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Research article

Local strong solutions to the compressible Oldroyd–B system without an initial compatibility condition

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Abstract: In this paper, we establish the local existence and uniqueness of strong solutions to the three-dimensional compressible Oldroyd–B system in a bounded domain, allowing the initial density to vanish without imposing an initial compatibility condition. The initial data satisfy $(\rho_0, \tau_0) \in H^1(\Omega) \cap W^{1,q}(\Omega)$ and $u_0 \in H_0^1(\Omega)$ with $q \in (3, 6)$. The main analytical ingredient is a family of a priori estimates independent of the compatibility condition and the artificial positive lower bound of the density. In particular, time-weighted energy estimates recover higher regularity of the velocity for positive times. These estimates, together with the elliptic estimates, allow us to propagate the $H^1 \cap W^{1,q}$ regularity of the nondiffusive stress, control both the nonlinear term $g_a(\tau, \nabla u)$ and the stress time derivative τ_t, and complete the compactness and uniqueness arguments.

Keywords: compressible Oldroyd–B system; existence and uniqueness; vacuum; compatibility condition; time-weighted energy estimates
Mathematics Subject Classification: 35Q35, 76A10, 76N10

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded connected domain with a C^2 boundary. We consider the following compressible Oldroyd–B system in $\Omega \times (0, T)$:

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\rho(u_t + u \cdot \nabla u) - \eta_s(\Delta u + \nabla \operatorname{div} u) + \nabla P(\rho) = \operatorname{div} \tau, \tag{1.2}$$

$$\lambda(\tau_t + u \cdot \nabla \tau + g_a(\tau, \nabla u)) + \tau = 2\eta_e D(u). \tag{1.3}$$

Here, ρ, u, and τ denote the density, velocity, and symmetric extra-stress tensor, respectively. The constant $\lambda > 0$ is the relaxation time, while $\eta_s > 0$ and $\eta_e > 0$ are the solvent and polymeric viscosities, respectively. The nonlinear constitutive term is defined by

$$g_a(\tau, \nabla u) := \tau W(u) - W(u)\tau - a(D(u)\tau + \tau D(u)), \qquad a \in [-1, 1],$$

where

$$D(u) := \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}}), \qquad W(u) := \frac{1}{2}(\nabla u - (\nabla u)^{\mathrm{T}})$$

are the symmetric and skew-symmetric parts of ∇u, respectively. The pressure law $P = P(\rho)$ is assumed to satisfy

$$P \in C^1([0, \infty)), \qquad P(0) = 0. \tag{1.4}$$

The mathematical analysis of Oldroyd–B fluids has attracted considerable attention. For incompressible viscoelastic flows and related models, numerous results are available concerning the existence, uniqueness, and long-time behavior of solutions. Local existence and uniqueness of strong solutions to the incompressible Oldroyd–B model were first established by Guillopé and Saut [1]. They subsequently proved global existence for sufficiently small initial data under weak coupling assumptions [2]. Further well-posedness results in Sobolev spaces were obtained by Chemin and Masmoudi [3], while Zi–Fang–Zhang [4] established global solutions in a critical framework. We refer to [5–8] for additional developments. For the full incompressible Oldroyd–B model, however, the global existence of weak solutions remains a challenging open problem. An important partial result was obtained by Lions–Masmoudi [9], who proved global existence of weak solutions for general initial data under a simplified constitutive law in which the strain-rate contribution is neglected.

Compared with the incompressible theory, the analysis of strong solutions to compressible Oldroyd–B systems is less developed. Guillopé–Salloum–Talhouk [10] studied the local existence and uniqueness of regular solutions to weakly compressible viscoelastic systems governed by an Oldroyd-type constitutive law. Hu–Wang established global existence for multidimensional compressible viscoelastic flows with small initial data [11], investigated singularity formation and blow-up criteria [12], and proved local well-posedness of three-dimensional strong solutions for large initial data [13]. More recently, Meng–Zhang [14] established global existence and uniqueness of strong solutions to a multidimensional compressible Oldroyd–B model for sufficiently small initial data. In the presence of vacuum, Fang–Zi [15] established local well-posedness of strong solutions with initial data satisfying

$$\rho_0 \geq 0, \qquad (\rho_0, \tau_0) \in H^1 \cap W^{1,6}, \qquad u_0 \in D_0^1 \cap D^2,$$

together with the compatibility condition

$$\eta_s(\Delta u_0 + \nabla \operatorname{div} u_0) + \operatorname{div} \tau_0 - \nabla P(\rho_0) = \sqrt{\rho_0}\, g \tag{1.5}$$

for some $g \in L^2$. Yin–Zhang [16] subsequently established local existence and uniqueness of classical solutions with vacuum under a natural compatibility condition of the form (1.5). Lu–Zhang [17] considered a related compressible Oldroyd–B model including a polymer number density equation and stress diffusion, and proved local well-posedness of strong solutions and weak–strong uniqueness in several space dimensions. Recent progress on compressible Oldroyd–B systems includes stability and exponential decay [18, 19], blow-up criteria in the presence of vacuum [20], and global small-data solutions and large-time behavior [21–23].

It should be pointed out that the compatibility condition (1.5) imposes restrictive constraints on the initial data in both the vacuum region and a neighborhood of the vacuum–nonvacuum interface. This motivates the development of an alternative well-posedness theory that does not require such a condition. For Navier–Stokes systems, the removal of the initial compatibility condition has received considerable attention. Li [24] initiated this line of research for the inhomogeneous incompressible Navier–Stokes equations and established local well-posedness without an initial compatibility condition. Compatibility-free local theories were subsequently developed for the isentropic compressible equations by Gong–Li–Liu–Zhang [25] and for the heat-conductive compressible equations by Li–Zheng [26]; see also [27–29] for related developments. To the best of our knowledge, a corresponding compatibility-free strong-solution theory for the compressible Oldroyd–B system with vacuum has not previously been established. Unlike the compatibility-free Navier–Stokes equations [24–26], the present system contains a nondiffusive stress equation. Consequently, the $H^1 \cap W^{1,q}$ regularity of τ must be propagated directly, and the nonlinear stretching term $g_a(\tau, \nabla u)$ requires additional coupled estimates.

The aim of this paper is to establish the local existence and uniqueness of strong solutions for the compressible Oldroyd–B system in the presence of vacuum, without any initial compatibility condition. Compared with the compatibility-based results of Fang–Zi [15] and Yin–Zhang [16], we retain the allowance of vacuum and comparable regularity assumptions on the initial density and stress, but lower the initial velocity regularity from $H^2 \cap H_0^1$ to H_0^1 and remove the compatibility condition (1.5).

Systems (1.1)–(1.3) are supplemented with the following initial and boundary conditions

$$(\rho, \rho u, \tau)|_{t=0} = (\rho_0, \rho_0 u_0, \tau_0), \qquad \text{in } \Omega, \tag{1.6}$$

$$u = 0, \qquad \text{on } \partial\Omega, \tag{1.7}$$

where ρ_0, u_0, and τ_0 are prescribed initial data.

We emphasize that the actual value of u_0 that we need is only in the non-vacuum region $\Omega_+ := \{x \in \Omega \mid \rho_0(x) > 0\}$ but not in the vacuum region $\Omega_0 := \{x \in \Omega \mid \rho_0(x) = 0\}$. More precisely, denote by u_0^{real} the initial velocity in the non-vacuum region Ω_+, and define $\mathscr{S}_{\mathrm{ext}}$ as

$$\mathscr{S}_{\mathrm{ext}} = \left\{ \tilde{u}_0 \;\middle|\; \tilde{u}_0 = u_0^{\mathrm{real}} \text{ on } \Omega_+, \tilde{u}_0 \in H_0^1(\Omega) \right\},$$

then any $u_0 \in \mathscr{S}_{\mathrm{ext}}$ can be chosen as the "initial" velocity without changing the initial condition (1.6). In fact, for any $u_0 \in \mathscr{S}_{\mathrm{ext}}$, it is clear that

$$\rho_0 u_0 = \begin{cases} \rho_0 u_0^{\mathrm{real}}, & \text{in } \Omega_+, \\ 0, & \text{in } \Omega_0. \end{cases}$$

Due to the above explanation, throughout this paper, we always assume that the "initial" velocity u_0 is defined on the whole domain such that $u_0 \in H_0^1(\Omega)$.

Before stating the main results, we first clarify some necessary notations used throughout the paper and state the definition of solutions to be established.

For $1 \leq p \leq \infty$ and positive integer m, we denote by $L^p = L^p(\Omega)$ and $W^{m,p} = W^{m,p}(\Omega)$ the standard Lebesgue and Sobolev spaces, respectively, and we write $H^m = W^{m,2}$. For simplicity, we also use notations L^p and H^m to denote the N product spaces $(L^p)^N$ and $(H^m)^N$, respectively. For matrices $A, B \in \mathbb{R}^{3\times3}$, we set $A : B := \sum_{i,j=1}^{3} A_{ij}B_{ij}, |A| := (A : A)^{1/2}$, and for $1 \leq p < \infty$, $\|A\|_{L^p} := \left(\int_\Omega |A(x)|^p \mathrm{d}x\right)^{1/p}$, with the usual modification for $p = \infty$. The divergence of a tensor is taken row-wise,

$(\operatorname{div} A)_i := \sum_{j=1}^{3} \partial_j A_{ij}$. We use $\|\cdot\|_p$ for the $L^p(\Omega)$ norm and abbreviate $\|\cdot\|_2$ as $\|\cdot\|$. For brevity, we sometimes use $\|(f_1, f_2, \cdots, f_N)\|_X$ to denote the norm $\left(\sum_{i=1}^{N} \|f_i\|_X^2\right)^{\frac{1}{2}}$.

Throughout the paper, the pressure law P is fixed. The generic constant C may also depend on P through its local C^1-norm on bounded density intervals. More precisely, denote $\bar{\rho}$ by an upper bound of the density, then C may depend on

$$\bar{P} := 1 + \|P'\|_{L^\infty(0,\bar{\rho})}.$$

For every $1 < p < \infty$ and a vector $F \in L^p$, the solution $v \in W_0^{1,p} \cap W^{2,p}$ of

$$-\eta_s(\Delta v + \nabla \operatorname{div} v) = F \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega,$$

satisfies

$$\|v\|_{W^{2,p}} \leq C(\Omega, p, \eta_s)\|F\|_p.$$

We now define the class of strong solutions used in this paper.

Definition 1.1. *Let $T > 0$ and $q \in (3, 6)$. Assume that*

$$\rho_0 \geq 0, \qquad (\rho_0, \tau_0) \in H^1(\Omega) \cap W^{1,q}(\Omega), \qquad u_0 \in H_0^1(\Omega), \qquad \tau_0 = \tau_0^{\mathsf{T}} \quad \textit{a.e. in } \Omega.$$

A triple (ρ, u, τ) is called a strong solution to systems (1.1)–(1.3) in $\Omega \times (0, T)$, subject to (1.6)–(1.7), if it possesses the regularities

$$\begin{aligned}
&\rho \geq 0, \qquad (\rho, \tau) \in C([0, T]; L^2) \cap L^\infty(0, T; H^1 \cap W^{1,q}), \qquad (\rho_t, \tau_t) \in L^\infty(0, T; L^2),\\
&\sqrt{\rho}u \in C([0, T]; L^2), \qquad u \in L^\infty(0, T; H_0^1) \cap L^2(0, T; H^2) \cap L^1(0, T; W^{2,q}),\\
&\sqrt{\rho}u_t \in L^2(0, T; L^2), \qquad \sqrt{t}u \in L^\infty(0, T; H^2) \cap L^2(0, T; W^{2,q}), \qquad \sqrt{t}u_t \in L^2(0, T; H_0^1).
\end{aligned}$$

Moreover, $\tau = \tau^{\mathsf{T}}$ a.e. in $\Omega \times (0, T)$. Equations (1.1)–(1.3) are required to hold a.e. in $\Omega \times (0, T)$, supplemented with the initial condition (1.6).

Remark 1.1. *The symmetry of the extra stress is preserved by the constitutive equation. Indeed,*

$$g_a(\tau, \nabla u)^{\mathsf{T}} = g_a(\tau^{\mathsf{T}}, \nabla u), \qquad D(u)^{\mathsf{T}} = D(u).$$

Consequently, the antisymmetric part $\tau - \tau^{\mathsf{T}}$ satisfies a homogeneous linear transport–relaxation equation with zero initial data. Therefore, $\tau(t) = \tau(t)^{\mathsf{T}}$ for all $t \in [0, T]$.

Remark 1.2. *From the regularities of ρ stated in Definition 1.1, together with the Gagliardo–Nirenberg inequality and Sobolev embedding theorem $W^{1,q} \hookrightarrow\hookrightarrow C(\overline{\Omega})$ for $q \in (3, 6)$, we obtain $\rho \in C([0, T]; C(\overline{\Omega}))$. Thanks to this and recalling that $\sqrt{\rho}u \in C([0, T]; L^2)$, it is clear that $\rho u \in C([0, T]; L^2)$. Therefore, the initial value of ρu is well-defined.*

Remark 1.3. *The solution class in Definition 1.1 is adapted to the regularity of the initial data and to the absence of an initial compatibility condition. Since $u_0 \in H_0^1(\Omega)$, no H^2 trace of the velocity is available at $t = 0$. Moreover, the momentum equation does not imply $\sqrt{\rho}\, u_t(0) \in L^2$ in the absence of* *the compatibility condition. It would therefore be inconsistent to require $u \in L^\infty(0,T;H^2)$ or $\sqrt{\rho}\,u_t \in L^\infty(0,T;L^2)$ up to the initial time. Instead, the time-weighted estimates*

$$\sqrt{t}\,u \in L^\infty(0,T;H^2) \cap L^2(0,T;W^{2,q}), \qquad \sqrt{t}\,u_t \in L^2(0,T;H_0^1)$$

describe the higher regularity generated for positive time without imposing an unsupported higher-order trace at $t = 0$. These weighted bounds are uniform with respect to the artificial compatibility quantities in the approximation and provide the time integrability needed for the compactness and uniqueness arguments.

Remark 1.4. *Formally, if the polymeric stress is neglected, the systems* (1.1)–(1.3) *reduces to the isentropic compressible Navier–Stokes equations, for which the local well-posedness theory has been established in [25] without any initial compatibility condition.*

We now state the main result of this paper.

Theorem 1.1. *Let $q \in (3,6)$ and assume that*

$$\rho_0 \geq 0, \qquad (\rho_0,\tau_0) \in H^1(\Omega) \cap W^{1,q}(\Omega), \qquad u_0 \in H_0^1(\Omega), \qquad \tau_0 = \tau_0^{\mathsf{T}} \quad a.e.\ in\ \Omega.$$

Then there exists a positive time T_0 depending only on $\Omega, a, \lambda, q, \eta_e, \eta_s, \bar{P}$, and Φ_0, such that systems (1.1)–(1.3), subject to (1.6)–(1.7), has a unique strong solution in the sense of Definition 1.1 on $\Omega \times (0,T_0)$, where $\Phi_0 := \|\rho_0\|_\infty + \|(\rho_0,\tau_0)\|_{H^1\cap W^{1,q}} + \|\nabla u_0\|^2$.

Remark 1.5. *The local-in-time result in Theorem 1.1 is established in three dimensions for a bounded connected domain with C^2 boundary and the no-slip condition $u|_{\partial\Omega} = 0$. It is proved for $q \in (3,6)$, $(\rho_0,\tau_0) \in H^1 \cap W^{1,q}$, and $u_0 \in H_0^1$, and it allows vacuum in the initial density.*

Remark 1.6. *Since $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$, the lower bound $q > 3$ is structural for the present functional framework. In contrast, the upper bound $q < 6$ is a technical restriction of the present argument. It arises from the time integral $\int_0^T t^{-\frac{3q-6}{2q}}\,\mathrm{d}t$, which is finite precisely when $q < 6$ in Proposition 2.2. Thus the condition $q < 6$ is required by the interpolation step used in this proof.*

The existence proof of Theorem 1.1 is based on compatibility-independent a priori estimates for the quantity

$$\Phi(t) := 1 + \sup_{0\leq s\leq t}\left(\|\rho\|_\infty + \|(\rho,\tau)\|_{H^1\cap W^{1,q}} + \|\nabla u\|^2\right) + \int_0^t \left(\|\sqrt{\rho}u_t\|^2 + \|\nabla^2 u\|^2 + \|\sqrt{s}\nabla u_t\|^2\right)\mathrm{d}s, \tag{1.8}$$

for approximate solutions (ρ,u,τ) to (1.1)–(1.3), subject to (1.6)–(1.7). The approximation and continuity framework follows the strategy developed in [25]. More precisely, we prove that there exist positive constants ϵ_0 and C, independent of the compatibility quantity and the artificial positive lower bound of the density, such that $\Phi(T) \leq C$ provided that $T^{\frac{6-q}{4q}}\Phi^2(T) \leq \epsilon_0$. The stress equation introduces difficulties that have no counterpart in the compressible Navier–Stokes equations [25, 26]. Since it contains no spatial diffusion, the $H^1 \cap W^{1,q}$ regularity of τ cannot be recovered through parabolic smoothing and must instead be propagated using the time integrability of $\|\nabla^2 u\|_q$. Moreover, differentiating $g_a(\tau,\nabla u)$ produces products of the form $\nabla\tau\,\nabla u$ and $\tau\nabla^2 u$. At the same time, $\operatorname{div}\tau$ enters the Lamé equation for u, while $\operatorname{div}\tau_t$ occurs in the time-differentiated momentum estimate.

Consequently, the stress estimate, the Lamé elliptic estimate, and the weighted estimate for u_t must be closed simultaneously. To overcome these difficulties, we address these couplings by first deriving compatibility-independent time-weighted estimates for the velocity, then using the constitutive equation to control τ_t, and finally closing the $H^1 \cap W^{1,q}$ estimate for τ together with the elliptic estimates for u.

2. A priori estimates independent of the compatibility condition

In this section, we derive some a priori estimates for strong solutions to systems (1.1)–(1.3), subject to (1.6)–(1.7), with initial data satisfying the compatibility condition (1.5). In particular, although the approximate solutions satisfy the initial compatibility condition, the a priori estimates established in this section do not depend on this condition. The density-transport estimates, continuity argument, and basic time-weighted velocity estimates in this section follow the strategy developed for compressible Navier–Stokes equations in [25, 26], where no initial compatibility condition is imposed. These arguments, however, do not directly control the extra stress. The estimates specific to the present compressible Oldroyd–B system include the propagation of τ in $H^1 \cap W^{1,q}$, the control of τ_t and $g_a(\tau, \nabla u)$, and their coupling with the Dirichlet Lamé elliptic estimates.

We now begin with the following local existence and uniqueness result for strong solutions to the compressible Oldroyd–B system, proved in [15], where the natural compatibility condition is required.

Proposition 2.1. *Let $q \in (3, 6]$, and assume that (ρ_0, u_0, τ_0) satisfies*

$$0 \le \rho_0, \qquad (\rho_0, \tau_0) \in H^1(\Omega) \cap W^{1,q}(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad \tau_0 = \tau_0^{\top} \ a.e. \ \textbf{\textit{in}} \ \Omega,$$

and the compatibility condition

$$\eta_s(\Delta u_0 + \nabla \operatorname{div} u_0) + \operatorname{div} \tau_0 - \nabla P(\rho_0) = \sqrt{\rho_0} g,$$

for some $g \in L^2(\Omega)$.

Then, there exists a positive time T_* depending on $\Omega, a, \lambda, q, \eta_e, \eta_s, \bar{P}, \Phi_0, \|\nabla^2 u_0\|$, and $\|g\|$ such that systems (1.1)–(1.3) admits a unique strong solution (ρ, u, τ) in $\Omega \times (0, T_*)$, satisfying

$$\begin{aligned} \rho \ge 0, \quad (\rho, \tau) \in C([0, T_*]; H^1 \cap W^{1,q}), \quad (\rho_t, \tau_t) \in C([0, T_*]; L^2 \cap L^q), \\ u \in C([0, T_*]; H_0^1 \cap H^2) \cap L^2(0, T_*; W^{2,q}), \\ u_t \in L^2(0, T_*; H_0^1), \qquad \sqrt{\rho} u_t \in L^\infty(0, T_*; L^2). \end{aligned}$$

It will be shown in this section that the existence time T_* in the above proposition can be chosen depending only on $\Omega, a, \lambda, q, \eta_e, \eta_s, \bar{P}$, and the upper bound of

$$\Phi_0 := \|\rho_0\|_\infty + \|(\rho_0, \tau_0)\|_{H^1 \cap W^{1,q}} + \|\nabla u_0\|^2.$$

In particular, T_* can be chosen independent of $\|\nabla^2 u_0\|$ and $\|g\|$. Let Φ be the quantity given by (1.8). The main content of this section is to obtain the local-in-time estimate of Φ independent of $\|\nabla^2 u_0\|$ and $\|g\|$, and therefore independent of the initial compatibility condition.

In the rest of this section until the last proposition, we always assume that (ρ, u, τ) is a solution to systems (1.1)–(1.3) in $\Omega \times (0, T)$, for some positive time $T \le 1$, satisfying the regularities in Proposition 2.1 with T_* there replaced by T. We emphasize again that C, which may vary from place to place, is a generic constant depending only on $\Omega, a, \lambda, q, \eta_e, \eta_s, \bar{P}$, and the upper bound of Φ_0.

Proposition 2.2. *It holds that*

$$\int_0^T \left(\|\nabla u\|_\infty + \|\nabla^2 u\|_q\right) \mathrm{d}t \le CT^{\frac{6-q}{4q}} \Phi^2(T).$$

Proof. Applying the elliptic estimates to (1.2), one obtains

$$\|\nabla^2 u\|_q \le C(\|\rho u_t\|_q + \|\rho(u\cdot\nabla)u\|_q + \|\nabla P\|_q + \|\operatorname{div}\tau\|_q).$$

It follows from the Hölder, Gagliardo–Nirenberg, Sobolev, and Poincaré inequalities together with (1.4) that

$$\begin{aligned}
\|\rho u_t\|_q &\le C\|\rho\|_\infty^{\frac{1}{2}} \|\sqrt{\rho} u_t\|^{\frac{6-q}{2q}} \|\sqrt{\rho} u_t\|_6^{\frac{3q-6}{2q}} \le C\|\rho\|_\infty^{\frac{5q-6}{4q}} \|\sqrt{\rho} u_t\|^{\frac{6-q}{2q}} \|\nabla u_t\|^{\frac{3q-6}{2q}},\\
\|\rho(u\cdot\nabla)u\|_q &\le \|\rho\|_\infty \|u\|_\infty \|\nabla u\|_q \le C\|\rho\|_\infty \|\nabla u\|^{\frac{1}{2}} \|\nabla^2 u\|^{\frac{3}{2}},\\
\|\nabla P\|_q &\le \|P'(\rho)\|_\infty \|\nabla\rho\|_q.
\end{aligned}$$

Integrating these estimates over $(0, T)$ and applying Hölder's inequality in time, we obtain

$$\begin{aligned}
\int_0^T \|\rho u_t\|_q \mathrm{d}t &\le C\int_0^T \|\rho\|_\infty^{\frac{5q-6}{4q}} \|\sqrt{\rho} u_t\|^{\frac{6-q}{2q}} \|\nabla u_t\|^{\frac{3q-6}{2q}} \mathrm{d}t\\
&\le C\Phi^{\frac{5q-6}{4q}}(T) \left(\int_0^T \|\sqrt{\rho} u_t\|^2 \mathrm{d}t\right)^{\frac{6-q}{4q}} \left(\int_0^T \|\sqrt{t}\nabla u_t\|^2 \mathrm{d}t\right)^{\frac{3q-6}{4q}} \left(\int_0^T t^{-\frac{3q-6}{2q}} \mathrm{d}t\right)^{\frac{1}{2}}\\
&\le CT^{\frac{6-q}{4q}} \Phi^{\frac{7q-6}{4q}}(T),
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \|\rho(u\cdot\nabla)u\|_q \mathrm{d}t &\le C\int_0^T \|\rho\|_\infty \|\nabla u\|^{\frac{1}{2}} \|\nabla^2 u\|^{\frac{3}{2}} \mathrm{d}t\\
&\le C\Phi^{\frac{5}{4}}(T) \left(\int_0^T \|\nabla^2 u\|^2 \mathrm{d}t\right)^{\frac{3}{4}} T^{\frac{1}{4}} \le CT^{\frac{1}{4}} \Phi^2(T),
\end{aligned}$$

as well as

$$\int_0^T (\|\nabla P\|_q + \|\operatorname{div}\tau\|_q) \mathrm{d}t \le C\int_0^T (\|\nabla\rho\|_q + \|\nabla\tau\|_q) \mathrm{d}t \le CT\Phi(T).$$

Consequently,

$$\int_0^T \|\nabla^2 u\|_q \mathrm{d}t \le C\left(T^{\frac{6-q}{4q}} \Phi^{\frac{7q-6}{4q}}(T) + T^{\frac{1}{4}} \Phi^2(T) + T\Phi(T)\right) \le CT^{\frac{6-q}{4q}} \Phi^2(T),$$

where $\frac{1}{4} > \frac{6-q}{4q}$ for $q \in (3, 6)$, $T \le 1$, and $\Phi(T) \ge 1$ were used. The Sobolev and the Poincaré inequalities then yield

$$\int_0^T \|\nabla u\|_\infty \mathrm{d}t \le C\int_0^T \|\nabla^2 u\|_q \mathrm{d}t \le CT^{\frac{6-q}{4q}} \Phi^2(T).$$

The proof is complete. □

Proposition 2.3. *It holds that*

$$\sup_{0\le t\le T}\|\rho\|_\infty \le \|\rho_0\|_\infty \exp\left\{CT^{\frac{6-q}{4q}}\Phi^2(T)\right\},$$
$$\sup_{0\le t\le T}\|(\rho, P(\rho))\|_{H^1\cap W^{1,q}} \le C\left(1+T^{\frac{6-q}{4q}}\Phi^2(T)\right)\exp\left\{CT^{\frac{6-q}{4q}}\Phi^2(T)\right\}.$$

Proof. For any given $x \in \Omega$ and $s \in [0, T]$, define $U(x, t; s)$ as

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}U(x,t;s) = u(U(x,t;s),t), & \forall\ t \in [0,T],\\ U(x,s;s) = x.\end{cases}$$

Note that since $u \in L^1(0, T; W^{1,\infty})$ is guaranteed by Proposition 2.2, U is well defined. Then, it follows from (1.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(U(x,t;s),t) = -\operatorname{div} u(U(x,t;s),t)\rho(U(x,t;s),t), \qquad \forall\ t \in (0,T).$$

Solving the above ordinary differential equation gives

$$\rho(U(x,t;s),t) = \rho(x,s)e^{-\int_s^t \operatorname{div} u(U(x,\sigma;s),\sigma)\mathrm{d}\sigma}, \qquad \forall\ s,t \in [0,T].$$

Setting $t = 0$ in the above, one gets

$$\rho(x,s) = \rho_0(U(x,0;s))e^{-\int_0^s \operatorname{div} u(U(x,\sigma;s),\sigma)\mathrm{d}\sigma}, \qquad \forall\ (x,s) \in \Omega\times[0,T].$$

It follows from Proposition 2.2 that

$$\sup_{0\le t\le T}\|\rho\|_\infty(t) \le \|\rho_0\|_\infty e^{\int_0^T \|\operatorname{div} u\|_\infty(\sigma)\mathrm{d}\sigma} \le \|\rho_0\|_\infty e^{CT^{\frac{6-q}{4q}}\Phi^2(T)},$$

proving the first conclusion.

Multiplying (1.1) by ρ^{q-1} and integrating over Ω, one deduces by integration by parts that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_q^q \le C\|\nabla u\|_\infty\|\rho\|_q^q.$$

Applying the operator ∇ to (1.1) and testing the resulting equation against $|\nabla\rho|^{q-2}\nabla\rho$, it follows from integration by parts that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_q^q &\le C\int_\Omega\left(|\nabla u||\nabla\rho|^q + |\rho||\nabla\rho|^{q-1}|\nabla^2 u|\right)\mathrm{d}x\\ &\le C(\|\nabla u\|_\infty\|\nabla\rho\|_q^q + \|\rho\|_\infty\|\nabla\rho\|_q^{q-1}\|\nabla^2 u\|_q).\end{aligned}$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{W^{1,q}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\|\rho\|_q + \|\nabla\rho\|_q\right) \le C(\|\nabla u\|_\infty\|\rho\|_{W^{1,q}} + \|\rho\|_\infty\|\nabla^2 u\|_q). \tag{2.1}$$

In a similar manner, one derives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{H^1} \le C(\|\nabla u\|_\infty \|\rho\|_{H^1} + \|\rho\|_\infty \|\nabla^2 u\|_2),$$

and combined with (2.1), it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{H^1\cap W^{1,q}} \le C(\|\nabla u\|_\infty \|\rho\|_{H^1\cap W^{1,q}} + \|\rho\|_\infty \|\nabla^2 u\|_{L^2\cap L^q}).$$

Applying the Grönwall inequality and using the first conclusion together with Proposition 2.2, one obtains

$$\begin{aligned}\sup_{0\le t\le T}\|\rho\|_{H^1\cap W^{1,q}} &\le \left(\|\rho_0\|_{H^1\cap W^{1,q}} + \int_0^T \|\rho\|_\infty \|\nabla^2 u\|_{L^2\cap L^q}\mathrm{d}t\right) e^{C\int_0^T \|\nabla u\|_\infty \mathrm{d}t} \\ &\le C\left(1 + T^{\frac{6-q}{4q}}\Phi^2(T)\right) e^{CT^{\frac{6-q}{4q}}\Phi^2(T)}. \end{aligned} \tag{2.2}$$

Finally, since $P(0) = 0$ and $P \in C^1([0,\infty))$, the Sobolev composition estimate gives

$$\sup_{0\le t\le T}\|P(\rho)\|_{H^1\cap W^{1,q}} \le C(1 + \|P'(\rho)\|_\infty)\|\rho\|_{H^1\cap W^{1,q}} \le C\left(1 + T^{\frac{6-q}{4q}}\Phi^2(T)\right) e^{CT^{\frac{6-q}{4q}}\Phi^2(T)}.$$

Combining this and (2.2) proves the second conclusion.

□

As a direct corollary of Proposition 2.3, one obtains:

Corollary 2.1. *There is a sufficiently small positive constant* $\epsilon_0 \le 1$ *depending only on* $\Omega, a, \lambda, q, \eta_e, \eta_s, \bar{P}$, *and* Φ_0, *such that*

$$\sup_{0\le t\le T}\|\rho\|_\infty \le 2\|\rho_0\|_\infty, \qquad \sup_{0\le t\le T}\|(\rho, P(\rho))\|_{H^1\cap W^{1,q}} \le C,$$

provided that

$$T^{\frac{6-q}{4q}}\Phi^2(T) \le \epsilon_0. \tag{2.3}$$

Since $\Phi(T) \ge 1$, it is easy to check that the following relations follow from (2.3):

$$T\Phi^3(T) = \left(T^{\frac{6-q}{4q}}\Phi^2(T)\right)^{\frac{3}{2}} T^{\frac{11q-18}{8q}} \le \epsilon_0^{\frac{3}{2}} \le 1, \tag{2.4}$$

$$T^{\frac{1}{4}}\Phi^{\frac{3}{2}}(T) = \left(T^{\frac{6-q}{4q}}\Phi^2(T)\right)^{\frac{3}{4}} T^{\frac{7q-18}{16q}} \le \epsilon_0^{\frac{3}{4}} \le 1. \tag{2.5}$$

These consequences will be used frequently in the rest of this section.

Proposition 2.4. *Let* ϵ_0 *be the number stated in Corollary 2.1 and assume that* (2.3) *holds. Then, the following estimate holds:*

$$\sup_{0\le t\le T}\|\tau\|_{H^1\cap W^{1,q}} \le C.$$

Proof. Multiplying (1.3) by $|\tau|^{q-2}\tau$ and integrating over Ω, one deduces by the Hölder and Sobolev inequalities

$$\begin{aligned}\frac{\lambda}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_q^q + \|\tau\|_q^q &\le C\int_\Omega \left(|\nabla u||\tau|^q + |\nabla u||\tau|^{q-1}\right)\mathrm{d}x\\ &\le C\left(\|\nabla u\|_\infty\|\tau\|_q^q + \|\nabla u\|_q\|\tau\|_q^{q-1}\right),\end{aligned}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_q \le C\left(\|\nabla u\|_\infty\|\tau\|_q + \|\nabla u\|_q\right). \tag{2.6}$$

Applying ∂_k to (1.3) gives

$$\lambda(\partial_k\partial_t\tau_{ij} + \partial_k u\cdot\nabla\tau_{ij} + u\cdot\nabla\partial_k\tau_{ij} + \partial_k g_a(\tau,\nabla u)_{ij}) + \partial_k\tau_{ij} = 2\eta_e\partial_k D(u)_{ij}.$$

Multiplying the resulting equation by $|\nabla\tau|^{q-2}\partial_k\tau_{ij}$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{\lambda}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\tau\|_q^q + \|\nabla\tau\|_q^q &\le C\int_\Omega \left(|\nabla u||\nabla\tau|^q + |\tau||\nabla^2 u||\nabla\tau|^{q-1} + |\nabla^2 u||\nabla\tau|^{q-1}\right)\mathrm{d}x\\ &\le C\|\nabla u\|_\infty\|\nabla\tau\|_q^q + C\left(1+\|\tau\|_\infty\right)\|\nabla^2 u\|_q\|\nabla\tau\|_q^{q-1}.\end{aligned}$$

Combining this estimate with (2.6), and using $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$, one deduces by the Poincaré inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_{W^{1,q}} &\le C\left(\|\nabla u\|_\infty\|\tau\|_{W^{1,q}} + \|\nabla^2 u\|_q\|\tau\|_{W^{1,q}} + \|\nabla u\|_q + \|\nabla^2 u\|_q\right)\\ &\le C\left(\|\nabla^2 u\|_q\|\tau\|_{W^{1,q}} + \|\nabla^2 u\|_q\right).\end{aligned} \tag{2.7}$$

Similarly, the H^1 estimate gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_{H^1} \le C\left(\|\nabla^2 u\|_q\|\tau\|_{H^1} + \|\nabla^2 u\|\|\tau\|_{W^{1,q}} + \|\nabla^2 u\|\right).$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tau\|_{H^1\cap W^{1,q}} \le C\left(\|\nabla^2 u\|_{L^2\cap L^q}\|\tau\|_{H^1\cap W^{1,q}} + \|\nabla^2 u\|_{L^2\cap L^q}\right).$$

By the Grönwall inequality and Corollary 2.1, this implies

$$\begin{aligned}\sup_{0\le t\le T}\|\tau\|_{H^1\cap W^{1,q}} &\le \left(\|\tau_0\|_{H^1\cap W^{1,q}} + C\int_0^T \|\nabla^2 u\|_{L^2\cap L^q}\,\mathrm{dt}\right)\exp\left\{C\int_0^T \|\nabla^2 u\|_{L^2\cap L^q}\,\mathrm{dt}\right\}\\ &\le C\left(1 + T^{\frac{6-q}{4q}}\Phi^2(T)\right)\exp\left\{CT^{\frac{6-q}{4q}}\Phi^2(T)\right\} \le C.\end{aligned}$$

This completes the proof. □

Proposition 2.5. *Under the conditions of Proposition 2.4, it holds that*

$$\sup_{0\le t\le T}\|(\nabla u, \tau_t)\|^2 + \int_0^T \left(\|\sqrt{\rho}u_t\|^2 + \|\nabla^2 u\|^2\right)\mathrm{dt} \le C.$$

Proof. By Corollary 2.1 and Proposition 2.4, it follows from (1.3) and the Hölder and Sobolev inequalities that

$$\begin{aligned}\|\tau_t\|^2 &\le C\left(\|u\cdot\nabla\tau\|^2 + \|g_a(\tau,\nabla u)\|^2 + \|\tau\|^2 + \|D(u)\|^2\right)\\ &\le C\left(\|u\|_6^2\|\nabla\tau\|_3^2 + \|\tau\|_\infty^2\|\nabla u\|^2 + \|\tau\|^2 + \|\nabla u\|^2\right) \le C\left(1+\|\nabla u\|^2\right).\end{aligned}\tag{2.8}$$

Applying the estimates to (1.2) and using the Young and Gagliardo–Nirenberg inequalities, one gets by Corollary 2.1 and Proposition 2.4

$$\begin{aligned}\|\nabla^2 u\|^2 &\le C(\|\rho u_t\|^2 + \|\rho(u\cdot\nabla)u\|^2 + \|\nabla P\|^2 + \|\operatorname{div}\tau\|^2)\\ &\le C(\|\rho\|_\infty\|\sqrt{\rho}u_t\|^2 + \|\rho\|_\infty^2\|u\|_6^2\|\nabla u\|_3^2 + \|P'(\rho)\|_\infty^2\|\nabla\rho\|^2 + \|\operatorname{div}\tau\|^2)\\ &\le C(\|\sqrt{\rho}u_t\|^2 + \|\nabla u\|^3\|\nabla^2 u\| + \|\nabla\rho\|^2 + \|\nabla\tau\|^2)\\ &\le \frac{1}{2}\|\nabla^2 u\|^2 + C(\|\sqrt{\rho}u_t\|^2 + \|\nabla u\|^6 + \|\nabla\rho\|^2 + \|\nabla\tau\|^2),\end{aligned}$$

and thus,

$$\|\nabla^2 u\|^2 \le C(\|\sqrt{\rho}u_t\|^2 + \|\nabla u\|^6 + \|\nabla\rho\|^2 + \|\nabla\tau\|^2).\tag{2.9}$$

Multiplying (1.2) by u_t, and integrating over Ω, one obtains by the Young, Sobolev, and Gagliardo–Nirenberg inequalities

$$\begin{aligned}\frac{\eta_s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + \|\operatorname{div}u\|^2\right) + \|\sqrt{\rho}u_t\|^2 &= -\int_\Omega \rho(u\cdot\nabla)u\cdot u_t\mathrm{d}x\\ &\quad -\int_\Omega \nabla P\cdot u_t\mathrm{d}x + \int_\Omega \operatorname{div}\tau\cdot u_t\mathrm{d}x.\end{aligned}$$

It follows from the Gagliardo–Nirenberg, Sobolev, and Young inequalities that for any positive number η,

$$\begin{aligned}\left|\int_\Omega \rho(u\cdot\nabla)u\cdot u_t\mathrm{d}x\right| &\le \|\rho\|_\infty^{\frac{1}{2}}\|u\|_6\|\nabla u\|_3\|\sqrt{\rho}u_t\| \le C\|\nabla u\|^{\frac{3}{2}}\|\nabla^2 u\|^{\frac{1}{2}}\|\sqrt{\rho}u_t\|\\ &\le \frac{1}{4}\|\sqrt{\rho}u_t\|^2 + \eta\|\nabla^2 u\|^2 + C_\eta\|\nabla u\|^6.\end{aligned}\tag{2.10}$$

To handle the pressure term, integration by parts gives

$$\int_\Omega \nabla P(\rho)\cdot u_t\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P(\rho)\mathrm{div}u\mathrm{d}x + \int_\Omega \partial_t(P(\rho))\mathrm{div}u\mathrm{d}x,$$

and, by (1.1) and Corollary 2.1, we further have

$$\begin{aligned}\left|\int_\Omega \partial_t(P(\rho))\mathrm{div}u\mathrm{d}x\right| &\le \int_\Omega |P'(\rho)|\left(|u||\nabla\rho| + \rho|\operatorname{div}u|\right)|\nabla u|\mathrm{d}x\\ &\le C\|P'(\rho)\|_\infty\left(\|u\|_6\|\nabla\rho\|_3\|\nabla u\| + \|\rho\|_\infty\|\nabla u\|^2\right) \le C\|\nabla u\|^2.\end{aligned}\tag{2.11}$$

For the stress term, integration by parts gives

$$\int_\Omega \operatorname{div}\tau\cdot u_t\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tau:\nabla u\mathrm{d}x + \int_\Omega \tau_t:\nabla u\mathrm{d}x$$

$$\leq -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tau : \nabla u \mathrm{d}x + C(\|\tau_t\|^2 + \|\nabla u\|^2). \tag{2.12}$$

Combining (2.8) and (2.10)–(2.12), adding the resultant with (2.9) multiplied by a small positive number ϵ_1, and choosing η sufficiently small, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\eta_s}{2}\|\nabla u\|^2 + \frac{\eta_s}{2}\|\operatorname{div} u\|^2 - \int_\Omega P(\rho)\operatorname{div} u \mathrm{d}x + \int_\Omega \tau : \nabla u \mathrm{d}x\right) + \frac{1}{2}\|\sqrt{\rho}u_t\|^2 + \frac{\epsilon_1}{2}\|\nabla^2 u\|^2$$
$$\leq C\left(1 + \|\nabla u\|^6 + \|\nabla\tau\|^2\right).$$

Integrating the above inequality over $(0, T)$, one deduces by the Cauchy inequality, Corollary 2.1, and Proposition 2.4

$$\begin{aligned}
&\eta_s \sup_{0\leq t\leq T}\|\nabla u\|^2 + \int_0^T \left(\|\sqrt{\rho}u_t\|^2 + \epsilon_1\|\nabla^2 u\|^2\right)\mathrm{dt}\\
&\leq 2\sup_{0\leq t\leq T}\left(\left|\int_\Omega P(\rho)\operatorname{div} u \mathrm{d}x\right| + \left|\int_\Omega \tau : \nabla u \mathrm{d}x\right|\right)\\
&\quad + C\left(\|\nabla u_0\|^2 + \left|\int_\Omega P(\rho_0)\operatorname{div} u_0 \mathrm{d}x\right| + \left|\int_\Omega \tau_0 : \nabla u_0 \mathrm{d}x\right|\right)\\
&\quad + C\int_0^T \left(1 + \|\nabla u\|^6 + \|\nabla\tau\|^2\right)\mathrm{d}t\\
&\leq \frac{\eta_s}{2}\sup_{0\leq t\leq T}\|\nabla u\|^2 + C\Big[1 + \sup_{0\leq t\leq T}\left(\|P\|^2 + \|\tau\|^2\right) + T\Phi^3(T)\Big]\\
&\leq \frac{\eta_s}{2}\sup_{0\leq t\leq T}\|\nabla u\|^2 + C\left(1 + T\Phi^3(T)\right),
\end{aligned}$$

which, by (2.4), the conclusion follows from this and (2.8). □

Proposition 2.6. *Under the conditions of Proposition 2.4, it holds that*

$$\sup_{0\leq t\leq T}\|(\sqrt{t}\sqrt{\rho}u_t, \sqrt{t}\nabla^2 u)\|^2 + \int_0^T \|\sqrt{t}\nabla u_t\|^2 \mathrm{dt} \leq C.$$

Proof. Differentiating (1.2) with respect to t yields

$$\rho(u_{tt} + u\cdot\nabla u_t) + \rho(u_t\cdot\nabla)u + \rho_t(u_t + (u\cdot\nabla)u) - \eta_s(\Delta u_t + \nabla\operatorname{div} u_t) + \nabla\partial_t(P(\rho)) = \operatorname{div}\tau_t.$$

Testing the resulting equation with tu_t, and integrating over $\Omega\times(0, T)$, one deduces after integration by parts

$$\begin{aligned}
&\frac{1}{2}\sup_{0\leq t\leq T}\|\sqrt{t}\sqrt{\rho}u_t\|^2 + \eta_s\int_0^T \left(\|\sqrt{t}\nabla u_t\|^2 + \|\sqrt{t}\operatorname{div} u_t\|^2\right)\mathrm{dt}\\
&= \frac{1}{2}\int_0^T \|\sqrt{\rho}u_t\|^2 \mathrm{dt} - \int_0^T t\int_\Omega \rho(u_t\cdot\nabla)u\cdot u_t \mathrm{d}x\mathrm{d}t + \int_0^T t\int_\Omega \partial_t(P(\rho))\operatorname{div} u_t \mathrm{d}x\mathrm{d}t\\
&\quad + \int_0^T t\int_\Omega \operatorname{div}\tau_t\cdot u_t \mathrm{d}x\mathrm{d}t - \int_0^T t\int_\Omega \rho u\cdot\nabla|u_t|^2 \mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$-\int_0^T t\int_\Omega \rho u\cdot\nabla((u\cdot\nabla)u\cdot u_t)\mathrm{d}x\mathrm{d}t =: \sum_{i=1}^6 J_i. \tag{2.13}$$

By the Young and Gagliardo–Nirenberg inequalities, it follows from Corollary 2.1 and Proposition 2.5 that $J_1 \le C$ and

$$\begin{aligned}
|J_2| &\le C\int_0^T t\|\sqrt{\rho}\|_\infty\|\nabla u\|\|\sqrt{\rho}u_t\|_3\|u_t\|_6\,\mathrm{dt}\\
&\le C\int_0^T t^{\frac14}\|\nabla u\|\|\sqrt{\rho}u_t\|^{\frac12}\|\sqrt{t}\nabla u_t\|^{\frac32}\,\mathrm{dt} \le CT^{\frac14}\Phi^{\frac32}(T),\\
|J_3| &\le \int_0^T t\int_\Omega |P'(\rho)|(|u||\nabla\rho| + \rho|\operatorname{div} u|)|\operatorname{div} u_t|\mathrm{d}x\,\mathrm{dt}\\
&\le C\int_0^T \sqrt{t}(\|u\|_6\|\nabla\rho\|_3 + \|\rho\|_\infty\|\nabla u\|)\|\sqrt{t}\nabla u_t\|\,\mathrm{dt}\\
&\le C\int_0^T \sqrt{t}\|\nabla u\|\|\sqrt{t}\nabla u_t\| \le CT\Phi(T),\\
|J_4| &\le C\int_0^T \sqrt{t}\|\tau_t\|\|\sqrt{t}\nabla u_t\|\,\mathrm{dt} \le CT\Phi(T),\\
|J_5| &\le C\int_0^T t\|\sqrt{\rho}\|_\infty\|u\|_6\|\sqrt{\rho}u_t\|_3\|\nabla u_t\|\,\mathrm{dt} \le CT^{\frac14}\Phi^{\frac32}(T),\\
|J_6| &\le \int_0^T t\int_\Omega \rho|u|(|\nabla u|^2|u_t| + |u||\nabla^2 u||u_t| + |u||\nabla u||\nabla u_t|)\mathrm{d}x\,\mathrm{dt}\\
&\le \int_0^T t\|\rho\|_\infty(\|u\|_6\|\nabla u\|_3^2\|u_t\|_6 + \|u\|_6^2\|\nabla^2 u\|\|u_t\|_6 + \|u\|_6^2\|\nabla u\|_6\|\nabla u_t\|)\,\mathrm{dt}\\
&\le C\int_0^T \sqrt{t}\|\nabla u\|^2\|\nabla^2 u\|\|\sqrt{t}\nabla u_t\|\,\mathrm{dt} \le CT^{\frac12}\Phi^2(T).
\end{aligned}$$

Substituting these bounds into (2.13), it follows from (2.5) that

$$\begin{aligned}
&\sup_{0\le t\le T}\|\sqrt{t}\sqrt{\rho}u_t\|^2 + 2\eta_s\int_0^T\|\sqrt{t}\nabla u_t\|^2\,\mathrm{dt}\\
&\le C\left(1 + T\Phi(T) + T^{\frac14}\Phi^{\frac32}(T) + T^{\frac12}\Phi^2(T)\right) \le C.
\end{aligned} \tag{2.14}$$

Combining this with (2.9), Corollary 2.1, and Propositions 2.4–2.5 gives

$$\sup_{0\le t\le T}\|\sqrt{t}\nabla^2 u\|^2 \le C\sup_{0\le t\le T}(\|\sqrt{t}\sqrt{\rho}u_t\|^2 + t\|\nabla u\|^6 + t\|\nabla\rho\|^2 + t\|\nabla\tau\|^2) \le C.$$

This finishes the proof. □

Corollary 2.1 and Propositions 2.4–2.6 yield the following.

Corollary 2.2. *Under the assumptions of Proposition 2.4, it holds that*

$$\Phi(T) + \sup_{0\le t\le T}\|(\tau_t,\ \sqrt{t}\nabla^2 u,\ \sqrt{t}\sqrt{\rho}u_t)\|^2 \le C,$$

equivalently,

$$\sup_{0\le t\le T}\left(\|\rho\|_\infty+\|(\rho,\tau)\|_{H^1\cap W^{1,q}}+\|(\tau_t,\nabla u,\ \sqrt{t}\nabla^2 u,\ \sqrt{t}\sqrt{\rho}u_t)\|^2\right)$$
$$+\int_0^T\left(\|(\sqrt{\rho}u_t,\nabla^2 u,\ \sqrt{t}\nabla u_t)\|^2\right)\mathrm{d}t\le C.$$

Proposition 2.7. *Under the assumptions of Proposition 2.4, it holds that*

$$\sup_{0\le t\le T}\|\rho_t\|+\int_0^T\|\sqrt{t}\nabla^2 u\|_q^2\mathrm{d}t\le C.$$

Proof. By (1.1), it follows from the Hölder and Sobolev inequalities and Corollary 2.2 that

$$\begin{aligned}\|\rho_t\|&=\|u\cdot\nabla\rho+\operatorname{div} u\rho\|\le\|u\|_6\|\nabla\rho\|_3+\|\nabla u\|\|\rho\|_\infty\\&\le C(\|\nabla\rho\|_3+\|\rho\|_\infty)\|\nabla u\|\le C.\end{aligned}$$

Next, applying the elliptic estimates to (1.2) and using Corollary 2.2, one obtains by the Hölder, Sobolev, and Poincaré inequalities that

$$\begin{aligned}\|\nabla^2 u\|_q&\le C\left(\|\rho u_t\|_q+\|\rho(u\cdot\nabla)u\|_q+\|\nabla P\|_q+\|\operatorname{div}\tau\|_q\right)\\&\le C\left(\|\rho\|_\infty\|u_t\|_6+\|\rho\|_\infty\|u\|_\infty\|\nabla u\|_6+\|P'(\rho)\|_\infty\|\nabla\rho\|_q+\|\nabla\tau\|_q\right)\\&\le C\left(1+\|\nabla u_t\|+\|\nabla^2 u\|^{\frac{3}{2}}\right),\end{aligned}$$

and thus, by Corollary 2.2, one gets

$$\int_0^T\|\sqrt{t}\nabla^2 u\|_q^2\mathrm{d}t\le C\int_0^T\left(1+\|\sqrt{t}\nabla u_t\|^2+\|\sqrt{t}\nabla^2 u\|\|\nabla^2 u\|^2\right)\mathrm{d}t\le C.$$

Together with the estimate for ρ_t, this proves the proposition. □

We conclude this section by showing that the existence time T_0 depends only on $\Omega, a, \lambda, q, \eta_e, \eta_s, \bar{P}$, and the upper bound of Φ_0 but is independent of the quantities $\|\nabla^2 u_0\|$ and $\|g\|$.

Proposition 2.8. *Let* $q\in(3,6)$ *and assume that* (ρ_0,u_0,τ_0) *satisfies*

$$0<\underline{\rho}\le\rho_0,\quad(\rho_0,\tau_0)\in H^1\cap W^{1,q},\quad u_0\in H_0^1\cap H^2,\quad\tau_0=\tau_0^{\top}\quad a.e.\ in\ \Omega,$$

for some positive number $\underline{\rho}$.

Then, there exist two positive constants T_0 and C depending only on $\Omega, a, \lambda, q, \eta_e, \eta_s, \bar{P}$, and the upper bound of Φ_0 such that systems (1.1)–(1.3), subject to (1.6) and (1.7), admits a unique solution (ρ,u,τ) in $\Omega\times(0,T_0)$, satisfying

$$\sup_{0\le t\le T_0}\left(\|\rho\|_\infty+\|(\rho,\tau)\|_{H^1\cap W^{1,q}}+\|(\rho_t,\tau_t,\nabla u,\ \sqrt{t}\nabla^2 u,\ \sqrt{t}\sqrt{\rho}u_t)\|^2\right)\le C,$$

and

$$\int_0^{T_0}\left(\|(\sqrt{\rho}u_t,\nabla^2 u,\ \sqrt{t}\nabla u_t)\|^2+\|\sqrt{t}\nabla^2 u\|_q^2\right)\mathrm{d}t\le C.$$

Proof. By Proposition 2.1, there is a unique local strong solution ρ, u, τ on $\Omega \times (0, T^*)$ satisfying the regularities stated in Proposition 2.1. By applying Proposition 2.1 inductively, one can extend the local solution uniquely to the maximal time of existence T_{max}. We claim that, if $T_{max} < \infty$, then the following holds:

$$\limsup_{t \to T_{\max}} \left(\left\| \frac{1}{\rho} \right\|_\infty + \|(\rho, \tau)\|_{W^{1,q}} + \|u\|_{H^2} \right) = \infty. \tag{2.15}$$

Indeed, suppose otherwise. There exists a constant $M > 0$ such that

$$\sup_{0 \le t < T_{\max}} \left(\left\| \frac{1}{\rho} \right\|_\infty + \|(\rho, \tau)\|_{W^{1,q}} + \|u\|_{H^2} \right) \le M.$$

For each $t_0 \in (0, T_{\max})$, define

$$g_{t_0} := \frac{\eta_s(\Delta u(t_0) + \nabla \operatorname{div} u(t_0)) + \operatorname{div} \tau(t_0) - \nabla P(\rho(t_0))}{\sqrt{\rho(t_0)}}.$$

Since $3 < q < 6$, the Sobolev embedding $W^{1,q} \hookrightarrow L^\infty$, the positive lower bound of $\rho(t_0)$, and the above uniform bounds imply $\|g_{t_0}\| \le C_M$, where C_M is independent of t_0. Hence the data $(\rho, u, \tau)(t_0)$ satisfy all the assumptions and the compatibility condition of Proposition 2.1, with norms bounded independently of t_0. By Proposition 2.1, there exists a number $\delta = \delta(M) > 0$, independent of t_0, such that the solution starting from $t = t_0$ exists on $[t_0, t_0 + \delta]$. Choosing $t_0 > T_{\max} - \delta/2$ extends the solution beyond $T_{\max}$, which contradicts the maximality of $T_{\max}$. This proves (2.15).

For any $T \in (0, T_{\max})$, (ρ, u, τ) satisfies the regularities in Proposition 2.1 with T_* there replaced by T. Let ϵ_0 be the constant stated in Corollary 2.1 and Φ the function given by (1.8), and set

$$T_0 = \sup \left\{ T \in (0, T_{\max}) \;\middle|\; T^{\frac{6-q}{4q}} \Phi^2(T) \le \epsilon_0 \right\}.$$

Claim: $T_0 < T_{\max}$. Assume by contradiction that $T_0 = T_{\max}$. Then, by definition,

$$T^{\frac{6-q}{4q}} \Phi^2(T) \le \epsilon_0, \quad \forall T \in (0, T_{\max}). \tag{2.16}$$

Since $\Phi(T) \ge 1$, it follows from (2.16) that $T_{\max} \le \epsilon_0^{\frac{4q}{6-q}}$. Thanks to (2.16), it follows from Corollary 2.2 that

$$\Phi(T) + \sup_{0 \le t \le T} \|\sqrt{t} \nabla^2 u\|^2 \le C, \quad \forall T \in (0, T_{\max}),$$

for a positive constant C independent of $T \in (0, T_{\max})$. Repeating the characteristic argument from Proposition 2.3, one deduces by Proposition 2.2 and (2.16) that

$$\inf_{\Omega \times (0,T)} \rho \ge \underline{\rho} e^{-\int_0^T \|\operatorname{div} u\|_\infty \mathrm{d}t} \ge \underline{\rho} e^{-C\epsilon_0}, \quad \forall T \in (0, T_{\max}),$$

for a positive constant C independent of $T \in (0, T_{\max})$. Combining the above two estimates yields

$$\limsup_{t \to T_{\max}} \left(\left\| \frac{1}{\rho} \right\|_\infty + \|(\rho, \tau)\|_{W^{1,q}} + \|u\|_{H^2} \right) \le C,$$

which contradicts (2.15). This contradiction proves the claim.

Since $T_0 < T_{\max}$ and $\Phi(T)$ is continuous on $[0, T_{\max})$, one gets by the definition of T_0 that

$$T_0^{\frac{6-q}{4q}} \Phi^2(T_0) = \epsilon_0. \tag{2.17}$$

Thanks to this and recalling that $\Phi(T) \geq 1$, it follows from Corollary 2.2 that $T_0 \leq \epsilon_0^{\frac{4q}{6-q}}$ and $\Phi(T_0) \leq C_0$ for a positive constant C_0 depending only on $\Omega, a, \lambda, q, \eta_e, \eta_s, \bar{P}$, and the upper bound of Φ_0. As a consequence, it follows from (2.17) that $T_0 \geq \left(\frac{\epsilon_0}{C_0^2}\right)^{\frac{4q}{6-q}}$. The corresponding estimates follow from Corollary 2.2 and Proposition 2.7. This completes the proof. □

3. Proof of Theorem 1.1

We now prove Theorem 1.1. The proof is divided into two parts: the construction of a strong solution by approximation and the uniqueness of the resulting solution.

3.1. Existence

We divide the existence proof into three steps.

Step 1. Construction of the initial data. Choose $\{u_{0n}\}_{n=1}^{\infty} \subseteq H_0^1 \cap H^2$ such that $u_{0n} \to u_0$ in H^1 as $n \to \infty$. Set $\rho_{0n} = \rho_0 + \frac{1}{n^2}$ and $\tau_{0n} = \tau_0$. Then, it is clear that

$$\|\rho_{0n}\|_{L^\infty} + \|(\rho_{0n}, \tau_{0n})\|_{H^1 \cap W^{1,q}} + \|\nabla u_{0n}\|_{L^2}^2 \leq \Phi_0 + \frac{1}{2}, \tag{3.1}$$

for all sufficiently large n. Since $\rho_{0n} > 0$, $\rho_{0n}, \tau_{0n} \in H^1 \cap W^{1,q}$, and $u_{0n} \in H_0^1 \cap H^2$, it follows directly that the approximate initial data $(\rho_{0n}, u_{0n}, \tau_{0n})$ satisfy the following compatibility condition:

$$g_n := \frac{\eta_s(\Delta u_{0n} + \nabla \operatorname{div} u_{0n}) + \operatorname{div} \tau_{0n} - \nabla P(\rho_{0n})}{\sqrt{\rho_{0n}}} \in L^2.$$

Although $\|u_{0n}\|_{H^2}$ and $\|g_n\|_2$ need not be bounded uniformly in n, Proposition 2.8 provides a common existence time T_0 and uniform estimates independent of n, $\|u_{0n}\|_{H^2}$, and $\|g_n\|_2$.

Step 2. Approximate solutions and convergence. Thanks to (3.1) and Proposition 2.8, there are two positive constants T_0 and C independent of n such that systems (1.1)–(1.3) admits a unique solution (ρ_n, u_n, τ_n) in $\Omega \times (0, T_0)$, and the following a priori estimates hold:

$$\left.\begin{aligned}
&\sup_{0 \leq t \leq T_0} \left(\|\rho_n\|_\infty + \|(\rho_n, \tau_n)\|_{H^1 \cap W^{1,q}} + \|(\partial_t \rho_n, \partial_t \tau_n, \nabla u_n)\|^2\right) \leq C, \\
&\sup_{0 \leq t \leq T_0} \|\sqrt{t}\nabla^2 u_n, \sqrt{t}\sqrt{\rho_n}\partial_t u_n\|^2 \leq C, \\
&\int_0^{T_0} \left(\|(\sqrt{\rho_n}\partial_t u_n, \nabla^2 u_n, \sqrt{t}\nabla \partial_t u_n)\|^2 + \|\sqrt{t}\nabla^2 u_n\|_q^2\right) \mathrm{d}t \leq C
\end{aligned}\right\} \tag{3.2}$$

for large n. Then, by the Banach–Alaoglu theorem and a diagonal argument, there exists a subsequence, still denoted by (ρ_n, u_n, τ_n), and (ρ, u, τ) satisfying

$$(\rho, \tau) \in L^\infty(0, T_0; H^1 \cap W^{1,q}), \qquad (\rho_t, \tau_t) \in L^\infty(0, T_0; L^2), \tag{3.3}$$

$$u \in L^\infty(0,T_0;H_0^1) \cap L^2(0,T_0;H^2), \qquad \sqrt{t}\nabla^2 u \in L^\infty(0,T_0;L^2), \tag{3.4}$$
$$\sqrt{t}\nabla u_t \in L^2(0,T_0;L^2), \qquad \sqrt{t}\nabla^2 u \in L^2(0,T_0;L^q), \tag{3.5}$$

such that

$$\rho_n \overset{*}{\rightharpoonup} \rho, \qquad \text{in } L^\infty(0,T_0;H^1 \cap W^{1,q}), \tag{3.6}$$
$$\partial_t\rho_n \overset{*}{\rightharpoonup} \rho_t, \qquad \text{in } L^\infty(0,T_0;L^2), \tag{3.7}$$
$$\tau_n \overset{*}{\rightharpoonup} \tau, \qquad \text{in } L^\infty(0,T_0;H^1 \cap W^{1,q}), \tag{3.8}$$
$$\partial_t\tau_n \overset{*}{\rightharpoonup} \tau_t, \qquad \text{in } L^\infty(0,T_0;L^2), \tag{3.9}$$
$$u_n \overset{*}{\rightharpoonup} u, \qquad \text{in } L^\infty(0,T_0;H_0^1), \tag{3.10}$$
$$u_n \rightharpoonup u, \qquad \text{in } L^2(0,T_0;H^2), \tag{3.11}$$
$$\partial_t u_n \rightharpoonup u_t, \qquad \text{in } L^2(\delta,T_0;H_0^1), \tag{3.12}$$

for any $\delta \in (0,T_0)$. Moreover, since $W^{1,q} \hookrightarrow\hookrightarrow C(\overline{\Omega})$ for $q \in (3,6)$, and $H^2 \hookrightarrow\hookrightarrow H^1 \hookrightarrow\hookrightarrow L^2$, it follows from the Aubin–Lions lemma and (3.6)–(3.12) that

$$\rho_n \to \rho, \qquad \text{in } C([0,T_0];C(\overline{\Omega})), \tag{3.13}$$
$$\tau_n \to \tau, \qquad \text{in } C([0,T_0];C(\overline{\Omega})), \tag{3.14}$$
$$u_n \to u, \qquad \text{in } C([\delta,T_0];L^2(\Omega)) \cap L^2(\delta,T_0;H_0^1(\Omega)). \tag{3.15}$$

We next establish the convergence of the nonlinear terms. From the strong convergence (3.15), we have

$$u_n \to u \quad \text{in } L^2(\delta,T_0;L^6), \qquad \nabla u_n \to \nabla u \quad \text{in } L^2(\delta,T_0;L^2).$$

Thus, recalling (3.12)–(3.13) and (3.15), we obtain

$$\begin{aligned}
&(\rho_n u_n,\ \sqrt{\rho_n}u_n) \to (\rho u,\ \sqrt{\rho}u) && \text{in } C([\delta,T_0];L^2),\\
&(\rho_n\partial_t u_n,\ \sqrt{\rho_n}\partial_t u_n) \rightharpoonup (\rho u_t,\ \sqrt{\rho}u_t), && \text{in } L^2(\delta,T_0;L^2).
\end{aligned}$$

For the convection term, we decompose

$$\begin{aligned}
&\rho_n(u_n\cdot\nabla)u_n - \rho(u\cdot\nabla)u\\
&\quad = (\rho_n-\rho)(u_n\cdot\nabla)u_n + \rho((u_n-u)\cdot\nabla)u_n + \rho(u\cdot\nabla)(u_n-u).
\end{aligned}$$

Using the uniform bounds of u_n in $L^\infty(0,T_0;H_0^1)$ and $L^2(0,T_0;H^2)$, together with the strong convergence $u_n \to u$ in $L^2(\delta,T_0;H_0^1)$, we get

$$\rho_n(u_n\cdot\nabla)u_n \to \rho(u\cdot\nabla)u \quad \text{in } L^1(\delta,T_0;L^1).$$

Moreover, we rewrite

$$u_n\cdot\nabla\tau_n - u\cdot\nabla\tau = (u_n-u)\cdot\nabla\tau_n + u\cdot\nabla(\tau_n-\tau).$$

Since $\nabla\tau_n \overset{*}{\rightharpoonup} \nabla\tau$ in $L^\infty(0,T_0;L^q)$ for $q \in (3,6)$, we have

$$\|(u_n-u)\cdot\nabla\tau_n\|_{L^2(\delta,T_0;L^2)} \le \|u_n-u\|_{L^2(\delta,T_0;L^6)}\|\nabla\tau_n\|_{L^\infty(\delta,T_0;L^3)} \to 0.$$

Furthermore, since

$$\nabla\tau_n \overset{*}{\rightharpoonup} \nabla\tau \quad \text{in } L^\infty(0,T_0;L^q),$$

for any $\varphi \in C_c^\infty(\Omega\times(\delta,T_0))$, one deduces

$$\int_\delta^{T_0}\int_\Omega [u\cdot\nabla(\tau_n-\tau)] : \varphi \mathrm{d}x\,\mathrm{dt} = \sum_{i,j,k=1}^{3}\int_\delta^{T_0}\int_\Omega (\partial_k\tau_{n,ij}-\partial_k\tau_{ij})u_k\varphi_{ij}\mathrm{d}x\,\mathrm{dt} \to 0.$$

Therefore,

$$u_n\cdot\nabla\tau_n \rightharpoonup u\cdot\nabla\tau \quad \text{in } L^2(\delta,T_0;L^2).$$

By the bilinearity of g_a,

$$\begin{aligned}&\|g_a(\tau_n,\nabla u_n)-g_a(\tau,\nabla u)\|_{L^2(\delta,T_0;L^2)}\\&\quad\le C\|\tau_n-\tau\|_{L^\infty(\delta,T_0;L^\infty)}\|\nabla u_n\|_{L^2(\delta,T_0;L^2)}\\&\qquad + C\|\tau\|_{L^\infty(\delta,T_0;L^\infty)}\|\nabla u_n-\nabla u\|_{L^2(\delta,T_0;L^2)} \to 0.\end{aligned}$$

Thanks to (3.14) and (3.15), we conclude

$$g_a(\tau_n,\nabla u_n) \to g_a(\tau,\nabla u) \text{ in } L^2(\delta,T_0;L^2).$$

Since $P \in C^1([0,\infty))$ and $\rho_n \to \rho$ uniformly,

$$P(\rho_n) \to P(\rho) \quad \text{in } C([0,T_0];C(\overline{\Omega})),$$

and

$$\nabla P(\rho_n) = P'(\rho_n)\nabla\rho_n \overset{*}{\rightharpoonup} P'(\rho)\nabla\rho = \nabla P(\rho) \quad \text{in } L^\infty(0,T_0;L^q).$$

Also, owing to (3.8), we have

$$\operatorname{div}\tau_n \overset{*}{\rightharpoonup} \operatorname{div}\tau \quad \text{in } L^\infty(0,T_0;L^q).$$

Moreover, by the weak lower semicontinuity of norms, it follows from (3.2) that

$$\int_\delta^{T_0}\|\sqrt{\rho}u_t\|^2\,\mathrm{dt} \le \liminf_{n\to\infty}\int_\delta^{T_0}\|\sqrt{\rho_n}\partial_t u_n\|^2\,\mathrm{dt} \le C,$$

for any $\delta \in (0,T_0)$ and for a positive constant C independent of δ, which implies

$$\sqrt{\rho}u_t \in L^2(0,T_0;L^2).$$

The regularity $u \in L^1(0,T_0;W^{2,q})$ can be proved in the same way as in Proposition 2.2.

Step 3. Existence. Thanks to the convergences (3.6)–(3.15), and the above convergence of the nonlinear terms, one can pass to the limit as $n\to\infty$ in the approximate equations of (ρ_n,u_n,τ_n) to show that (ρ,u,τ) satisfies Eqs (1.1)–(1.3) in the sense of distributions. Due to the regularities (3.3)–(3.5), one can further show that (ρ,u,τ) satisfies (1.1)–(1.3) a.e. in $\Omega\times(0,T_0)$. Moreover, since $\tau_0=\tau_0^{\mathsf{T}}$, the argument in Remark 1.1 applied to the limit stress equation yields $\tau=\tau^{\mathsf{T}}$ a.e. in $\Omega\times(0,T_0)$. The initial conditions $\rho|_{t=0}=\rho_0$ and $\tau|_{t=0}=\tau_0$ follow from (3.13) and (3.14), because $\rho_n|_{t=0}=\rho_0+\frac{1}{n^2}$ and

$\tau_n|_{t=0} = \tau_0$. We turn to the proof of the continuity of ρu. From the regularities of the limit solution in Step 2 and the embedding $u \in L^2(0, T_0; H^2) \hookrightarrow L^2(0, T_0; L^\infty)$, we have

$$\begin{aligned}\|(\rho u)_t\|_{L^2(0,T_0;L^2)} &\le \|\rho_t u\|_{L^2(0,T_0;L^2)} + \|\rho u_t\|_{L^2(0,T_0;L^2)} \\ &\le \|\rho_t\|_{L^\infty(0,T_0;L^2)}\|u\|_{L^2(0,T_0;L^\infty)} + \|\sqrt{\rho}\|_{L^\infty(0,T_0;L^\infty)}\|\sqrt{\rho}u_t\|_{L^2(0,T_0;L^2)}.\end{aligned}$$

Hence,

$$\rho u \in H^1(0, T_0; L^2) \hookrightarrow C([0, T_0]; L^2).$$

Moreover, the uniform estimates imply

$$\|\partial_t(\rho_n u_n)\|_{L^2(0,T_0;L^2)} \le C.$$

Consequently,

$$\|\rho_n u_n(t) - \rho_{0n} u_{0n}\| \le C\sqrt{t}.$$

For every fixed $t > 0$, $\rho_n u_n(t) \to \rho u(t)$ in L^2, while $\rho_{0n} u_{0n} \to \rho_0 u_0$ in L^2. Hence,

$$\|\rho u(t) - \rho_0 u_0\|_2 \le C\sqrt{t}.$$

Therefore,

$$\rho u \in C([0, T_0]; L^2), \qquad (\rho u)(0) = \rho_0 u_0.$$

It remains to verify the continuity of the weighted velocity $\sqrt{\rho}\, u$. For every $\delta \in (0, T_0)$, the weighted regularity of u_t gives

$$u_t \in L^2(\delta, T_0; H_0^1).$$

Hence,

$$u \in H^1(\delta, T_0; H_0^1) \hookrightarrow C([\delta, T_0]; H_0^1).$$

Since $\rho \in C([0, T_0]; C(\overline{\Omega}))$, we also have

$$\sqrt{\rho} \in C([0, T_0]; L^\infty).$$

It follows that

$$\sqrt{\rho}\, u \in C((0, T_0]; L^2).$$

It suffices to prove the continuity at $t = 0$. For $\varepsilon > 0$, define

$$E_\varepsilon := \{x \in \Omega : \rho_0(x) \ge \varepsilon\}.$$

Since $\rho(t) \to \rho_0$ in L^∞, for all sufficiently small $t > 0$,

$$\rho(t) \ge \frac{\varepsilon}{2} \quad \text{on } E_\varepsilon, \qquad \rho(t) \le 2\varepsilon \quad \text{on } \Omega \setminus E_\varepsilon.$$

On E_ε,

$$\|(\sqrt{\rho}u)(t) - \sqrt{\rho_0}u_0\|_{L^2(E_\varepsilon)} \le \sqrt{\frac{2}{\varepsilon}}\|(\rho u)(t) - \rho_0 u_0\|_{L^2(E_\varepsilon)} + \|\rho_0 u_0\|_{L^2(E_\varepsilon)} \left\|\rho(t)^{-1/2} - \rho_0^{-1/2}\right\|_{L^\infty(E_\varepsilon)}.$$

Since $\left|a^{-1/2} - b^{-1/2}\right| \le C\varepsilon^{-3/2}|a-b|$, $a, b \ge \frac{\varepsilon}{2}$, it follows that

$$\|(\sqrt{\rho}u)(t) - \sqrt{\rho_0}u_0\|_{L^2(E_\varepsilon)} \to 0 \qquad \text{as } t \to 0^+.$$

On $\Omega \setminus E_\varepsilon$, using $u \in L^\infty(0, T_0; H_0^1)$, we obtain

$$\|(\sqrt{\rho}u)(t) - \sqrt{\rho_0}u_0\|_{L^2(\Omega\setminus E_\varepsilon)} \le \sqrt{2\varepsilon}\,\|u(t)\| + \sqrt{\varepsilon}\,\|u_0\| \le C\sqrt{\varepsilon}.$$

Consequently,

$$\limsup_{t\to 0^+} \|(\sqrt{\rho}u)(t) - \sqrt{\rho_0}u_0\|_2 \le C\sqrt{\varepsilon}.$$

Letting $\varepsilon \to 0^+$ gives

$$(\sqrt{\rho}u)(t) \to \sqrt{\rho_0}u_0 \quad \text{in } L^2 \quad \text{as } t \to 0^+.$$

Therefore,

$$\sqrt{\rho}\,u \in C([0, T_0]; L^2).$$

3.2. Uniqueness

We proceed to prove the uniqueness of strong solutions established in Subsection 3.1. We use the following singular Grönwall-type inequality stated in [25].

Lemma 3.1. *Let $T > 0$ and let f, h, and G be nonnegative functions on $[0, T]$, with f and h being absolutely continuous on $[0, T]$. Assume that*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} f(t) \le \alpha(t) f(t) + A\sqrt{G(t)}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} h(t) + G(t) \le \beta(t) h(t) + \gamma(t) f^2(t), \\ f(0) = 0, \end{cases}$$

where A is a positive constant and α, β, and γ are nonnegative functions satisfying $\alpha, \beta, t\gamma \in L^1((0, T))$.

Then the following estimates hold:

$$f(t) \le AB\sqrt{th(0)} \exp\left\{\frac{1}{2}\int_0^t (\beta(s) + A^2B^2 s\gamma(s))\mathrm{d}s\right\},$$
$$h(t) + \int_0^t G(s)\mathrm{d}s \le h(0) \exp\left\{\int_0^t (\beta(s) + A^2B^2 s\gamma(s))\mathrm{d}s\right\},$$

with $B = 1 + e^{\int_0^T \alpha(\xi)\mathrm{d}\xi}$. In particular, if $h(0) = 0$, then $f \equiv h \equiv 0$ on $[0, T]$.

Let (ρ_1, u_1, τ_1) and (ρ_2, u_2, τ_2) be two solutions to systems (1.1)–(1.3), with the same initial data in $\Omega \times (0, T_0)$. Set

$$\delta\rho = \rho_1 - \rho_2, \qquad \delta u = u_1 - u_2, \qquad \delta\tau = \tau_1 - \tau_2.$$

Subtracting the corresponding equations, we obtain

$$\delta\rho_t + (u_1 \cdot \nabla)\delta\rho + (\delta u \cdot \nabla)\rho_2 + \delta\rho \operatorname{div} u_1 + \rho_2 \operatorname{div} \delta u = 0, \tag{3.16}$$

$$\rho_1(\delta u_t + (u_1 \cdot \nabla)\delta u) - \eta_s(\Delta \delta u + \nabla \operatorname{div} \delta u) + \nabla(P(\rho_1) - P(\rho_2))$$
$$= -\delta\rho(\partial_t u_2 + (u_2 \cdot \nabla)u_2) - \rho_1(\delta u \cdot \nabla)u_2 + \operatorname{div} \delta\tau, \tag{3.17}$$
$$\lambda(\delta\tau_t + (u_1 \cdot \nabla)\delta\tau + (\delta u \cdot \nabla)\tau_2 + g_a(\tau_1, \nabla u_1) - g_a(\tau_2, \nabla u_2)) + \delta\tau = 2\eta_e D(\delta u). \tag{3.18}$$

Multiplying (3.16) by $\delta\rho$ and integrating by parts, one deduces by the Hölder inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\rho\|^2 &\leq \left|\int_\Omega (u_1 \cdot \nabla\delta\rho + \delta u \cdot \nabla\rho_2 + \delta\rho \operatorname{div} u_1 + \rho_2 \operatorname{div} \delta u)\delta\rho \mathrm{d}x\right| \\ &\leq C(\|\nabla u_1\|_\infty \|\delta\rho\|^2 + \|\delta u\|_6 \|\nabla\rho_2\|_3 \|\delta\rho\| + \|\rho_2\|_\infty \|\nabla\delta u\|\|\delta\rho\|) \\ &\leq C\|\nabla u_1\|_\infty \|\delta\rho\|^2 + C(\|\rho_2\|_\infty + \|\nabla\rho_2\|_3)\|\nabla\delta u\|\|\delta\rho\|,\end{aligned}$$

thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\rho\| \leq C\|\nabla u_1\|_\infty \|\delta\rho\| + C(\|\rho_2\|_\infty + \|\nabla\rho_2\|_3)\|\nabla\delta u\|. \tag{3.19}$$

Similarly, testing (3.16) against $\operatorname{sgn}(\delta\rho)|\delta\rho|^{\frac{1}{2}}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\rho\|_{3/2} \leq C\|\nabla u_1\|_\infty \|\delta\rho\|_{3/2} + C\left(\|\nabla\rho_2\|_2 + \|\rho_2\|_6\right)\|\nabla\delta u\|.$$

Therefore, combining this and (3.19), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\rho\|_{L^{3/2}\cap L^2} \leq C\|\nabla u_1\|_\infty \|\delta\rho\|_{L^{3/2}\cap L^2} + C(\|\rho_2\|_\infty + \|\nabla\rho_2\|_{L^2\cap L^3})\|\nabla\delta u\|. \tag{3.20}$$

Multiplying (3.17) by δu and integrating over Ω, one gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_1}\delta u\|^2 + \eta_s(\|\nabla\delta u\|^2 + \|\operatorname{div}\delta u\|^2) = -\int_\Omega \nabla(P(\rho_1) - P(\rho_2)) \cdot \delta u \mathrm{d}x$$
$$-\int_\Omega \delta\rho(\partial_t u_2 + (u_2 \cdot \nabla)u_2) \cdot \delta u \mathrm{d}x - \int_\Omega \rho_1(\delta u \cdot \nabla)u_2 \cdot \delta u \mathrm{d}x + \int_\Omega \operatorname{div}\delta\tau \cdot \delta u \mathrm{d}x. \tag{3.21}$$

For the pressure term, integration by parts and the local Lipschitz continuity of P on the density range give

$$\left|\int_\Omega \nabla(P(\rho_1) - P(\rho_2)) \cdot \delta u \mathrm{d}x\right| \leq \frac{\eta_s}{12}\|\nabla\delta u\|^2 + C\|\delta\rho\|^2.$$

By the Hölder, Gagliardo–Nirenberg, and Young inequalities, one deduces

$$\begin{aligned}&\left|\int_\Omega \delta\rho(\partial_t u_2 + (u_2 \cdot \nabla)u_2) \cdot \delta u \mathrm{d}x\right| \\ &\leq \|\partial_t u_2\|_6 \|\delta u\|_6 \|\delta\rho\|_{\frac{3}{2}} + \|u_2\|_6 \|\nabla u_2\|_6 \|\delta u\|_6 \|\delta\rho\| \\ &\leq C(\|\nabla\partial_t u_2\| + \|\nabla u_2\|\|\nabla^2 u_2\|)\|\nabla\delta u\|\|\delta\rho\|_{L^{3/2}\cap L^2} \\ &\leq \frac{\eta_s}{12}\|\nabla\delta u\|^2 + C(\|\nabla\partial_t u_2\|^2 + \|\nabla u_2\|^2\|\nabla^2 u_2\|^2)\|\delta\rho\|^2_{L^{3/2}\cap L^2},\end{aligned}$$

and

$$\left|\int_\Omega \rho_1(\delta u\cdot\nabla)u_2\cdot\delta u \mathrm{d}x\right| \le \|\nabla u_2\|_\infty\|\sqrt{\rho_1}\delta u\|^2.$$

Similar to the pressure term, integrating by parts yields

$$\left|\int_\Omega \operatorname{div}\delta\tau\cdot\delta u \mathrm{d}x\right| \le \frac{\eta_s}{12}\|\nabla\delta u\|^2 + C\|\delta\tau\|^2.$$

Substituting these estimates into (3.21), we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho_1}\delta u\|^2 + \frac{3\eta_s}{4}\|\nabla\delta u\|^2 &\le \|\nabla u_2\|_\infty\|\sqrt{\rho_1}\delta u\|^2 + C\|\delta\tau\|^2\\ &+ C(1+\|\nabla\partial_t u_2\|^2 + \|\nabla u_2\|^2\|\nabla^2 u_2\|^2)\|\delta\rho\|^2_{L^{3/2}\cap L^2}.\end{aligned}\tag{3.22}$$

Testing (3.18) against $\delta\tau$ and integrating over Ω, one has

$$\begin{aligned}\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\tau\|^2 + \|\delta\tau\|^2 &= -\lambda\int_\Omega (u_1\cdot\nabla)\delta\tau : \delta\tau \mathrm{d}x - \lambda\int_\Omega(\delta u\cdot\nabla)\tau_2 : \delta\tau\mathrm{d}x\\ -\lambda\int_\Omega (g_a(\tau_1,\nabla u_1) - g_a(\tau_2,\nabla u_2)) &: \delta\tau\mathrm{d}x + 2\eta_e\int_\Omega D(\delta u) : \delta\tau\mathrm{d}x =: \sum_{k=1}^4 I_k.\end{aligned}\tag{3.23}$$

The terms on the right-hand side are estimated by integration by parts and the Hölder, Gagliardo–Nirenberg, and Cauchy inequalities as follows:

$$\begin{aligned}|I_1| &\le \|\nabla u_1\|_\infty\|\delta\tau\|^2,\\ |I_2| &\le \|\delta u\|_6\|\nabla\tau_2\|_3\|\delta\tau\| \le \frac{\eta_s}{12}\|\nabla\delta u\|^2 + C\|\nabla\tau_2\|_3^2\|\delta\tau\|^2,\\ |I_3| &\le C\int_\Omega(|\tau_2||\nabla\delta u| + |\delta\tau||\nabla u_1|)|\delta\tau|\mathrm{d}x\\ &\le C(\|\tau_2\|_\infty\|\nabla\delta u\|\|\delta\tau\| + \|\nabla u_1\|_\infty\|\delta\tau\|^2)\\ &\le \frac{\eta_s}{12}\|\nabla\delta u\|^2 + C(\|\tau_2\|^2_{W^{1,q}} + \|\nabla u_1\|_\infty)\|\delta\tau\|^2,\\ |I_4| &\le \frac{\eta_s}{12}\|\nabla\delta u\|^2 + C\|\delta\tau\|^2.\end{aligned}$$

Substituting the above estimates into (3.23), one obtains

$$\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\tau\|^2 + \|\delta\tau\|^2 \le \frac{\eta_s}{4}\|\nabla\delta u\|^2 + C(1+\|\tau_2\|^2_{W^{1,q}} + \|\nabla u_1\|_\infty)\|\delta\tau\|^2.$$

Therefore, combining this with (3.20) and (3.22) leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\rho_1}\delta u\|^2 + \lambda\|\delta\tau\|^2\right) + \eta_s\|\nabla\delta u\|^2 + 2\|\delta\tau\|^2\\ &\le C(1+\|\tau_2\|^2_{W^{1,q}} + \|(\nabla u_1,\nabla u_2)\|_\infty)\left(\|\sqrt{\rho_1}\delta u\|^2 + \|\delta\tau\|^2\right)\\ &\quad + C(1+\|\nabla\partial_t u_2\|^2 + \|\nabla u_2\|^2\|\nabla^2 u_2\|^2)\|\delta\rho\|^2_{L^{3/2}\cap L^2}.\end{aligned}\tag{3.24}$$

Denote

$$
\begin{aligned}
&f(t) = \|\delta\rho\|_{L^{3/2}\cap L^2}, \qquad h(t) = \|\sqrt{\rho_1}\delta u\|^2 + \lambda\|\delta\tau\|^2, \qquad G(t) = \eta_s\|\nabla\delta u\|^2 + 2\|\delta\tau\|^2,\\
&\alpha(t) = C\|\nabla u_1\|_\infty, \qquad A = C\sup_{0\le t\le T_0}(\|\rho_2\|_\infty + \|\nabla\rho_2\|_{L^2\cap L^3})(t),\\
&\beta(t) = C(1 + \|\tau_2\|^2_{W^{1,q}} + \|(\nabla u_1, \nabla u_2)\|_\infty), \quad \gamma(t) = C(1 + \|\nabla\partial_t u_2\|^2 + \|\nabla u_2\|^2\|\nabla^2 u_2\|^2).
\end{aligned}
$$

Recalling the regularities of (ρ_1, u_1, τ_1) and (ρ_2, u_2, τ_2) in Definition 1.1, one obtains

$$
\begin{aligned}
&\int_0^{T_0}\alpha(t)\mathrm{d}t = C\int_0^{T_0}\|\nabla u_1\|_\infty\mathrm{d}t \le C\int_0^{T_0}\|\nabla^2 u_1\|_q\mathrm{d}t \le C,\\
&\int_0^{T_0}\beta(t)\mathrm{d}t = C\int_0^{T_0}\left(1 + \|\tau_2\|^2_{W^{1,q}} + \|(\nabla u_1, \nabla u_2)\|_\infty\right)\mathrm{d}t \le C,\\
&\int_0^{T_0} t\gamma(t)\mathrm{d}t = \int_0^{T_0} C\left(t + \|\sqrt{t}\nabla\partial_t u_2\|^2 + t\|\nabla u_2\|^2\|\nabla^2 u_2\|^2\right)\mathrm{d}t \le C,
\end{aligned}
$$

which implies that $\alpha, \beta, t\gamma \in L^1((0,T))$. Moreover, it remains to verify that $f(t)$ is absolutely continuous on $[0,T]$. By Definition 1.1, we get that $\delta\rho_t \in L^\infty(0,T;L^2)$, and therefore $\delta\rho \in W^{1,\infty}(0,T;L^2)$. Since $L^2 \hookrightarrow L^{3/2}$, thus $\delta\rho \in W^{1,\infty}(0,T;L^{3/2})$. Indeed, for $0 \le s \le t \le T$, the reverse triangle inequality and $\|v\|_{3/2} \le C\|v\|_2$ give

$$
|f(t) - f(s)| \le \|\delta\rho(t) - \delta\rho(s)\|_{3/2} + \|\delta\rho(t) - \delta\rho(s)\|_2 \le C\int_s^t \|\delta\rho_t(r)\|_2\mathrm{d}r,
$$

which implies that f is absolutely continuous on $[0,T]$, and inequality (3.20) holds for almost every $t \in (0,T)$. Since $G \in L^1(0,T)$ and $f(0) = 0$, it follows from (3.20) and the Grönwall inequality that

$$
f(t) \le C\int_0^t \sqrt{G(s)}\mathrm{d}s \le C\sqrt{t}\left(\int_0^t G(s)\mathrm{d}s\right)^{1/2},
$$

which implies

$$
f^2(t) \le Ct.
$$

Together with $t\gamma \in L^1(0,T)$, it follows that $\gamma f^2 \in L^1(0,T)$. Therefore, the energy relation implies that h is absolutely continuous on $[0,T]$. Then, combining (3.20) and (3.24), one deduces by Lemma 3.1 that $\delta\rho = \delta u = \delta\tau = 0$ on $[0,T]$. Since $T \in (0, T_0]$ is arbitrary, the uniqueness follows on $[0, T_0]$.

4. Conclusions

We have proved the local existence and uniqueness of strong solutions to the three-dimensional compressible Oldroyd–B initial-boundary-value problem in a bounded connected C^2 domain, allowing vacuum and assuming no initial compatibility condition. The main ingredient is a family of a priori estimates whose lifespan and constants are independent of the positive lower bound on the density and the compatibility condition introduced. The time-weighted estimates recover the H^2 regularity of the velocity and the corresponding time-derivative control for every positive time, while the stress equation is handled through coupled $H^1 \cap W^{1,q}$ estimates.

The result is local in time and is restricted to three dimensions, the no-slip boundary condition, $q \in (3,6)$, $(\rho_0, \tau_0) \in H^1 \cap W^{1,q}$, and $u_0 \in H^1_0$. In particular, we have not proved global existence or a Cauchy problem. These directions are left for future work.

Use of Generative-AI tools declaration

Generative AI tools were used solely to assist with language editing and presentation. The author independently verified all mathematical arguments, statements, and references and assumes full responsibility for the content of the article.

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Conflict of interest

The author declares no conflict of interest.

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