetilde{X}_2 = i(\widetilde{Z}_1 - \overline{\widetilde{Z}}_1)$. Then $\widetilde{Z}_1 = \frac{1}{2}(\widetilde{X}_1 - i\widetilde{X}_2)$. We choose the sub-Riemannian metric on M_0 for which $\widetilde{X}_1$ and $\widetilde{X}_2$ are orthonormal. The corresponding Carnot–Carathéodory distance is denoted by $d_{\mathrm{cc}}^{M_0}$.

Define $T : M_0 \to G$ by $T(z,\omega_1,\ldots,\omega_{n-1}) = (x_1,x_2,x_3,\ldots,x_{n+1})$, where $z = x_1 + ix_2$, and, for $1 \le j \le n-1$,

$$x_{j+2} = -\frac{1}{2^{j+1}j!}\operatorname{Re}\omega_j + \frac{1}{2^j(j+1)!}\operatorname{Im} z^{j+1}. \tag{3.6}$$

The inverse map is explicitly given as follows. Given $(x_1,x_2,x_3,\ldots,x_{n+1}) \in G$, set $z = x_1 + ix_2$. Then define

$$\operatorname{Re}\omega_j = -2^{j+1}j!\,x_{j+2} + \frac{2}{j+1}\operatorname{Im} z^{j+1}, \qquad 1 \le j \le n-1, \tag{3.7}$$

and define $\operatorname{Im}\omega_j$ by Eq (3.4). This defines a smooth inverse of T. Hence T is a diffeomorphism.

Proposition 3.1. *The map $T : M_0 \to G$ is a CR diffeomorphism and a Carnot–Carathéodory isometry. More precisely,*

$$T_*\widetilde{T}_{1,0}M_0 = T_{1,0}G,$$

and

$$d_{\mathrm{cc}}^{G}(T(q_1),T(q_2)) = d_{\mathrm{cc}}^{M_0}(q_1,q_2), \qquad q_1,q_2 \in M_0.$$

Proof. By the chain rule,

$$T_*\widetilde{Z}_1 = \frac{\partial}{\partial z} + \sum_{j=1}^{n-1}(\widetilde{Z}_1 x_{j+2})\frac{\partial}{\partial x_{j+2}}. \tag{3.8}$$

From (3.5) and (3.7), we have

$$\tilde{Z}_1(\operatorname{Re}\omega_j) = i\left((z+\overline{z})^j - z^j\right), \qquad \tilde{Z}_1(\operatorname{Im} z^{j+1}) = \frac{j+1}{2i}z^j.$$

Using (3.6), we obtain

$$\begin{aligned}
\widetilde{Z}_1 x_{j+2} &= -\frac{i}{2^{j+1} j!}((z+\overline{z})^j - z^j) + \frac{1}{2^j (j+1)!} \cdot \frac{j+1}{2i} z^j \\
&= -\frac{i}{2^{j+1} j!}((z+\overline{z})^j - z^j) - \frac{i}{2^{j+1} j!} z^j \\
&= -\frac{i}{2^{j+1} j!}(z+\overline{z})^j = -\frac{i}{2j!}(\operatorname{Re} z)^j.
\end{aligned}$$

Substituting this into (3.8), we get

$$T_* \widetilde{Z}_1 = \frac{\partial}{\partial z} - \sum_{j=1}^{n-1} \frac{i}{2j!} (\operatorname{Re} z)^j \frac{\partial}{\partial x_{j+2}}.$$

Comparing with (3.3), we conclude that

$$T_* \widetilde{Z}_1 = Z_1. \tag{3.9}$$

Taking complex conjugates gives

$$T_* \overline{\widetilde{Z}}_1 = \overline{Z}_1. \tag{3.10}$$

Therefore

$$T_* \widetilde{T}_{1,0} M_0 = T_* \operatorname{span}_{\mathbb{C}}\{\widetilde{Z}_1\} = \operatorname{span}_{\mathbb{C}}\{Z_1\} = T_{1,0} G.$$

Thus T is a CR diffeomorphism. Recall that

$$\widetilde{X}_1 = \widetilde{Z}_1 + \overline{\widetilde{Z}}_1, \qquad \widetilde{X}_2 = i(\widetilde{Z}_1 - \overline{\widetilde{Z}}_1),$$

and similarly,

$$X_1 = Z_1 + \overline{Z}_1, \qquad X_2 = i(Z_1 - \overline{Z}_1).$$

Using (3.9) and (3.10), we get

$$T_* \widetilde{X}_1 = X_1, \qquad T_* \widetilde{X}_2 = X_2. \tag{3.11}$$

Applying the same argument to the inverse map T^{-1} gives

$$(T^{-1})_* X_1 = \widetilde{X}_1, \qquad (T^{-1})_* X_2 = \widetilde{X}_2.$$

Hence T preserves horizontal vectors and their lengths and

$$d_{\mathrm{cc}}^{G}(T(q_1), T(q_2)) = d_{\mathrm{cc}}^{M_0}(q_1, q_2), \qquad q_1, q_2 \in M_0.$$

This proves that T is a Carnot–Carathéodory isometry. □

3.2. The Carnots group of type $(2, 1, \ldots, 1, 2)$

We now consider the second class of Carnot groups, namely the groups of type $(2, 1, \ldots, 1, 2)$. Compared with the groups of type $(2, 1, \ldots, 1)$, the top layer is two-dimensional.

Let $r \geq 3$ be odd, and let $\mathfrak{g}$ be the nilpotent Lie algebra with basis $X_1, X_2, \ldots, X_{r+2}$. Its nonzero brackets between basis elements are

$$[X_1, X_j] = X_{j+1}, \qquad 2 \leq j \leq r,$$

and

$$[X_i, X_j] = (-1)^i X_{r+2}, \qquad 2 \leq i < j \leq r, \quad i + j = r + 2.$$

Let G be the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, and set

$$\mathfrak{n} = \operatorname{span}\{X_2, \ldots, X_{r+2}\}.$$

Define

$$\begin{aligned} \Psi : \mathbb{R}^{r+2} &\longrightarrow G, \\ (x_1, \ldots, x_{r+2}) &\longmapsto \exp\left(\sum_{j=2}^{r+2} x_j X_j\right) \exp(x_1 X_1). \end{aligned}$$

Then Ψ is a global diffeomorphism. For $x = (x_1, \ldots, x_{r+2})$ and $x' = (x'_1, \ldots, x'_{r+2})$, define

$$\widetilde{x}'_j(x_1) = \sum_{\ell=2}^{j} \frac{x_1^{j-\ell}}{(j-\ell)!} x'_\ell, \qquad 2 \leq j \leq r + 1.$$

Let $\mathfrak{n} = \operatorname{span}\{X_2, \ldots, X_{r+2}\}$. The bracket relations imply $[X_1, \mathfrak{n}] \subseteq \mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathbb{R}X_{r+2}$. Moreover, X_{r+2} is central. Hence

$$[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = \{0\}.$$

Thus $\mathfrak{n}$ is a two-step nilpotent ideal of $\mathfrak{g}$. For $x = (x_1, \ldots, x_{r+2})$ and $x' = (x'_1, \ldots, x'_{r+2})$, let

$$v = \sum_{j=2}^{r+2} x_j X_j, \qquad v' = \sum_{j=2}^{r+2} x'_j X_j.$$

Using the conjugation identity

$$\exp(x_1 X_1) \exp(v') \exp(-x_1 X_1) = \exp\left(e^{x_1 \operatorname{ad} X_1} v'\right),$$

we obtain

$$\begin{aligned} \Psi(x)\Psi(x') &= \exp(v) \exp(x_1 X_1) \exp(v') \exp(x'_1 X_1) \\ &= \exp(v) \exp\left(e^{x_1 \operatorname{ad} X_1} v'\right) \exp\left((x_1 + x'_1) X_1\right). \end{aligned}$$

For $2 \leq \ell \leq r + 1$, the relations

$$[X_1, X_j] = X_{j+1}, \qquad 2 \leq j \leq r,$$

give

$$(\operatorname{ad} X_1)^m X_\ell = \begin{cases} X_{\ell+m}, & 0 \le m \le r+1-\ell, \\ 0, & m > r+1-\ell. \end{cases}$$

Since X_{r+2} is central, $e^{x_1 \operatorname{ad} X_1} X_{r+2} = X_{r+2}$. Therefore

$$\begin{aligned} e^{x_1 \operatorname{ad} X_1} v' &= \sum_{\ell=2}^{r+1} x'_\ell \sum_{m=0}^{r+1-\ell} \frac{x_1^m}{m!} X_{\ell+m} + x'_{r+2} X_{r+2} \\ &= \sum_{j=2}^{r+1} \left(\sum_{\ell=2}^{j} \frac{x_1^{j-\ell}}{(j-\ell)!} x'_\ell \right) X_j + x'_{r+2} X_{r+2} \\ &= \sum_{j=2}^{r+1} \widetilde{x}'_j(x_1) X_j + x'_{r+2} X_{r+2}. \end{aligned}$$

Set $\widetilde{v}' = e^{x_1 \operatorname{ad} X_1} v'$. Since $\mathfrak{n}$ is two-step nilpotent, the Baker–Campbell–Hausdorff formula on $\mathfrak{n}$ terminates after the first commutator:

$$\exp(v) \exp(\widetilde{v}') = \exp\left(v + \widetilde{v}' + \frac{1}{2}[v, \widetilde{v}'] \right).$$

It remains to calculate the commutator $[v, \widetilde{v}']$. The terms containing X_{r+1} or X_{r+2} do not contribute, because these vectors are central in $\mathfrak{n}$. Hence

$$[v, \widetilde{v}'] = \left[\sum_{i=2}^{r} x_i X_i, \sum_{j=2}^{r} \widetilde{x}'_j(x_1) X_j \right] = \sum_{i=2}^{r} \sum_{j=2}^{r} x_i \widetilde{x}'_j(x_1) [X_i, X_j].$$

Only the pairs satisfying $i + j = r + 2$ give nonzero terms. Since r is odd, $r + 2$ is odd, and consequently $i \neq j$ whenever $i + j = r + 2$. Thus the ordered pairs can be grouped into pairs with $i < j$. For such a pair,

$$\begin{aligned} x_i \widetilde{x}'_j(x_1)[X_i, X_j] &+ x_j \widetilde{x}'_i(x_1)[X_j, X_i] \\ &= (-1)^i x_i \widetilde{x}'_j(x_1) X_{r+2} - (-1)^i x_j \widetilde{x}'_i(x_1) X_{r+2} \\ &= (-1)^i \left(x_i \widetilde{x}'_j(x_1) - x_j \widetilde{x}'_i(x_1) \right) X_{r+2}. \end{aligned}$$

Therefore

$$[v, \widetilde{v}'] = \sum_{\substack{2 \le i < j \le r \\ i+j=r+2}} (-1)^i \left(x_i \widetilde{x}'_j(x_1) - x_j \widetilde{x}'_i(x_1) \right) X_{r+2}.$$

Combining the preceding formulas gives

$$v + \widetilde{v}' + \frac{1}{2}[v, \widetilde{v}'] = \sum_{j=2}^{r+1} \left(x_j + \widetilde{x}'_j(x_1) \right) X_j + \left[x_{r+2} + x'_{r+2} + \frac{1}{2} \sum_{\substack{2 \le i < j \le r \\ i+j=r+2}} (-1)^i \left(x_i \widetilde{x}'_j(x_1) - x_j \widetilde{x}'_i(x_1) \right) \right] X_{r+2}.$$

Consequently,

$$\Psi(x)\Psi(x') = \exp\left(\sum_{j=2}^{r+2} y_j X_j\right) \exp\left((x_1 + x_1')X_1\right),$$

and

$$\begin{aligned} y_1 &= x_1 + x_1', \\ y_j &= x_j + \widetilde{x}_j'(x_1) = x_j + \sum_{\ell=2}^{j} \frac{x_1^{j-\ell}}{(j-\ell)!} x_\ell', \qquad 2 \le j \le r+1, \\ y_{r+2} &= x_{r+2} + x_{r+2}' + \frac{1}{2} \sum_{\substack{2\le i<j\le r \\ i+j=r+2}} (-1)^i \left(x_i \widetilde{x}_j'(x_1) - x_j \widetilde{x}_i'(x_1)\right). \end{aligned} \tag{3.12}$$

We identify G with $\mathbb{R}^{r+2}$ through the above coordinates and use the multiplication (3.12). It is convenient to write

$$x = x_1, \quad y = x_2, \quad u_\alpha = x_{\alpha+2}, \quad 1 \le \alpha \le r.$$

Thus $(x, y, u_1, \dots, u_r)$ are global coordinates on G. A direct calculation gives the following horizontal left-invariant vector fields:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + \sum_{\alpha=1}^{r-1} \frac{x^\alpha}{\alpha!} \frac{\partial}{\partial u_\alpha} + P_r(x, y, u) \frac{\partial}{\partial u_r},$$

where

$$P_r(x, y, u) = \frac{1}{2} \frac{y x^{r-2}}{(r-2)!} + \frac{1}{2} \sum_{\alpha=1}^{r-2} (-1)^\alpha \frac{u_\alpha x^{r-\alpha-2}}{(r-\alpha-2)!}. \tag{3.13}$$

More generally, the remaining left-invariant fields are obtained by iterated commutators:

$$X_{j+1} = [X_1, X_j], \qquad 2 \le j \le r, \quad X_{r+2} = [X_2, X_r].$$

The horizontal distribution is $H_pG = \operatorname{span}\{X_1(p), X_2(p)\}$. We choose the left-invariant sub-Riemannian metric for which X_1 and X_2 are orthonormal. This gives the sub-Riemannian metric and the associated Carnot–Carathéodory distance d_{cc}^G.

Define the complex horizontal vector field $Z_1 = \frac{1}{2}(X_1 - iX_2)$. The induced CR structure on G is $T_{1,0}G = \operatorname{span}_{\mathbb{C}}\{Z_1\}$.

We now realize G as a CR submanifold of a complex Euclidean space. Let $(z, \omega_1, \dots, \omega_r) \in \mathbb{C}^{r+1}$, where

$$z = x + iy, \qquad \omega_\alpha = u_\alpha + iv_\alpha, \qquad 1 \le \alpha \le r.$$

Define real-valued functions $\Phi_1, \dots, \Phi_r$ by

$$\Phi_\alpha(x, y, u) = -\frac{x^{\alpha+1}}{(\alpha+1)!}, \qquad 1 \le \alpha \le r-1, \tag{3.14}$$

and

$$\Phi_r(x, y, u) = -\frac{1}{2} \frac{y x^{r-1}}{(r-1)!} - \frac{1}{2} \sum_{\alpha=1}^{r-2} (-1)^\alpha \frac{u_\alpha x^{r-\alpha-1}}{(r-\alpha-1)!} - \frac{1}{2} u_{r-1}. \tag{3.15}$$

Set

$$\rho_\alpha = v_\alpha - \Phi_\alpha, \qquad 1 \le \alpha \le r.$$

Equivalently,

$$\rho_\alpha = v_\alpha + \frac{x^{\alpha+1}}{(\alpha+1)!}, \qquad 1 \le \alpha \le r-1,$$

and

$$\rho_r = v_r + \frac{1}{2}\frac{yx^{r-1}}{(r-1)!} + \frac{1}{2}\sum_{\alpha=1}^{r-2}(-1)^\alpha \frac{u_\alpha x^{r-\alpha-1}}{(r-\alpha-1)!} + \frac{1}{2}u_{r-1}.$$

Now define

$$M_0 = \left\{(z, \omega_1, \ldots, \omega_r) \in \mathbb{C}^{r+1} : \rho_1 = \cdots = \rho_r = 0\right\}. \tag{3.16}$$

Since

$$\frac{\partial \rho_\alpha}{\partial v_\beta} = \delta_{\alpha\beta},$$

the functions $\rho_1, \ldots, \rho_r$ have linearly independent differentials on M_0. Hence M_0 is a smooth real submanifold of $\mathbb{C}^{r+1}$ of real dimension $r+2$.

Let $T : M_0 \to G$ be the map

$$T(z, \omega_1, \ldots, \omega_r) = (x, y, u_1, \ldots, u_r). \tag{3.17}$$

Since M_0 is the graph

$$v_\alpha = \Phi_\alpha(x, y, u), \qquad 1 \le \alpha \le r,$$

the map T is a global diffeomorphism. Its inverse is

$$T^{-1}(x, y, u_1, \ldots, u_r) = (x + iy, u_1 + i\Phi_1(x, y, u), \ldots, u_r + i\Phi_r(x, y, u)).$$

Define the following $(1,0)$ vector field

$$\widetilde{Z}_1 = \frac{\partial}{\partial z} - i\sum_{\alpha=1}^{r} P_\alpha(x, y, u)\frac{\partial}{\partial \omega_\alpha}, \tag{3.18}$$

where

$$P_\alpha(x, y, u) = \frac{x^\alpha}{\alpha!}, \qquad 1 \le \alpha \le r-1,$$

and P_r is given by (3.13).

Lemma 3.1. *The vector field $\widetilde{Z}_1$ is tangent to M_0. More precisely,*

$$\widetilde{Z}_1\rho_\beta = 0, \qquad \beta = 1, \ldots, r.$$

Consequently,

$$\widetilde{T}_{1,0}M_0 = \mathrm{span}_{\mathbb{C}}\{\widetilde{Z}_1\}.$$

Proof. Since $\rho_\beta = v_\beta - \Phi_\beta$, we have

$$\frac{\partial \rho_\beta}{\partial z} = -\frac{1}{2}\Phi_{\beta,x} + \frac{i}{2}\Phi_{\beta,y}, \quad \frac{\partial \rho_\beta}{\partial \omega_\alpha} = -\frac{1}{2}\Phi_{\beta,u_\alpha} - \frac{i}{2}\delta_{\alpha\beta}.$$

Therefore

$$\begin{aligned}\widetilde{Z}_1\rho_\beta &= -\frac{1}{2}\Phi_{\beta,x} + \frac{i}{2}\Phi_{\beta,y} - i\sum_{\alpha=1}^{r} P_\alpha\left(-\frac{1}{2}\Phi_{\beta,u_\alpha} - \frac{i}{2}\delta_{\alpha\beta}\right)\\ &= -\frac{1}{2}(\Phi_{\beta,x} + P_\beta) + \frac{i}{2}\left(\Phi_{\beta,y} + \sum_{\alpha=1}^{r} P_\alpha \Phi_{\beta,u_\alpha}\right).\end{aligned}$$

Thus it is enough to prove

$$\Phi_{\beta,x} = -P_\beta, \qquad \Phi_{\beta,y} + \sum_{\alpha=1}^{r} P_\alpha \Phi_{\beta,u_\alpha} = 0. \tag{3.19}$$

For $1 \le \beta \le r-1$, (3.14) gives

$$\Phi_{\beta,x} = -\frac{x^\beta}{\beta!} = -P_\beta.$$

Moreover Φ_β is independent of y and $u_1, \ldots, u_r$, so the second identity in (3.19) is immediate. It remains to check $\beta = r$. From (3.15),

$$\Phi_{r,x} = -\frac{1}{2}\frac{yx^{r-2}}{(r-2)!} - \frac{1}{2}\sum_{\alpha=1}^{r-2}(-1)^\alpha \frac{u_\alpha x^{r-\alpha-2}}{(r-\alpha-2)!} = -P_r, \qquad \Phi_{r,y} = -\frac{1}{2}\frac{x^{r-1}}{(r-1)!},$$

$$\Phi_{r,u_\alpha} = -\frac{1}{2}(-1)^\alpha \frac{x^{r-\alpha-1}}{(r-\alpha-1)!}, 1 \le \alpha \le r-2, \qquad \Phi_{r,u_{r-1}} = -(-1)^{r-1}\frac{1}{2} = -\frac{1}{2}, \qquad \Phi_{r,u_r} = 0.$$

Therefore

$$\begin{aligned}\Phi_{r,y} + \sum_{\alpha=1}^{r} P_\alpha \Phi_{r,u_\alpha} &= -\frac{1}{2}\frac{x^{r-1}}{(r-1)!} - \frac{1}{2}\sum_{\alpha=1}^{r-2}(-1)^\alpha \frac{x^\alpha}{\alpha!}\frac{x^{r-\alpha-1}}{(r-\alpha-1)!} - \frac{1}{2}\frac{x^{r-1}}{(r-1)!}\\ &= -\frac{1}{2}x^{r-1}\sum_{\alpha=0}^{r-1}\frac{(-1)^\alpha}{\alpha!(r-1-\alpha)!} = -\frac{x^{r-1}}{2(r-1)!}\sum_{\alpha=0}^{r-1}(-1)^\alpha\binom{r-1}{\alpha}\\ &= -\frac{x^{r-1}}{2(r-1)!}(1-1)^{r-1} = 0.\end{aligned}$$

The oddness of r is used here to identify the last endpoint term with the $\alpha = r-1$ term of the alternating binomial sum. This proves $\widetilde{Z}_1\rho_\beta = 0$ for every β. Finally, the matrix $\left(\frac{\partial \rho_\beta}{\partial \omega_\alpha}\right)_{\alpha,\beta}$ is triangular with nonzero diagonal entries $-i/2$. Hence it is invertible. Thus the space of $(1,0)$ tangent vectors has complex dimension one. Since $\widetilde{Z}_1$ is a nonzero tangent $(1,0)$ vector field, it spans $\widetilde{T}_{1,0}M_0$. □

The complex conjugate field is

$$\overline{\widetilde{Z}}_1 = \frac{\partial}{\partial \bar{z}} + i\sum_{\alpha=1}^{r} P_\alpha(x,y,u)\frac{\partial}{\partial \bar{\omega}_\alpha}.$$

Proposition 3.2. *The map $T : M_0 \to G$ is a CR diffeomorphism. If M_0 is equipped with the horizontal metric for which*

$$\widetilde{X}_1 = 2\,\mathrm{Re}\,\widetilde{Z}_1, \qquad \widetilde{X}_2 = -2\,\mathrm{Im}\,\widetilde{Z}_1$$

are orthonormal, then T is also a Carnot–Carathéodory isometry.

Proof. Using (3.17), we have

$$T_*\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}, \qquad T_*\frac{\partial}{\partial \omega_\alpha} = \frac{1}{2}\frac{\partial}{\partial u_\alpha}, 1 \le \alpha \le r.$$

Therefore, by (3.18),

$$\begin{aligned} T_*\widetilde{Z}_1 &= \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y} - \frac{i}{2}\sum_{\alpha=1}^{r} P_\alpha \frac{\partial}{\partial u_\alpha} \\ &= \frac{1}{2}(X_1 - iX_2) = Z_1. \end{aligned}$$

Thus $T_*(\widetilde{T}_{1,0}M_0) = T_{1,0}G$. Thus T is a CR diffeomorphism. Taking real and imaginary parts gives $T_*\widetilde{X}_1 = X_1, T_*\widetilde{X}_2 = X_2$. Hence T maps the horizontal distribution of M_0 onto the horizontal distribution of G and preserves the length of horizontal tangent vectors. Therefore, if γ is a horizontal curve in M_0, then $T \circ \gamma$ is a horizontal curve in G and

$$\ell_G(T \circ \gamma) = \ell_{M_0}(\gamma).$$

Applying the same argument to T^{-1} gives the reverse inequality for the length distance. Hence

$$d_{cc}^{G}(T(q_1), T(q_2)) = d_{cc}^{M_0}(q_1, q_2), \qquad q_1, q_2 \in M_0.$$

Thus T is a Carnot–Carathéodory isometry. □

4. Conformal mappings and CR mappings on the models M_0

In Section 3, for each of the two Carnot groups, we constructed a CR manifold M_0 and a diffeomorphism $T : M_0 \to G$ which is both CR and Carnot–Carathéodory isometric. More precisely, if $\widetilde{Z}_1$ denotes the generator of $\widetilde{T}_{1,0}M_0$ and $Z_1 = \frac{1}{2}(X_1 - iX_2)$ denotes the generator of $T_{1,0}G$, then

$$T_*\widetilde{Z}_1 = Z_1, \qquad T_*\overline{\widetilde{Z}}_1 = \overline{Z}_1.$$

Moreover,

$$d_{cc}^{G}(T(q_1), T(q_2)) = d_{cc}^{M_0}(q_1, q_2), \qquad q_1, q_2 \in M_0.$$

We now prove that conformal mappings on G induce CR or anti-CR mappings on the corresponding CR model M_0.

Theorem 4.1. *Let G be one of the two Carnot groups in Section 3, and let $T : M_0 \to G$ be the corresponding CR diffeomorphism and Carnot–Carathéodory isometry. Let $\Omega \subset G$ be a connected domain, and let $f : \Omega \to f(\Omega) \subset G$ be a conformal mapping with respect to the Carnot–Carathéodory distance on G. Set $\widehat{\Omega} = T^{-1}(\Omega) \subset M_0$ and define $\widehat{f} = T^{-1} \circ f \circ T : \widehat{\Omega} \to T^{-1}(f(\Omega)) \subset M_0$. Then $\widehat{f}$ is either a CR mapping or an anti-CR mapping.*

Proof. Since conformal mappings on these Carnot groups are smooth, and since the inverse of a conformal mapping is also conformal, f is a smooth diffeomorphism. By Lemma 2.1,

$$f_{*p}(H_pG) = H_{f(p)}G, \qquad p \in \Omega.$$

We first show that the horizontal differential of f is a similarity at every point. For $p \in \Omega$, set

$$A_p := f_{*p}\big|_{H_pG} : H_pG \longrightarrow H_{f(p)}G.$$

Since f is a diffeomorphism, A_p is a linear isomorphism. We shall use the following first-order behavior of the Carnot–Carathéodory distance along horizontal curves. Let $\eta : [0, \varepsilon) \to G$ be a C^1 horizontal curve with $\eta(0) = q$. Then

$$\lim_{t\to 0} \frac{d_{cc}(\eta(t), \eta(0))}{|t|} = |\dot{\eta}(0)|.$$

Therefore every C^1 horizontal curve satisfies

$$d_{cc}(q, \eta(t)) = t|\dot{\eta}(0)| + o(t), \qquad t \to 0^+. \tag{4.1}$$

Define the maximal and minimal singular values of A_p by

$$\sigma_{\max}(p) = \max_{\substack{U\in H_pG \\ |U|=1}} |A_pU|, \qquad \sigma_{\min}(p) = \min_{\substack{U\in H_pG \\ |U|=1}} |A_pU|.$$

Since A_p is an isomorphism,

$$0 < \sigma_{\min}(p) \le \sigma_{\max}(p).$$

Choose unit vectors $U_{\max}, U_{\min} \in H_pG$ such that

$$|A_pU_{\max}| = \sigma_{\max}(p), \qquad |A_pU_{\min}| = \sigma_{\min}(p).$$

For a unit horizontal vector

$$U = (\tau_p)_{*e}X, \qquad X \in \mathfrak{g}_1, \quad |X| = 1,$$

consider the horizontal curve

$$\gamma_U(t) = p * \exp(tX).$$

Its length on $[0, t]$ is t. Moreover,

$$d_{cc}(p, \gamma_U(t)) = t, \qquad t \ge 0. \tag{4.2}$$

By Lemma 2.1, the curve $f \circ \gamma_U$ is horizontal, and

$$(f \circ \gamma_U)'(0) = A_pU.$$

Applying (4.1), we obtain

$$d_{cc}(f(p), f(\gamma_U(t))) = t|A_pU| + o(t), \qquad t \to 0^+. \tag{4.3}$$

In particular,

$$d_{cc}(f(p), f(\gamma_{U_{\max}}(t))) = t\sigma_{\max}(p) + o(t),$$
$$d_{cc}(f(p), f(\gamma_{U_{\min}}(t))) = t\sigma_{\min}(p) + o(t).$$

For all sufficiently small $t > 0$, both $\gamma_{U_{\max}}(t)$ and $\gamma_{U_{\min}}(t)$ belong to Ω. By (4.2), they lie on the Carnot–Carathéodory sphere of radius t centered at p. Therefore,

$$L_f(p, t) \geq d_{cc}(f(p), f(\gamma_{U_{\max}}(t))),$$
$$l_f(p, t) \leq d_{cc}(f(p), f(\gamma_{U_{\min}}(t))).$$

It follows that

$$\frac{L_f(p,t)}{l_f(p,t)} \geq \frac{t\sigma_{\max}(p) + o(t)}{t\sigma_{\min}(p) + o(t)}.$$

Taking the upper limit as $t \to 0^+$ gives

$$H_f(p) \geq \frac{\sigma_{\max}(p)}{\sigma_{\min}(p)}. \tag{4.4}$$

Since f is conformal, there exists a full-measure subset $E \subset \Omega$ such that

$$H_f(p) = 1, \qquad p \in E.$$

For every $p \in E$, (4.4) therefore implies

$$1 \geq \frac{\sigma_{\max}(p)}{\sigma_{\min}(p)} \geq 1.$$

Thus

$$\sigma_{\max}(p) = \sigma_{\min}(p), \qquad p \in E. \tag{4.5}$$

The bundle map $p \mapsto A_p$ is continuous with respect to the smooth orthonormal frames X_1, X_2. Hence the matrices $A_p^T A_p$ depend continuously on p. Therefore their eigenvalues, and hence the singular values $\sigma_{\max}(p)$ and $\sigma_{\min}(p)$, also depend continuously on p. Since the equality $\sigma_{\max}(p) = \sigma_{\min}(p)$ holds on a full-measure, hence dense, subset of Ω, it holds at every point of Ω. Thus the horizontal differential of f is a similarity at every point of Ω and there exist a positive smooth function $\mu(p) > 0$ and a matrix $A(p) \in O(2)$ such that $f_{*p}|_{H_pG} = \mu(p)A(p)$. The horizontal Jacobian is

$$J_H f(p) = \det(f_{*p}|_{H_pG}) = \mu(p)^2 \det A(p).$$

Since f is a diffeomorphism, $J_H f(p) \neq 0$. Since f is smooth, $J_H f$ is continuous. Since Ω is connected, the sign of $J_H f$ is constant on Ω.

If $J_H f(p) > 0$ on Ω, then $A(p) \in SO(2)$. Hence there exist smooth functions $a(p)$ and $b(p)$, with

$$a(p)^2 + b(p)^2 = 1,$$

such that

$$A(p) = \begin{pmatrix} a(p) & -b(p) \\ b(p) & a(p) \end{pmatrix}.$$

Thus

$$f_{*p}X_1(p) = \mu(p)\big(a(p)X_1(f(p)) + b(p)X_2(f(p))\big), \quad f_{*p}X_2(p) = \mu(p)\big(-b(p)X_1(f(p)) + a(p)X_2(f(p))\big).$$

Since $Z_1 = \frac{1}{2}(X_1 - iX_2)$, we obtain $f_{*p}Z_1(p) = \mu(p)(a(p)+ib(p))Z_1(f(p))$. Hence $f_{*p}(T_{1,0,p}G) \subset T_{1,0,f(p)}G$.

If $J_H f(p) < 0$ on Ω, then $A(p) \in O(2) \setminus SO(2)$. Hence there exist smooth functions $a(p)$ and $b(p)$, with

$$a(p)^2 + b(p)^2 = 1,$$

such that

$$A(p) = \begin{pmatrix} a(p) & b(p) \\ b(p) & -a(p) \end{pmatrix}.$$

Thus

$$f_{*p}X_1(p) = \mu(p)\big(a(p)X_1(f(p)) + b(p)X_2(f(p))\big), \quad f_{*p}X_2(p) = \mu(p)\big(b(p)X_1(f(p)) - a(p)X_2(f(p))\big).$$

A direct computation gives $f_{*p}Z_1(p) = \mu(p)(a(p) - ib(p))\overline{Z}_1(f(p))$. Hence $f_{*p}(T_{1,0,p}G) \subset T_{0,1,f(p)}G$.

Let $q \in \widehat{\Omega}$ and put $p = T(q)$. Since $T_*\widetilde{T}_{1,0,q}M_0 = T_{1,0,p}G$ and $\widehat{f} = T^{-1} \circ f \circ T$, we have

$$\widehat{f}_*(\widetilde{T}_{1,0,q}M_0) = (T^{-1})_*\, f_{*p}\, T_*(\widetilde{T}_{1,0,q}M_0).$$

If $J_H f > 0$, then $f_{*p}(T_{1,0,p}G) \subset T_{1,0,f(p)}G$. Since $f(p) = T(\widehat{f}(q))$, and since $T_*\widetilde{T}_{1,0,\widehat{f}(q)}M_0 = T_{1,0,f(p)}G$, we get

$$\widehat{f}_*(\widetilde{T}_{1,0,q}M_0) \subset \widetilde{T}_{1,0,\widehat{f}(q)}M_0.$$

Thus $\widehat{f}$ is CR.

If $J_H f < 0$, then $f_{*p}(T_{1,0,p}G) \subset T_{0,1,f(p)}G$. We obtain

$$\widehat{f}_*(\widetilde{T}_{1,0,q}M_0) \subset \widetilde{T}_{0,1,\widehat{f}(q)}M_0.$$

Thus $\widehat{f}$ is anti-CR. This proves the theorem. □

We finally express the two alternatives in ambient complex coordinates. Let $M_0 \subset \mathbb{C}^N$ have ambient complex coordinates $\zeta = (\zeta_1, \dots, \zeta_N)$, and write

$$\widehat{f_\alpha} = \zeta_\alpha \circ \widehat{f}, \qquad \alpha = 1, \dots, N.$$

If $\widehat{f}$ is CR, then $\widehat{f}_*(\widetilde{T}_{1,0}M_0) \subset \widetilde{T}_{1,0}M_0$. Taking complex conjugates gives $\widehat{f}_*(\widetilde{T}_{0,1}M_0) \subset \widetilde{T}_{0,1}M_0$. Since the ambient coordinate functions ζ_α are annihilated by $(0,1)$ vector fields, we obtain

$$\overline{\widetilde{Z}}_1\widehat{f_\alpha} = d\zeta_\alpha\left(\widehat{f}_*\overline{\widetilde{Z}}_1\right) = 0, \qquad \alpha = 1, \dots, N.$$

If $\widehat{f}$ is anti-CR, then $\widehat{f}_*(\widetilde{T}_{1,0}M_0) \subset \widetilde{T}_{0,1}M_0$. Thus

$$\widetilde{Z}_1\widehat{f_\alpha} = d\zeta_\alpha\left(\widehat{f}_*\widetilde{Z}_1\right) = 0, \qquad \alpha = 1, \dots, N.$$

5. Conclusions

In this paper, we studied conformal mappings on two classes of rank-two Carnot groups, namely those of type $(2, 1, \dots, 1)$ and type $(2, 1, \dots, 1, 2)$. For each class, we constructed an explicit embedded CR submanifold M_0 and a global diffeomorphism $T : M_0 \to G$. By identifying the corresponding complex horizontal vector fields, we proved that T is a CR diffeomorphism. Moreover, since T preserves the orthonormal horizontal frames, it is also an isometry with respect to the associated Carnot–Carathéodory distances. These constructions provide explicit CR realizations of the two families of Carnot groups. Using these realizations, we further characterized conformal mappings on connected domains of G. We showed that the horizontal differential of a conformal mapping is a similarity at every point. Since the sign of its horizontal Jacobian is constant on a connected domain, the mapping either preserves or reverses the horizontal complex structure. Consequently, after transferring the mapping to the CR model through T, the induced mapping $T^{-1} \circ f \circ T$ is either CR or anti-CR. Thus, the conformal rigidity of these Carnot groups admits a natural interpretation in terms of CR geometry.

Author contributions

The authors equally contributed to the present research at all stages, from the formulation of the problem to the final findings and solution. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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