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Research article

The improvement of local limit theorem without assuming finite third moment

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Abstract: In this paper, we refined the local limit theorem for sums of independent integer-valued random variables under the assumption of a finite $(2 + \delta)$-moment, where $0 < \delta < 1$. This class includes sums of discrete Pareto random variables, which arise naturally in various applications. Moreover, Kammoo et al. (2023) established a local limit theorem in this setting with an explicit error bound. However, the associated constants become excessively large as $\delta \to 0$. By combining the method of Siripraparat and Neammanee (2021) with a non-integer-order Taylor expansion, we obtained substantially sharper constants in the corresponding error bound.

Keywords: local limit theorem; discrete Pareto random variable; characteristic function and Taylor expansion for non-integer order
Mathematics Subject Classification: 60F05

1. Motivation

Many real-world data sets, including those arising from economic activity and financial transactions, exhibit heavy-tailed behavior. In such settings, a random variable may possess a finite $(2+\delta)$-th moment for some $0 < \delta < 1$, while its third moment is infinite. A simple and widely used example is the discrete Pareto random variable with parameter $s \in (3, 4)$, defined by

$$P(X = k) = \frac{k^{-s}}{\zeta(s)}, \quad k = 1, 2, \ldots,$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ ([1], p. 2). This model is commonly used to describe socioeconomic quantities with heavy-tailed behavior [2]. Empirical studies support this modeling choice; for example, firm and establishment sizes in the United States exhibit clear heavy-tailed behavior [3], and the distribution of global billionaire wealth over 2010–2020 displays a robust power-law upper tail [4].

Let $X_1, X_2, \ldots, X_n$ be independent random variables with the discrete Pareto distribution, and define

$$S_n = X_1 + X_2 + \cdots + X_n.$$

An explicit closed-form expression for $P(S_n = k)$ is not available. Therefore, we seek suitable approximations. A first candidate is the Poisson approximation. However, the Poisson approximation is appropriate when the sum is driven by many small and rare contributions. In contrast, heavy-tailed random variables enable occasional large values that have a substantial impact on the sum S_n, making the Poisson approximation unsuitable in this setting.

For $s \in (3, 4)$, the discrete Pareto random variable has finite $(2+\delta)$-th moments for all $\delta < s-3$, while its third moment is infinite (see Section 5). In particular, its mean and variance are finite. Consequently, the central limit theorem (CLT) applies and yields a normal approximation for the distribution of S_n. However, the CLT concerns convergence in distribution of a normalized sum and does not directly provide accurate approximations for individual point probabilities. Local limit theorems (LLTs) refine the CLT by approximating the point probabilities $P(S_n = k)$ using the normal density, and therefore yield more accurate approximations to the exact values.

The classical de Moivre–Laplace theorem is a foundational result establishing an LLT for the binomial distribution (see [5]). This theorem was further analyzed in [6], where an explicit uniform error bound was obtained. Beyond the binomial case, uniform bounds for local limit theorems have been investigated for various classes of random variables, including Poisson binomial random variables [7, 8], collective risk models [9], and general lattice random variables [10–12]. Motivated by the discrete Pareto distribution, which is integer-valued and has a finite $(2 + \delta)$-th moment, we study local limit approximations under this condition. Kammoo et al. [12] established a uniform bound for integer-valued random variables under the assumption of a finite $(2 + \delta)$-th moment in 2023. Their main result is stated below.

Theorem 1.1. *Let $X_1, X_2, \ldots, X_n$ be independent integer-valued random variables with means μ_j and variances σ_j^2 for $j = 1, 2, \ldots, n$. Define*

$$S_n = \sum_{j=1}^{n} X_j, \quad \mu = \sum_{j=1}^{n} \mu_j, \quad \sigma^2 = \sum_{j=1}^{n} \sigma_j^2.$$

Let $\alpha = \sum_{j=1}^{n} \alpha_j$, where $\alpha_j = 2 \sum_{l=-\infty}^{\infty} p_{jl} p_{j(l+1)}$ and $p_{jl} = P(X_j = l)$. If there exists a $\delta \in (0, 1)$ such that $E|X_j|^{2+\delta} < \infty$ and $\alpha_j > 0$ and $j = 1, 2, \ldots, n$, then

$$\sup_{k \in \mathbb{Z}} \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \leq A_1 + B_1,$$

where

$$A_1 = \frac{0.0020}{\left(\sum_{j=1}^{n} E|X_j|^{2+\delta}\right)^{\frac{1+\delta}{2}}} + \frac{4.6171(3^{\frac{1}{\delta}})}{\sigma^4}\left(\sum_{j=1}^{n} E|X_j|^{2+\delta}\right)^{\frac{3-\delta}{2}},$$

$$B_1 = \frac{0.3184}{\sigma^2\tau_1}e^{-\frac{\sigma^2\tau_1^2}{2}} + \frac{1.5708}{\tau_1\alpha}e^{-\frac{\tau_1^2\alpha}{\pi^2}} \quad \textit{and} \quad \tau_1 = \frac{1}{3^{\frac{1}{\delta}}\left(\sum_{j=1}^{n} E|X_j|^{2+\delta}\right)^{\frac{1}{2+\delta}}}.$$

Since Theorem 1.1 assumes only $E|X_j|^{2+\delta} < \infty$, we can apply it to our problem. However, the term $3^{\frac{1}{\delta}}$ in A_1 tends to ∞ when $\delta \to 0$. This situation gives a very large constant in the error bound, which is not suitable for our situation. In this work, we improve constants under the same $(2+\delta)$-moment condition.

2. Statement of main result

Let $X_1, X_2, \ldots, X_n$ be independent integer-valued random variables with means μ_j, variances ${\sigma_j}^2$, characteristic functions ψ_j for $j = 1, 2, \ldots, n$, and define $S_n = \sum_{j=1}^{n} X_j$, with $\mu = E[S_n]$ and $\sigma^2 = \mathrm{Var}(S_n)$. In this section, we present an improved error bound for Theorem 1.1. Our main result is stated in the following theorem.

Theorem 2.1. *Under the assumption of Theorem 1.1, we have*

$$\sup_{k\in\mathbb{Z}}\left|P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}}\right| \le A_2 + B_2,$$

where

$$A_2 = \frac{2^{\frac{1+\delta}{2}}}{\sigma^{3+\delta}}\left[1.4464 + 0.761(2^\delta)\left(\frac{2.3907(2^\delta)}{3^{2+\delta}} + 1\right)\right]\sum_{j=1}^{n} E|X_j|^{2+\delta},$$

$$B_2 = \frac{0.3184}{\sigma^2\tau_2}e^{-\frac{\sigma^2\tau_2^2}{2}} + \frac{1.5708}{\tau_2\alpha}e^{-\frac{\tau_2^2\alpha}{\pi^2}} \quad \textit{and} \quad \tau_2 = \frac{1}{3\left(\sum_{j=1}^{n} E|X_j|^{2+\delta}\right)^{\frac{1}{2+\delta}}}.$$

Remark 2.1. *From Theorem 2.1, by considering the independent and identically distributed (i.i.d.) case and using the bound*

$$\alpha_1 \le 2\sigma_1^2,$$

([11], p. 14), we obtain

$$\sup_{k\in\mathbb{Z}}\left|P(S_n = k) - \frac{1}{\sigma_1\sqrt{2\pi n}}e^{-\frac{(k-n\mu_1)^2}{2n\sigma_1^2}}\right| \le \frac{7.554}{n^{\frac{1+\delta}{2}}\sigma_1^{3+\delta}}E|X_1|^{2+\delta} + \frac{2.2076}{n\tau_2\alpha_1}e^{-\frac{n\tau_2^2\alpha_1}{\pi^2}}, \tag{2.1}$$

where

$$\tau_2 = \frac{1}{3\left(nE|X_1|^{2+\delta}\right)^{\frac{1}{2+\delta}}}, \qquad \alpha_1 = 2\sum_{l=-\infty}^{\infty} p_l p_{l+1}, \quad p_l = P(X_1 = l).$$

Remark 2.2. *From the expression for the tail bound B_2, the bound decreases as α increases. Moreover, since*

$$\alpha_j = 2\sum_{l=-\infty}^{\infty} p_{jl} p_{j(l+1)},$$

a larger value of α_j indicates that more probability mass is concentrated on consecutive lattice points. Hence, parameter α may be viewed as a measure of the smoothness (or aperiodicity) of the lattice distribution.

To illustrate the result, we give some applications to discrete Pareto and maximal lattice in Section 5.

3. Auxiliary results

Using the same setting in Section 2, for each $j = 1, 2, \ldots, n$, let $\psi_j(t) = |\psi_j(t)|e^{i\theta(t)}$, where

$$\theta_j(t) = \arctan\left(\frac{\sum_{r\in\mathbb{Z}} p_{jr}\sin(rt)}{\sum_{s\in\mathbb{Z}} p_{js}\cos(st)}\right), \quad p_{jr} = P(X_j = r). \tag{3.1}$$

To prove Theorem 1.1, Kammoo et al. [12] write $P(S_n = k)$ in the form of

$$P(S_n = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\psi(t)dt,$$

where $\psi(t) = \prod_{j=1}^{n} \psi_j(t)$ is the characteristic function of S_n and use a Taylor expansion for non-integer degree to give a bound of

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\psi(t)dt - \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}}\right|.$$

However, in this work, we use the representation introduced by Siripraparat and Neammanee [8]

$$P(S_n = k) = \frac{1}{\pi}\int_0^{\pi} |\psi(t)|\cos((k-\mu)t - \gamma(t))\, dt, \tag{3.2}$$

where

$$\gamma(t) = \theta(t) - \mu t \qquad \text{and} \qquad \theta(t) = \sum_{j=1}^{n} \theta_j(t). \tag{3.3}$$

To bound

$$\left|\frac{1}{\pi}\int_0^{\pi} |\psi(t)|\cos\left((k-\mu)t - \gamma(t)\right) dt - \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}}\right|,$$

we need Lemmas 3.1 and 3.2.

Lemma 3.1. *Suppose that there exists* $\delta \in (0,1)$ *such that* $E|X_j|^{2+\delta} < \infty$. *Then for all* $|t| \leq \dfrac{1}{3(E|X_j|^{2+\delta})^{\frac{1}{2+\delta}}}$,

$$|\psi_j(t)| = e^{-\frac{1}{2}\sigma_j{}^2t^2+\epsilon_j(t)},$$

where

$$\left|\epsilon_j(t)\right| \leq 3.1825E|X_j|^{2+\delta}|t|^{2+\delta}.$$

Proof. First, we note that

$$2\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\sin^2\left((r-s)\frac{t}{2}\right)p_{jr}p_{js} \leq t^2\sigma_j{}^2, \quad ([11], \text{p. } 7) \tag{3.4}$$

and

$$\ln|\psi_j(t)| \geq -\frac{1}{2}\sigma_j{}^2t^2 - \frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k}\left[2\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\sin^2\left((r-s)\frac{t}{2}\right)p_{jr}p_{js}\right]^k \ ([11], \text{p. } 8). \tag{3.5}$$

Combining (3.4) with the inequality

$$\sigma_j{}^2 \leq E(X_j^2) \leq (E|X_j|^{2+\delta})^{\frac{2}{2+\delta}}, \tag{3.6}$$

we obtain that for $|t| \leq \dfrac{1}{3(E|X_j|^{2+\delta})^{\frac{1}{2+\delta}}}$,

$$0 \leq 2\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\sin^2\left((r-s)\frac{t}{2}\right)p_{jr}p_{js} \leq \frac{1}{9},$$

and hence

$$\frac{8}{9} \leq 1 - 2\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\sin^2\left((r-s)\frac{t}{2}\right)p_{jr}p_{js} \leq 1. \tag{3.7}$$

From this fact, (3.4)–(3.6), we obtain

$$\begin{aligned}
\ln|\psi_j(t)| &\geq -\frac{1}{2}\sigma_j{}^2t^2 - \frac{1}{4}\sum_{k=2}^{\infty}\left[2\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\sin^2\left((r-s)\frac{t}{2}\right)p_{jr}p_{js}\right]^k \\
&\geq -\frac{1}{2}\sigma_j{}^2t^2 - \frac{9}{32}\left[2\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\sin^2\left((r-s)\frac{t}{2}\right)p_{jr}p_{js}\right]^2 \\
&\geq -\frac{1}{2}\sigma_j{}^2t^2 - \frac{9}{32}\sigma_j^4t^4 \\
&\geq -\frac{1}{2}(E|X_j|^{2+\delta})^{\frac{2}{2+\delta}}t^2 - \frac{9}{32}(E|X_j|^{2+\delta})^{\frac{4}{2+\delta}}t^4.
\end{aligned}$$

Therefore,

$$|\psi_j(t)| \geq e^{-\frac{17}{288}} \quad \text{for} \quad |t| \leq \frac{1}{3(E|X_j|^{2+\delta})^{\frac{1}{2+\delta}}}. \tag{3.8}$$

This lower bound ensures that $\psi_j(t) \neq 0$ on the interval, which justifies dividing by $\psi_j(u)$ in (3.10). Next, by Lemma 2.1 in [12], the characteristic function ψ_j can be represented as

$$\psi_j(t) = e^{i\mu_j t - \frac{1}{2}\sigma_j{}^2 t^2 + \int_0^t \frac{g_j(u)}{\psi_j(u)} du}, \tag{3.9}$$

where the remainder term $g_j(t)$ satisfies

$$|g_j(t)| \leq \frac{2^{1-\delta}(3+\delta)}{1+\delta} E|X_j|^{2+\delta}|t|^{1+\delta}.$$

Hence, by (3.8), we obtain

$$\begin{aligned}\int_0^t \left|\frac{g_j(u)}{\psi_j(u)}\right| du &\leq \frac{2^{1-\delta}(3+\delta)}{e^{-\frac{17}{288}}(1+\delta)(2+\delta)} E|X_j|^{2+\delta}|t|^{2+\delta} \\ &\leq 3.1825 E|X_j|^{2+\delta}|t|^{2+\delta},\end{aligned} \tag{3.10}$$

where we use the fact that

$$h(\delta) = \frac{2^{1-\delta}(3+\delta)}{e^{-\frac{17}{288}}(1+\delta)(2+\delta)}$$

is decreasing on $[0, 1]$ in the second inequality.

Applying (3.9), (3.10) and the inequality $|e^z| \leq e^{|z|}$ for any complex z, we obtain

$$\begin{aligned}\left|\psi_j(t)\right| &= \left|e^{i\mu_j t}\right| \left|e^{-\frac{1}{2}\sigma_j{}^2 t^2}\right| \left|e^{\int_0^t \frac{g_j(u)}{\psi_j(u)} du}\right| \\ &= e^{-\frac{1}{2}\sigma_j{}^2 t^2 + \epsilon_j(t)},\end{aligned}$$

where

$$\left|\epsilon_j(t)\right| \leq 3.1825 E|X_j|^{2+\delta}|t|^{2+\delta}.$$

□

Lemma 3.2. *Let τ_2 be defined in Theorem 2.1 and $\gamma(t)$ be defined in (3.3). Then, for $t \in [0, \tau_2]$, we have*

$$|\gamma(t)| \leq 2.3907 \times 2^\delta \sum_{j=1}^n E|X_j|^{2+\delta}\, |t|^{2+\delta}.$$

Proof. Since

$$\theta_j(t) = \theta_j^{(1)}(0)\, t + \frac{1}{2}\theta_j^{(2)}(u) t^2 \;\; \text{for} \;\; 0 \leq u \leq t \;\; \text{and} \;\; \theta_j^{(1)}(0) = \mu_j \quad ([11], \text{p. } 8),$$

we get

$$|\gamma(t)| \leq \frac{1}{2}\sum_{j=1}^{n}|\theta_j^{(2)}(u)|t^2. \tag{3.11}$$

By (3.1), we obtain

$$\theta_j^{(2)}(t) = \frac{M_j(t)}{Q_j(t)} + \frac{N_j(t)}{[Q_j(t)]^2}, \tag{3.12}$$

where

$$\begin{aligned}
M_j(t) &= -\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} r\sin((r-s)t)p_{jr}p_{js},\\
N_j(t) &= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} m(r-s)\sin((r-s)t)\cos((m-l)t)p_{jr}p_{js}p_{jm}p_{jl},\\
Q_j(t) &= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\cos((r-s)t)\, p_{jr}p_{js},
\end{aligned}$$

Using the inequalities $|\sin x| \leq |x|^{\delta}$ and $(a+b)^k \leq 2^{k-1}(a^k + b^k)$ for $a, b \geq 0$, $k \geq 1$, we obtain

$$\begin{aligned}
|M_j(t)| &\leq |t|^{\delta}\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}|r||r-s|^{1+\delta}p_{jr}p_{js}\\
&\leq |t|^{\delta}\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}|r|(|r|+|s|)^{1+\delta}p_{jr}p_{js}\\
&\leq 2^{\delta}|t|^{\delta}\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}|r|\left(|r|^{1+\delta}+|s|^{1+\delta}\right)p_{jr}p_{js}\\
&= 2^{\delta}|t|^{\delta}\left(E|X_j|^{2+\delta} + E|X_j|E|X_j|^{1+\delta}\right)\\
&\leq 2^{1+\delta}|t|^{\delta}E|X_j|^{2+\delta}
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
|N_j(t)| &\leq |t|^{\delta}\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}|r-s|^{1+\delta}p_{jr}p_{js}\sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}|m|p_{jm}p_{jl}\\
&\leq |t|^{\delta}E|X_j|\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}(|r|+|s|)^{1+\delta}p_{jr}p_{js}\\
&\leq 2^{\delta}|t|^{\delta}E|X_j|\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\left(|r|^{1+\delta}+|s|^{1+\delta}\right)p_{jr}p_{js}\\
&= 2^{1+\delta}|t|^{\delta}E|X_j|^{1+\delta}E|X_j|\\
&\leq 2^{1+\delta}|t|^{\delta}E|X_j|^{2+\delta}.
\end{aligned} \tag{3.14}$$

From (3.7), we obtain

$$1 \geq Q_j(t) = \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\left(1 - 2\sin^2\!\left(\frac{(r-s)t}{2}\right)\right)p_{jr}p_{js}$$

$$
\begin{aligned}
&= 1 - 2\sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\sin^2\left(\frac{(r-s)t}{2}\right)p_{jr}p_{js}\\
&\geq \frac{8}{9}.
\end{aligned}
\tag{3.15}
$$

Combining (3.11)–(3.15), it follows that

$$
|\theta_j^{(2)}(t)| \leq 4.7813 \times 2^\delta |t|^\delta E|X_j|^{2+\delta}
$$

which implies that

$$
|\gamma(t)| \leq 2.3907 \times 2^\delta \sum_{j=1}^{n} E|X_j|^{2+\delta}\,|t|^{2+\delta}.
$$

□

Remark 3.1. *(1) The bound in Lemma 3.1 is derived under the finite $(2+\delta)$-moment condition, which relaxes the finite third-moment assumption used by Siripraparat and Neammanee [11]. Moreover, by employing the sharper lower bound $|\psi_j(t)| \geq e^{-\frac{17}{288}}$, instead of $|\psi_j(t)| \geq \frac{1}{3}$ used in Kammoo et al. [12], we obtain significantly smaller constants in the final error bound.*

(2) In Lemma 3.2, we use a second-order Taylor expansion of $\theta_j(t)$, whereas Siripraparat and Neammanee [11] employed a third-order expansion. Consequently, the required bound can be derived under the finite $(2+\delta)$-moment condition instead of the finite third-moment assumption.

4. Proof of Theorem 2.1

Most of the proof follows the technique of Siripraparat and Neammanee [11]. However, instead of assuming finite third moments, we work under the weaker finite $(2+\delta)$-moment condition, which requires several modifications to the argument, as shown below.

Proof. From (3.2) and the fact that

$$
\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}} = \frac{1}{\pi}\int_0^\infty e^{-\frac{1}{2}\sigma^2t^2}\cos\left((k-\mu)t\right)dt, \qquad ([11],\ \text{p. 15}),
$$

we have

$$
\left|P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(k-\mu)^2}{2\sigma^2}}\right| \leq \varepsilon_1 + \varepsilon_2,
$$

where

$$
\begin{aligned}
\varepsilon_1 &= \frac{1}{\pi}\left|\int_0^{\tau_2}|\psi(t)|\cos\left((k-\mu)t-\gamma(t)\right)dt - \int_0^{\tau_2}e^{-\frac{1}{2}\sigma^2t^2}\cos\left((k-\mu)t\right)dt\right|,\\
\varepsilon_2 &= \frac{1}{\pi}\left|\int_{\tau_2}^{\infty}e^{-\frac{1}{2}\sigma^2t^2}\cos\left((k-\mu)t\right)dt - \int_{\tau_2}^{\pi}|\psi(t)|\cos\left((k-\mu)t-\gamma(t)\right)dt\right|.
\end{aligned}
$$

We divide the proof into two steps.

Step 1. We show that $\varepsilon_1 \leq A_2$.

From Lemma 3.1 together with $e^x - 1 \leq |x|e^{|x|}$ for $x \in \mathbb{R}$

$$\begin{aligned}
\left||\psi(t)| - e^{-\frac{1}{2}\sigma^2 t^2}\right| &= \left|\prod_{j=1}^{n} |\psi_j(t)| - e^{-\frac{1}{2}\sigma^2 t^2}\right| \\
&= e^{-\frac{1}{2}\sigma^2 t^2} \times |e^{\sum_{j=1}^{n} \epsilon_j(t)} - 1| \\
&\leq e^{-\frac{1}{2}\sigma^2 t^2} e^{\sum_{j=1}^{n} |\epsilon_j(t)|} \sum_{j=1}^{n} |\epsilon_j(t)| \\
&\leq 3.1825 e^{-\frac{1}{2}\sigma^2 t^2} e^{3.1825 \sum_{j=1}^{n} E|X_j|^{2+\delta}|t|^{2+\delta}} \sum_{j=1}^{n} E|X_j|^{2+\delta}|t|^{2+\delta} \\
&\leq 3.1825 e^{0.3561} \sum_{j=1}^{n} E|X_j|^{2+\delta}|t|^{2+\delta}\, e^{-\frac{1}{2}\sigma^2 t^2} \\
&\leq 4.5439 e^{-\frac{1}{2}\sigma^2 t^2} \sum_{j=1}^{n} E|X_j|^{2+\delta}|t|^{2+\delta},
\end{aligned}$$

for $|t| \leq \dfrac{1}{3\left(\sum_{j=1}^{n} E|X_j|^{2+\delta}\right)^{\frac{1}{2+\delta}}}$. This implies that

$$\frac{1}{\pi}\int_0^{\tau_2} |\psi(t)| \cos((k-\mu)t - \gamma(t))\, \mathrm{d}t = \frac{1}{\pi}\int_0^{\tau_2} e^{-\frac{1}{2}\sigma^2 t^2} \cos((k-\mu)t - \gamma(t)) dt + \Delta_1, \tag{4.1}$$

where

$$\begin{aligned}
|\Delta_1| &\leq \frac{4.5439}{\pi} \sum_{j=1}^{n} E|X_j|^{2+\delta} \int_0^{\tau_2} t^{2+\delta} e^{-\frac{1}{2}\sigma^2 t^2} dt \\
&\leq \frac{4.5439}{\pi} \sum_{j=1}^{n} E|X_j|^{2+\delta} \int_0^{\infty} t^{2+\delta} e^{-\frac{1}{2}\sigma^2 t^2} dt \\
&\leq \frac{4.5439 \times 2^{\frac{1+\delta}{2}}}{\pi\sigma^{3+\delta}} \sum_{j=1}^{n} E|X_j|^{2+\delta},
\end{aligned} \tag{4.2}$$

where the last inequality follows from the fact that

$$\int_0^{\infty} t^{2+\delta} e^{-\frac{1}{2}\sigma^2 t^2} dt \leq \frac{2^{\frac{1+\delta}{2}}}{\sigma^{3+\delta}} \Gamma\left(\frac{3+\delta}{2}\right).$$

We next analyze the term

$$\frac{1}{\pi}\int_0^{\tau_2} e^{-\frac{1}{2}\sigma^2 t^2} \cos((k-\mu)t - \gamma(t)) dt.$$

Since

$$\cos((k-\mu)t-\gamma(t)) = \cos((k-\mu)t) + \Delta, \quad \text{where } |\Delta| \leq |\gamma(t)| + \gamma^2(t), \qquad ([11], \text{pp. } 12-13),$$

$$\frac{1}{\pi}\int_0^{\tau_2} e^{-\frac{1}{2}\sigma^2 t^2}\cos((k-\mu)t-\gamma(t))dt = \frac{1}{\pi}\int_0^{\tau_2} e^{-\frac{1}{2}\sigma^2 t^2}\cos((k-\mu)t)dt + \Delta_2, \tag{4.3}$$

where

$$|\Delta_2| \leq \frac{1}{\pi}\int_0^{\tau_2} e^{-\frac{1}{2}\sigma^2 t^2}(|\gamma(t)| + \gamma^2(t))dt.$$

Using Lemma 3.2, we have

$$|\gamma(t)| \leq 2.3907 \times 2^{\delta} \sum_{j=1}^{n} E|X_j|^{2+\delta}|t|^{2+\delta} \quad \text{and}$$

$$|\gamma^2(t)| \leq \frac{2.3907 \times 2^{\delta}}{3^{2+\delta}}\left(2.3907 \times 2^{\delta} \sum_{j=1}^{n} E|X_j|^{2+\delta}|t|^{2+\delta}\right),$$

which implies that

$$\begin{aligned}
|\Delta_2| &\leq \frac{2.3907 \times 2^{\delta}}{\pi}\left(\frac{2.3907 \times 2^{\delta}}{3^{2+\delta}} + 1\right)\sum_{j=1}^{n} E|X_j|^{2+\delta}\int_0^{\infty} t^{2+\delta}e^{-\frac{1}{2}\sigma^2 t^2}dt \\
&\leq \frac{2.3907 \times 2^{\delta}}{\pi}\left(\frac{2.3907 \times 2^{\delta}}{3^{2+\delta}} + 1\right)\frac{2^{\frac{1+\delta}{2}}}{\sigma^{3+\delta}}\sum_{j=1}^{n} E|X_j|^{2+\delta}. \qquad (4.4)
\end{aligned}$$

From (4.1)–(4.4), it follows that

$$\begin{aligned}
\varepsilon_1 &\leq |\Delta_1| + |\Delta_2| \\
&= \frac{2^{\frac{1+\delta}{2}}}{\sigma^{3+\delta}}\left[\frac{4.5439}{\pi} + \frac{2.3907 \times 2^{\delta}}{\pi}\left(\frac{2.3907 \times 2^{\delta}}{3^{2+\delta}} + 1\right)\right]\sum_{j=1}^{n} E|X_j|^{2+\delta} \\
&= A_2.
\end{aligned}$$

Step 2. We show that $\varepsilon_2 \leq B_2$.

Since

$$|\psi(t)| \leq e^{-\frac{1}{\pi^2}\alpha t^2}, \quad t \in [0,\pi] \qquad ([11], \text{p. } 5),$$

we obtain

$$\begin{aligned}
\varepsilon_2 &\leq \frac{1}{\pi}\int_{\tau_2}^{\infty} e^{-\frac{1}{2}\sigma^2 t^2}dt + \frac{1}{\pi}\int_{\tau_2}^{\pi}|\psi(t)|dt \\
&\leq \frac{1}{\pi\tau_2}\int_{\tau_2}^{\infty} te^{-\frac{1}{2}\sigma^2 t^2}dt + \frac{1}{\pi\tau_2}\int_{\tau_2}^{\infty} te^{-\frac{1}{\pi^2}\alpha t^2}dt \\
&= \frac{1}{\pi\tau_2\sigma^2}e^{-\frac{\sigma^2\tau_2^2}{2}} + \frac{\pi}{2\tau_2\alpha}e^{-\frac{\tau_2^2\alpha}{\pi^2}} \\
&\leq \frac{0.3184}{\sigma^2\tau_2}e^{-\frac{\sigma^2\tau_2^2}{2}} + \frac{1.5708}{\tau_2\alpha}e^{-\frac{\tau_2^2\alpha}{\pi^2}} \\
&= B_2.
\end{aligned}$$

Combining the estimates from Steps 1 and 2 completes the proof. □

5. Applications

In this section, we present several applications of Theorem 2.1. In particular, we apply the theorem to discrete Pareto and maximal lattice distributions, illustrating the effectiveness of the result under the finite $(2 + \delta)$-moment condition, even when the third moment does not exist.

5.1. Discrete Pareto

The Pareto distribution, introduced by Vilfredo Pareto, is commonly associated with the "80–20 rule", which states that a large proportion of outcomes is often generated by a relatively small proportion of causes. Owing to its ability to model highly unequal and heavy-tailed behavior, it has been widely applied in areas such as economics, productivity analysis, and population studies (see [13] for more details). Motivated by these applications, we apply Theorem 2.1 to discrete Pareto random variables. In particular, if X_1 is a discrete Pareto random variable with parameter $3 + \delta_0$ for some $\delta_0 \in (0, 1)$, then for any $r > 0$,

$$E|X_1|^r = \sum_{k=1}^{\infty} k^r P(X_1 = k) = \frac{\zeta(3 + \delta_0 - r)}{\zeta(3 + \delta_0)}.$$

Hence,

$$E|X_1|^p < \infty \quad \text{if and only if} \quad p < 2 + \delta_0,$$

$$\mu_1 = E(X_1) = \frac{\zeta(2 + \delta_0)}{\zeta(3 + \delta_0)}, \quad \sigma_1^2 = \mathrm{Var}(X_1) = \frac{\zeta(1 + \delta_0)}{\zeta(3 + \delta_0)} - \left(\frac{\zeta(2 + \delta_0)}{\zeta(3 + \delta_0)}\right)^2,$$

and

$$\begin{aligned}\alpha_1 = 2\sum_{l=1}^{\infty} P(X_1 = l)P(X_1 = l + 1) &= \frac{2}{\zeta(3 + \delta_0)^2} \sum_{l=1}^{\infty} \frac{1}{l^{3+\delta_0}(l + 1)^{3+\delta_0}} \\ &\geq \frac{2}{\zeta(3 + \delta_0)^2}\left(\frac{1}{2^{3+\delta_0}}\right) \\ &\geq \frac{1}{8\zeta(3 + \delta_0)^2}.\end{aligned}$$

Hence, all assumptions of Theorem 2.1 are satisfied, and Theorem 5.1 follows directly from Theorem 2.1.

Theorem 5.1. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. discrete Pareto random variables with parameter $(3 + \delta_0)$ for some $\delta_0 \in (0, 1)$. Then, for $0 < \delta < \delta_0$,*

$$\sup_{k\in\mathbb{Z}} \left| P(S_n = k) - \frac{1}{\sigma_1 \sqrt{2\pi n}} e^{-\frac{(k-n\mu_1)^2}{2n\sigma_1^2}} \right| \leq \frac{7.554}{n^{\frac{1+\delta}{2}} \sigma_1^{3+\delta}} E|X_1|^{2+\delta} + \frac{17.6608\, \zeta(3 + \delta_0)^2}{n\tau_2} e^{-\frac{n\tau_2^2}{8\pi^2\zeta(3+\delta_0)^2}},$$

where

$$\tau_2 = \frac{1}{3\left(n\frac{\zeta(1 + \delta_0 - \delta)}{\zeta(3 + \delta_0)}\right)^{\frac{1}{2+\delta}}}.$$

5.2. Maximal lattice distributions

A random variable X is said to be lattice if all support points are of the form $a + bk$, where $k \in \mathbb{Z}$, $a \in \mathbb{R}$, and $b > 0$. The distribution is said to be maximal lattice if b is the largest positive number for which such a representation is possible. Lattice and maximal lattice distributions arise naturally in integer-valued stochastic models such as random walks and queueing systems [14, 15], as well as in network degree distributions [2].

Suppose that $X_1, X_2, \ldots, X_n$ are maximal lattice random variables with common lattice parameters a and $b > 1$. Then Theorem 2.1 cannot be applied directly, since

$$\alpha_j := 2 \sum_{k \in \mathbb{Z}} P(X_j = k) P(X_j = k + 1) = 0.$$

However, by defining

$$Y_j = \frac{X_j - a}{b},$$

we obtain

$$P\left(\sum_{j=1}^{n} X_j = x\right) = P\left(\sum_{j=1}^{n} Y_j = \frac{x - na}{b}\right).$$

If

$$\beta_j := 2 \sum_{k \in \mathbb{Z}} P(Y_j = k) P(Y_j = k + 1) > 0,$$

then Theorem 2.1 can be applied to $Y_1, Y_2, \ldots, Y_n$. The following example illustrates the application of Theorem 2.1 to maximal lattice distributions for which $\beta_j > 0$.

Example 5.1. *Let $X, X_1, X_2, \ldots, X_n$ be i.i.d. random variables with*

$$P(X = 2k) = \frac{90}{\pi^4 k^4}, \qquad k \in \mathbb{N},$$

and define

$$S_n = \sum_{j=1}^{n} X_j.$$

Then, for every $0 < \delta < 1$,

$$\sup_{x \in 2\mathbb{Z}} \left| P(S_n = x) - \frac{2}{\sigma_X \sqrt{2\pi n}} e^{-\frac{(x - n\mu_X)^2}{2n\sigma_X^2}} \right| \leq \frac{15.108}{n^{\frac{1+\delta}{2}} \sigma_X^{3+\delta}} E|X|^{2+\delta} + \frac{10.3449}{n\tau_{2,X}} e^{-0.04324 n\tau_{2,X}^2}, \tag{5.1}$$

where

$$\mu_X = \frac{180\zeta(3)}{\pi^4}, \sigma_X^2 = \frac{60}{\pi^2} - \left(\frac{180\zeta(3)}{\pi^4}\right)^2,$$

$$E|X|^{2+\delta} = \frac{90 \cdot 2^{2+\delta}}{\pi^4} \zeta(2 - \delta) \quad \textit{and} \quad \tau_{2,X} = \frac{1}{3\left(\frac{90 \cdot 2^{2+\delta} n}{\pi^4} \zeta(2 - \delta)\right)^{\frac{1}{2+\delta}}}.$$

Proof. Note that the support of X is contained in $2\mathbb{N}$. Thus,

$$\alpha_X := 2\sum_{k\in\mathbb{Z}} P(X = k)P(X = k + 1) = 0,$$

and Theorem 2.1 cannot be applied directly.

Let $Y = \dfrac{X}{2}$ and for $j = 1, 2, \ldots,$

$$Y_j = \frac{X_j}{2}.$$

Then

$$P(Y = k) = P(X = 2k) = \frac{90}{\pi^4 k^4}, \qquad k \in \mathbb{N},$$

$$P(S_n = x) = P\left(\sum_{j=1}^{n} Y_j = \frac{x}{2}\right), \qquad x \in 2\mathbb{N},$$

$$\mu_Y = \frac{\mu_X}{2}, \sigma_Y^2 = \frac{\sigma_X^2}{4}, \tau_{2,Y} = 2\tau_{2,X},$$

and for every $0 < \delta < 1$,

$$E|Y|^{2+\delta} = \frac{1}{2^{2+\delta}} E|X|^{2+\delta}.$$

Furthermore,

$$\begin{aligned}
\beta_Y := 2\sum_{k=1}^{\infty} P(Y = k)P(Y = k + 1) &= \frac{16200}{\pi^8} \sum_{k=1}^{\infty} \frac{1}{k^4(k + 1)^4} \\
&\geq \frac{16200}{\pi^8}\left(\frac{1}{2^4}\right) \\
&> 0.1067.
\end{aligned}$$

Therefore, all assumptions of Theorem 2.1 are satisfied for $Y_1, Y_2, \ldots, Y_n$.

Hence, applying (2.1) to $Y_1, \ldots, Y_n$, we obtain

$$\begin{aligned}
\left|P(S_n = x) - \frac{2}{\sigma_X\sqrt{2\pi n}} e^{-\frac{(x-n\mu_X)^2}{2n\sigma_X^2}}\right| &= \left|P\left(\sum_{j=1}^{n} Y_j = \frac{x}{2}\right) - \frac{1}{\sigma_Y\sqrt{2\pi n}} e^{-\frac{\left(\frac{x}{2}-n\mu_Y\right)^2}{2n\sigma_Y^2}}\right| \\
&\leq \frac{7.554}{n^{\frac{1+\delta}{2}}\sigma_Y^{3+\delta}} E|Y|^{2+\delta} + \frac{20.6898}{n\tau_{2,Y}} e^{-0.01081 n\tau_{2,Y}^2} \\
&= \frac{15.108}{n^{\frac{1+\delta}{2}}\sigma_X^{3+\delta}} E|X|^{2+\delta} + \frac{10.3449}{n\tau_{2,X}} e^{-0.04324 n\tau_{2,X}^2}.
\end{aligned}$$

□

6. Discussion

In this section, we compare the error bound obtained in Theorem 2.1 with the corresponding bound in Theorem 1.1. We show that the constants appearing in our estimate are significantly smaller, yielding a sharper local limit approximation under the same finite $(2+\delta)$-moment condition. More precisely, we establish that

$$A_2 \le A_1 \quad \text{and} \quad B_2 \le B_1.$$

Proof. Since

$$\sigma^2 \le \sum_{j=1}^{n} E|X_j|^2 \le \sum_{j=1}^{n}\sum_{i=1}^{\infty} |i|^{2+\delta} P(X_j = \pm i) = \sum_{j=1}^{n} E|X_j|^{2+\delta},$$

it follows that

$$\sigma^{1-\delta} \le (\sum_{j=1}^{n} E|X_j|^{2+\delta})^{\frac{1-\delta}{2}}. \tag{6.1}$$

We will use this fact to show that $A_2 \le A_1$. Note that

$$A_1 = T_1 + T_2,$$

where T_1 and T_2 are nonnegative terms such that

$$T_1 = \frac{0.0020}{\left(\sum_{j=1}^{n} E|X_j|^{2+\delta}\right)^{\frac{1+\delta}{2}}} \quad \text{and} \quad T_2 = \frac{4.6171(3^{\frac{1}{\delta}})}{\sigma^4}\left(\sum_{j=1}^{n} E|X_j|^{2+\delta}\right)^{\frac{3-\delta}{2}}.$$

From (6.1),

$$\begin{aligned}
T_2 &\ge \frac{13.8513}{\sigma^4}\left(\sum_{j=1}^{n} E|X_j|^{2+\delta}\right)^{\frac{3-\delta}{2}} \\
&\ge \frac{13.8513}{\sigma^{3+\delta}} \sum_{j=1}^{n} E|X_j|^{2+\delta} \\
&\ge \frac{2^{\frac{1+\delta}{2}}}{\sigma^{3+\delta}}\left[1.4464 + 0.761(2^\delta)\left(\frac{2.3907(2^\delta)}{3^{2+\delta}} + 1\right)\right] \sum_{j=1}^{n} E|X_j|^{2+\delta} \\
&= A_2.
\end{aligned}$$

Therefore, $A_2 \le T_2 \le T_1 + T_2 = A_1$.

Since $\tau_2 \ge \tau_1$, we have $B_2 \le B_1$. □

The above comparison shows that Theorem 2.1 provides a sharper error bound than Theorem 1.1. The improvement arises from the use of the polar representation

$$\psi(t) = |\psi(t)|e^{i\theta(t)},$$

whereas Theorem 1.1 estimates $\psi(t)$ directly. By separating the modulus $|\psi(t)|$ from the argument $\theta(t)$, we obtain more precise control of the inversion integral, which ultimately leads to substantially smaller constants in the final estimate. This improvement becomes even more pronounced for small values of δ, as illustrated in the following example.

Example 6.1. *Let*

$$P(X_j = 1) = P(X_j = 2) = 0.45,$$
$$P(X_j = 2^m) = \frac{2^{4.3}(1 - 2^{-2.15})}{10 \cdot 2^{2.15m}}, \qquad \text{for } \ j = 1, 2, \ldots, n, \quad m \geq 2.$$

Note that

$$E|X_j|^3 = 0.45 + 0.45(2)^3 + \frac{2^{4.3}(1 - 2^{-2.15})}{10} \sum_{m=2}^{\infty} \frac{2^{3m}}{2^{2.15m}} = \infty.$$

Choose $\delta = 0.1$. *Then*

$$E|X_j|^{2.1} = 0.45 + 0.45(2)^{2.1} + \frac{2^{4.3}(1 - 2^{-2.15})}{10} \sum_{m=2}^{\infty} \frac{2^{2.1m}}{2^{2.15m}} = 44.1777,$$

and

$$\tau_1 = \frac{2.7884 \times 10^{-6}}{n^{\frac{1}{2.1}}}, \qquad \sigma = 3.3121\sqrt{n}.$$

Since the support of X_j *is* $\{1, 2\} \cup \{2^m : m \geq 2\}$, *the only pair of consecutive support points is* $(1, 2)$. *Hence,*

$$\alpha_j = 2P(X_j = 1)P(X_j = 2) = 2(0.45)(0.45) = 0.405,$$

which implies

$$\alpha = 0.405n.$$

By Theorem 1.1,

$$\sup_{k\in\mathbb{Z}} \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \leq \frac{550443.2657}{n^{\frac{1.1}{2}}} + \frac{10409.0502}{n^{\frac{1.1}{2.1}}} e^{-4.2647\times 10^{-11} n^{\frac{0.1}{2.1}}} + \frac{1390947.6830}{n^{\frac{1.1}{2.1}}} e^{-3.1905\times 10^{-13} n^{\frac{0.1}{2.1}}}. \tag{6.2}$$

The constants in (6.2) *are extremely large. By direct computation from the definition of* τ_2 *in Theorem 2.1, we have*

$$\tau_2 = \frac{0.0549}{n^{\frac{1}{2.1}}}.$$

Applying Theorem 2.1, we obtain

$$\sup_{k\in\mathbb{Z}} \left| P(S_n = k) - \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-\mu)^2}{2\sigma^2}} \right| \leq \frac{3.9009}{n^{\frac{1.1}{2}}} + \frac{0.5287}{n^{\frac{1.1}{2.1}}} e^{-0.0165 n^{\frac{0.1}{2.1}}} + \frac{70.6470}{n^{\frac{1.1}{2.1}}} e^{-1.2368\times 10^{-4} n^{\frac{0.1}{2.1}}}.$$

Hence, the constants are significantly smaller than those obtained from Theorem 1.1, demonstrating the sharper estimate provided by Theorem 2.1.

7. Conclusions

Local limit theorems for integer-valued random variables are widely studied under a finite third-moment assumption. However, this assumption does not cover some important cases, such as the discrete Pareto distribution, for which the third moment may be infinite while a finite $(2+\delta)$-moment exists for some $0 < \delta < 1$. Kammoo et al. [12] addressed this setting by establishing a local limit theorem with an explicit error bound, as given in Theorem 1.1. However, the constants in their bound can become very large as $\delta \to 0$. In this work, we established Theorem 2.1 under the same moment condition and obtained a sharper error bound. We also showed that the constants appearing in our estimate are significantly smaller than those in Theorem 1.1. Finally, applications to discrete Pareto and maximal lattice distributions were presented to illustrate the usefulness of the result.

Author contributions

Jirapat Trakankitnukul: Methodology, validation, investigation; Kritsana Neammanee: Conceptualization, methodology, supervision, validation, writing–original draft, writing–review and editing; Tatpon Siripraparat: Conceptualization, methodology, supervision, validation, writing–original draft, writing–review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflicts of interest.

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