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Research article

Controllability of implicit neutral tempered fractional systems with state and control delays under explicit parameter bounds

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Abstract: This research was carried out to analyze the controllability of tempered Caputo fractional differential equations with state and control delay. The well established concept of fixed point theory and tempered α resolvent families associated with an analytic C_0 semi-group operator have been incorporated to find the existence of a control function corresponding to the mild solution, which is a form of equivalent integral equation of the proposed fractional problem. The illustration focuses on verifying the applicability of the proposed controllability theorem and investigating the sensitivity of the controllability constant with respect to the fractional order and the tempering parameter. The obtained results demonstrate how these parameters influence the sufficient controllability condition and the corresponding admissible parameter region.

Keywords: controllability; fractional derivative; fixed point theorem; fractional-resolvent operator
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1. Introduction

Over the past few decades, fractional differential equations (FDEs) have drawn a lot of attention because they can explain memory and inherited characteristics that traditional integer-order models are unable to. Fractional derivatives offer a more flexible framework for modeling complicated dynamical processes than ordinary and partial differential equations because they permit the order of differentiation to take any actual value. The monographs of [1, 2] provide a solid foundation in fractional calculus theory and its applications to differential equations. A thorough basis for the development of fractional dynamical systems and their analytical properties is provided by these publications.

Tempered fractional calculus has become a significant extension of conventional fractional operators in recent years. In order to improve their applicability to physical processes with finite memory, tempered fractional derivatives contain an exponential tempering parameter that essentially truncates the large memory of traditional fractional models. The literature has thoroughly examined the theoretical underpinnings of tempered fractional calculus and its relationship to anomalous diffusion and stochastic processes. Specifically, the probabilistic basis for fractional dynamics was established by the limit theorems for continuous-time random walks created by Meerschaert and Scheffler [3], and the tempered fractional calculus formulation was thoroughly examined by Meerschaert et al. [4]. The theoretical aspects of tempered fractional differential equations, such as existence and uniqueness results, operator properties, and analytical frameworks, have been further investigated in a number of recent studies; see, for example, [5–7].

Numerous efforts have been made to develop numerical techniques for tempered fractional differential equations in parallel with theoretical advancements. Effective numerical algorithms are necessary for practical implementation because fractional operators are nonlocal. To approximate tempered fractional derivatives and solve the resulting equations, a number of numerical procedures have been presented, such as finite difference approaches, graded mesh techniques, and spectral methods. The numerical algorithms presented in [8, 9], which offer precise and reliable computational frameworks for tempered fractional models, are representative contributions in this direction.

The increasing theoretical maturity of fractional calculus has led to its successful application in a wide spectrum of real-world problems [10, 11]. In epidemiology, fractional-order models have been employed to describe the transmission dynamics and sensitivity of COVID–TB coinfection, demonstrating the capability of memory-dependent models to capture complex disease evolution more accurately than classical approaches [12]. The introduction of piecewise modified Atangana–Baleanu–Caputo (mABC) fractional derivatives has further enhanced the modeling of systems exhibiting abrupt transitions and heterogeneous memory characteristics [13]. Likewise, generalized hybrid coupled modified ABC-fractional differential equations have been investigated using rigorous existence theory together with efficient numerical algorithms, illustrating their applicability to nonlinear engineering and physical systems [14]. Recently, deep learning techniques based on neural operators have emerged as powerful computational tools for solving high-dimensional partial differential equations, providing an efficient bridge between fractional mathematical modeling and modern scientific machine learning [15].

Beyond these developments, tempered fractional systems have found numerous applications in scientific and engineering problems, including fractional-order neural networks with delays [16], fractional biomechanical modeling in medical prognosis [17], nonlinear Schrödinger equations and anomalous diffusion processes [18], thermo-elastic models involving nonlocal memory effects [19, 20], and fractional damping phenomena in wave equations [21, 22]. Further applications have been reported in heterogeneous materials, bio-heat transfer, and continuum mechanics; see, for example, [23–26]. These studies collectively demonstrate that tempered fractional models provide a versatile framework for accurately describing complex systems with memory, hereditary characteristics, and nonlocal interactions.

The controllability analysis of fractional dynamical systems is another significant area of study. In order to determine if a system can be guided from a given initial state to a desired final state utilizing admissible control inputs, controllability is essential. Semi-group theory and fixed point methods are examples of functional analytic techniques that have been used to establish various controllability results for fractional systems. For instance, [27] examined the controllability of tempered fractional dynamical systems, whereas [28] examined the controllability of fractional delayed systems. Furthermore, utilizing fixed point techniques, the controllability of nonlinear neutral fractional integro-differential equations with delay was recently established in [29–31].

Recent developments in controllability and optimal control of fractional and distributed parameter systems have attracted considerable attention. Hammad et al. [32] established optimal control and controllability results for impulsive Hilfer fractional integro-differential inclusions with supporting numerical simulations. Hammad and Aboelenen [33] further investigated the controllability of impulsive fractional stochastic integro-differential equations involving hemivariational inequalities and demonstrated the effectiveness of their theoretical findings through numerical examples. In the context of distributed parameter systems, Dekhkonov et al. [34] analyzed the time-optimal control problem for a fourth-order parabolic equation in three-dimensional space and derived optimality results. Moreover, Gao et al. [22] studied the long-term dynamics of wave equations with nonlinear localized damping connected in parallel, providing significant insights into the stability and asymptotic behavior of such systems.

Although substantial progress has been achieved in the theory, numerical analysis, applications, and controllability of fractional dynamical systems, several important challenges remain open. In particular, controllability results for nonlinear neutral implicit tempered fractional systems involving simultaneously state delay, control delay, and nonlinear perturbations are still scarce. Existing studies mainly address either classical fractional operators, non-neutral systems, or models without delay, while the analytical treatment of tempered fractional neutral systems with delayed controls has received comparatively little attention. These observations motivate the present investigation.

Motivated by the above observations, this paper investigates the controllability of a class of nonlinear neutral tempered fractional dynamical systems with state and control delays. The considered model involves the tempered Caputo fractional derivative together with nonlinear perturbations and delay effects, resulting in a mathematically challenging framework. By constructing a suitable controllability operator and employing appropriate fixed point techniques, sufficient conditions guaranteeing controllability are established. The obtained results generalize and extend several existing controllability results available in the literature by incorporating tempered memory, neutral terms, and delayed control mechanisms into a unified analytical framework.

The mathematical model considered throughout this paper is the following impulsive neutral tempered fractional control system with state and control delays (1.1):

$$\begin{cases} {}_0^{TC}D_t^{\alpha,\lambda}[x(t) - \Re(t, x(t))] = Ax(t) + Bu(t - \mathfrak{f}) + f\left(t, x_t, {}_0^{TC}D_t^{\alpha,\lambda}x(t), u(t)\right), t \in [0, \top], \\ x(t) = \varphi(t), \quad t \in [-\mathfrak{f}, 0], \\ u(t) = u_0, \quad t \in [-\mathfrak{f}, 0]. \end{cases} \tag{1.1}$$

Here, $x(t) \in \Omega$ denotes the state variable and $u(t) \in U$ represents the control function, where U

is a Banach space. The operator $A : D(A) \subset \Omega \to \Omega$ is assumed to be densely defined and closed, generating an analytic C_0- semigroup and its associated tempered α-resolvent family $\{\mathcal{T}_{\alpha,\lambda}(t)\}_{t\geq 0}$ on Ω. The operator ${}_0^{TC}D_t^{\alpha,\lambda}$ denotes the tempered Caputo fractional derivative of order $0 < \alpha < 1$ with tempering parameter $\lambda > 0$. The mapping $\mathfrak{R} : [0, \top] \times \Omega \to \Omega$ represents the neutral term, while $f : [0, \top] \times C \times \Omega \times U \to \Omega$ is a nonlinear function depending on the current state, delayed state, fractional derivative, and control input. Moreover, $B : U \to \Omega$ is a bounded linear control operator. For each $t \in [0, T]$, the history segment of the state is denoted by $x_t \in C$ and is defined as $x_t(\theta) = x(t+\theta)$ for $-\mathfrak{t} \leq \theta \leq 0$, representing the past state of the system over the interval $[t - \mathfrak{t}, t]$.

This article is organized as follows:

- **Section 2** presents basic definitions, Laplace transform of the tempered Caputo fractional derivative, function spaces, and corollaries used throughout the paper.
- **Section 3** is devoted to the existence and uniqueness ofmild solution corresponding to the defined control function for the proposed implicit neutral functional differential system, established with the help of Darbo-point theorem.
- **Section 4** deals a parameter sensitivity analysis of the controllability condition, illustrating the influence of the fractional order α and the tempering parameter λ on the controllability constant and the admissible controllability region.
- **Section 5** provides an example to demonstrate the applicability of proven theoretical results for controllability.
- **Section 6** comprises the overall theoretical findings and the outcomes.

2. Preliminaries

In this section, we recall some basic definitions and auxiliary results from tempered fractional calculus that will be used throughout this paper. The presentation is mainly based on the works [5, 6, 35].

Functional framework

Let $(\Omega, \|\cdot\|)$ be a real Banach space, referred to as the state space. For each $t \in [0, T]$, the system state satisfies $x(t) \in \Omega$. The linear operator $A : D(A) \subset \Omega \to \Omega$ is assumed to be densely defined and closed and generates a tempered α-resolvent family $\{\mathcal{T}_{\alpha,\lambda}(t)\}_{t\geq 0}$ on Ω.

Phase space. To incorporate the state delay, we define the phase space

$$C = C([-\mathfrak{t}, 0], \Omega),$$

equipped with the norm

$$\|\varphi\|_C = \sup_{\theta\in[-\mathfrak{t},0]} \|\varphi(\theta)\|.$$

For each $t \in [0, \top]$, the history segment of the state is given by

$$x_t(\theta) = x(t + \theta), \qquad -\mathfrak{t} \leq \theta \leq 0,$$

so that $x_t \in C$. The prescribed initial history satisfies $\varphi \in C$.

Solution space. The space of admissible trajectories is

$$\mathbb{X} = C([-\mathfrak{k}, \top], \Omega),$$

equipped with the norm

$$\|x\|_{\mathbb{X}} = \sup_{t\in[-\mathfrak{k},\top]} \|x(t)\|.$$

Control space. The admissible control functions belong to the Hilbert space

$$\mathbb{U} = L^2([0, \top]; U),$$

where U is a Banach space. The control operator

$$B : U \to \Omega$$

is assumed to be bounded and linear.

2.1. Tempered fractional integral

Definition 2.1 ([5,35]). *Let $\alpha > 0$ and $\lambda \geq 0$. The tempered fractional integral of order α of a function $x \in \mathbb{X}$ is defined by*

$$({}^{\lambda}I^{\alpha}_{0^+}x(t)) = \frac{1}{\Gamma(\alpha)}\int_0^t e^{-\lambda(t-\mathfrak{z})}(t-\mathfrak{z})^{\alpha-1}x(\mathfrak{z})\,d\mathfrak{z}, \quad t \in J. \tag{2.1}$$

For $\lambda = 0$, the operator ${}^{\lambda}I^{\alpha}_{0^+}$ reduces to the classical Riemann–Liouville fractional integral.

2.2. Tempered Caputo fractional derivative

Definition 2.2 ([5,6]). *Let $\alpha \in (0, 1)$ and $\lambda \geq 0$. The tempered Caputo fractional derivative of order α of a function $x \in \mathbb{X}$ is defined by*

$$({}^{TC}_{0}D^{\alpha,\lambda}_{t}x)(t) = e^{-\lambda t}\,{}^{C}D^{\alpha}_{0^+}(e^{\lambda t}x(t)), \quad t \in J, \tag{2.2}$$

where ${}^{C}D^{\alpha}_{0^+}$ denotes the classical Caputo fractional derivative, that is,

$${}^{C}D^{\alpha}_{0^+}x(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\mathfrak{z})^{-\alpha}x'(\mathfrak{z})\,d\mathfrak{z}.$$

Equivalently, the tempered Caputo derivative can be written as

$$({}^{TC}_{0}D^{\alpha,\lambda}_{t}x)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t e^{-\lambda(t-\mathfrak{z})}(t-\mathfrak{z})^{-\alpha}\frac{d}{d\mathfrak{z}}(e^{\lambda\mathfrak{z}}x(\mathfrak{z}))\,d\mathfrak{z}. \tag{2.3}$$

2.3. Basic properties

The following properties are well known and can be found in [5, 6, 35].

Lemma 2.3 ([5]). *Let $x \in \mathbb{X}$ and $\alpha \in (0, 1)$. Then,*

$${}^{TC}D^{\alpha,\lambda}_{0^+}({}^{\lambda}I^{\alpha}_{0^+}x)(t) = x(t), \quad t \in J. \tag{2.4}$$

Lemma 2.4 ([5,6]). *Let $x \in \mathbb{X}$ and $\alpha \in (0,1)$. Then,*

$$ {}^{\lambda}I^{\alpha}_{0^+}({}^{TC}_{0}D^{\alpha,\lambda}_{t}x)(t) = x(t) - x(0)e^{-\lambda t}, \quad t \in J. \tag{2.5}$$

Lemma 2.5. *[36] Let $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be the infinitesimal generator of a bounded analytic C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space $\mathbb{X}$. Then, A is a sectorial operator of angle $\omega < \pi/2$. Hence, for every $\psi \in (\omega, \pi)$ there exists a constant $C_\psi > 0$ such that*

$$\sup_{\nu \in \mathbb{C}\setminus\overline{\Sigma_\psi}} |\nu| \, \|(\nu - A)^{-1}\| \leq C_\psi,$$

where

$$\Sigma_\psi = \{\nu \in \mathbb{C} \setminus \{0\} : |\arg(\nu)| < \psi\}.$$

Consequently, for any $\lambda > 0$ and $0 < \alpha \leq 1$, if Γ is a contour such that $\nu^\alpha + \lambda \in \mathbb{C} \setminus \overline{\Sigma_\psi}$ for all $\nu \in \Gamma$, then

$$\|(\nu^\alpha + \lambda - A)^{-1}\| \leq \frac{C_\psi}{|\nu^\alpha + \lambda|}, \qquad \nu \in \Gamma.$$

Moreover, there exists a constant $C > 0$, depending only on α, λ, and the contour Γ, such that

$$|\nu^\alpha + \lambda| \geq C\big(|\nu|^\alpha + \lambda\big),$$

and therefore

$$\|(\nu^\alpha + \lambda - A)^{-1}\| \leq \frac{C_\psi}{C\big(|\nu|^\alpha + \lambda\big)}, \qquad \nu \in \Gamma.$$

2.4. Laplace transform properties of the tempered Caputo derivative [37,38]

- $$\mathcal{L}\Big[{}^{TC}_{0}D^{\alpha,\lambda}_{t}x(t)\Big](\omega) = (\omega+\lambda)^\alpha\,\hat{x}(\omega) - (\omega+\lambda)^{\alpha-1}\,x(0),$$
  where $\hat{x}(\omega) = \mathcal{L}[x(t)](\omega)$ and $0 < \alpha < 1$.
- $$\mathcal{L}\Big[e^{-\lambda t}t^{\alpha-1}\Big](\omega) = \frac{\Gamma(\alpha)}{(\omega+\lambda)^\alpha}, \qquad \alpha > 0.$$
- $$\mathcal{L}\Big[e^{-\lambda t}t^{\alpha}\Big](\omega) = \frac{\Gamma(\alpha+1)}{(\omega+\lambda)^{\alpha+1}}, \qquad \alpha > -1.$$
- $$\mathcal{L}[f(t) * \Psi(t)]\,(\omega) = \mathcal{L}[f(t)](\omega)\,\mathcal{L}[\Psi(t)](\omega),$$
  where $*$ denotes the convolution
  $$(f * \Psi)(t) = \int_0^t f(\tau)\Psi(t-\tau)\,d\tau.$$
- $$\mathcal{L}\Big[e^{-\lambda t}f(t)\Big](\omega) = \hat{f}(\omega+\lambda).$$

2.5. Mild solution representation

Definition 2.6. *Consider the linear tempered fractional differential equation*

$$ {}_0^{TC}D_t^{\alpha,\lambda} x(t) = f(t), \quad t \in \mathcal{J}, \qquad x(0) = x_0 \in \mathbb{X}, \tag{2.6}$$

where $f \in \complement(\mathcal{J}, \mathbb{X})$. Then, the unique solution is given by [5, 6]

$$ x(t) = x_0 e^{-\lambda t} + \frac{1}{\Gamma(\alpha)} \int_0^t e^{-\lambda(t-\mathfrak{z})}(t-\mathfrak{z})^{\alpha-1} f(\mathfrak{z})\, d\mathfrak{z}, \quad t \in \mathcal{J}. \tag{2.7}$$

Lemma 2.7 (Properties of the tempered α-resolvent families [39])**.** *Let $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be the sectorial operator generates a analytic C_0 - semigroup $\{T(t)\}_{t\geq 0}$, which is acting on a Banach space $\mathbb{X}$. Let $\{\mathbb{P}_\alpha^\lambda(t)\}_{t\geq 0}$ and $\{\mathbb{Q}_\alpha^\lambda(t)\}_{t\geq 0}$ be the associated tempered α-resolvent families defined by*

$$ \mathbb{P}_\alpha^\lambda(t) = e^{-\lambda t} t^{\alpha-1} E_{\alpha,\alpha}(At^\alpha), \qquad t > 0, \qquad \mathbb{Q}_\alpha^\lambda(t) = e^{-\lambda t} E_{\alpha,1}(At^\alpha), \qquad t \geq 0, $$

where $E_{\alpha,\beta}(\cdot)$ is the Mittag–Leffler function with two parameter and it satisfies the following properties:

(i) The families $\{\mathbb{P}_\alpha^\lambda(t)\}_{t\geq 0}$ and $\{\mathbb{Q}_\alpha^\lambda(t)\}_{t\geq 0}$ are strongly continuous operator families on $\mathbb{X}$.

(ii) $\mathbb{Q}_\alpha^\lambda(0) = I$.

(iii) $\exists\, \mathscr{R}_\alpha^{\mathbb{P}} > 0$ and $\mathscr{R}_\alpha^{\mathbb{Q}} \geq 1$ such that

$$ \|\mathbb{P}_\alpha^\lambda(t)\| \leq \mathscr{R}_\alpha^{\mathbb{P}} e^{-\lambda t} t^{\alpha-1}, \qquad t \in (0, \top], $$

and

$$ \|\mathbb{Q}_\alpha^\lambda(t)\| \leq \mathscr{R}_\alpha^{\mathbb{Q}} e^{-\lambda t}, \qquad t \in [0, \top]. $$

(iv) For every $g \in C([0,\top], \mathbb{X})$, the mild solution of

$$ {}_0^{TC}D_t^{\alpha,\lambda} x(t) = Ax(t) + g(t), \qquad x(0) = x_0, $$

is given by

$$ x(t) = \mathbb{Q}_\alpha^\lambda(t) x_0 + \int_0^t \mathbb{P}_\alpha^\lambda(t-\mathfrak{z})\, g(\mathfrak{z})\, d\mathfrak{z}. $$

(v) The convolution estimate

$$ \int_0^t \|\mathbb{P}_\alpha^\lambda(t-s)\|\, ds \leq \mathscr{R}_\alpha^{\mathbb{P}} \int_0^t e^{-\lambda(t-s)}(t-s)^{\alpha-1}\, ds \leq \frac{\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}, \qquad t \in [0, \top], $$

holds.

Definition 2.8. *For the system* (1.1)

$$ \begin{cases} {}_0^{TC}D_t^{\alpha,\lambda}[x(t) - \mathfrak{N}(t, x(t))] & = Ax(t) + Bu(t-\mathfrak{k}) + f(t, x_t, {}_0^{TC}D_t^{\alpha,\lambda}x(t), u(t)), t \in [0,\top] = \mathcal{J}, \quad 0 < \alpha \leq 1, \\ x(t) = \varphi(t), \quad t \in [-\mathfrak{k}, 0], \\ u(t) = u_0(t), \quad t \in [-\mathfrak{k}, 0], \end{cases} $$

a function $x \in \mathbb{X}$ is called a mild solution *if it satisfies the following integral equation:*

$$x(t)=\begin{cases}\varphi(t), & t\in[-\mathfrak{k},0],\\ \mathfrak{R}(t,x(t))+\mathbb{Q}_\alpha^\lambda(t)[x_0-\mathfrak{R}(0,x(0))]+\int_0^t\mathbb{P}_\alpha^\lambda(t-\tau)A\mathfrak{R}(\mathfrak{z},x(\mathfrak{z}))\,d\mathfrak{z} & \\ +\displaystyle\int_{-\mathfrak{k}}^0\mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})\,Bu_0(\mathfrak{z})\,d\mathfrak{z}+\int_0^{t-\mathfrak{k}}\mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})\,Bu_x(\mathfrak{z})\,d\mathfrak{z} & \\ +\displaystyle\int_0^t\mathbb{P}_\alpha^\lambda(t-\mathfrak{z})\,f(\mathfrak{z},x_\mathfrak{z},{}_0^{TC}D_\mathfrak{z}^{\alpha,\lambda}x(\mathfrak{z}),u(\mathfrak{z}))\,d\mathfrak{z}, & t\in[0,\top].\end{cases} \tag{$\star$}$$

Here, the tempered α–resolvent families $\{\mathbb{P}_\alpha^\lambda(t)\}_{t\geq 0}$ and $\{\mathbb{Q}_\alpha^\lambda(t)\}_{t\geq 0}$ are defined by

$$\mathbb{P}_\alpha^\lambda(t)=e^{-\lambda t}\,t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)=\frac{e^{-\lambda t}}{2\pi i}\int_\Gamma e^{\nu t}\,\nu^{\alpha-1}(\nu^\alpha-A)^{-1}d\nu,$$

$$\mathbb{Q}_\alpha^\lambda(t)=e^{-\lambda t}\,E_{\alpha,1}(At^\alpha)=\frac{e^{-\lambda t}}{2\pi i}\int_\Gamma e^{\nu t}\,(\nu^\alpha-A)^{-1}d\nu,\ \text{where}\ \nu=\omega+\lambda$$

where Γ is a suitable contour enclosing the spectrum of the operator A.

Lemma 2.9. *Let $A : D(A)\subset\Omega\to\Omega$ be a sectorial operator generating the tempered fractional resolvent families $\{\mathbb{Q}_\alpha^\lambda(t)\}_{t\geq 0}$ and $\{\mathbb{P}_\alpha^\lambda(t)\}_{t\geq 0}$. Assume that $\mathfrak{R} : [0,\top]\times\Omega\to D(A)$, that is, $\mathfrak{R}(t,x(t))\in D(A)$ for every $t\in[0,\top]$.*

Then, every solution of

$$\begin{cases}{}_0^{TC}D_t^{\alpha,\lambda}[x(t)-\mathfrak{R}(t,x(t))] & =Ax(t)+Bu(t-\mathfrak{k})+f(t,x_t,{}_0^{TC}D_t^{\alpha,\lambda}x(t),u(t)),t\in[0,\top]=\mathcal{J},\quad 0<\alpha\leq 1,\\ x(t)=\varphi(t),\quad t\in[-\mathfrak{k},0], & \\ u(t)=u_0(t),\quad t\in[-\mathfrak{k},0], & \end{cases}$$

satisfies the integral equation

$$\begin{aligned}x(t)=&\mathfrak{R}(t,x(t))+\mathbb{Q}_\alpha^\lambda(t)[x_0-\mathfrak{R}(0,x(0))]+\int_0^t\mathbb{P}_\alpha^\lambda(t-\mathfrak{z})A\mathfrak{R}(\mathfrak{z},x(\mathfrak{z}))\,d\mathfrak{z}+\int_{-\mathfrak{k}}^0\mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})Bu_0(\mathfrak{z})\,d\mathfrak{z}\\ &+\int_0^{t-\mathfrak{k}}\mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})Bu_x(\mathfrak{z})\,d\mathfrak{z}+\int_0^t\mathbb{P}_\alpha^\lambda(t-\mathfrak{z})f\Big(\mathfrak{z},x_\mathfrak{z},{}_0^{TC}D_\mathfrak{z}^{\alpha,\lambda}x(\mathfrak{z}),u(\mathfrak{z})\Big)\,d\mathfrak{z}.\end{aligned}$$

Consequently, the above integral equation is the mild solution of the system.

Proof. Let, $y(t)=x(t)-\mathfrak{R}(t,x(t))$. Then, $x(t)=y(t)+\mathfrak{R}(t,x(t))$, and the differential equation becomes,

$${}_0^{TC}D_t^{\alpha,\lambda}y(t)=Ay(t)+A\mathfrak{R}(t,x(t))+Bu(t-\mathfrak{k})+f\Big(t,x_t,{}_0^{TC}D_t^{\alpha,\lambda}x(t),u(t)\Big), \tag{2.8}$$

with the initial value, $y(0)=x_0-\mathfrak{R}(0,x(0))$. Applying the Laplace transform to (2.8), we obtain

$$(s+\lambda)^\alpha Y(s)-(s+\lambda)^{\alpha-1}y(0)=AY(s)+A\mathcal{L}\{\mathfrak{R}(t,x(t))\}+\mathcal{L}\{Bu(t-\mathfrak{k})\}+F(s),$$

where

$$F(s)=\mathcal{L}\left\{f\Big(t,x_t,{}_0^{TC}D_t^{\alpha,\lambda}x(t),u(t)\Big)\right\}.$$

Hence,

$$\Big((s+\lambda)^{\alpha}I-A\Big)Y(s)=(s+\lambda)^{\alpha-1}y(0)+A\mathcal{L}\{\mathfrak{N}(t,x(t))\}+\mathcal{L}\{Bu(t-\mathfrak{k})\}+F(s).$$

Now,

$$\begin{aligned}Y(s)=&\Big((s+\lambda)^{\alpha}I-A\Big)^{-1}(s+\lambda)^{\alpha-1}y(0)+\Big((s+\lambda)^{\alpha}I-A\Big)^{-1}A\mathcal{L}\{\mathfrak{N}(t,x(t))\}\\&+\Big((s+\lambda)^{\alpha}I-A\Big)^{-1}\mathcal{L}\{Bu(t-\mathfrak{k})\}+\Big((s+\lambda)^{\alpha}I-A\Big)^{-1}F(s).\end{aligned}$$

Taking the inverse Laplace transform and using

$$\mathcal{L}^{-1}\left\{(s+\lambda)^{\alpha-1}\Big((s+\lambda)^{\alpha}I-A\Big)^{-1}\right\}=\mathbb{Q}_{\alpha}^{\lambda}(t),$$

and

$$\mathcal{L}^{-1}\left\{\Big((s+\lambda)^{\alpha}I-A\Big)^{-1}\right\}=\mathbb{P}_{\alpha}^{\lambda}(t),$$

the convolution theorem yields

$$\begin{aligned}y(t)=&\mathbb{Q}_{\alpha}^{\lambda}(t)y(0)+\int_0^t\mathbb{P}_{\alpha}^{\lambda}(t-\mathfrak{z})A\mathfrak{N}(\mathfrak{z},x(\mathfrak{z}))\,d\mathfrak{z}+\int_0^t\mathbb{P}_{\alpha}^{\lambda}(t-\mathfrak{z})Bu(\mathfrak{z}-\mathfrak{k})\,d\mathfrak{z}\\&+\int_0^t\mathbb{P}_{\alpha}^{\lambda}(t-\mathfrak{z})f\Big(\mathfrak{z},x_{\mathfrak{z}},{}_0^{TC}D_{\mathfrak{z}}^{\alpha,\lambda}x(\mathfrak{z}),u(\mathfrak{z})\Big)\,d\mathfrak{z}.\end{aligned}$$

Since

$$u(\mathfrak{z}-\mathfrak{k})=\begin{cases}u_0(\mathfrak{z}-\mathfrak{k}), & 0\le\mathfrak{z}\le\mathfrak{k},\\ u_x(\mathfrak{z}-\mathfrak{k}), & \mathfrak{z}\ge\mathfrak{k},\end{cases}$$

setting $z=\mathfrak{z}-\mathfrak{k}$, we obtain,

$$\int_0^t\mathbb{P}_{\alpha}^{\lambda}(t-\mathfrak{z})Bu(\mathfrak{z}-\mathfrak{k})\,d\mathfrak{z}=\int_{-\mathfrak{k}}^0\mathbb{P}_{\alpha}^{\lambda}(t-z-\mathfrak{k})Bu_0(z)\,dz+\int_0^{t-\mathfrak{k}}\mathbb{P}_{\alpha}^{\lambda}(t-z-\mathfrak{k})Bu_x(z)\,dz.$$

Finally, we conclude that,

$$\begin{aligned}x(t)=&\mathfrak{N}(t,x(t))+\mathbb{Q}_{\alpha}^{\lambda}(t)[x_0-\mathfrak{N}(0,x(0))]+\int_0^t\mathbb{P}_{\alpha}^{\lambda}(t-\mathfrak{z})A\mathfrak{N}(\mathfrak{z},x(\mathfrak{z}))\,d\mathfrak{z}+\int_{-\mathfrak{k}}^0\mathbb{P}_{\alpha}^{\lambda}(t-\mathfrak{z}-\mathfrak{k})Bu_0(\mathfrak{z})\,d\mathfrak{z}\\&+\int_0^{t-\mathfrak{k}}\mathbb{P}_{\alpha}^{\lambda}(t-\mathfrak{z}-\mathfrak{k})Bu_x(\mathfrak{z})\,d\mathfrak{z}+\int_0^t\mathbb{P}_{\alpha}^{\lambda}(t-\mathfrak{z})f\Big(\mathfrak{z},x_{\mathfrak{z}},{}_0^{TC}D_{\mathfrak{z}}^{\alpha,\lambda}x(\mathfrak{z}),u(\mathfrak{z})\Big)\,d\mathfrak{z}.\end{aligned}$$

Therefore, the above integral equation is the desired mild solution of the given tempered neutral fractional system. □

2.6. Kuratowski measure of noncompactness

Let $\mathbb{Y}$ be a Banach space and let $\mathcal{B} \subset \mathbb{Y}$ be bounded. The Kuratowski measure of noncompactness [40, 41] χ is defined by

$$\chi(\mathcal{B}) = \inf\Big\{\epsilon > 0 : \mathcal{B} \subset \bigcup_{j=1}^{m} B_j,\ \mathrm{diam}(B_j) \leq \epsilon\Big\}.$$

The measure χ satisfies the following properties:

1) If $\mathcal{B}_1 \subset \mathcal{B}_2$, then $\chi(\mathcal{B}_1) \leq \chi(\mathcal{B}_2)$.
2) $\chi(\mathcal{B}) = 0$ if and only if $\mathcal{B}$ is relatively compact in $\mathbb{Y}$.
3) For every $z \in \mathbb{Y}$ $\chi(\{z\} \cup \mathcal{B}) = \chi(\mathcal{B})$.
4) $\chi(\mathcal{B}_1 \cup \mathcal{B}_2) \leq \max\{\chi(\mathcal{B}_1), \chi(\mathcal{B}_2)\}$.
5) $\chi(\mathcal{B}_1 + \mathcal{B}_2) \leq \chi(\mathcal{B}_1) + \chi(\mathcal{B}_2)$.
6) For $\lambda \in \mathbb{R}$ $\chi(\lambda\mathcal{B}) = |\lambda|\chi(\mathcal{B})$.

Proposition 2.10. *[29–31]*
Let $\mathcal{M} \subset C(\mathbb{J}, \mathbb{Y})$ be bounded and equicontinuous. Then, the function

$$t \mapsto \chi(\mathcal{M}(t))$$

is continuous on $\mathbb{J}$ and

$$\chi(\mathcal{M}) = \max_{t\in\mathbb{J}} \chi(\mathcal{M}(t)).$$

Moreover, if $v : \mathbb{J} \to \mathbb{Y}$ is integrable, then

$$\chi\left(\int_0^t v(s)ds\right) \leq \int_0^t \chi(v(s))ds.$$

Proposition 2.11. *[29–31]*
Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions $v_n : \mathbb{J} \to \mathbb{Y}$ satisfying

$$\|v_n(t)\| \leq m(t) \quad a.e.$$

where $m \in L^1(\mathbb{J}, \mathbb{R}^+)$. If

$$\xi(t) = \chi(\{v_n(t)\}_{n=1}^{\infty}) \in L^1(\mathbb{J}, \mathbb{R}^+),$$

then

$$\chi\left(\left\{\int_0^t v_n(s)ds : n \in \mathbb{N}\right\}\right) \leq 2\int_0^t \xi(s)ds.$$

Proposition 2.12. *[29–31]*
Let $\mathcal{M}$ be a bounded subset of $\mathbb{Y}$. Then, for each $\zeta > 0$ there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset \mathcal{M}$ such that

$$\chi(\mathcal{M}) \leq 2\chi(\{v_n\}_{n=1}^{\infty}) + \zeta.$$

2.7. Auxiliary fixed point result

We recall the following classical result, which will be used to establish the existence of solutions.

Theorem 2.13 (Darbo fixed–point theorem)**.** *[40, 41], Let Ω be a Banach space and let Ω_r be a nonempty, bounded, closed, and convex subset of Ω. Let $\Psi : \Omega_r \to \Omega_r$ be a continuous mapping. Assume that for every bounded subset $\mathcal{D} \subset \Omega_r$, the following inequality holds:*

$$\chi(\Psi(\mathcal{D})) \leq k\chi(\mathcal{D}),$$

where χ denotes the Kuratowski measure of noncompactness and $0 \leq k < 1$. Then, the operator Ψ has at least one fixed point in Ω_r. Moreover, such a mapping Ψ is called a k–set contraction.

Theorem 2.14 (Arzelà–Ascoli theorem [1, 42])**.** *Let Ω be a Banach space and let $\mathbb{X} = C([-\mathfrak{t}, T]; \Omega)$ endowed with the supremum norm. A subset $\Omega_r \subset \mathbb{X}$ is relatively compact in $\mathbb{X}$ if the following conditions hold:*

(i) Ω_r is uniformly bounded, that is,

$$\sup_{x\in\Omega_r} \|x(t)\| < \infty \quad \textit{for all } t \in [-\mathfrak{t}, T],$$

(ii) Ω_r is equicontinuous on $[-\mathfrak{t}, T]$,

(iii) for each $t \in [-\mathfrak{t}, T]$, the set

$$\{x(t) : x \in \Omega_r\}$$

is relatively compact in Ω.

Then, Ω_r is relatively compact in $\mathbb{X}$.

Theorem 2.15 (Dominated convergence theorem)**.** *[43] Let $(\Omega, \|\cdot\|)$ be a Banach space and let $\{x_n\}_{n\in\mathbb{N}} \subset \Omega$ be a sequence such that*

$$x_n(t) \to x(t) \quad \textit{for all } t \in [0, \top].$$

Assume that there exists a function $\eta \in L^1([0, \top], \mathbb{R}_+)$ such that

$$\|x_n(t)\|_\Omega \leq \eta(t), \quad \textit{for all } t \in [0, \top],\ n \in \mathbb{N}.$$

Then, $x \in \Omega$ and

$$\lim_{n\to\infty} \int_0^t \|x_n(\mathfrak{z}) - x(\mathfrak{z})\|_\Omega \, d\mathfrak{z} = 0, \quad \forall t \in [0, \top].$$

In particular,

$$\lim_{n\to\infty} \int_0^t s_n(\mathfrak{z})\, d\mathfrak{z} = \int_0^t s(\mathfrak{z})\, d\mathfrak{z}, \quad \forall t \in [0, \top].$$

Definition 2.16. *[29] System (1.1) is said to be controllable on $[0, T]$ if for every initial state x_0 and target state x_T, there exists a control function $u(t)$ such that the corresponding solution satisfies $x(T) = x_T$.*

Postulates

Throughout this article, the following postulates are imposed.

(P1) The operator $A : D(A) \subset \Omega \to \Omega$ is a sectorial operator generating an analytic C_0-semigroup $\{T(t)\}_t \geq 0$. The associated tempered fractional resolvent families $\{\mathbb{Q}_\alpha^\lambda(t)\}_{t\geq 0}$ and $\{\mathbb{P}_\alpha^\lambda(t)\}_{t\geq 0}$ satisfy the lemma (2.7)

$$\|\mathbb{Q}_\alpha^\lambda(t)\| \leq \mathscr{R}_\alpha^{\mathbb{Q}} e^{-\lambda t}, \qquad t \geq 0, \quad \text{and} \quad \|\mathbb{P}_\alpha^\lambda(t)\| \leq \mathscr{R}_\alpha^{\mathbb{P}} e^{-\lambda t} t^{\alpha-1}, \qquad t \in (0, \top],$$

for some constants $\mathscr{R}_\alpha^{\mathbb{Q}} \geq 1$ and $\mathscr{R}_\alpha^{\mathbb{P}} > 0$.

(P2) The neutral function $\mathfrak{N} : \mathcal{J} \times \Omega \to D(A)$ is continuous and there exist constants $\mathscr{R}_{\mathfrak{N}}, \mathscr{R}_{A\mathfrak{N}} \geq 0$ such that

$$\|\mathfrak{N}(t, x_1) - \mathfrak{N}(t, x_2)\| \leq \mathscr{R}_{\mathfrak{N}} \|x_1 - x_2\|,$$

and

$$\|A\mathfrak{N}(t, x_1) - A\mathfrak{N}(t, x_2)\| \leq \mathscr{R}_{A\mathfrak{N}} \|x_1 - x_2\|.$$

for some constant $\mathscr{R}_{\mathfrak{N}} \geq 0$ with $\|\mathfrak{N}(t, x(t)\| = \mathscr{R}_{\mathfrak{N}}\|x\| + \mathscr{B}_{\mathfrak{N}}$
and for some constant $\mathscr{R}_{A\mathfrak{N}} \geq 0$ with $\|A\,\mathfrak{N}(t, x(t)\| = \mathscr{R}_{A\mathfrak{N}}\|x\| + \mathscr{B}_{A\mathfrak{N}}$.

(P3) The nonlinear mapping

$$f : \mathcal{J} \times \Omega \times \Omega \times U \to \Omega$$

is continuous and there exist constants $\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3 \geq 0$ such that

$$\|f(t, x_1, \rho_1, u_1) - f(t, x_2, \rho_2, u_2)\| \leq \mathscr{R}_1\|x_1 - x_2\| + \mathscr{R}_2\|\rho_1 - \rho_2\| + \mathscr{R}_3\|u_1 - u_2\|,$$

for all $t \in \mathcal{J}$.

(P4) For every admissible trajectory $x \in \Omega_r$, the tempered Caputo fractional derivative ${}_0^{TC}D_t^{\alpha,\lambda}x(t)$ exists. Furthermore, there exists a constant $\mathscr{R}_D > 0$ such that

$$\left\|{}_0^{TC}D_t^{\alpha,\lambda}x_1(t) - {}_0^{TC}D_t^{\alpha,\lambda}x_2(t)\right\| \leq \mathscr{R}_D\|x_1 - x_2\|_\Omega,$$

for all $x_1, x_2 \in \Omega_r$ and $t \in [0, T]$.

(P5) There exists a function $\tilde{m} \in L^1(\mathcal{J}, \mathbb{R}_+)$ and a constant $\mathscr{R}_f := \max\{\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3\} > 0$ such that

$$\|f(t, x, \rho, u)\| \leq \tilde{m}(t) + \mathscr{R}_f(1 + \|x\| + \|\rho\| + \|u\|), \quad \forall t \in \mathcal{J}.$$

(P6) The control operator $B : U \to \Omega$ is a bounded linear operator and $u \in L^2([0, T]; U)$, that is, $B \in \mathcal{L}(U, \Omega)$. and $\|u(t)\|_{L^2} \leq \mathscr{R}_u, \quad \forall t \in \mathcal{J}$. Also, there exists a constant $\mathscr{R}_{\geq}$ 0 such that $\|u_0(t)\|_{L^2} \leq \mathscr{R}_{u_0}, \quad \forall t \in \mathcal{J}$.

(P7) Define the linear operator $\mathbb{W} : L^2([0, \top]; \Omega) \to \Omega$ by

$$\mathbb{W}u = \int_0^{\top - \mathfrak{k}} \mathbb{P}_\alpha^\lambda(\top - \mathfrak{z} - \mathfrak{k})\, B\, u(\mathfrak{z})\, d\mathfrak{z}.$$

The operator $\mathbb{W}$ is assumed to be bounded and possesses a bounded inverse $\mathbb{W}^{-1}$ on $\mathcal{R}(\mathbb{W})$ and $\|\mathbb{W}^{-1}\| = \mathscr{R}_w$.

(P8) There exists three constants, $M_{\mathcal{D}_1}, M_{\mathcal{D}_2}$, and $M_{\mathcal{D}_3} > 0$, such that

$$\chi\left(f(t, \mathcal{D}_1(t), \mathcal{D}_2(t), \mathcal{D}_3(t))\right) \leq M_{\mathcal{D}_1}\chi(\mathcal{D}_1) + M_{\mathcal{D}_2}\chi(\mathcal{D}_2) + M_{\mathcal{D}_3}\chi(\mathcal{D}_3).$$

3. Controllability analysis via Darbo fixed point theorem

Theorem 3.1. *Suppose that postulates* $(P1)$–$(P8)$ *hold. Then, the system* (1.1) *is controllable on* $[0, T]$ *provided that*

$$\mathscr{R}_{\mathfrak{N}} < 1$$

and

$$R_C = \mathscr{R}_{\mathfrak{N}} + \mathscr{R}_\alpha^{\mathbb{Q}}\mathscr{R}_{\mathfrak{N}} + \frac{2\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}\Big(\mathscr{R}_{A\mathfrak{N}} + \|B\|\mathscr{R}_u + M_{\mathcal{D}_1} + M_{\mathcal{D}_2} + \mathscr{R}_u M_{\mathcal{D}_3}\Big) < 1.$$

Proof. Let $\Omega = C([-\mathfrak{k}, \top]; \Omega)$ endowed with the norm

$$\|x\|_\Omega = \sup_{t \in [-\mathfrak{k}, \top]} \|x(t)\|.$$

Step 1: Control function formulation:

For any $x \in \Omega$, define the control

$$\begin{aligned} u_x(t) = \mathbb{W}^{-1}\Bigg[& x_\top - \mathfrak{N}(\top, x(\top)) - \mathbb{Q}_\alpha^\lambda(\top)(x_0 - \mathfrak{N}(0, x(0))) - \int_0^\top \mathbb{P}_\alpha^\lambda(t - \mathfrak{z})A\mathfrak{N}(\mathfrak{z}, x(\mathfrak{z}))\, d\mathfrak{z} \\ & - \int_{-\mathfrak{k}}^0 \mathbb{P}_\alpha^\lambda(\top - \mathfrak{z} - \mathfrak{k})\, Bu_0(\mathfrak{z})\, d\mathfrak{z} - \int_0^\top \mathbb{P}_\alpha^\lambda(\top - \mathfrak{z}) f(\mathfrak{z}, x_\mathfrak{z}, {}_0^{TC}D_\mathfrak{z}^{\alpha,\lambda} x(\mathfrak{z}), u(\mathfrak{z}))d\mathfrak{z}\Bigg](t). \end{aligned} \tag{3.1}$$

Step 2: Operator formulation

By (3.1) control into the mild solution ($\star$) yields $x(\top) = x_\top$. Now, the controllability problem reduces to finding a fixed point of an operator.

Define the operator $\Psi : \Omega_r \to \Omega_r$ by

$$(\Psi x)(t) = \begin{cases} \varphi(t), & t \in [-\mathfrak{k}, 0], \\ \mathfrak{N}(t, x(t)) + \mathbb{Q}_\alpha^\lambda(t)[x_0 - \mathfrak{N}(0, x(0))] + \displaystyle\int_0^t \mathbb{P}_\alpha^\lambda(t - \mathfrak{z})A\mathfrak{N}(\mathfrak{z}, x(\mathfrak{z}))\, d\mathfrak{z} & \\ + \displaystyle\int_{-\mathfrak{k}}^0 \mathbb{P}_\alpha^\lambda(t - \mathfrak{z} - \mathfrak{k})\, Bu_0(\mathfrak{z})\, d\mathfrak{z} + \int_0^{t-\mathfrak{k}} \mathbb{P}_\alpha^\lambda(t - \mathfrak{z} - \mathfrak{k})\, Bu_x(\mathfrak{z})\, d\mathfrak{z} & \\ + \displaystyle\int_0^t \mathbb{P}_\alpha^\lambda(t - \mathfrak{z})\, f(\mathfrak{z}, x_\mathfrak{z}, {}_0^{TC}D_\mathfrak{z}^{\alpha,\lambda} x(\mathfrak{z}), u(\mathfrak{z}))\, d\mathfrak{z}, & t \in [0, \top]. \end{cases}$$

Next, consider the closed ball

$$\Omega_r = \left\{x \in \Omega : \|x\|_\Omega \leq r,\ {}_0^{TC}D_t^{\alpha,\lambda} x(t) \text{ exists for all } t \in [0, T]\right\},$$

$$r = \frac{\mathscr{B}_{\mathfrak{N}} + \mathscr{R}_\alpha^{\mathbb{Q}}[\varphi(0) + \mathscr{B}_{\mathfrak{N}}] + \frac{\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}[\mathscr{B}_{A\mathfrak{N}} + \|B\|(\mathscr{R}_{u_0} + \mathscr{R}_u)]}{1 - \left\{\mathscr{R}_{\mathfrak{N}} + \frac{\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}\Big(\mathscr{R}_{A\mathfrak{N}} + \tilde{m}(t) + \mathscr{R}_f(1 + \mathscr{R}_D)\Big)\right\}}$$

Step 3: Ψ is continuous on Ω_r

The result is trivial for $t \in [-\mathfrak{k}, 0]$.
For $t \in [0, \top]$,
Let $\{x_n\} \subset \Omega_r$ be a sequence such that $x_n \to x$ in Ω_r as $n \to \infty$. Then,

$$\|x_n - x\|_\Omega = \sup_{t\in[-\mathfrak{k},\top]} \|x_n(t) - x(t)\| \longrightarrow 0.$$

We prove that

$$\|\Psi x_n - \Psi x\|_\Omega \to 0.$$

For $t \in [-\mathfrak{k}, 0]$, we have $\Psi x_n(t) = \Psi x(t) = \varphi(t)$, hence

$$\|\Psi x_n(t) - \Psi x(t)\| = 0.$$

Now, let $t \in [0, \top]$. Using the definition of Ψ,

$$\begin{aligned}\|\Psi x_n(t) - \Psi x(t)\| &\leq \|\mathfrak{N}(t, x_n(t)) - \mathfrak{N}(t, x(t))\| + \|\mathbb{Q}^\lambda_\alpha(t)\|\,\|(x_n(0) - \mathfrak{N}(0, x_n(0))) - (x(0) - \mathfrak{N}(0, x(0)))\| \\ &+ \int_0^t \|\mathbb{P}^\lambda_\alpha(t-\mathfrak{z})\|\,\|A\mathfrak{N}(\mathfrak{z}, x_n(\mathfrak{z})) - A\mathfrak{N}(\mathfrak{z}, x(\mathfrak{z}))\|\,d\mathfrak{z} + \int_0^{t-\mathfrak{k}} \|\mathbb{P}^\lambda_\alpha(t-\mathfrak{z}-\mathfrak{k})\|\,\|B\|\,\|u_{x_n}(\mathfrak{z}) - u_x(\mathfrak{z})\|\,d\mathfrak{z} \\ &+ \int_0^t \|\mathbb{P}^\lambda_\alpha(t-\mathfrak{z})\|\Big\|f(\mathfrak{z}, x_{n,\mathfrak{z}}, {}^{TC}D^{\alpha,\lambda}x_n(\mathfrak{z}), u_{x_n}(\mathfrak{z})) - f(\mathfrak{z}, x_\mathfrak{z}, {}^{TC}D^{\alpha,\lambda}x(\mathfrak{z}), u_x(\mathfrak{z}))\Big\|\,d\mathfrak{z},\end{aligned}$$

$$\begin{aligned}\|\Psi x_n(t) - \Psi x(t)\| &\leq \|\mathfrak{N}(t, x_n(t)) - \mathfrak{N}(t, x(t))\| + \|\mathbb{Q}^\lambda_\alpha(t)\|\,\|(x_n(0) - \mathfrak{N}(0, x_n(0))) - (x(0) - \mathfrak{N}(0, x(0)))\| \\ &+ \int_0^t \|\mathbb{P}^\lambda_\alpha(t-\mathfrak{z})\|\,\|A\mathfrak{N}(\mathfrak{z}, x_n(\mathfrak{z})) - A\mathfrak{N}(\mathfrak{z}, x(\mathfrak{z}))\|\,d\mathfrak{z} + \int_0^{t-\mathfrak{k}} \|\mathbb{P}^\lambda_\alpha(t-\mathfrak{z}-\mathfrak{k})\|\,\|B\|\Bigg\|\mathbb{W}^{-1}\Big[x_\top \\ &- \Big\{\mathfrak{N}(\top, x_n(\top)) - \mathfrak{N}(\top, x(\top)) + \mathbb{Q}^\lambda_\alpha(\top)(x_n(0) - x(0)) - \mathbb{Q}^\lambda_\alpha(\top)(\mathfrak{N}(0, x_n(0)) - \mathfrak{N}(0, x(0))) \\ &+ \int_0^\top \mathbb{P}^\lambda_\alpha(\top-\xi)(A\mathfrak{N}(\xi, x_n(\xi)) - A\mathfrak{N}(\xi, x(\xi)))\,d\xi + \int_{-\mathfrak{k}}^0 \mathbb{P}^\lambda_\alpha(\top-\xi-\mathfrak{k})B(u_{n,0}(\xi) - u_0(\xi))\,d\xi \\ &+ \int_0^\top \mathbb{P}^\lambda_\alpha(\top-\xi)\Big(f(\xi, x_{n,\xi}, {}^{TC}_0D^{\alpha,\lambda}_\xi x_n(\xi), u_n(\xi)) - f(\xi, x_\xi, {}^{TC}_0D^{\alpha,\lambda}_\xi x(\xi), u(\xi))\Big)\,d\xi\Big\}\Big]\Bigg\|\,d\mathfrak{z} \\ &+ \int_0^t \|\mathbb{P}^\lambda_\alpha(t-\mathfrak{z})\|\Big\|f(\mathfrak{z}, x_{n,\mathfrak{z}}, {}^{TC}D^{\alpha,\lambda}x_n(\mathfrak{z}), u_{x_n}(\mathfrak{z})) - f(\mathfrak{z}, x_\mathfrak{z}, {}^{TC}D^{\alpha,\lambda}x(\mathfrak{z}), u_x(\mathfrak{z}))\Big\|\,d\mathfrak{z}.\end{aligned}$$

Consider,

$$\begin{aligned}\|u_{x_n} - u_x\| = \|\mathbb{W}^{-1}\Big[x_\top &- \{\mathfrak{N}(\top, x_n(\top)) - \mathfrak{N}(\top, x(\top))\} - \mathbb{Q}^\lambda_\alpha(\top)(x_n(0) - x(0)) + \mathbb{Q}^\lambda_\alpha(\top)(\mathfrak{N}(0, x_n(0)) - \mathfrak{N}(0, x(0))) \\ &- \int_0^\top \mathbb{P}^\lambda_\alpha(\top-\xi)(A\mathfrak{N}(\xi, x_n(\xi)) - A\mathfrak{N}(\xi, x(\xi)))\,d\xi - \int_{-\mathfrak{k}}^0 \mathbb{P}^\lambda_\alpha(\top-\xi-\mathfrak{k})B(u_{n,0}(\xi) - u_0(\xi))\,d\xi \\ &- \int_0^\top \mathbb{P}^\lambda_\alpha(\top-\xi)\Big(f(\xi, x_{n,\xi}, {}^{TC}_0D^{\alpha,\lambda}_\xi x_n(\xi), u_n(\xi)) - f(\xi, x_\xi, {}^{TC}_0D^{\alpha,\lambda}_\xi x(\xi), u(\xi))\Big)\,d\xi\|.\end{aligned}$$

Now,

$$\|u_{x_n}-u_x\| \le \|\mathbb{W}^{-1}\|\Big(r+\mathscr{R}_{\mathfrak{R}}\|x_n-x\|+\mathscr{R}_{\alpha}^{\mathbb{Q}}\, e^{-\lambda\top}\|x_n(0)-x(0)\|+\mathscr{R}_{\alpha}^{\mathbb{Q}}\, e^{-\lambda\top}\mathscr{R}_{\mathfrak{R}}\|x_n(0)-x(0)\|$$
$$+\mathscr{R}_{\alpha}^{\mathbb{P}}\mathscr{R}_{A\mathfrak{R}}\int_0^{\top} e^{-\lambda(\top-\xi)}(\top-\xi)^{\alpha-1}\|x_n(\xi)-x(\xi)\|d\xi+\mathscr{R}_{\alpha}^{\mathbb{P}}\|B\|\int_{-\mathfrak{t}}^{0} e^{-\lambda(\top-\xi)}(\top-\xi)^{\alpha-1}\|u_{n,0}(\xi)-u_0(\xi)\|d\xi$$
$$+\mathscr{R}_{\alpha}^{\mathbb{P}}\int_0^{\top} e^{-\lambda(\top-\xi)}(\top-\xi)^{\alpha-1}\Big[\mathscr{R}_1\|x_{n,\xi}-x_{\xi}\|+\mathscr{R}_2\|^{TC}D^{\alpha,\lambda}x_n(\xi)-{}^{TC}D^{\alpha,\lambda}x(\xi)\|+\mathscr{R}_3\|u_{x_n}(\xi)-u_x(\xi)\|\Big]d\xi\Big).$$

By the results of norm, $\|x_n(0)-x(0)\|\le\|x_n-x\|$, $\|x_n(\xi)-x(\xi)\le\|x_n-x\|$, $\|x_{n,\xi}-x_{\xi}\|=\|x_n(\xi+t)-x(\xi+t)\|\le\|x_n-x\|$ and from the postulate $(P1)$–$(P7)$, we obtain

$$\|u_{x_n}-u_x\| \le \mathscr{R}_w\Big(r+\mathscr{R}_{\mathfrak{R}}\|x_n-x\|+\mathscr{R}_{\alpha}^{\mathbb{Q}}e^{-\lambda\top}\,\|x_n-x\|+\mathscr{R}_{\alpha}^{\mathbb{Q}}\,\mathscr{R}_{\mathfrak{R}}e^{-\lambda\top}\|x_n-x\|+\mathscr{R}_{\alpha}^{\mathbb{P}}\mathscr{R}_{A\mathfrak{R}}\|x_n-x\|$$
$$\times\int_0^{\top} e^{-\lambda(\top-\xi)}(\top-\xi)^{\alpha-1}d\xi+\mathscr{R}_{\alpha}^{\mathbb{P}}\|\Big[\mathscr{R}_1\|x_n-x\|+\mathscr{R}_2\mathscr{R}_D\|x_n-x\|+\mathscr{R}_3\|u_{x_n}-u_x\|\Big]$$
$$\times\int_0^{\top} e^{-\lambda(\top-\xi)}(\top-\xi)^{\alpha-1}d\xi\Big).$$

Since $e^{-\lambda\top}\le 1$, $\int_0^{\top} e^{-\lambda(\top-\xi)}(\top-\xi)^{\alpha-1}d\xi\le\frac{\Gamma(\alpha)}{\lambda^{\alpha}}$ and

$$\|u_{x_n}-u_x\| \le \mathscr{R}_w\Big(r+\mathscr{R}_{\mathfrak{R}}\|x_n-x\|+\mathscr{R}_{\alpha}^{\mathbb{Q}}\,\|x_n-x\|+\mathscr{R}_{\alpha}^{\mathbb{Q}}\,\mathscr{R}_{\mathfrak{R}}\|x_n-x\|+\frac{\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)\mathscr{R}_{A\mathfrak{R}}}{\lambda^{\alpha}}\|x_n-x\|$$
$$+\frac{\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)}{\lambda^{\alpha}}\|\Big[\mathscr{R}_1\|x_n-x\|+\mathscr{R}_2\mathscr{R}_D\|x_n-x\|+\mathscr{R}_3\|u_{x_n}-u_x\|\Big]\Big)$$

$$\|u_{x_n}-u_x\| \le \frac{\mathscr{R}_w\Big(r+\mathscr{R}_{\mathfrak{R}}+\mathscr{R}_{\alpha}^{\mathbb{Q}}+\mathscr{R}_{\alpha}^{\mathbb{Q}}\,\mathscr{R}_{\mathfrak{R}}+\frac{\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)\mathscr{R}_{A\mathfrak{R}}}{\lambda^{\alpha}}+\frac{\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)}{\lambda^{\alpha}}\|\Big[\mathscr{R}_1+\mathscr{R}_2\mathscr{R}_D\Big]\Big)}{1-\Big(\frac{\mathscr{R}_w\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)\mathscr{R}_3}{\lambda^{\alpha}}\Big)}\|x_n-x\|$$
$$\le \frac{\mathscr{R}_w\Big(r+\mathscr{R}_{\mathfrak{R}}+\mathscr{R}_{\alpha}^{\mathbb{Q}}(1+\mathscr{R}_{\mathfrak{R}})+\frac{\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)}{\lambda^{\alpha}}[\mathscr{R}_{A\mathfrak{R}}+\mathscr{R}_1+\mathscr{R}_2\mathscr{R}_D]\Big)}{1-\Big(\frac{\mathscr{R}_w\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)\mathscr{R}_3}{\lambda^{\alpha}}\Big)}\|x_n-x\|$$
$$\le \frac{\mathscr{R}_w\Big(\lambda^{\alpha}[r+\mathscr{R}_{\mathfrak{R}}+\mathscr{R}_{\alpha}^{\mathbb{Q}}(1+\mathscr{R}_{\mathfrak{R}})]+\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)[\mathscr{R}_{A\mathfrak{R}}+\mathscr{R}_1+\mathscr{R}_2\mathscr{R}_D]\Big)}{\lambda^{\alpha}-\mathscr{R}_w\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)\mathscr{R}_3}\|x_n-x\|$$
$$\le \hat{H}\|x_n-x\|.$$

Where,

$$\hat{H}=\frac{\mathscr{R}_w\Big(\lambda^{\alpha}[r+\mathscr{R}_{\mathfrak{R}}+\mathscr{R}_{\alpha}^{\mathbb{Q}}(1+\mathscr{R}_{\mathfrak{R}})]+\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)[\mathscr{R}_{A\mathfrak{R}}+\mathscr{R}_1+\mathscr{R}_2\mathscr{R}_D]\Big)}{\lambda^{\alpha}-\mathscr{R}_w\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)\mathscr{R}_3},$$

$$
\begin{aligned}
\|\Psi x_n(t) - \Psi x(t)\| \leq{}& \|\mathfrak{N}(t, x_n(t)) - \mathfrak{N}(t, x(t))\| + \|\mathbb{Q}_\alpha^\lambda(t)\| \, \|(x_n(0) - \mathfrak{N}(0, x_n(0))) - (x(0) - \mathfrak{N}(0, x(0)))\| \\
&+ \int_0^t \|\mathbb{P}_\alpha^\lambda(t-\mathfrak{z})\| \, \|A\mathfrak{N}(\mathfrak{z}, x_n(\mathfrak{z})) - A\mathfrak{N}(\mathfrak{z}, x(\mathfrak{z}))\| \, d\mathfrak{z} + \int_0^{t-\mathfrak{k}} \|\mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})\| \, \|B\| \, \|u_{x_n}(\mathfrak{z}) - u_x(\mathfrak{z})\| \, d\mathfrak{z} \\
&+ \int_0^t \|\mathbb{P}_\alpha^\lambda(t-\mathfrak{z})\| \left\| f(\mathfrak{z}, x_{n,\mathfrak{z}}, {}^{TC}D^{\alpha,\lambda} x_n(\mathfrak{z}), u_{x_n}(\mathfrak{z})) - f(\mathfrak{z}, x_{\mathfrak{z}}, {}^{TC}D^{\alpha,\lambda} x(\mathfrak{z}), u_x(\mathfrak{z})) \right\| d\mathfrak{z} \\
\leq{}& \|\mathfrak{N}(t, x_n(t)) - \mathfrak{N}(t, x(t))\| + \|\mathbb{Q}_\alpha^\lambda(t)\| \, \|(x_n(0) - \mathfrak{N}(0, x_n(0))) - (x(0) - \mathfrak{N}(0, x(0)))\| \\
&+ \int_0^t \|\mathbb{P}_\alpha^\lambda(t-\mathfrak{z})\| \, \|A\mathfrak{N}(\mathfrak{z}, x_n(\mathfrak{z})) - A\mathfrak{N}(\mathfrak{z}, x(\mathfrak{z}))\| \, d\mathfrak{z} + \int_0^{t-\mathfrak{k}} \|\mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})\| \, \|B\| \hat{H} \|x_n - x\| \, d\mathfrak{z} \\
&+ \int_0^t \|\mathbb{P}_\alpha^\lambda(t-\mathfrak{z})\| \left\| f(\mathfrak{z}, x_{n,\mathfrak{z}}, {}^{TC}D^{\alpha,\lambda} x_n(\mathfrak{z}), u_{x_n}(\mathfrak{z})) - f(\mathfrak{z}, x_{\mathfrak{z}}, {}^{TC}D^{\alpha,\lambda} x(\mathfrak{z}), u_x(\mathfrak{z})) \right\| d\mathfrak{z} \\
\leq{}& \mathscr{R}_{\mathfrak{N}} + \mathscr{R}_\alpha^{\mathbb{Q}}[\|x_n(0) - x(0)\| - \mathscr{R}_{\mathfrak{N}} \|x_n(0) - x(0)\|] + \frac{\mathscr{R}_\alpha^{\mathbb{P}} \Gamma(\alpha) \mathscr{R}_{A\mathfrak{N}}}{\lambda^\alpha} \|x_n - x\| \\
&+ \frac{\mathscr{R}_\alpha^{\mathbb{P}} \Gamma(\alpha) \, \|B\| \hat{H}}{\lambda^\alpha} \|x_n - x\| + \frac{\mathscr{R}_\alpha^{\mathbb{P}} \Gamma(\alpha)}{\lambda^\alpha} \Big[\mathscr{R}_1 \|x_n - x\| + \mathscr{R}_2 \mathscr{R}_D \|x_n - x\| + \mathscr{R}_3 \hat{H} \|x_n - x\|. \Big] \\
\leq{}& \left(\mathscr{R}_{\mathfrak{N}} + \mathscr{R}_\alpha^{\mathbb{Q}}[1 - \mathscr{R}_{\mathfrak{N}}] + \frac{\mathscr{R}_\alpha^{\mathbb{P}} \Gamma(\alpha)}{\lambda^\alpha} \Big[\mathscr{R}_{A\mathfrak{N}} + \|B\| \hat{H} (\mathscr{R}_1 + \mathscr{R}_2 \mathscr{R}_D + \mathscr{R}_3 \hat{H}) \Big] \right) \|x_n - x\| \\
\leq{}& \mathscr{H}_c \|x_n - x\|_{\Omega_r}.
\end{aligned}
$$

By Postulate $(P4)$, the tempered Caputo derivative is well defined on the admissible set Ω_r and satisfies the Lipschitz estimate, and thus we obtain

$$
\|\Psi x_n - \Psi x\|_{\Omega_r} \leq \mathscr{H}_c \|x_n - x\|_{\Omega_r}
$$

Here,

$$
\mathscr{H}_c = \mathscr{R}_{\mathfrak{N}} + \mathscr{R}_\alpha^{\mathbb{Q}}[1 - \mathscr{R}_{\mathfrak{N}}] + \frac{\mathscr{R}_\alpha^{\mathbb{P}} \Gamma(\alpha)}{\lambda^\alpha} \Big[\mathscr{R}_{A\mathfrak{N}} + \|B\| \hat{H} (\mathscr{R}_1 + \mathscr{R}_2 \mathscr{R}_D + \mathscr{R}_3 \hat{H}) \Big].
$$

Therefore, Ψ is continuous on Ω_r.

Step 4: $\Psi(\Omega_r) \subset \Omega_r$

Consider, $x(t) \in \Omega_r$
Using postulates $(P1)$–$(P7)$, we yield

$$
\begin{aligned}
\|u_x(t)\| ={}& \bigg\| \mathbb{W}^{-1} \bigg[x_\top - \mathfrak{N}(\top, x(\top)) - \mathbb{Q}_\alpha^\lambda(\top)\big(x_0 - \mathfrak{N}(0, x(0))\big) - \int_0^\top \mathbb{P}_\alpha^\lambda(t-\mathfrak{z}) A\mathfrak{N}(\mathfrak{z}, x(\mathfrak{z})) \, d\mathfrak{z} \\
&- \int_{-\mathfrak{k}}^0 \mathbb{P}_\alpha^\lambda(\top-\mathfrak{z}-\mathfrak{k}) \, Bu_0(\mathfrak{z}) \, d\mathfrak{z} - \int_0^\top \mathbb{P}_\alpha^\lambda(\top-\mathfrak{z}) f\big(\mathfrak{z}, x_{\mathfrak{z}}, {}_0^{TC}D_{\mathfrak{z}}^{\alpha,\lambda} x(\mathfrak{z}), u(\mathfrak{z})\big) d\mathfrak{z} \bigg](t) \bigg\| \\
\leq{}& \mathscr{R}_w \bigg[\|x_\top\| + \mathscr{R}_{\mathfrak{N}} \|x_\top\| + \mathscr{B}_{\mathfrak{N}} + \mathscr{R}_\alpha^{\mathbb{Q}} e^{-\lambda\top} (\|x_0\| - \mathscr{B}_{\mathfrak{N}}) \\
&+ \frac{\mathscr{R}_\alpha^{\mathbb{P}} \Gamma(\alpha)}{\lambda^\alpha} \Big[\mathscr{R}_{A\mathfrak{N}} \|x(\mathfrak{z})\| + \mathscr{B}_{A\mathfrak{N}} + \|B\| \mathscr{R}_{u_0} + \tilde{m}(t) + \mathscr{R}_f \big(1 + \|x_{\mathfrak{z}}\| + \|{}^{TC}D^{\alpha,\lambda} x(\mathfrak{z})\| + \|u_x(\mathfrak{z})\|\big) \Big] \bigg] \\
\leq{}& \mathscr{R}_w \bigg[\|x_\top\| + \mathscr{R}_{\mathfrak{N}} \|x_\top\| + \mathscr{B}_{\mathfrak{N}} + \mathscr{R}_\alpha^{\mathbb{Q}} (\|\varphi(0)\| + \mathscr{B}_{\mathfrak{N}})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}\Big[\mathscr{R}_{A\mathfrak{N}}\|x_\top\|+\mathscr{B}_{A\mathfrak{N}}+\|B\|\mathscr{R}_{u_0}+\tilde{m}(t)+\mathscr{R}_f\big(1+\|x_\top\|+\mathscr{R}_D\|x_\top\|+\|u_{x_\top}\|\big)\Big]\Bigg]\\
&\le \mathscr{R}_w\Bigg[r(1+\mathscr{R}_{\mathfrak{N}}+\frac{\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}(\mathscr{R}_{A\mathfrak{N}}+\mathscr{R}_f(1+\mathscr{R}_D)))+\mathscr{B}_{\mathfrak{N}}+\mathscr{R}_\alpha^{\mathbb{Q}}(\|\varphi(0)\|+\mathscr{B}_{\mathfrak{N}})\\
&+\frac{\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}\Big[\mathscr{B}_{A\mathfrak{N}}+\|B\|\mathscr{R}_{u_0}+\tilde{m}(t)+\mathscr{R}_f\big(1+\|u_x(t)\|\|\big)\Big]\Bigg]\\
&\le \frac{\mathscr{R}_w\Bigg[r\bar{\mu}+\lambda^\alpha\Big[\mathscr{B}_{\mathfrak{N}}+\mathscr{R}_\alpha^{\mathbb{Q}}(\|\varphi(0)\|+\mathscr{B}_{\mathfrak{N}})\Big]+\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)\Big[\mathscr{B}_{A\mathfrak{N}}+\|B\|\mathscr{R}_{u_0}+\tilde{m}(t)+\mathscr{R}_f\Big]\Bigg]}{\lambda^\alpha-\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)\mathscr{R}_f}\\
&=\mathscr{R}_u,
\end{aligned}
$$

$$
\bar{\mu}:=\lambda^\alpha(1+\mathscr{R}_{\mathfrak{N}})+\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)(\mathscr{R}_{A\mathfrak{N}}+\mathscr{R}_f(1+\mathscr{R}_D)).
$$

Also,

$$
(\Psi x)(t)=\begin{cases}
\varphi(t), & t\in[-\mathfrak{k},0],\\
\mathfrak{N}(t,x(t))+\mathbb{Q}_\alpha^\lambda(t)[x_0-\mathfrak{N}(0,x(0))]+\int_0^t\mathbb{P}_\alpha^\lambda(t-\mathfrak{z})A\mathfrak{N}(\mathfrak{z},x(\mathfrak{z}))\,d\mathfrak{z} & \\
+\int_{-\mathfrak{k}}^0\mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})\,Bu_0(\mathfrak{z})\,d\mathfrak{z}+\int_0^{t-\mathfrak{k}}\mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})\,Bu_x(\mathfrak{z})\,d\mathfrak{z} & \\
+\int_0^t\mathbb{P}_\alpha^\lambda(t-\mathfrak{z})\,f\big(\mathfrak{z},x_{\mathfrak{z}},{}_0^{TC}D_{\mathfrak{z}}^{\alpha,\lambda}x(\mathfrak{z}),u(\mathfrak{z})\big)\,d\mathfrak{z}, & t\in[0,\top].
\end{cases}
$$

$$
\|(\Psi x)(t)\|=\begin{cases}
\|\varphi(t)\|, & t\in[-\mathfrak{k},0],\\
\|\mathfrak{N}(t,x(t))+\mathbb{Q}_\alpha^\lambda(t)[x_0-\mathfrak{N}(0,x(0))]+\int_0^t\mathbb{P}_\alpha^\lambda(t-\mathfrak{z})A\mathfrak{N}(\mathfrak{z},x(\mathfrak{z}))\,d\mathfrak{z} & \\
+\int_{-\mathfrak{k}}^0\mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})\,Bu_0(\mathfrak{z})\,d\mathfrak{z}+\int_0^{t-\mathfrak{k}}\mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})\,Bu_x(\mathfrak{z})\,d\mathfrak{z} & \\
+\int_0^t\mathbb{P}_\alpha^\lambda(t-\mathfrak{z})\,f\big(\mathfrak{z},x_{\mathfrak{z}},{}_0^{TC}D_{\mathfrak{z}}^{\alpha,\lambda}x(\mathfrak{z}),u(\mathfrak{z})\big)\,d\mathfrak{z}\|, & t\in[0,\top].
\end{cases}
$$

Now,

$$
\|(\Psi x)(t)\|\le\begin{cases}
\|\varphi(t)\|, & t\in[-\mathfrak{k},0],\\
\mathscr{R}_{\mathfrak{N}}\|x(t))\|+\mathscr{B}_{\mathfrak{N}}+\mathscr{R}_\alpha^{\mathbb{Q}}e^{-\lambda t}[\|x_0\|+\mathscr{B}_{\mathfrak{N}}]+\frac{\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}\Bigg[\mathscr{R}_{A\mathfrak{N}}\|x\|+\mathscr{B}_{A\mathfrak{N}}+\|B\|\mathscr{R}_{u_0} & \\
+\|B\|\mathscr{R}_u+\tilde{m}(t)+\mathscr{R}_f\big(1+\|x_t\|+\mathscr{R}_D\|x(t)\|+\mathscr{R}_u\big)\Bigg], & t\in[0,\top].
\end{cases}
$$

Hence,

$$
\|\Psi x\|_\Omega\le\mathscr{H}_1+\mathscr{H}_2 r
$$

where, $\mathscr{H}_1 = \mathscr{R}_{\mathfrak{N}} + \frac{\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}\Bigg[\mathscr{R}_{A\mathfrak{N}} + \mathscr{R}_f(1+\mathscr{R}_D)\Bigg]$, and

$\mathscr{H}_2 = \mathscr{B}_{\mathfrak{N}} + \mathscr{R}_\alpha^{\mathbb{Q}}\|\varphi(0)\| + \mathscr{B}_{\mathfrak{N}} + \frac{\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}\Bigg[\mathscr{B}_{A\mathfrak{N}} + \|B\|\mathscr{R}_{u_0} + \|B\|\mathscr{R}_u + \tilde{m}(t) + \mathscr{R}_f + \mathscr{R}_f\mathscr{R}_u\Bigg]$.

By choosing $\frac{\mathscr{H}_1}{1-\mathscr{H}_2} \leq r$ for the constants $\mathscr{H}_1$ & $\mathscr{H}_2 > 0$, we get

$$\Psi(\Omega_r) \subset \Omega_r.$$

Step 5: $\{\Psi(x) : x \in \Omega_r\}$ *is a set of equi-continuous function*

Let $t_1, t_2 \in [0, \top]$ with $t_1 < t_2$ and let $x \in \Omega_r$.
Now,

$$\begin{aligned}
&\|(\Psi x)(t_2) - (\Psi x)(t_1)\| \\
&\leq \|\mathfrak{N}(t_2, x(t_2)) - \mathfrak{N}(t_1, x(t_1))\| + \left\|\left(\mathbb{Q}_\alpha^\lambda(t_2) - \mathbb{Q}_\alpha^\lambda(t_1)\right)[x_0 - \mathfrak{N}(0, x(0))]\right\| + \left\|\int_0^{t_2} \mathbb{P}_\alpha^\lambda(t_2 - \mathfrak{z})A\mathfrak{N}(\mathfrak{z}, x(\mathfrak{z}))\,d\mathfrak{z}\right. \\
&\quad \left. - \int_0^{t_1} \mathbb{P}_\alpha^\lambda(t_1 - \mathfrak{z})A\mathfrak{N}(\mathfrak{z}, x(\mathfrak{z}))\,d\mathfrak{z}\right\| + \left\|\int_{-\mathfrak{k}}^0 \Big[\mathbb{P}_\alpha^\lambda(t_2 - \mathfrak{z} - \mathfrak{k}) - \mathbb{P}_\alpha^\lambda(t_1 - \mathfrak{z} - \mathfrak{k})\Big]Bu_0(\mathfrak{z})\,d\mathfrak{z}\right\| \\
&\quad + \left\|\int_0^{t_2-\mathfrak{k}} \mathbb{P}_\alpha^\lambda(t_2 - \mathfrak{z} - \mathfrak{k})Bu_x(\mathfrak{z})\,d\mathfrak{z} - \int_0^{t_1-\mathfrak{k}} \mathbb{P}_\alpha^\lambda(t_1 - \mathfrak{z} - \mathfrak{k})Bu_x(\mathfrak{z})\,d\mathfrak{z}\right\| \\
&\quad + \left\|\int_0^{t_2} \mathbb{P}_\alpha^\lambda(t_2 - \mathfrak{z})f(\mathfrak{z}, x_{\mathfrak{z}}, {}_0^{TC}D_{\mathfrak{z}}^{\alpha,\lambda}x(\mathfrak{z}), u(\mathfrak{z}))\,d\mathfrak{z} - \int_0^{t_1} \mathbb{P}_\alpha^\lambda(t_1 - \mathfrak{z})f(\mathfrak{z}, x_{\mathfrak{z}}, {}_0^{TC}D_{\mathfrak{z}}^{\alpha,\lambda}x(\mathfrak{z}), u(\mathfrak{z}))\,d\mathfrak{z}\right\| \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

$$I_1 = \|\mathfrak{N}(t_2, x(t_2)) - \mathfrak{N}(t_1, x(t_1))\| \longrightarrow 0, \qquad t_2 \to t_1.$$

$$I_2 \leq \|\mathbb{Q}_\alpha^\lambda(t_2) - \mathbb{Q}_\alpha^\lambda(t_1)\|\left(\|x_0\| + \mathscr{R}_{\mathfrak{N}}\|x\| + \mathscr{B}_{\mathfrak{N}}\right) \longrightarrow 0.$$

$$\begin{aligned}
I_3 &\leq \int_0^{t_1} \|\mathbb{P}_\alpha^\lambda(t_2 - \mathfrak{z}) - \mathbb{P}_\alpha^\lambda(t_1 - \mathfrak{z})\|\,\|A\mathfrak{N}(\mathfrak{z}, x(\mathfrak{z}))\|\,d\mathfrak{z} + \int_{t_1}^{t_2} \|\mathbb{P}_\alpha^\lambda(t_2 - \mathfrak{z})\|\,\|A\mathfrak{N}(\mathfrak{z}, x(\mathfrak{z}))\|\,d\mathfrak{z} \\
&\leq (\mathscr{R}_{A\mathfrak{N}}\|x\| + \mathscr{B}_{A\mathfrak{N}})\int_0^{t_1} \|\mathbb{P}_\alpha^\lambda(t_2 - \mathfrak{z}) - \mathbb{P}_\alpha^\lambda(t_1 - \mathfrak{z})\|\,d\mathfrak{z} + \mathscr{R}_\alpha^{\mathbb{P}}(\mathscr{R}_{A\mathfrak{N}}\|x\| + \mathscr{B}_{A\mathfrak{N}})\lambda^{-\alpha}\gamma(\alpha, \lambda(t_2 - t_1)).
\end{aligned}$$

$$I_4 \leq \|B\|\,\mathscr{R}_{u_0}\int_{-\mathfrak{k}}^0 \|\mathbb{P}_\alpha^\lambda(t_2 - \mathfrak{z} - \mathfrak{k}) - \mathbb{P}_\alpha^\lambda(t_1 - \mathfrak{z} - \mathfrak{k})\|\,d\mathfrak{z}.$$

$$I_5 \leq \|B\|\,\mathscr{R}_u\int_0^{t_1-\mathfrak{k}} \|\mathbb{P}_\alpha^\lambda(t_2 - \mathfrak{z} - \mathfrak{k}) - \mathbb{P}_\alpha^\lambda(t_1 - \mathfrak{z} - \mathfrak{k})\|\,d\mathfrak{z} + \mathscr{R}_\alpha^{\mathbb{P}}\|B\|\mathscr{R}_u\lambda^{-\alpha}\gamma(\alpha, \lambda(t_2 - t_1)).$$

$$I_6 \leq \int_0^{t_1} \|\mathbb{P}_\alpha^\lambda(t_2 - \mathfrak{z}) - \mathbb{P}_\alpha^\lambda(t_1 - \mathfrak{z})\|\,\|f(\mathfrak{z}, x_{\mathfrak{z}}, {}_0^{TC}D_{\mathfrak{z}}^{\alpha,\lambda}x(\mathfrak{z}), u(\mathfrak{z}))\|\,d\mathfrak{z}$$

$$+ \mathscr{R}_\alpha^{\mathbb{P}} \int_{t_1}^{t_2} e^{-\lambda(t_2-\mathfrak{z})}(t_2-\mathfrak{z})^{\alpha-1} \|f(\mathfrak{z}, x_{\mathfrak{z}}, {}_0^{TC}D_{\mathfrak{z}}^{\alpha,\lambda}x(\mathfrak{z}), u(\mathfrak{z}))\| \, d\mathfrak{z}.$$

Hence,

$$\lim_{t_2 \to t_1} \|(\Psi x)(t_2) - (\Psi x)(t_1)\| = 0.$$

Therefore, the family $\Psi(\Omega_r)$ is equicontinuous on $[0, \top]$.

Since

$$(\Psi x)(t) = \varphi(t), \qquad t \in [-\mathfrak{k}, 0],$$

for every $x \in \Omega_r$, the family $\{\Psi x : x \in \Omega_r\}$ is trivially equicontinuous on $[-\mathfrak{k}, 0]$. Moreover,

$$(\Psi x)(0) = \varphi(0) = x_0,$$

so the continuity at $t = 0$ is preserved. Hence, combining this with the equicontinuity established on $[0, \top]$, we conclude that $\Psi(\Omega_r)$ is equicontinuous on the whole interval $[-\mathfrak{k}, \top]$.

Step 6: $\Psi : \Omega_r \to \Omega_r$ is a k-set contraction

Let $\mathcal{D}$ be an arbitrary bounded subset of Ω_r. By Proposition 3.12, for every $\zeta > 0$, there exists a countable subset $\{x_\ell\}_{\ell=1}^\infty \subset \mathcal{D}$ such that

$$\chi(\Psi(\mathcal{D})) \leq 2\chi(\{\Psi x_\ell\}_{\ell=1}^\infty) + \zeta,$$

where $\chi(\cdot)$ denotes the Kuratowski measure of noncompactness.

Since every admissible trajectory in Ω_r possesses a tempered Caputo fractional derivative, the image $\{{}_0^{TC}D_t^{\alpha,\lambda}x : x \in \mathcal{D}\}$ is well defined for every bounded subset $\mathcal{D} \subset \Omega_r$. Moreover, by Postulate (P4),

$$\chi\Big({}_0^{TC}D_t^{\alpha,\lambda}(\mathcal{D})\Big) \leq \mathscr{R}_D\chi(\mathcal{D}).$$

Since $\Psi(\Omega_r)$ is equicontinuous on $\mathcal{J} = [-\mathfrak{k}, \top]$ by Step 5, the family $\{\Psi x_\ell\}_{\ell=1}^\infty$ is also equicontinuous. Therefore, it is sufficient to estimate

$$\chi\left(\{(\Psi x_\ell)(t)\}_{\ell=1}^\infty\right), \qquad t \in \mathcal{J}.$$

For $t \in [-\mathfrak{k}, 0]$, we have

$$(\Psi x_\ell)(t) = \varphi(t), \qquad \ell = 1, 2, \ldots,$$

and therefore

$$\chi\big((\Psi\mathcal{D})(t)\big) = \chi(\{\varphi(t)\}) = 0.$$

Hence, it remains to estimate

$$\chi\big((\Psi\mathcal{D})(t)\big), \qquad t \in (0, \top].$$

For each fixed $t \in \mathcal{J}$, define

$$\mathcal{D}_1(t) = \{x(t) : x \in D\},\ \mathcal{D}_2(t) = \left\{{}_0^{TC}D_t^{\alpha,\lambda}x(t) : x \in \mathcal{D}\right\},\ \text{and}\ \mathcal{D}_3(t) = \{u_x(t) : x \in \mathcal{D}\}.$$

Since the evaluation operator $E_t : \Omega_r \to \mathbb{X}$, $E_t(x) = x(t)$ is continuous, it follows from the basic properties of the Kuratowski measure of noncompactness that

$$\chi(\mathcal{D}_1(t)) = \chi(E_t(\mathcal{D})) \le \chi(\mathcal{D}).$$

Similarly, by Postulate (P3), the tempered Caputo fractional derivative operator $x \longmapsto {}_0^{TC}D_t^{\alpha,\lambda}x(t)$ is Lipschitz continuous on Ω_r, and hence it is continuous.

Therefore,

$$\chi(\mathcal{D}_2(t)) = \chi\Big(\big\{{}_0^{TC}D_t^{\alpha,\lambda}x(t) : x \in \mathcal{D}\big\}\Big) \le \chi(\mathcal{D}).$$

Moreover, by the continuity of the control mapping $x \longmapsto u_x(t)$, established in the previous step, we have

$$\chi(\mathcal{D}_3(t)) = \chi(\{u_x(t) : x \in \mathcal{D}\}) \le \chi(\mathcal{D}).$$

Consequently, by the Postulate $(P8)$

$$M_{\mathcal{D}_1}\chi(\mathcal{D}_1(t)) + M_{\mathcal{D}_2}\chi(\mathcal{D}_2(t)) + M_{\mathcal{D}_3}\chi(\mathcal{D}_3(t)) \le \left(M_{\mathcal{D}_1} + M_{\mathcal{D}_2} + M_{\mathcal{D}_3}\right)\chi(\mathcal{D}).$$

Moreover, since the control mapping $x \longmapsto u_x(t)$ is Lipschitz continuous, it is continuous. Hence,

$$\chi(\mathcal{D}_3(t)) = \chi(\{u_x(t) : x \in \mathcal{D}\}) \le \mathscr{R}_u\chi(\mathcal{D}).$$

For every $t \in (0, \top]$, we have

$$\begin{aligned}
\chi((\Psi\mathcal{D})(t)) &= \chi\left(\{(\Psi x_\ell)(t)\}_{\ell=1}^\infty\right) \\
&= \chi\Bigg(\Bigg\{\mathfrak{N}(t, x_\ell(t)) + \mathbb{Q}_\alpha^\lambda(t)[x_0 - \mathfrak{N}(0, x_\ell(0))] + \int_0^t \mathbb{P}_\alpha^\lambda(t-\mathfrak{z})A\mathfrak{N}(\mathfrak{z}, x_\ell(\mathfrak{z}))\,d\mathfrak{z} \\
&\quad + \int_{-\mathfrak{k}}^0 \mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})Bu_0(\mathfrak{z})\,d\mathfrak{z} + \int_0^{t-\mathfrak{k}} \mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})Bu_{x_\ell}(\mathfrak{z})\,d\mathfrak{z} \\
&\quad + \int_0^t \mathbb{P}_\alpha^\lambda(t-\mathfrak{z})f\Big(\mathfrak{z}, x_{\ell,\mathfrak{z}}, {}_0^{TC}D_\mathfrak{z}^{\alpha,\lambda}x_\ell(\mathfrak{z}), u_{x_\ell}(\mathfrak{z})\Big)\,d\mathfrak{z}\Bigg\}_{\ell=1}^\infty\Bigg).
\end{aligned}$$

Since the term $\int_{-\mathfrak{k}}^0 \mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})Bu_0(\mathfrak{z})\,d\mathfrak{z}$ is independent of x_ℓ, it does not contribute to the Kuratowski measure of noncompactness.

Using the subadditivity of the Kuratowski measure of noncompactness, the boundedness of the linear operators $\mathbb{Q}_\alpha^\lambda$, $\mathbb{P}_\alpha^\lambda$, and B, together with the estimates established above, we obtain

$$\begin{aligned}
\chi((\Psi\mathcal{D})(t)) \le& \chi(\mathfrak{N}(\mathcal{D}_1)) + \chi\Big(\mathbb{Q}_\alpha^\lambda(t)\mathfrak{N}(\mathcal{D}_1)\Big) + \chi\Bigg(\int_0^t \mathbb{P}_\alpha^\lambda(t-\mathfrak{z})A\mathfrak{N}(\mathfrak{z}, \mathcal{D}_1)\,d\mathfrak{z}\Bigg) \\
&+ \chi\Bigg(\int_0^{t-\mathfrak{k}} \mathbb{P}_\alpha^\lambda(t-\mathfrak{z}-\mathfrak{k})B\mathcal{D}_3\,d\mathfrak{z}\Bigg) + \chi\Bigg(\int_0^t \mathbb{P}_\alpha^\lambda(t-\mathfrak{z})f(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)\,d\mathfrak{z}\Bigg).
\end{aligned}$$

Since

$$\chi(\mathfrak{N}(\mathcal{D}_1)) \le \mathscr{R}_\mathfrak{N}\chi(\mathcal{D}_1), \quad \text{and} \quad \chi\Big(\mathbb{Q}_\alpha^\lambda(t)\mathfrak{N}(\mathcal{D}_1)\Big) \le \mathscr{R}_\alpha^\mathbb{Q}\mathscr{R}_\mathfrak{N}\chi(\mathcal{D}_1),$$

From Proposition 3.10 and 3.11, we obtain

$$\chi((\Psi\mathcal{D})(t)) \le \Big(\mathscr{R}_\mathfrak{N} + \mathscr{R}_\alpha^\mathbb{Q}\mathscr{R}_\mathfrak{N}\Big)\chi(\mathcal{D}_1) + \frac{2\mathscr{R}_\alpha^\mathbb{P}\Gamma(\alpha)}{\lambda^\alpha}\mathscr{R}_{A\mathfrak{N}}\chi(\mathcal{D}_1) + \frac{2\mathscr{R}_\alpha^\mathbb{P}\Gamma(\alpha)}{\lambda^\alpha}\|B\|\chi(\mathcal{D}_3)$$

$$+\frac{2\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)}{\lambda^{\alpha}}\left(M_{\mathcal{D}_1}\chi(\mathcal{D}_1)+M_{\mathcal{D}_2}\chi(\mathcal{D}_2)+M_{\mathcal{D}_3}\chi(\mathcal{D}_3)\right).$$

Using

$$\chi(\mathcal{D}_1)\leq\chi(\mathcal{D}),\qquad \chi(\mathcal{D}_2)\leq\chi(\mathcal{D}),\qquad \chi(\mathcal{D}_3)\leq\mathscr{R}_u\chi(\mathcal{D}),$$

where $\mathscr{R}_u$ is the Lipschitz constant of the control mapping $x\mapsto u_x$, we obtain

$$\chi((\Psi\mathcal{D})(t))\leq\left[\mathscr{R}_{\mathfrak{N}}+\mathscr{R}_{\alpha}^{\mathbb{Q}}\mathscr{R}_{\mathfrak{N}}+\frac{2\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)}{\lambda^{\alpha}}\Big[\mathscr{R}_{A\mathfrak{N}}\|B\|\mathscr{R}_u+\left(M_{\mathcal{D}_1}+M_{\mathcal{D}_2}+\mathscr{R}_uM_{\mathcal{D}_3}\right)\Big]\right]\chi(\mathcal{D}).$$

Therefore, $\chi((\Psi\mathcal{D})(t))\leq R_C\,\chi(\mathcal{D})$,

where the constant R_C is determined after substituting the estimate of

$$R_C=\mathscr{R}_{\mathfrak{N}}+\mathscr{R}_{\alpha}^{\mathbb{Q}}\mathscr{R}_{\mathfrak{N}}+\frac{2\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)}{\lambda^{\alpha}}\Big(\mathscr{R}_{A\mathfrak{N}}+\|B\|\mathscr{R}_u+M_{\mathcal{D}_1}+M_{\mathcal{D}_2}+\mathscr{R}_uM_{\mathcal{D}_3}\Big).$$

Hence, $\chi((\Psi D)(t))\leq R_C\,\chi(D),\qquad t\in[0,\top]$.

Assume that $R_C<1$.

Then, $\Psi:\Omega_r\to\Omega_r$ is a k-set contraction with respect to the Hausdorff measure of noncompactness. Therefore, by Darbo's fixed-point theorem, the operator Ψ possesses a fixed point $x\in\Omega_r$ such that $\Psi x=x$.

Consequently, x is a mild solution of the nonlinear tempered fractional control system. Moreover, by the construction of the control function u_x, we have

$$x(\top)=x_{\top},$$

which shows that the system (1.1) is controllable on the interval $[0,\top]$. □

4. Parameter sensitivity analysis

Based on the theoretical postulates, controllability constant $\mathscr{R}_c$ is involving a set of constants and the range of these play a vital role of controllability of the prescribed system. This section provides an idea to choose the appropriate constant corresponding to our application problem for ensuring the controllability results.

4.1. Sensitivity analysis of the controllability constant

Theorem 4.1. *Suppose that postulates* $(P1)$–$(P7)$ *hold and let*

$$\mathscr{R}_C=\mathscr{R}_{\mathfrak{N}}+\mathscr{R}_{\alpha}^{\mathbb{Q}}\mathscr{R}_{\mathfrak{N}}+\frac{2\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)}{\lambda^{\alpha}}\left(\mathscr{R}_{A\mathfrak{N}}+\|B\|\mathscr{R}_u+M_{\mathcal{D}_1}+M_{\mathcal{D}_2}+\mathscr{R}_uM_{\mathcal{D}_3}\right),$$

where $\mathscr{R}_u$ *denotes the Lipschitz constant of the control mapping* $x\mapsto u_x$*.*

If

$$\mathscr{R}_C<1,$$

then system (2.1) *is controllable on* $[0,\top]$*.*

Moreover, the following properties hold.

1) *The controllability constant $\mathscr{R}_C$ is strictly increasing with respect to the neutral Lipschitz constant $\mathscr{R}_{\mathfrak{N}}$:*

$$\frac{\partial \mathscr{R}_C}{\partial \mathscr{R}_{\mathfrak{N}}} = 1 + \mathscr{R}_\alpha^{\mathbb{Q}} > 0.$$

2) *The controllability constant is strictly increasing with respect to $\mathscr{R}_{A\mathfrak{N}}$:*

$$\frac{\partial \mathscr{R}_C}{\partial \mathscr{R}_{A\mathfrak{N}}} = \frac{2\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha} > 0.$$

3) *The controllability constant is strictly increasing with respect to the control sensitivity constant $\mathscr{R}_u$:*

$$\frac{\partial \mathscr{R}_C}{\partial \mathscr{R}_u} = \frac{2\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}\left(\|B\| + M_{\mathcal{D}_3}\right) > 0.$$

4) *The controllability constant is strictly increasing with respect to each nonlinear measure coefficient $M_{\mathcal{D}_i}$ $(i = 1, 2, 3)$:*

$$\frac{\partial \mathscr{R}_C}{\partial M_{\mathcal{D}_1}} = \frac{2\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha} > 0,$$

$$\frac{\partial \mathscr{R}_C}{\partial M_{\mathcal{D}_2}} = \frac{2\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha} > 0,$$

$$\frac{\partial \mathscr{R}_C}{\partial M_{\mathcal{D}_3}} = \frac{2\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^\alpha}\mathscr{R}_u > 0.$$

5) *The controllability constant decreases monotonically with respect to the tempering parameter λ:*

$$\frac{\partial \mathscr{R}_C}{\partial \lambda} = -\frac{2\alpha\mathscr{R}_\alpha^{\mathbb{P}}\Gamma(\alpha)}{\lambda^{\alpha+1}}\left(\mathscr{R}_{A\mathfrak{N}} + \|B\|\mathscr{R}_u + M_{\mathcal{D}_1} + M_{\mathcal{D}_2} + \mathscr{R}_u M_{\mathcal{D}_3}\right) < 0.$$

Hence, increasing the tempering parameter enlarges the admissible controllability region.

Proof. The controllability condition follows directly from Theorem 4.1, where the operator Ψ is shown to be a k-set contraction whenever $\mathscr{R}_C < 1$.

The remaining assertions follow by direct differentiation of the expression defining $\mathscr{R}_C$.

Since all parameters

$$\mathscr{R}_\alpha^{\mathbb{P}}, \quad \mathscr{R}_\alpha^{\mathbb{Q}}, \quad \Gamma(\alpha), \quad \lambda, \quad \|B\|, \quad \mathscr{R}_u, \quad M_{\mathcal{D}_i}$$

are nonnegative, the partial derivatives with respect to $\mathscr{R}_{\mathfrak{N}}$, $\mathscr{R}_{A\mathfrak{N}}$, $\mathscr{R}_u$, $M_{\mathcal{D}_1}$, $M_{\mathcal{D}_2}$, and $M_{\mathcal{D}_3}$ are strictly positive. Therefore, increasing any of these quantities enlarges the controllability constant and consequently restricts the controllability region.

On the other hand,

$$\frac{\partial \mathscr{R}_C}{\partial \lambda} < 0,$$

which shows that $\mathscr{R}_C$ decreases monotonically as the tempering parameter increases. Hence, larger values of λ enlarge the admissible parameter region for which $\mathscr{R}_C < 1$.

This completes the proof. □

Remark 4.2. *The sensitivity analysis presented in Theorem 5.1 provides practical guidelines for selecting admissible model parameters in applications.*

- *Since*
$$\frac{\partial \mathscr{R}_C}{\partial \lambda} < 0,$$
increasing the tempering parameter λ decreases the controllability constant $\mathscr{R}_C$ and consequently enlarges the admissible controllability region.
- *Since*
$$\frac{\partial \mathscr{R}_C}{\partial \mathscr{R}_{\mathfrak{N}}} > 0, \qquad \frac{\partial \mathscr{R}_C}{\partial \mathscr{R}_{A\mathfrak{N}}} > 0,$$
stronger neutral effects make the controllability condition more restrictive. Hence, systems with weaker neutral nonlinearities are more likely to satisfy $\mathscr{R}_C < 1$.
- *The control contribution enters through the quantity $\|B\|\mathscr{R}_u$. Therefore, improving the regularity of the control mapping (smaller $\mathscr{R}_u$) or reducing the operator norm $\|B\|$ enhances the controllability property.*
- *The nonlinear measure constants $M_{\mathcal{D}_1}$, $M_{\mathcal{D}_2}$, and $M_{\mathcal{D}_3}$ quantify the influence of the nonlinear perturbation. Smaller values of these constants reduce $\mathscr{R}_C$, thereby enlarging the set of admissible parameters for which controllability is guaranteed.*

Consequently, the condition $\mathscr{R}_C < 1$ provides a practical criterion for selecting suitable parameters in concrete applications, as illustrated in the numerical examples.

4.2. Sensitivity analysis

The controllability condition established in Theorem 3.1 depends explicitly on several system parameters, including the fractional order α, the tempering parameter λ, the neutral Lipschitz constant $\mathscr{R}_{\mathfrak{N}}$, the control operator norm $\|B\|$, and the bound of the admissible control $\mathscr{R}_u$. Although the theoretical results provide sufficient conditions for controllability, it is also useful to investigate the sensitivity of the controllability constant

$$R_C = \mathscr{R}_{\mathfrak{N}} + \mathscr{R}_{\alpha}^{\mathbb{Q}}\mathscr{R}_{\mathfrak{N}} + \frac{2\mathscr{R}_{\alpha}^{\mathbb{P}}\Gamma(\alpha)}{\lambda^{\alpha}} \left(\mathscr{R}_{A\mathfrak{N}} + \|B\|\mathscr{R}_u + M_{\mathcal{D}_1} + M_{\mathcal{D}_2} + \mathscr{R}_u M_{\mathcal{D}_3}\right)$$

with respect to the principal fractional parameters.

It should be emphasized that the following plots are not numerical approximations of the solution of the fractional control system. Instead, they provide a parameter sensitivity analysis of the sufficient controllability condition derived in Theorem 3.1. Throughout the study, all parameters except the one under consideration are fixed at representative admissible values satisfying the postulates of the theorem. Consequently, the figures illustrate how variations in the fractional order and the tempering parameter influence the controllability threshold $R_C < 1$.

Figures 1 and 2 illustrate the sensitivity of the controllability constant with respect to the fractional order α and the tempering parameter λ, respectively. It is observed that the sufficient controllability condition is strongly influenced by these fractional parameters. In particular, increasing the tempering parameter λ reduces the contribution of the factor $\Gamma(\alpha)/\lambda^{\alpha}$, thereby decreasing the controllability constant and enlarging the admissible controllability region. Likewise, variations in the fractional order α modify the memory characteristics of the system, leading to corresponding changes in the controllability threshold. These figures serve as a parameter sensitivity study of the theoretical controllability criterion rather than numerical simulations of the state trajectories.

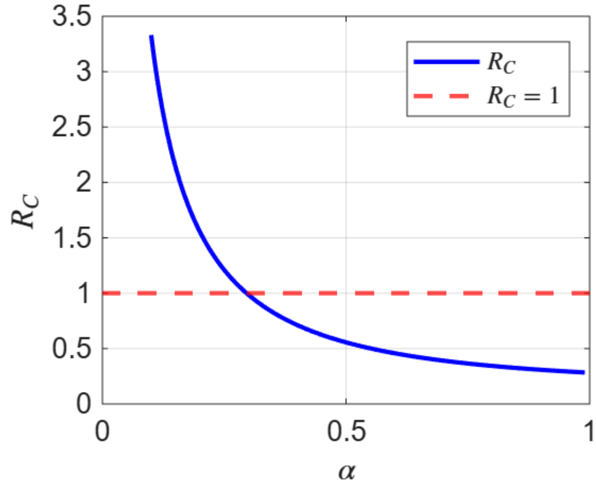

Figure 1. Sensitivity analysis of the controllability constant R_C with respect to the fractional order α. The remaining parameters are fixed at $\mathscr{R}_{\mathfrak{N}} = 0.05$, $\mathscr{R}_{\alpha}^{\mathbb{Q}} = 1$, $\mathscr{R}_{\alpha}^{\mathbb{P}} = 1$, $\mathscr{R}_{A\mathfrak{N}} = 0.05$, $\|B\| = 1$, $\mathscr{R}_u = 0.10$, $M_{\mathcal{D}_1} = 0.02$, $M_{\mathcal{D}_2} = 0.01$, $M_{\mathcal{D}_3} = 0.02$, and $\lambda = 2$. The horizontal dashed line represents the threshold $R_C = 1$, below which the sufficient controllability condition of Theorem 3.1 is satisfied.

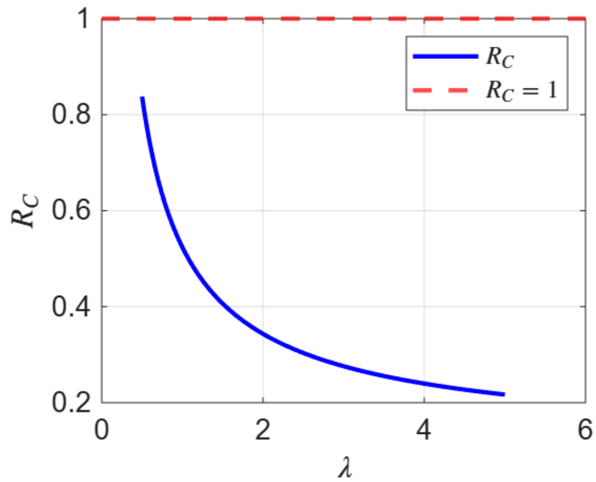

Figure 2. Sensitivity analysis of the controllability constant R_C with respect to the tempering parameter λ. The remaining parameters are fixed at $\mathscr{R}_{\mathfrak{N}} = 0.05$, $\mathscr{R}_{\alpha}^{\mathbb{Q}} = 1$, $\mathscr{R}_{\alpha}^{\mathbb{P}} = 1$, $\mathscr{R}_{A\mathfrak{N}} = 0.05$, $\|B\| = 1$, $\mathscr{R}_u = 0.10$, $M_{\mathcal{D}_1} = 0.02$, $M_{\mathcal{D}_2} = 0.01$, $M_{\mathcal{D}_3} = 0.02$, and $\alpha = 0.8$. The dashed line indicates the critical value $R_C = 1$, separating the admissible controllability region from the non-admissible parameter region.

To further examine the combined influence of the fractional order and the tempering parameter, a two-parameter sensitivity analysis of the controllability constant is performed. Figure 3 presents the surface variation of $R_C(\alpha, \lambda)$, where the region satisfying $R_C < 1$ corresponds to the admissible parameter regime guaranteed by Theorem 3.1.

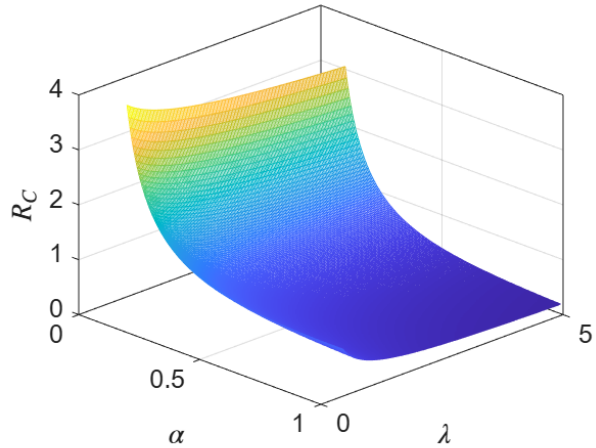

Figure 3. Surface representation of the sensitivity of the controllability constant $R_C(\alpha,\lambda)$ with respect to the fractional order α and the tempering parameter λ. The fixed parameters are chosen as $\mathscr{R}_{\mathfrak{R}} = 0.05$, $\mathscr{R}_\alpha^{\mathbb{Q}} = 1$, $\mathscr{R}_\alpha^{\mathbb{P}} = 1$, $\mathscr{R}_{A\mathfrak{R}} = 0.05$, $\|B\| = 1$, $\mathscr{R}_u = 0.10$, $M_{\mathcal{D}_1} = 0.02$, $M_{\mathcal{D}_2} = 0.01$, and $M_{\mathcal{D}_3} = 0.02$. The surface illustrates the variation of the sufficient controllability bound with respect to the fractional parameters. The plane $R_C = 1$ represents the threshold separating the admissible parameter region satisfying the controllability condition from the non-admissible region.

5. Application

Example 5.1. *Consider the nonlinear tempered fractional neutral control system*

$$\begin{cases} {}_0^{TC}D_t^{\alpha,\lambda}[x(t)-\mathfrak{R}(t,x(t))] = Ax(t) + Bu(t-\mathfrak{k}) + f\left(t, x_t, {}_0^{TC}D_t^{\alpha,\lambda}x(t), u(t)\right), \\ x(\theta) = \varphi(\theta), \qquad \theta \in [-\mathfrak{k}, 0], \end{cases} \tag{5.1}$$

where $\Omega = L^2(0,\pi)$, $\quad \mathfrak{k} = 0.5$, *and* $T = 1$.

Let $A : \Omega \to \Omega$, $\quad (Ax)(\xi) = -x(\xi)$, *and* $\xi \in (0,\pi)$.

Since $A = -I$ *is a bounded linear operator, it generates the analytic semigroup* $T(t) = e^{-t}I$, $\quad t \geq 0$.

Consequently, the associated tempered fractional resolvent families $\{\mathbb{Q}_\alpha^\lambda(t)\}_{t\geq0}, \{\mathbb{P}_\alpha^\lambda(t)\}_{t\geq0}$, *exist and satisfy the estimates in Postulate (P1).*

Choose $\alpha = 0.8$, $\lambda = 2$. *Since* $A = -I$, *the analytic semigroup generated by* A *is* $T(t) = e^{-t}I$, $t \geq 0$. *The associated tempered fractional resolvent families are given by* $\mathbb{Q}_\alpha^\lambda(t) = e^{-\lambda t}E_\alpha(-t^\alpha)\, I$, *and* $\mathbb{P}_\alpha^\lambda(t) = e^{-\lambda t}t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha)\, I$.

Hence, $\|\mathbb{Q}_\alpha^\lambda(t)\| = e^{-\lambda t}\,|E_\alpha(-t^\alpha)|$, *and* $\|\mathbb{P}_\alpha^\lambda(t)\| = e^{-\lambda t}t^{\alpha-1}\left|E_{\alpha,\alpha}(-t^\alpha)\right|$.

Since, for $0 < \alpha < 1$, *the Mittag–Leffler functions satisfy [1]*

$$0 < E_\alpha(-x) \leq 1, \qquad 0 < E_{\alpha,\alpha}(-x) \leq 1, \qquad x \geq 0,$$

it follows that $\|\mathbb{Q}_\alpha^\lambda(t)\| \leq e^{-\lambda t}$, *and* $\|\mathbb{P}_\alpha^\lambda(t)\| \leq e^{-\lambda t}\, t^{\alpha-1}$.

Therefore, the estimates in Postulate (P1) *hold with* $\mathscr{R}_\alpha^{\mathbb{Q}} = 1, \mathscr{R}_\alpha^{\mathbb{P}} = 1$.

Choose the neutral function $\mathfrak{R}(t,x) = 0.05x$.

Then, $\quad \|\mathfrak{R}(t,x) - \mathfrak{R}(t,y)\| = 0.05\|x-y\|$, *so that* $\quad \mathscr{R}_{\mathfrak{R}} = 0.05$.

Moreover, $A\mathfrak{N}(t,x) = -0.05x$, *and therefore* $\|A\mathfrak{N}(t,x) - A\mathfrak{N}(t,y)\| = 0.05\|x-y\|$, *which yields* $\mathscr{R}_{A\mathfrak{N}} = 0.05$.

Hence, Postulate (P2) is satisfied.

Next, define $f(t,\phi,\rho,u) = 0.02\phi(0) + 0.01\rho + 0.02u$. *Then,* $\mathscr{R}_1 = 0.02$, $\mathscr{R}_2 = 0.01$, $\mathscr{R}_3 = 0.02$, *and* $\mathscr{R}_f = 0.02$.

Furthermore, for every admissible trajectory $x \in \Omega_r$, *the tempered Caputo derivative exists and satisfies Postulate (P4).*

Hence, Postulates (P3) and (P4) hold.

Also, $\|f(t,\phi,\rho,u)\| \leq 0.02\|\phi(0)\| + 0.01\|\rho\| + 0.02\|u\|$,
so Postulate (P5) is fulfilled with $M_{\mathcal{D}_1} = 0.02$, $M_{\mathcal{D}_2} = 0.01$, *and* $M_{\mathcal{D}_3} = 0.02$.

Choose the control operator $B = I_\Omega$, *the identity operator on* Ω. *Clearly,* $\|B\| = 1$,
so Postulate (P6) holds.

The controllability operator is $\mathbb{W}u = \int_0^{T-\mathfrak{k}} \mathbb{P}_\alpha^\lambda(T-s-\mathfrak{k})u(s)\,ds$.

Since $A = -I$, *the fractional resolvent operator is simply a scalar multiple of the identity,* $\mathbb{P}_\alpha^\lambda(t) = e^{-\lambda t}t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha)\,I$.

Hence, $\mathbb{W} = c\,I$, *where* $c = \int_0^{T-\mathfrak{k}} e^{-\lambda s}s^{\alpha-1}E_{\alpha,\alpha}(-s^\alpha)\,ds$.

Since $e^{-\lambda s} > 0$, $s^{\alpha-1} > 0$, *and* $E_{\alpha,\alpha}(-s^\alpha) > 0$, $0 < \alpha < 1$, *it follows that* $c > 0$.

Therefore, $\mathbb{W}^{-1} = \frac{1}{c}I$, *and* $\|\mathbb{W}^{-1}\| = \frac{1}{c} < \infty$.

Thus, Postulate (P7) is verified.

Finally, the estimate $\chi(f(t,\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3)) \leq 0.02\chi(\mathcal{D}_1) + 0.01\chi(\mathcal{D}_2) + 0.02\chi(\mathcal{D}_3)$ *shows that Postulate (P8) is satisfied.*

Hence, all the postulates of Theorem 3.1 are fulfilled. Consequently, whenever the controllability constant $\mathscr{R}_C < 1$, *system* (5.1) *is controllable on* $[0,T]$.

6. Conclusions

This paper investigates the controllability of a class of tempered fractional neutral control systems with state and control delays within the Banach space framework. By combining the theory of tempered fractional resolvent families with suitable fixed point techniques, sufficient conditions for the controllability of the considered system are established. The obtained results extend several existing controllability results for fractional and classical dynamical systems by incorporating both tempering effects and delay components into a unified framework.

An example is provided to illustrate the applicability of the theoretical results. The example verifies that the proposed postulates and controllability criterion are satisfied for a specific class of tempered fractional neutral control systems, ensuring the existence of an admissible control that steers the system from an initial state to a prescribed terminal state.

Furthermore, a parameter sensitivity analysis is carried out to investigate the influence of the fractional order α and the tempering parameter λ on the controllability constant $\mathscr{R}_C$. The presented graphical illustrations are not intended as numerical approximations of the state trajectories; rather, they provide a qualitative assessment of the theoretical controllability condition and identify the variation of the admissible parameter region. The analysis shows that the controllability bound is strongly affected by the fractional parameters, where the tempering parameter reduces the contribution of the memory kernel through the factor $\lambda^{-\alpha}$, thereby improving the controllability condition. The fractional order also modifies the memory characteristics of the system and consequently influences the sufficient controllability region.

The results obtained in this work provide a theoretical framework for analyzing controllability of delayed tempered fractional systems and contribute to the ongoing development of fractional control theory. Future investigations may focus on extending these results to more general settings, including optimal control problems, stochastic tempered fractional systems, impulsive fractional systems, and other classes of nonlocal dynamical models.

Author contributions

Sivaranjani Ramasamy, Kulandhaivel Karthikeyan, Wafa F Alfwzan, Hasib Khan, and Jehad Alzabut: Writing original draft, Investigation, Conceptualization. Kulandhaivel Karthikeyan, and Jehad Alzabut: Supervision, Review manuscript, Editing. Sivaranjani Ramasamy, Kulandhaivel Karthikeyan, Hasib Khan, and Jehad Alzabut: validated the results, Conceptualization, Software.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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